ADDING A LOT OF COHEN REALS BY ADDING A FEW II

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ABSTRACT. We study pairs (V, V_1), $V \subseteq V_1$, of models of ZFC such that adding $\kappa-$many Cohen reals over V_1 adds $\lambda-$many Cohen reals over V for some $\lambda > \kappa$.

1. Introduction

We continue our study from [3]. We study pairs (V, V_1), $V \subseteq V_1$, of models of ZFC with the same ordinals, such that adding $\kappa-$many Cohen reals over V_1 adds $\lambda-$many Cohen reals over V for some $\lambda > \kappa$. We are mainly interested when V and V_1 have the same cardinals and reals. We prove that for such models, adding $\kappa-$many Cohen reals over V_1 can not produce more Cohen reals over V for κ below the first fixed point of the $\aleph-$function, but the situation at the first fixed point of the $\aleph-$function is different. We also reduce the large cardinal assumptions from [1, 3] to the optimal ones.

2. Adding many Cohen reals by adding a few: a general result

In this section we prove the following general result.

Theorem 2.1. *Suppose $\kappa < \lambda$ are infinite cardinals, and let V_1 be an extension of V. Suppose that in V_1 :*

(*a*) *$\kappa < \lambda$ are still infinite cardinals,*

(*b*) *there exists an increasing sequence $\langle \kappa_n : n < \omega \rangle$ cofinal in κ. In particular $cf(\kappa) = \omega$,*

(*c*) *there is an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in the product $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$,*

(*d*) *there is a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ into sets of size λ such that for every countable set $I \in V$ and every $\sigma < \kappa$ we have $|I \cap S_\sigma| < \aleph_0$.*

Then adding $\kappa-$many Cohen reals over V_1 produces $\lambda-$many Cohen reals over V.

Remark 2.2. *Condition (c) holds automotically for $\lambda = \kappa^+$; given any collection $\mathcal{F}$ of κ-many elements of $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$, there exists f such that for each $g \in \mathcal{F}, f(n) > g(n)$ for* *all large n. Thus we can define by induction on $\alpha < \kappa^+$, an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$.*

Proof. Force to add $\kappa-$many Cohen reals over V_1. Split them into $\langle r_{i,\sigma} : i, \sigma < \kappa \rangle$ and $\langle r'_\sigma : \sigma < \kappa \rangle$. Also in V, split κ into $\kappa-$blocks $B_\sigma, \sigma < \kappa$, each of size κ, and let $\langle f_\alpha : \alpha < \lambda \rangle \in V_1$ be an increasing (mod finite) sequence in $\prod\limits_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$. Let $\alpha < \lambda$. We define a real s_α as follows. Pick $\sigma < \kappa$ such that $\alpha \in S_\sigma$. Let $k_\alpha = min\{k < \omega : r'_\sigma(k)\} = 1$ and set

$$\forall n < \omega,\ s_\alpha(n) = r_{f_\alpha(n+k_\alpha),\sigma}(0).$$

The following lemma completes the proof.

Lemma 2.3. *$\langle s_\alpha : \alpha < \lambda \rangle$ is a sequence of $\lambda-$many Cohen reals over V.*

Notation 2.4. (a) *For a forcing notion $\mathbb{P}$ and $p, q \in \mathbb{P}$, we let $p \leq q$ to mean p is stronger than q.*

(b) *For each set I, let $\mathbb{C}(I)$ be the Cohen forcing notion for adding $I-$many Cohen reals. Thus $\mathbb{C}(I) = \{p : p$ is a finite partial function from $I \times \omega$ into $2\}$, ordered by reverse inclusion.*

Proof. First note that $\langle \langle r_{i,\sigma} : i, \sigma < \kappa \rangle, \langle r'_\sigma : \sigma < \kappa \rangle \rangle$ is $\mathbb{C}(\kappa \times \kappa) \times \mathbb{C}(\kappa)-$generic over V_1. By the c.c.c of $\mathbb{C}(\lambda)$ it suffices to show that for any countable set $I \subseteq \lambda$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I \rangle$ is $\mathbb{C}(I)-$generic over V. Thus it suffices to prove the following

(*) For every $(p, q) \in \mathbb{C}(\kappa \times \kappa) \times \mathbb{C}(\kappa)$ and every open dense subset $D \in V$ of $\mathbb{C}(I)$, there is $(\bar{p}, \bar{q}) \leq (p, q)$ such that $(\bar{p}, \bar{q}) \Vdash \ulcorner \langle \underset{\sim}{s}_\alpha : \alpha \in I \rangle$ extends some element of $D \urcorner$.

Let p and D be as above and for simplicity suppose that $p = q = \emptyset$. Let $b \in D$, and let $\alpha_1, ..., \alpha_m$ be an enumeration of the components of b, i.e. those α such that $(\alpha, n) \in domb$ for some n. Also let $\sigma_1, ..., \sigma_m < \kappa$ be such that $\alpha_i \in S_{\sigma_i}, i = 1, ..., m$. By (d) each $I \cap S_{\sigma_i}$ is finite, thus by (c) we can find $n^* < \omega$ such that for all $n \geq n^*, 1 \leq i \leq m$ and $\alpha^*_1 < \alpha^*_2$ in $I \cap S_{\sigma_i}$ we have $f_{\alpha^*_1}(n) < f_{\alpha^*_2}(n)$. Let

$$\bar{q} = \{\langle \sigma_i, n, 0 \rangle : 1 \leq i \leq m, n < n^*\}.$$

Then $\bar{q} \in \mathbb{C}(\kappa)$ and $(\emptyset, \bar{q}) \Vdash \ulcorner k_{\alpha_i} \geq n^* \urcorner$ for all $1 \leq i \leq m$. Let

$$\bar{p} = \{\langle f_{\alpha_i}(n + k_{\alpha_i}), \sigma_i, 0, b(\alpha_i, n)\rangle : 1 \leq i \leq m, (\alpha_i, n) \in domb\}.$$

Then $\bar{p} \in \mathbb{C}(\kappa \times \kappa)$ is well-defined and for $(\alpha_i, n) \in domb, 1 \leq i \leq m$ we have

$$(\bar{p}, \bar{q}) \Vdash \ulcorner \underset{\sim}{s}_{\alpha_i}(n) = \underset{\sim}{r}_{f_{\alpha_i}(n+k_{\alpha_i}), \sigma_i}(0) = \bar{p}(f_{\alpha_i}(n + k_{\alpha_i}), \sigma_i, 0) = b(\alpha_i, n)\urcorner$$

and hence

$$(\bar{p}, \bar{q}) \Vdash \ulcorner \langle \underset{\sim}{s}_\alpha : \alpha \in I\rangle \text{ extends } b\urcorner.$$

