ADDING A LOT OF RANDOM REALS BY ADDING A FEW

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Abstract. We study pairs (V, V_1) of models of ZFC such that adding κ-many random reals over V_1 adds λ-many random reals over V, for some $\lambda > \kappa$.

1. Introduction

In [1] and [2], we studied pairs (V, V_1) of models of ZFC such that adding κ-many Cohen reals over V_1 adds λ-many Cohen reals over V, for some $\lambda > \kappa$. In this paper we prove similar results for random forcing, by producing pairs (V, V_1) of models of ZFC such that adding κ-many random reals over V_1 adds λ-many random reals over V, where by κ-random reals over V, we mean a sequence $\langle r_i : i < \kappa \rangle$ which is $\mathbb{R}(\kappa)$-generic over V and $\mathbb{R}(\kappa)$ is the usual forcing notion for adding κ-many random reals (see Section 2). The proofs are more involved than those given in [1] and [2] for Cohen reals. This is because random reals, in contrast to Cohen reals, may depend on ω-many coordinates, instead of finitely many as in the Cohen case. Also the proofs in [1] and [2] were based on the fact that the product of Cohen forcing with itself is essentially the same as Cohen forcing, while this is not true in the case of random forcing.

2. Random real forcing

In this section we briefly review random forcing and refer to [3] for more details. Suppose I is a non-empty set and consider the product measure space $2^{I\times\omega}$ with the standard product measure μ_I on it. Let $\mathbb{B}(I)$ denote the class of Borel subsets of $2^{I\times\omega}$. Note that sets of the form

$$[s] = \{x \in 2^{I\times\omega} : x \restriction \mathrm{dom}(s) = s\},$$

where $s : I \times \omega \to 2$ is a finite partial function form a basis of open sets of $2^{I\times\omega}$.

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For Borel sets $S, T \in \mathbb{B}(I)$ set

$$S \sim T \iff S \bigtriangleup T \text{ is null},$$

where $S \bigtriangleup T$ denotes the symmetric difference of S and T. The relation $\sim$ is easily seen to be an equivalence relation on $\mathbb{B}(I)$. Then $\mathbb{R}(I)$, the forcing for adding $|I|$-many random reals, is defined as

$$\mathbb{R}(I) = \mathbb{B}(I)/\sim .$$

Thus elements of $\mathbb{R}(I)$ are equivalent classes $[S]$ of Borel sets modulo null sets. The order relation is defined by

$$[S] \leq [T] \iff \mu(S \setminus T) = 0.$$

The following fact is standard.

Lemma 2.1. *$\mathbb{R}(I)$ is c.c.c.*

Using the above lemma, we can easily show that $\mathbb{R}(I)$ is in fact a complete Boolean algebra. Let $\underset{\sim}{F}$ be an $\mathbb{R}(I)$-name for a function from $I \times \omega$ to 2 such that for each $i \in I, n \in \omega$ and $k < 2$, $\| \underset{\sim}{F}(i,n) = k \|_{\mathbb{R}(I)} = p_k^{i,n}$, where

$$p_k^{i,n} = [x \in 2^{I \times \omega} : x(i,n) = k].$$

This defines $\mathbb{R}(I)$-names $\underset{\sim}{r}_i \in 2^\omega, i \in I$, such that

$$\| \forall n < \omega,\ \underset{\sim}{r}_i(n) = \underset{\sim}{F}(i,n) \|_{\mathbb{R}(I)} = 1_{\mathbb{R}(I)} = [2^{I \times \omega}].$$

Lemma 2.2. *Assume G is $\mathbb{R}(I)$-generic over V and for each $i \in I$ set $r_i = \underset{\sim}{r}_i[G]$. Then each $r_i \in 2^\omega$ is a new real and for $i \neq j$ in $I, r_i \neq r_j$. Further, $V[G] = V[\langle r_i : i \in I \rangle]$.*

The reals r_i are called random reals. By κ-random reals over V, we mean a sequence $\langle r_i : i < \kappa \rangle$ which is $\mathbb{R}(\kappa)$-generic over V.