(*) follows and we are done. □

The theorem follows. □

3. Getting results from optimal hypotheses

Theorem 3.1. *Suppose GCH holds and κ is a cardinal of countable cofinality and there are $\kappa-$many measurable cardinals below κ. Then there is a cardinal preserving not adding a real extension V_1 of V in which there is a splitting $\langle S_\sigma : \sigma < \kappa\rangle$ of κ^+ into sets of size κ^+ such that for every countable set $I \in V$ and every $\sigma < \kappa, |I \cap S_\sigma| < \aleph_0$.*

Proof. Let X be a set of measurable cardinals below κ of size κ which is discrete, i.e., contains none of its limit points, and for each $\xi \in X$ fix a normal measure U_ξ on ξ. For each $\xi \in X$ let $\mathbb{P}_\xi$ be the Prikry forcing associated with the measure U_ξ and let $\mathbb{P}_X$ be the Magidor iteration of $\mathbb{P}_\xi$'s, $\xi \in X$ (cf. [2, 5]). Since X is discrete, each condition in $\mathbb{P}_X$ can be seen as $p = \langle\langle s_\xi, A_\xi\rangle : \xi \in X\rangle$ where for $\xi \in X, \langle s_\xi, A_\xi\rangle \in \mathbb{P}_\xi$ and $supp(p) = \{\xi \in X : s_\xi \neq \emptyset\}$ is finite. We may further suppose that for each $\xi \in X$ the Prikry sequence for ξ is contained in $(sup(X \cap \xi), \xi)$. Let G be $\mathbb{P}_X-$generic over V. Note that G is uniquely determined by a sequence $(x_\xi : \xi \in X)$, where each x_ξ is an $\omega-$sequence cofinal in ξ, V and $V[G]$ have the same cardinals, and that GCH holds in $V[G]$.

Work in $V[G]$. We now force $\langle S_\sigma : \sigma < \kappa\rangle$ as follows. The set of conditions $\mathbb{P}$ consists of pairs $p = \langle \tau, \langle s_\sigma : \sigma < \kappa\rangle\rangle \in V[G]$ such that:

(1) $\tau < \kappa^+$,

(2) $\langle s_\sigma : \sigma < \kappa\rangle$ is a splitting of τ,

(3) for every countable set $I \in V$ and every $\sigma < \kappa, |I \cap s_\sigma| < \aleph_0$.

Remark 3.2. *(a) Given a condition $p \in \mathbb{P}$ as above, p decides an initial segment of S_σ, namely $S_\sigma \cap \tau$, to be s_σ. Condition (3) guarantees that each component in this initial segment has finite intersection with countable sets from the ground model.*

(b) Let $t_0 = \bigcup_{\xi \in X} x_\xi$. By genericity arguments, it is easily seen that t_0 is a subset of κ of size κ such that for all countable sets $I \in V, |I \cap t_0| < \aleph_0$. For each $i < \kappa$ set $t_i = t_0 + i = \{\alpha + i : \alpha \in t_0\}$. Then clearly for every countable set $I \in V, |I \cap t_i| < \aleph_0$. Define $s_i, i < \kappa$ by recursion as $s_0 = t_0$ and $s_i = t_i \setminus \bigcup_{j<i} t_j$ for $i > 0$. Then $p = \langle \kappa, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in \mathbb{P}$, and hence $\mathbb{P}$ is non-trivial.

We call τ the height of p and denote it by $ht(p)$. For $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle$ and $q = \langle \nu, \langle t_\sigma : \sigma < \kappa \rangle \rangle$ in $\mathbb{P}$ we define $p \leq q$ iff

(1) $\tau \geq \nu$,

(2) for every $\sigma < \kappa, s_\sigma \cap \nu = t_\sigma$, i.e. each s_σ end extends t_σ.

Lemma 3.3. *(a) $\mathbb{P}$ satisfies the $\kappa^{++} - c.c$,*

(b) $\mathbb{P}$ is $< \kappa-$distributive.

Proof. (a) is trivial, as $|\mathbb{P}| \leq 2^\kappa = \kappa^+$. For (b), fix $\delta < \kappa, \delta$ regular, and let $p \in \mathbb{P}$ and $\underset{\sim}{g} \in V[G]^{\mathbb{P}}$ be such that

$$p \Vdash \ulcorner \underset{\sim}{g} : \delta \to on \urcorner.$$

We find $q \leq p$ which decides $\underset{\sim}{g}$. Fix in V a splitting of κ into $\delta-$many sets of size κ, $\langle Z_i : i < \delta \rangle$ (note that this is possible, as $\delta < \kappa$ are cardinals in V). Let θ be a large enough regular cardinal. Pick an increasing continuous sequence $\langle M_i : i \leq \delta \rangle$ of elementary submodels of $\langle H(\theta), \in \rangle$ of size $< \kappa$ such that:

(1) $\langle M_i : i \leq \delta \rangle \in V[G]$,

(2) $p, \mathbb{P}, \underset{\sim}{g}, \langle Z_i : i < \delta \rangle \in M_0$,

(3) if $i < \delta$ is a limit ordinal, then ${}^i M_{i+1} \subseteq M_{i+1}$,

(4) $cf(M_\delta \cap \kappa^+) = \delta$,

(5) if i is not a limit ordinal, then $cf^V(M_{i+1} \cap \kappa^+) = \xi_i$ for a measurable ξ_i of V in X,

(6) $i < j \Rightarrow \xi_i < \xi_j$,

(7) $\langle M_i \cap V : i \leq \delta \rangle \in V$

For each non-limit $i < \delta, M_{i+1} \cap V$ is in V by clause (7), and so by clause (5), $cf^V(M_{i+1} \cap \kappa^+) = \xi_i$, where $\xi_i \in X$, so we can pick a cofinal in $M_{i+1} \cap \kappa^+$ sequence $\langle \eta^i_\alpha : \alpha < \xi_i \rangle$, where $\eta^i_\alpha > M_i \cap \kappa^+$, for all $\alpha < \xi_i$.

Denote by $\xi_i^{'}$ the first element of the Prikry sequence of ξ_i. We define a descending sequence $p_i = \langle \tau_i, \langle s_{i,\sigma} : \sigma < \kappa \rangle\rangle$ of conditions by induction as follows:

i=0. Set $p_0 = p$,

i=j+1. Assume p_j is constructed such that $p_j \in M_j$ if j is not a limit ordinal, and $p_j \in M_{j+1}$ if j is a limit ordinal and p_j decides $\underset{\sim}{g} \restriction j$. Fix a bijection [1] $f_j : Z_j \to (ht(p_j), \eta^j_{\xi'_j})$ in M_{j+1} and set

$$p'_{j+1} = \langle \eta^j_{\xi'_j}, \langle s_{j,\sigma} \cup \{f_j(\sigma)\} : \sigma \in Z_j\rangle^\frown \langle s_{j,\sigma} : \sigma \in \kappa \setminus Z_j\rangle\rangle$$

Clearly $p'_{j+1} \in M_{j+1}$. Let $p_{j+1} \in M_{j+1}$ be an extension of $p^{'}_{j+1}$ which decides $\underset{\sim}{g}(j)$,

limit(i). Let $p_i = \langle sup_{j<i} ht(p_j), \langle \bigcup_{j<i} s_{j,\sigma} : \sigma < \kappa \rangle\rangle$.