Given $b = [T] \in \mathbb{R}(I)$ and $|I|$-random reals $\langle r_i : i \in I \rangle$ over V, we say $\langle r_i : i \in I \rangle$ extends b, if

$$\forall i \in I, \forall n < \omega, \exists x \in T(\mu_I(T \cap [x \restriction \{(i,m) : m < n\}]) > 0 \text{ and } \forall m < n,\ x(i,m) = r_i(m)\ ).$$

This simply says that if i and n are given, then we can extend b to some

$$\bar{b} = [T \cap [x \restriction \{(i,m) : m < n\}]]$$

such that $\bar{b}$ decides $r_i \restriction n$. In fact, $\bar{b} \Vdash$ "$\forall m < n,\ \underset{\sim}{r}_i(m) = x(i,m)$". Note that if $\langle r_i : i < \kappa \rangle$ is a sequence of $|I|$-random reals generated by G, then

$$G = \{[T] \in \mathbb{R}(I) : \langle r_i : i \in I \rangle \text{ extends } [T]\}$$

The next lemma follows from Lemma 2.1.

Lemma 2.3. *The sequence $\langle r_i : i < \kappa \rangle$ is $\mathbb{R}(\kappa)$-generic over V iff for each countable set $I \subseteq \kappa, I \in V$, the sequence $\langle r_i : i \in I \rangle$ is $\mathbb{R}(I)$-generic over V.*

3. The first general fact about adding many random reals

In this section we prove the following theorem, which is an analogue of Theorem 2.1 from [1], and use it to get some consequences.

Theorem 3.1. *Let V_1 be an extension of V. Suppose that in V_1 :*

- (a) *$\kappa < \lambda$ are infinite cardinals,*
- (b) *λ is regular,*
- (c) *there exists an increasing sequence $\langle \kappa_n : n < \omega \rangle$ cofinal in κ. In particular $cf(\kappa) = \omega$,*
- (d) *there exists an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in the product $\prod_{n<\omega} (\kappa_{n+1} \setminus \kappa_n)$,*
- (e) *there exists a club $C \subseteq \lambda$ which avoids points of countable $V-$cofinality.*

Then adding $\kappa-$many random reals over V_1 produces $\lambda-$many random reals over V.

Proof. There are two cases to consider: $(1) : \lambda = \kappa^+$ and $(2) : \lambda > \kappa^+$. We give a proof for the first case, as the second case can be proved similarly, using ideas from [1, Theorem 2.1] (combined with the proof of the first case given below). We assume that $\min(C) = 0$.

Thus assume that $\lambda = \kappa^+$, and force to add $\kappa-$many random reals over V_1. We denote them by $\langle r_{\imath,\tau} : \imath, \tau < \kappa \rangle$. Also let $\langle f_\alpha : \alpha < \kappa^+ \rangle \in V_1$ be an increasing (mod finite) sequence in $\prod_{n<\omega} (\kappa_{n+1} \setminus \kappa_n)$. We define a sequence $\langle s_\alpha : \alpha < \kappa^+ \rangle$ of reals as follows:

Assume $\alpha < \kappa^+$. Let α^* and α^{**} be two successor points of C so that $\alpha^* \leq \alpha < \alpha^{**}$. Let $\langle \alpha_\imath : \imath < \kappa \rangle$ be some fixed enumeration of the interval $[\alpha^*, \alpha^{**})$ with $\alpha_0 = \alpha^*$. Then for some $\imath < \kappa$, $\alpha = \alpha_\imath$. Let $k(\imath) = \min\{k < \omega : r_{\imath,\imath}(k) = 1\}$. Set

$$\forall n < \omega,\ s_\alpha(n) = r_{f_\alpha(k(\imath)+n), f_\alpha(k(\imath)+n)}(0).$$

The following lemma completes the proof.

Lemma 3.2. *$\langle s_\alpha : \alpha < \kappa^+ \rangle$ is a sequence of κ^+-many random reals over V.*

Proof. First note that $\langle r_{\imath,\tau} : \imath, \tau < \kappa \rangle$ is $\mathbb{R}(\kappa \times \kappa)$-generic over V_1. By Lemma 2.3, it suffices to show that for any countable set $I \subseteq \kappa^+$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I \rangle$ is $\mathbb{R}(I)-$generic over V. Thus it suffices to prove the following:

(∗) for every $p \in \mathbb{R}(\kappa \times \kappa)$ and every open dense subset $D \in V$ of $\mathbb{R}(I)$, there is $\bar{p} \leq p$ such that $\bar{p} \Vdash$ "$\langle \underset{\sim}{s}_\alpha : \alpha \in I \rangle$ extends some element of D".

Let p and D be as above. For simplicity suppose that $p = 1_{\mathbb{R}(\kappa \times \kappa)} = [2^{(\kappa \times \kappa) \times \omega}]$. By (e) there are only finitely many $\alpha^* \in C$ such that $I \cap [\alpha^*, \alpha^{**}) \neq \emptyset$, where $\alpha^{**} = \min(C \setminus (\alpha^* + 1))$. For simplicity suppose that there are two $\alpha_1^* < \alpha_2^*$ in C with this property. Let $n^* < \omega$ be such that for all $n \geq n^*$, $f_{\alpha_1^*}(n) < f_{\alpha_2^*}(n)$.

Let $b = [T_b] \in D$, where $T_b \subseteq 2^{I \times \omega}$. For $j \in \{1, 2\}$, let $\{\alpha_{j_l} : l < k_j \leq \omega\}$ be an enumeration of $I \cap [\alpha_j^*, \alpha_j^{**})$. For $j \in \{1, 2\}$ and $l < k_j$ let $\alpha_{jl} = \alpha_{\imath_{jl}}$ where $\imath_{jl} < \kappa$ is the index of α_{jl} in the enumeration of the interval $[\alpha_j^*, \alpha_j^{**})$ considered above.

For every $j_1, j_2 \in \{1, 2\}, l_1 < k_{j_1}, l_2 < k_{j_2}$ and $n_1, n_2 < \omega$ set

$$c(j_1, j_2, l_1, l_2, n_1, n_2) = \parallel \underset{\sim}{s}_{\alpha_{j_1,l_1}}(n_1) \neq \underset{\sim}{s}_{\alpha_{j_2,l_2}}(n_2) \parallel.$$

Claim 3.3. *The set $\Delta = \{(j_1, j_2, l_1, l_2, n_1, n_2) : b \leq c(j_1, j_2, l_1, l_2, n_1, n_2)\}$ is finite. Also $(j_1, j_2, l_1, l_2, n_1, n_2) \in \Delta$ implies $(j_2, j_1, l_2, l_1, n_2, n_1) \in \Delta$.*

Proof. Recall that $b = [T_b]$. By shrinking T_b if necessary, we can assume that T_b is closed. Then $2^{I \times \omega} \setminus T_b$ is open, so there are finite partial functions $t_k : I \times \omega \to 2$ such that $2^{I \times \omega} \setminus T_b = \bigcup_{k<\omega} [t_k]$ and for $k \neq l$, $[t_k] \cap [t_l] = \emptyset$. For each k set $\Omega_k = \{t : \mathrm{dom}(t) = \mathrm{dom}(t_k)$

and $t \neq t_k\}$. Then each Ω_k is finite and $2^{I\times\omega} \setminus [t_k] = \bigcup_{t\in\Omega_k}[t]$. So

$$T_b = \bigcap_{k<\omega}(2^{I\times\omega} \setminus [t_k]) = \bigcap_{k<\omega}(\bigcup_{t\in\Omega_k}[t]).$$

Also, as $\mu_I(T_b) > 0$, we have

$$\mu_I(2^{I\times\omega} \setminus T_b) = \sum_{k<\omega} 2^{-|t_k|} < 1.$$

Note that $\mu_I(T_b) = 1 - \sum_{k<\omega} 2^{-|t_k|} > 0$. Fix an increasing sequence $\langle \eta_k : k < \omega\rangle$ of natural numbers such that

$$(\dagger) \qquad \textstyle\sum_{k<\omega} 2^{-\eta_k} < \dfrac{1-\mu_I(2^\omega \setminus T_b)}{1+\mu_I(2^\omega \setminus T_b)}.$$