Let us show that the above sequence is well-defined. Thus we need to show that for each $i \le \delta, p_i \in \mathbb{P}$. We prove this by induction on i. The successor case is trivial. Thus fix a limit ordinal $i \le \delta$. If $p_i \notin \mathbb{P}$, we can find a countable set $I \in V$ and $\sigma < \kappa$ such that $I \cap s_{i,\sigma}$ is infinite. Define the sequence $\langle \alpha(j) : j < i \rangle$ as follows:

- if $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$, then $\alpha(j) \in [sup(X \cap \xi_j), \xi_j]$ is the least such that $\eta^j_{\alpha(j)} > sup(I \cap (M_{j+1} \setminus M_j))$,
- $\alpha(j) = sup(X \cap \xi_j)$ otherwise. Note that in this case $\alpha(j) < \xi'_j$ (because the Prikry sequence for ξ was chosen in the interval $(sup(X \cap \xi), \xi)$).

Clearly $\langle \alpha(j) : j < i \rangle \in V$.

Lemma 3.4. *The set $K = \{j < i : \xi^{'}_j \le \alpha(j)\}$ is finite.*

Proof. Suppose not. Let $p \in \mathbb{P}_X, p = \langle\langle s_\xi, A_\xi\rangle : \xi \in X\rangle$, be such that

$$p \Vdash \ulcorner \underset{\sim}{K} \text{ is infinite } \urcorner.$$

Then $p \Vdash \ulcorner \{\xi_j : j \in \underset{\sim}{K}\} \setminus supp(p)$ is infinite $\urcorner$, so by the maximum principle we can pick $j \in X \setminus supp(p)$ such that $p \Vdash \ulcorner \xi_j \in \underset{\sim}{K} \urcorner$. Extend p to $q = \langle\langle t_\xi, B_\xi\rangle : \xi \in X\rangle$ by setting

[1] It is easily seen by induction on $j \le i$ that $ht(p_j) < \eta^j_{\xi'_j}$, using the facts that $\eta^j_{\xi'_j} > M_j \cap \kappa^+$, if j is not a limit ordinal, and that $ht(p_j) = sup_{k<j} ht(p_k)$, if j is a limit ordinal.

- $t_\xi = s_\xi$ and $B_\xi = A_\xi$ for $\xi \neq \xi_j$,
- $t_{\xi_j} = \langle min(A_{\xi_j} \backslash (\alpha(j)+1))\rangle$, and $B_{\xi_j} = A_{\xi_j} \backslash (max(t_{\xi_j})+1)$.

Then $q \leq p$ and $q \Vdash \ulcorner \xi_j^{'} > \alpha(j) \urcorner$ which is a contradiction. $\square$

Take $i_0 < i$ large enough so that no point $\geq i_0$ is in K. Then for all $j \geq i_0$ we have $\xi_j^{'} > \alpha(j)$, hence $\eta^j_{\xi_j^{'}} > sup(I \cap (M_{j+1}))$ [2].

Claim 3.5. *We have*

$$I \cap s_{i,\sigma} \subseteq I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$$

where i_1 is the unique ordinal less than δ so that $\sigma \in Z_{i_1}$.

Proof. Assume toward contradiction that the inclusion fails, and let $t \in I \cap s_{i,\sigma}$ be such that $t \notin I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$. As i is a limit ordinal, $I \cap s_{i,\sigma} = I \cap \bigcup_{j<i} s_{j,\sigma}$. Let $j < i$ be the least such that $t \in s_{j+1,\sigma}$. Then as $t \in I \cap M_{j+1}$ and $j \geq i_0$ we have $t < \eta^j_{\xi_j^{'}}$, so that by our definition of $p^{'}_{j+1}, t$ must be of the form $f_j(\sigma)$, where $\sigma \in Z_j$. But then $j = i_1$ and hence $t = f_{i_1}(\sigma)$. This is a contradiction, and the result follows. $\square$

Thus we must have $I \cap s_{i_0,\sigma}$ is infinite, and this is in contradiction with our inductive assumption.

It then follows that $q = p_\delta \in \mathbb{P}$ and it decides $\underset{\sim}{g}$. $\square$

Let H be $\mathbb{P}-$generic over $V[G]$ and set $V_1 = V[G][H]$. It follows from Lemma 3.3 that all cardinals $\leq \kappa$ and $\geq \kappa^{++}$ are preserved. Also note that κ^+ is preserved, as otherwise it would have cofinality less that κ, which is impossible by $< \kappa-$distributivity of $\mathbb{P}$. Hence V_1 is a cardinal preserving and not adding reals forcing extension of $V[G]$ and hence of V. For $\sigma < \kappa$ set $S_\sigma = \bigcup_{\langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H} s_\sigma$.

Lemma 3.6. *The sequence $\langle S_\sigma : \sigma < \kappa \rangle$ is as required.*

Proof. For each $\tau < \kappa^+$, it is easily seen that the set of all conditions p such that $ht(p) \geq \tau$ is dense, so $\langle S_\sigma : \sigma < \kappa \rangle$ is a partition of κ^+. Now suppose that $I \in V$ is a countable subset

[2]This is trivial if $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$, as then $\eta^j_{\xi_j^{'}} > \eta^j_{\alpha(j)} > sup(I \cap (M_{j+1} \setminus M_j)) = sup(I \cap M_{j+1})$. If $I \cap (M_{j+1} \setminus M_j) = \emptyset$, then $\eta^j_{\xi_j^{'}} > M_j \cap \kappa^+ \geq sup(I \cap M_j)$ and $sup(I \cap M_{j+1}) = sup(I \cap M_j)$, and hence again $\eta^j_{\xi_j^{'}} > sup(I \cap (M_{j+1}))$.

of κ^+. Find $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H$ such that $\tau \supseteq I$. Then for all $\sigma < \kappa, S_\sigma \cap I = s_\sigma \cap I$, and hence $|S_\sigma \cap I| = |s_\sigma \cap I| < \aleph_0$. □

The theorem follows. □

Remark 3.7. *(a) The size of a set I in V can be changed from countable to any fixed $\eta < \kappa$. Given such η, we start with the Magidor iteration of Prikry forcings above η.*[3] *The rest of the conclusions are the same.*

(b) It is possible to add one element Prikry sequence to each $\xi \in X$.[4] *Then V_1 will be a cofinality preserving generic extension of V.*

The next corollary follows from Theorem 3.1 and Remark 2.2.

Corollary 3.8. *Suppose that κ is a cardinal of countable cofinality and there are $\kappa-$many measurable cardinals below κ. Then there is a cardinal preserving not adding a real extension V_1 of V such that adding $\kappa-$many Cohen reals over V_1 produces κ^+-many Cohen reals over V.*

Theorem 3.9. *Assume there is no inner model with a strong cardinal. Suppose $V_1 \subseteq V_2$ have the same cardinals and reals and there is a set $S \in V_2, S \subseteq \kappa$ of size κ which does not contain an infinite subset which is in V_1. Then there is $\delta \leq \kappa$ which is a limit of $\delta-$many measurable cardinals of $\mathcal{K}(V_2)$, where $\mathcal{K}(V_2)$ is the core model of V_2 below the strong cardinal.*

Proof. Note that $\mathcal{K}(V_1) = \mathcal{K}(V_2)$ since the models V_1 and V_2 agree about cardinals. We denote this common core model by $\mathcal{K}$. First suppose that the measurables of $\mathcal{K}$ are bounded below κ. Let δ be a bound. Then by the Covering Theorem (see [6]), every set of ordinals of size δ^+ can be covered by a set in $\mathcal{K}$, and hence in V_1, of the same size. Pick $X \subseteq S$

[3]The reason for starting the iteration above η is to add no subsets of η. This will guarantee that if t_0 is defined as in Remark 3.2(b), then t_0 has finite intersection with sets from V of size η. Using this fact we can show as before that there is a splitting of κ into κ sets, each of them having finite intersection with ground model sets of size η. This makes the second step of the above forcing construction well-behaved.

[4]Conditions in the forcing are of the form $\langle p_\xi : \xi \in X \rangle$, where for each $\xi \in X, p_\xi$ is either of the form A_ξ for some $A_\xi \in U_\xi$, or α_ξ for some $\alpha_\xi < \xi$. We also require that there are only finitely many p_ξ's of the form α_ξ. When extending a condition, we allow either A_ξ to become thinner, or replace it by some ordinal $\alpha_\xi \in A_\xi$.

of size δ^+. Let $X^* \in V_1$ be its cover of size δ^+. Let $f_{X^*} : \delta^+ \leftrightarrow X^*$ be in V_1 and Consider $X' = f_{X^*}^{-1''} X$. Note that $X' \in V_2$ is a subset of δ^+ of size δ^+, and it does not contain an infinite subset from V_1. Now deal with δ^+ and X' instead of κ and S.

So suppose that the measurables of $\mathcal{K}$ are unbounded below κ. If κ is regular in $\mathcal{K}$, then we are done. Assume that $cf^{\mathcal{K}}(\kappa) \leq \eta < \kappa$ and the set of measurables in $\mathcal{K}$ below κ has order type η. Pick in V_2 some $X \subseteq S$ of size η^+. Let $\alpha = sup(X)$. Pick $f_\alpha : \kappa \leftrightarrow \alpha$ in V_1. Set $X^* = f_\alpha^{-1''} X$. Find the least $\delta < \kappa$ with $|X^* \cap \delta| = \eta^+$. Without loss of generality we can assume that δ is a limit cardinal (just use the fact that V_1 and V_2 have the same cardinals). So $cf\delta = \eta^+$. The order type of the measurables in $\mathcal{K}$ below δ is some $\eta_1 < \eta$. Suppose first that $\eta_1 > 0$, i.e., there are measurables below δ in $\mathcal{K}$. Use the Covering Theorem and find $Y \subseteq \delta, Y \in \mathcal{K} \subseteq V_1, |Y| = sup\{\nu < \delta : \nu \text{ is a measurable cardinal }\}$ such that $Y \supseteq X^*$. Denote $|Y|$ by η_1^*. Then η_1^* is a measurable or a singular cardinal of cofinality $cf\eta_1$. Move X^* to a subset of η_1^* by a function $f_Y : Y \leftrightarrow \eta_1^*$ which is in V_1. Now again pick $\delta_1 < \eta_1^*$ to be the least such that $X_1^* \cap \delta_1 = \eta^+$ and repeat the process.

Finally we will get into a situation where there are no measurables below δ (or one of the δ_n's defined above). Then by the Covering Theorem, every countable subset Y of δ in V_2 can be covered by a set Z in $\mathcal{K}$ (and hence in V_1) of cardinality $\aleph_1$. Since V_1 and V_2 have the same cardinals and reals, we must have $Y \in V_1$. But this is a contradiction. □

Theorem 3.10. *Suppose that $V_1 \supseteq V$ are such that:*

(a) *V_1 and V have the same cardinals and reals,*

(b) *$\kappa < \lambda$ are infinite cardinals of V_1,*

(c) *there is no splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ in V_1 as in Theorem 2.1(d).*

Then adding $\kappa-$many Cohen reals over V_1 cannot produce $\lambda-$many Cohen reals over V.

Proof. Suppose not. Let $\langle r_\alpha : \alpha < \lambda \rangle$ be a sequence of $\lambda-$many Cohen reals over V added after forcing with $\mathbb{C}(\kappa)$ over V_1. Let G be $\mathbb{C}(\kappa)-$generic over V_1. For each $p \in \mathbb{C}(\kappa)$ set

$$C_p = \{\alpha < \lambda : p \text{ decides } \underset{\sim}{r}_\alpha(0)\}.$$

Then by genericity $\lambda = \bigcup_{p \in G} C_p$. Fix an enumeration $\langle p_\xi : \xi < \kappa \rangle$ of G, and define a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ in $V_1[G]$ by setting $S_\sigma = C_{p_\sigma} \setminus \bigcup_{\xi < \sigma} C_{p_\xi}$. By (a) and (c) we can find a countable $I \in V$ and $\sigma < \kappa$ such that $I \subseteq S_\sigma$. [5] Suppose for simplicity that $\forall \alpha \in S_\sigma, p_\sigma \Vdash \ulcorner \underset{\sim}{r}_\alpha(0) = 0 \urcorner$. Let $q \in \mathbb{C}(\kappa)$ be such that

$$q \Vdash^{V} \ulcorner I \in V \text{ is countable and } \forall \alpha \in I, \underset{\sim}{r}_\alpha(0) = 0 \urcorner.$$

Pick $\langle 0, \alpha \rangle \in \omega \times I$ such that $\langle 0, \alpha \rangle \notin supp(q)$. Let $\bar{q} = q \cup \{\langle \langle 0, \alpha \rangle, 1 \rangle\}$. Then $\bar{q} \in \mathbb{C}(\kappa), \bar{q} \leq q$ and $\bar{q} \Vdash \ulcorner \underset{\sim}{r}_\alpha(0) = 1 \urcorner$ which is a contradiction. $\square$

The following corollary answers a question from [1].