Assume on the contrary that the set Δ is infinite. For each $k \in \omega$, choose

$$X_k = \{(j_1^{k,u}, j_2^{k,u}, l_1^{k,u}, l_2^{k,u}, n_1^{k,u}, n_2^{k,u}) : u < \eta_k\} \subseteq \Delta$$

such that for each u,$(\alpha_{j_1^{k,u}, l_1^{k,u}}, n_1^{k,u}), (\alpha_{j_2^{k,u}, l_2^{k,u}}, n_2^{k,u}) \notin \mathrm{dom}(t_k)$. Set

$$Y_k = \mathrm{dom}(t_k) \cup \{(\alpha_{j_1^{k,u}, l_1^{k,u}}, n_1^{k,u}), (\alpha_{j_2^{k,u}, l_2^{k,u}}, n_2^{k,u}) : (j_1^{k,u}, j_2^{k,u}, l_1^{k,u}, l_2^{k,u}, n_1^{k,u}, n_2^{k,u}) \in X_k\}.$$

This is possible, as Δ is infinite. We also assume all $(j_1^{k,u}, j_2^{k,u}, l_1^{k,u}, l_2^{k,u}, n_1^{k,u}, n_2^{k,u})$'s, for $k < \omega, u < \eta_k$ are different. For each $t \in \Omega_k$ let

$$\Lambda_{k,t} = \{t' : Y_k \to 2 : t' \supseteq t \text{ and for some } u < \eta_k, t'(\alpha_{j_1^{k,u}, l_1^{k,u}}, n_1^{k,u}) = t'(\alpha_{j_2^{k,u}, l_2^{k,u}}, n_2^{k,u})\}.$$

Set $\bar{T} = \bigcap_{k<\omega}(\bigcup_{t\in\Omega_k}(\bigcup_{t'\in\Lambda_{k,t}}[t']))$. Clearly, $|\Omega_k| = 2^{|t_k|} - 1$, $|\Lambda_{k,t}| = 2^{2\eta_k} - 2^{\eta_k} = 2^{\eta_k}(2^{\eta_k} - 1)$ and for each $t' \in \Lambda_{k,t}, |t'| = |t| + 2\eta_k = |t_k| + 2\eta_k$, and so

$$\mu_I(\bigcup_{t\in\Omega_k}(\bigcup_{t'\in\Lambda_{k,t}}[t'])) = \sum_{t\in\Omega_k}(\sum_{t'\in\Lambda_{k,t}}\mu_I([t'])) = (2^{|t_k|} - 1)2^{\eta_k}(2^{\eta_k} - 1)2^{-(|t_k|+2\eta_k)}.$$

But we have

$$(2^{|t_k|} - 1)2^{\eta_k}(2^{\eta_k} - 1)2^{-(|t_k|+2\eta_k)} = (1 - 2^{-\eta_k})(1 - 2^{-|t_k|}),$$

and so

$$\mu_I(\bigcup_{t\in\Omega_k}(\bigcup_{t'\in\Lambda_{k,t}}[t'])) = (1 - 2^{-\eta_k})(1 - 2^{-|t_k|}).$$

It follows that

$$\begin{aligned} \mu_I(2^\omega \setminus \bar{T}) &\leq \textstyle\sum_{k<\omega}(1 - (1 - 2^{-\eta_k})(1 - 2^{-|t_k|})) \\ &= \textstyle\sum_{k<\omega}(2^{-|t_k|} + 2^{-\eta_k} - 2^{-|t_k|-\eta_k}) \\ &= \textstyle\sum_{k<\omega} 2^{-|t_k|} + \sum_{k<\omega} 2^{-\eta_k} + \sum_{k<\omega} 2^{-|t_k|-\eta_k} \end{aligned}$$

$$\leq \sum_{k<\omega} 2^{-|t_k|} + \sum_{k<\omega} 2^{-\eta_k} + (\sum_{k<\omega} 2^{-|t_k|})(\sum_{k<\omega} 2^{-\eta_k})$$
$$\leq \mu_I(2^\omega \setminus T_b) + \sum_{k<\omega} 2^{-\eta_k} + \mu_I(2^\omega \setminus T_b)(\sum_{k<\omega} 2^{-\eta_k})$$
$$< 1 \text{ (by (†))}.$$