Corollary 3.11. *The following are equiconsistent:*

(a) *There exists a pair (V_1, V_2) of models of set theory with the same cardinals and reals and a cardinal κ of cofinality ω (in V_2) such that adding $\kappa-$many Cohen reals over V_2 adds more than $\kappa-$many Cohen reals over V_1,*

(b) *there exists a cardinal $\delta \leq \kappa$ which is a limit of $\delta-$many measurable cardinals of some inner model of V_2.*

Proof. Assume (a) holds for some pair (V_1, V_2) of models of set theory which have the same cardinals and reals. Then by Theorem 3.10 there exists a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ^+ in V_2 such that for every countable set $I \in V_1$ and $\sigma < \kappa, I \cap S_\sigma$ is finite. So by Theorem 3.9, we get the consistency of (b).

Conversely if (b) is consistent, then by Corollary 3.8 the consistency of (a) follows. $\square$

4. Below the first fixed point of the $\aleph-$function

Theorem 4.1. *Suppose that $V_1 \supseteq V$ are such that V_1 and V have the same cardinals and reals. Suppose $\aleph_\delta <$ the first fixed point of the $\aleph-$function, $X \subseteq \aleph_\delta, X \in V_1$ and $|X| \geq \delta^+$ (in V_1). Then X has a countable subset which is in V.*

Proof. By induction on $\delta <$ the first fixed point of the $\aleph-$function.

Case 1. $\delta = 0$. Then $X \in V$ by the fact that V_1 and V have the same reals.

Case 2. $\delta = \delta^{'} + 1$. We have $\delta^{'} < \aleph_{\delta^{'}}$, hence $\delta^+ < \aleph_\delta$, thus we may suppose that $|X| \leq \aleph_{\delta^{'}}$. Let $\eta = sup(X) < \aleph_\delta$. Pick $f_\eta : \aleph_{\delta^{'}} \leftrightarrow \eta, f_\eta \in V$. Set $Y = f_\eta^{-1''} X$. Then

[5] In fact, by (c) there exist a countable $I \in V$ and some $\sigma < \kappa$ such that $I \cap S_\sigma$ is infinite. By (a), V and V_1 have the same reals, and hence $I \cap S_\sigma \in V$. So by replacing I with $I \cap S_\sigma$, if necessary, we can assume that $I \subseteq S_\sigma$.

$Y \subseteq \aleph_{\delta'}, \delta' < \aleph_{\delta'}$ and $|Y| \geq \delta^+ = \delta'^+$. Hence by induction there is a countable set $B \in V$ such that $B \subseteq Y$. Let $A = f_\eta'' B$. Then $A \in V$ is a countable subset of X.

Case 3. *limit*(δ). Let $\langle \delta_\xi : \xi < cf\delta \rangle$ be increasing and cofinal in δ. Pick $\xi < cf\delta$ such that $|X \cap \aleph_{\delta_\xi}| \geq \delta^+$. By induction there is a countable set $A \in V$ such that $A \subseteq X \cap \aleph_{\delta_\xi} \subseteq X$. □

The following corollary gives a negative answer to another question from [1].

Corollary 4.2. *Suppose V_1, V and δ are as in Theorem 4.1. Then adding $\aleph_\delta-$many Cohen reals over V_1 cannot produce $\aleph_{\delta+1}-$many Cohen reals over V.*

Proof. Toward contradiction suppose that adding $\aleph_\delta-$many Cohen reals over V_1 produces $\aleph_{\delta+1}-$many Cohen reals over V. Then by Theorem 3.10, there exists $X \subseteq \aleph_{\delta+1}, X \in V_1$ such that $|X| = \aleph_{\delta+1} (\geq \delta^+)$ and X does not contain any countable subset from V, which is in contradiction with Theorem 4.1. □

5. At the first fixed point of the $\aleph-$function

The next theorem shows that Theorem 4.1 does not extend to the first fixed point of the $\aleph-$function.

Theorem 5.1. *Suppose GCH holds and κ is the least singular cardinal of cofinality ω which is a limit of $\kappa-$many measurable cardinals. Then there is a pair $(V[G], V[H])$ of generic extensions of V with $V[G] \subseteq V[H]$ such that:*

(*a*) *$V[G]$ and $V[H]$ have the same cardinals and reals,*

(*b*) *κ is the first fixed point of the $\aleph-$function in $V[G]$ (and hence in V[H]),*

(*c*) *in $V[H]$ there exists a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ into sets of size κ such that for every countable $I \in V[G]$ and $\sigma < \kappa, |I \cap S_\sigma| < \aleph_0$.*

Proof. We first give a simple observation.

Claim 5.2. *Suppose there is $S \subseteq \kappa$ of size κ in $V[H] \supseteq V[G]$ such that for every countable $A \in V[G], |A \cap S| < \aleph_0$. Then there is a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ as in (c).*

Proof. Let $\langle \alpha_i : i < \kappa \rangle$ be an increasing enumeration of S. We may further suppose that $\alpha_0 = 0$, each $\alpha_i, i > 0$ is measurable in V and is not a limit point of S.[6] Note that for all $i < \kappa, sup_{j<i}\alpha_j < \alpha_i \setminus sup_{j<i}\alpha_j$. Now set:

$S_0 = S$,

$S_\sigma = \{\alpha_l + \sigma : i \leq l < \kappa\}$, for $0 < \sigma \in [sup_{j<i}\alpha_j, \alpha_i)$.

Then $\langle S_\sigma : \sigma < \kappa \rangle$ is as required (note that for $\sigma > 0, S_\sigma \subseteq S + \sigma = \{\alpha + \sigma : \alpha \in S\}$, and clearly $S + \sigma$, and hence S_σ, has finite intersection with countable sets from $V[G]$). □

Thus it is enough to find a pair $(V[G], V[H])$ of generic extensions of V satisfying (a) and (b) with $V[G] \subseteq V[H]$ such that in $V[H]$ there is $S \subseteq \kappa$ of size κ composed of inaccessibles, such that for every countable $A \in V, |A \cap S| < \aleph_0$.

Let X be a discrete set of measurable cardinals below κ of size κ, and for each $\xi \in X$ fix a normal measure U_ξ on ξ. For each $\xi \in X$ we define two forcing notions $\mathbb{P}_\xi$ and $\mathbb{Q}_\xi$ as follows.

Remark 5.3. *In the following definitions we let $sup(X \cap \xi) = \omega$ for $\xi = minX$.*

A condition in $\mathbb{P}_\xi$ is of the form $p = \langle s_\xi, A_\xi, f_\xi \rangle$ where

(1) $s_\xi \in [\xi \backslash sup(X \cap \xi)^+]^{<2}$,
(2) if $s_\xi \neq \emptyset$ then $s_\xi(0)$ is an inaccessible cardinal,
(3) $A_\xi \in U_\xi$,
(4) $maxs_\xi < minA_\xi$,
(5) $s_\xi = \emptyset \Rightarrow f_\xi \in Col(sup(X \cap \xi)^+, < \xi)$, where $Col(sup(X \cap \xi)^+, < \xi)$ is the Levy collapse for collapsing all cardinals less than ξ to $sup(X \cap \xi)^+$, and making ξ to become the successor of $sup(X \cap \xi)^+$,
(6) $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f^1_\xi, f^2_\xi \rangle$ where $f^1_\xi \in Col(sup(X \cap \xi)^+, < s_\xi(0))$ and $f^2_\xi \in Col((s_\xi(0))^+, < \xi)$.

For $p, q \in \mathbb{P}_\xi, p = \langle s_\xi, A_\xi, f_\xi \rangle$ and $q = \langle t_\xi, B_\xi, g_\xi \rangle$ we define $p \leq q$ iff

(1) s_ξ end extends t_ξ,

[6] Let $f \in V$ be such that $f : \kappa \to X$ is a bijection, where X is a discrete set of measurable cardinals of V below κ of size κ. Then if $S \subseteq \kappa$ satisfies the claim, so does $f[S]$, hence we can suppose all non-zero elements of S are measurable in V, and are not a limit point of S.

(2) $A_\xi \cup (s_\xi \backslash t_\xi) \subseteq B_\xi$,

(3) $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \leq g_\xi$.

(4) $t_\xi = \emptyset$ and $s_\xi \neq \emptyset \Rightarrow sup(ran(g_\xi)) < s_\xi(0)$ and $f^1_\xi \leq g_\xi$.

(5) $t_\xi \neq \emptyset \Rightarrow f^1_\xi \leq g^1_\xi$ and $f^2_\xi \leq g^2_\xi$ (note that in this case we have $s_\xi = t_\xi$).

We also define $p \leq^* q$ (p is a Prikry or a direct extension of q) iff

(1) $p \leq q$,

(2) $s_\xi = t_\xi$.

The proof of the following lemma is essentially the same as in the proofs in [2, 5].

Lemma 5.4. *(GCH) (a) $\mathbb{P}_\xi$ satisfies the $\xi^+ - c.c.$*

(b) Suppose $p = \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{P}_\xi$ and $l(s_\xi) = 1$ (where $l(s_\xi)$ is the length of s_ξ). Then $\mathbb{P}_\xi / p = \{q \in \mathbb{P}_\xi : q \leq p\}$ satisfies the $\xi - c.c.$

(c) $(\mathbb{P}_\xi, \leq, \leq^*)$ satisfies the Prikry property, i.e given $p \in \mathbb{P}$ and a sentence σ of the forcing language for $(\mathbb{P}, \leq)$, there exists $q \leq^* p$ which decides σ.

(d) Let G_ξ be $\mathbb{P}_\xi -$generic over V and let $\langle s_\xi(0) \rangle$ be the one element sequence added by G_ξ. Then in $V[G_\xi], GCH$ holds, and the only cardinals which are collapsed are the cardinals in the intervals $(sup(X \cap \xi)^{++}, s_\xi(0))$ and $(s_\xi(0)^{++}, \xi)$, which are collapsed to $sup(X \cap \xi)^+$ and $s_\xi(0)^+$ respectively.

We now define the forcing notion $\mathbb{Q}_\xi$. A condition in $\mathbb{Q}_\xi$ is of the form $p = \langle s_\xi, A_\xi, f_\xi \rangle$ where

(1) $s_\xi \in [\xi \backslash sup(X \cap \xi)^+]^{<3}$,

(2) if $s_\xi \neq \emptyset$ then for all $i < l(s_\xi), s_\xi(i)$ is an inaccessible cardinal,

(3) $A_\xi \in U_\xi$,

(4) $max s_\xi < min A_\xi$,

(5) $s_\xi = \emptyset \Rightarrow f_\xi \in Col(sup(X \cap \xi)^+, < \xi)$,

(6) $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f^1_\xi, f^2_\xi \rangle$ where $f^1_\xi \in Col(sup(X \cap \xi)^+, < s_\xi(0))$ and $f^2_\xi \in Col((s_\xi(0))^+, < \xi)$.

For $p, q \in \mathbb{Q}_\xi, p = \langle s_\xi, A_\xi, f_\xi \rangle$ and $q = \langle t_\xi, B_\xi, g_\xi \rangle$ we define $p \leq q$ iff

(1) s_ξ end extends t_ξ,

(2) $A_\xi \cup (s_\xi \backslash t_\xi) \subseteq B_\xi$,

(3) $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \leq g_\xi$.

(4) $t_\xi = \emptyset$ and $s_\xi \neq \emptyset \Rightarrow sup(ran(g_\xi)) < s_\xi(0)$ and $f^1_\xi \leq g_\xi$.

(5) $t_\xi \neq \emptyset$ and $s_\xi = t_\xi \Rightarrow f^1_\xi \leq g^1_\xi$ and $f^2_\xi \leq g^2_\xi$,

(6) $t_\xi \neq \emptyset$ and $s_\xi \neq t_\xi \Rightarrow sup(ran(g^2_\xi)) < s_\xi(1), f^1_\xi \leq g^1_\xi$ and $f^2_\xi \leq g^2_\xi$.

We also define $p \leq^* q$ iff

(1) $p \leq q$,

(2) $s_\xi = t_\xi$.

As above we have the following.

Lemma 5.5. *(GCH) (a) $\mathbb{Q}_\xi$ satisfies the $\xi^+ - c.c.$*

(b) Suppose $p = \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{Q}_\xi, l(s_\xi) = 2$. Then $\mathbb{Q}_\xi / p = \{q \in \mathbb{Q}_\xi : q \leq p\}$ satisfies the $\xi - c.c.$.

(c) $(\mathbb{Q}_\xi, \leq, \leq^*)$ satisfies the Prikry property.

(d) Let H_ξ be $\mathbb{Q}_\xi-$generic over V and let $\langle s_\xi(0), s_\xi(1) \rangle$ be the two element sequence added by H_ξ. Then in $V[H_\xi], GCH$ holds, and the only cardinals which are collapsed are the cardinals in the intervals $(sup(X \cap \xi)^{++}, s_\xi(0))$ and $(s_\xi(0)^{++}, \xi)$, which are collapsed to $sup(X \cap \xi)^+$ and $s_\xi(0)^+$ respectively.

Now let $\mathbb{P}$ be the Magidor iteration of the forcings $\mathbb{P}_\xi, \xi \in X$, and $\mathbb{Q}$ be the Magidor iteration of the forcings $\mathbb{Q}_\xi, \xi \in X$. Since the set X is discrete we can view each condition in $\mathbb{P}$ as a sequence $p = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$ where for each $\xi \in X, \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{P}_\xi$ and $supp(p) = \{\xi : s_\xi \neq \emptyset\}$ is finite. Similarly each condition in $\mathbb{Q}$ can be viewed as a sequence $p = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$ where for each $\xi \in X, \langle s_\xi, A_\xi, f_\xi \rangle \in \mathbb{Q}_\xi$ and $supp(p) = \{\xi : s_\xi \neq \emptyset\}$ is finite (for more information see [2, 4, 5]).

Notation 5.6. *If p is as above, then we write $p(\xi)$ for $\langle s_\xi, A_\xi, f_\xi \rangle$.*

We also define

$$\pi : \mathbb{Q} \to \mathbb{P}$$

by

$$\pi(\langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle) = \langle \langle s_\xi \restriction 1, A_\xi, f_\xi \rangle : \xi \in X \rangle.$$

It is clear that π is well-defined.

Lemma 5.7. *π is a projection i.e.*

(a) $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$,

(b) *π is order preserving,*

(c) *if $p \in \mathbb{Q}, q \in \mathbb{P}$ and $q \leq \pi(p)$ then there is $r \leq p$ in $\mathbb{Q}$ such that $\pi(r) \leq q$.*

Now let H be $\mathbb{Q}-$generic over V and let $G = \pi'' H$ be the filter generated by $\pi'' H$. Then G is $\mathbb{P}-$generic over V.

Lemma 5.8. *(a) if $\langle \tau_\xi : \xi \in X \rangle$ and $\langle \langle \eta^0_\xi, \eta^1_\xi \rangle : \xi \in X \rangle$ are the Prikry sequences added by G and H respectively, then $\tau_\xi = \eta^0_\xi$ for all $\xi \in X$.*

(b) The models $V[G]$ and $V[H]$ satisfy the GCH, have the same cardinals and reals, and furthermore the only cardinals of V below κ which are preserved are $\{\omega, \omega_1\} \cup limX \cup \{\tau_\xi, \tau^+_\xi, \xi, \xi^+ : \xi \in X\}$.

(c) κ is the first fixed point of the $\aleph-$function in $V[G]$ (and hence in $V[H]$).

Proof. (a) and (b) follow easily from Lemmas 5.4 and 5.5 and the definition of the projection π. Let's prove (c). It is clear that κ is a fixed point of the $\aleph-$function in $V[G]$. On the other hand, by (b), the only cardinals of V below κ which are preserved in $V[G]$ are $\{\omega, \omega_1\} \cup limX \cup \{\tau_\xi, \tau^+_\xi, \xi, \xi^+ : \xi \in X\}$, and so if $\lambda < \kappa$ is a limit cardinal in $V[G]$, then $\lambda \in limX$. But by our assumption on κ, if $\lambda \in limX$, then $X \cap \lambda$ has order type less than λ, and hence $(\{\omega, \omega_1\} \cup limX \cup \{\tau_\xi, \tau^+_\xi, \xi, \xi^+ : \xi \in X\}) \cap \lambda$ has order type less than $\aleph_\lambda$. Thus $\lambda < \aleph_\lambda$. $\square$

Let $\mathbb{Q}/G = \{p \in \mathbb{Q} : \pi(p) \in G\}$. Then $V[H]$ can be viewed as a generic extension of $V[G]$ by $\mathbb{Q}/G$.

Lemma 5.9. *$\mathbb{Q}/G$ is cone homogenous: given p and q in $\mathbb{Q}/G$ there exist $p^* \leq p, q^* \leq q$ and an isomorphism $\rho : (\mathbb{Q}/G)/p^* \to (\mathbb{Q}/G)/q^*$.*

Proof. Suppose $p, q \in \mathbb{Q}/G$. Extend p and q to $p^* = \langle \langle s_\xi, A_\xi, f_\xi \rangle : \xi \in X \rangle$ and $q^* = \langle \langle t_\xi, B_\xi, g_\xi \rangle : \xi \in X \rangle$ respectively so that the following conditions are satisfied:

(1) $supp(p^*) = supp(q^*)$. Call this common support K.

(2) For every $\xi \in K, l(s_\xi) = l(t_\xi) = 2$. Note that then for every $\xi \in K, s_\xi(0) = t_\xi(0) = \tau_\xi$, $f_\xi = \langle f^1_\xi, f^2_\xi \rangle$ and $g_\xi = \langle g^1_\xi, g^2_\xi \rangle$ where $f^1_\xi, g^1_\xi \in Col(sup(X \cap \xi)^+, < \tau_\xi)$ and $f^2_\xi, g^2_\xi \in Col((\tau^+_\xi, < \xi)$.

(3) For every $\xi \in K, A_\xi = B_\xi$.

(4) For every $\xi \in K, dom(f^1_\xi) = dom(g^1_\xi)$ and $dom(f^2_\xi) = dom(g^2_\xi)$.

(5) For every $\xi \in K$, there exists an automorphism ρ^1_ξ of $Col(sup(X \cap \xi)^+, < \tau_\xi)$ such that $\rho^1_\xi(f^1_\xi) = g^1_\xi$.

(6) For every $\xi \in K$, there exists an automorphism ρ^2_ξ of $Col(\tau^+_\xi, < \xi)$ such that $\rho^2_\xi(f^2_\xi) = g^2_\xi$.

We now define $\rho : (\mathbb{Q}/G)/p^* \rightarrow (\mathbb{Q}/G)/q^*$ as follows. Suppose $r \in \mathbb{Q}/G, r \leq p^*$. Let $r = \langle \langle r_\xi, C_\xi, h_\xi \rangle : \xi \in X \rangle$. Then for every $\xi \in K, r_\xi = s_\xi$, and $h_\xi = \langle h^1_\xi, h^2_\xi \rangle$ where where $h^1_\xi \in Col(sup(X \cap \xi)^+, < \tau_\xi)$ and $h^2_\xi \in Col((\tau^+_\xi, < \xi)$. Let

$$\rho(r) = \langle \langle t_\xi, C_\xi, \langle \rho^1_\xi(h^1_\xi), \rho^2_\xi(h^2_\xi) \rangle \rangle : \xi \in K \rangle ^\frown \langle \langle r_\xi, C_\xi, h_\xi \rangle : \xi \in X \setminus K \rangle.$$

It is easily seen that ρ is an isomorphism from $(\mathbb{Q}/G)/p^*$ to $(\mathbb{Q}/G)/q^*$.

$\square$

The following lemma completes the proof.

Lemma 5.10. *Let $S = \{\eta^1_\xi : \xi \in X\}$. Then S is a subset of κ of size κ and $|A \cap S| < \aleph_0$ for every countable set $A \in V[G]$.*

Remark 5.11. (*a*) *Since $V[G]$ and $V[H]$ have the same reals, it suffices to prove the lemma for $A \subseteq S$. In fact suppose that the lemma is true for all countable $A \subseteq S$. If the lemma fails, then for some countable set $A \in V[G], |A \cap S| = \aleph_0$. Let $g : \omega \rightarrow A$ be a bijection in $V[G]$. Then $g^{-1}[A \cap S]$ is a subset of ω which is in $V[H]$, and hence in $V[G]$. Thus $A \cap S \in V[G]$. Hence we find a countable subset of S in $V[G]$, namely $A \cap S$, for which the lemma fails, which is in contradiction with our initial assumption.*

(*b*) *In what follows we say A codes ξ (for $\xi \in X$), if $\eta^1_\xi \in A$.*

Proof. Let $\underset{\sim}{S}$ be a $\mathbb{Q}/G-$name for S. Also let $p_0 \in H \cap \mathbb{Q}/G$ be such that $p_o \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner \check{A} \subseteq \underset{\sim}{S}$ is countable$\urcorner$.

Claim 5.12. *For every $p \in \mathbb{Q}/G$ and every $\xi \in X \setminus supp(p)$ there is $q \leq p$ in $\mathbb{Q}/G$ such that $\xi \in supp(q)$ and if $q(\xi) = \langle s_\xi, A_\xi, f_\xi \rangle$, then $l(s_\xi) = 2$ and $q \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A} \urcorner$.*

Proof. Let p and ξ be as in the claim. First pick $\langle\langle\langle t_\xi(0)\rangle, A_\xi, f_\xi\rangle\rangle \in G$, and then let $q = p^\frown \langle\langle s_\xi, A_\xi, f_\xi\rangle\rangle$, where $s_\xi(0) = t_\xi(0) = \tau_\xi$, $s_\xi(1) < \xi$ is large enough so that $s_\xi(1) \notin A$, $sup(ran(f^2_\xi)) < s_\xi(1)$ and $s_\xi(1)$ is inaccessible. Then $\pi(\langle\langle s_\xi, A_\xi, f_\xi\rangle\rangle) = \langle\langle\langle t_\xi(0)\rangle, A_\xi, f_\xi\rangle\rangle \in G$. On the other hand $\pi(p) \in G$. Let $r \in G, r \leq \pi(p), \langle\langle\langle t_\xi(0)\rangle, A_\xi, f_\xi\rangle\rangle$. Then $r \leq \pi(q)$, hence $\pi(q) \in G$. This implies that $q \in \mathbb{Q}/G$. Clearly q satisfies the requirements of the Claim. $\square$

It follows that the set

$$D = \{p \in \mathbb{Q}/G : \forall \xi \in X \setminus supp(p) \text{ there exists } q \leq p \text{ as in the above Claim}\}$$

is dense open in $\mathbb{Q}/G$. Let $p \in H \cap D$. We can assume that $p \leq p_0$. We show that $p \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner$if $\check{A}$ codes ξ then $\xi \in supp(p)\urcorner$. To see this suppose that $\xi \in X \setminus supp(p)$. Thus by Claim 5.12 we can find $q \leq p$ in $\mathbb{Q}/G$ such that $\xi \in supp(q)$ and if $q(\xi) = \langle s_\xi, A_\xi, f_\xi\rangle$, then $l(s_\xi) = 2$ and $q \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$. It then follows that $\sim p \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \in \check{A}\urcorner$. But then by the cone homogeneity of $\mathbb{Q}/G$ we have $p \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$ [7]. Hence $p \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner \check{A}$ does not code $\xi\urcorner$. This means that $p \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner \check{A} \subseteq \{s_\xi(1) : \xi \in supp(p)\} = \{\eta^1_\xi : \xi \in supp(p)\}\urcorner$. Lemma 5.10 follows by noting that $p \in H$ and since the Magidor iteration is used, the support of any condition is finite. $\square$

Theorem 5.1 follows. $\square$

The following theorem can be proved by combining the methods of the proofs of Theorems 3.1 and 5.1.

Theorem 5.13. *Suppose GCH holds and κ is the least singular cardinal of cofinality ω which is a limit of $\kappa-$many measurable cardinals. Also let $V[G]$ and $V[H]$ be the models constructed in the proof of Theorem 5.1. Then there is a cardinal preserving, not adding a real generic extension $V[H][K]$ of $V[H]$ such that in $V[H][K]$ there exists a splitting*

[7] If not, then for some $p' \leq p, p' \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \in \check{A}\urcorner$. By cone homogeneity of $\mathbb{Q}/G$ we can find $q^* \leq q, p^* \leq p'$ and an isomorphism $\rho : (\mathbb{Q}/G)/p^* \to (\mathbb{Q}/G)/q^*$. But then by standard forcing arguments and the fact that $q^* \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$, we can conclude that $p^* \|\!-^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$, which is impossible, as $p^* \leq p'$ and $p' \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \in \check{A}\urcorner$.

$\langle S_\sigma : \sigma < \kappa \rangle$ of κ^+ into sets of size κ^+ such that for every countable set $I \in V[G]$ and $\sigma < \kappa, |I \cap S_\sigma| < \aleph_0$.

Proof. Work over $V[H]$ and force the splitting $\langle S_\sigma : \sigma < \kappa \rangle$ as in the proof of Theorem 3.1, with $V, V[G]$ used there are replaced by $V[G], V[H]$ here respectively. The role of the sequence $\bigcup_{\xi \in X} x_\xi$ in the proof of Theorem 3.1 is now played by the sequence $S = \{\eta^1_\xi : \xi \in X\}$. □

Corollary 5.14. *Suppose GCH holds and there exists a cardinal κ which is of cofinality ω and is a limit of $\kappa-$many measurable cardinals. Then there is pair (V_1, V_2) of models of ZFC such that:*

(a) V_1 and V_2 have the same cardinals and reals.

(b) κ is the first fixed point of the $\aleph-$function in V_1 (and hence in V_2).

(c) Adding $\kappa-$many Cohen reals over V_2 adds κ^+-many Cohen reals over V_1.

Proof. Let $V_1 = V[G]$ and $V_2 = V[H][K]$, where $V[G], V[H][K]$ are as in Theorem 5.13. The result follows using Remark 2.2. and the above theorem. □

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