Hence

$$\mu_I(\bar{T}) = 1 - \mu_I(2^\omega \setminus \bar{T}) > 0.$$

Set $\bar{b} = [\bar{T}]$, Then $\bar{b} \in \mathbb{R}(I)$ and $\bar{b} \leq b$. Also note that:

$$\forall x \in \bar{T}, \forall k < \omega, \exists u < 2^k,\ x(\alpha_{j_1^{k,u}, l_1^{k,u}}, n_1^{k,u}) = x(\alpha_{j_2^{k,u}, l_2^{k,u}}, n_2^{k,u}).$$

Let S' consists of those $y \in 2^{(\kappa\times\kappa)\times\omega}$ such that for some $k < \omega$, some $u < 2^k$ and some $x \in \bar{T}$

(1) $y(f_{\alpha_{j_1^{k,u} l_1^{k,u}}}(n_1^{k,u}), f_{\alpha_{j_1^{k,u} l_1^{k,u}}}(n_1^{k,u}), n_1^{k,u}) = x(\alpha_{j_1^{k,u} l_1^{k,u}}, n_1^{k,u})$.
(2) $y(f_{\alpha_{j_2^{k,u} l_2^{k,u}}}(n_2^{k,u}), f_{\alpha_{j_2^{k,u} l_2^{k,u}}}(n_2^{k,u}), n_2^{k,u}) = x(\alpha_{j_2^{k,u} l_2^{k,u}}, n_2^{k,u})$.
(3) $x(\alpha_{j_1^{k,u} l_1^{k,u}}, n_1^{k,u}) = x(\alpha_{j_2^{k,u} l_2^{k,u}}, n_2^{k,u})$.

Clearly, $\mu_{\kappa\times\kappa}(S') > 0$. For each $y \in S'$ let k_y denote the least k as above. Similarly, let u_y denote the least u as above. For some $\bar{k} < \omega$ and $\bar{u} < 2^{\bar{k}}$, the set $S'' = \{y \in S' : k_y = \bar{k}$ and $u_y = \bar{u}\}$ has positive measure. Let

$$\bar{S} = \{y \in S'' : y(\imath_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}, \imath_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}, 0) = y(\imath_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}, \imath_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}, 0) = 1\}.$$

Then $\mu_{\kappa\times\kappa}(\bar{S}) = \frac{1}{4}\mu_{\kappa\times\kappa}(S'') > 0$ and if $\bar{p} = [\bar{S}]$, then $\bar{p} \in \mathbb{R}(\kappa\times\kappa)$ and

$$\bar{p} \Vdash \text{“}\underset{\sim}{k}(\imath_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}) = \underset{\sim}{k}(\imath_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}) = 0\text{”}.$$

For each $y \in \bar{S}$, if x (with $\bar{k}$ and $\bar{u}$) is a witness as above, then

$$\begin{aligned}
\bar{p} \Vdash \text{“}\underset{\sim}{s}_{\alpha_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}}(n_1^{\bar{k},\bar{u}}) &= \underset{\sim}{r}_{f_{\alpha_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}}(n_1^{\bar{k},\bar{u}}), f_{\alpha_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}}(n_1^{\bar{k},\bar{u}})}(0) \\
&= y(f_{\alpha_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}}(n_1^{\bar{k},\bar{u}}), f_{\alpha_{j_1^{\bar{k},\bar{u}} l_1^{\bar{k},\bar{u}}}}(n_1^{\bar{k},\bar{u}}), n_1^{\bar{k},\bar{u}}) \\
&= x(\alpha_{j_1^{\bar{k},\bar{u}}, l_1^{\bar{k},\bar{u}}}, n_1^{\bar{k},\bar{u}}) \text{ (by (1))} \\
&= x(\alpha_{j_2^{\bar{k},\bar{u}}, l_2^{\bar{k},\bar{u}}}, n_2^{\bar{k},\bar{u}}) \text{ (by (3))} \\
&= y(f_{\alpha_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}}(n_2^{\bar{k},\bar{u}}), f_{\alpha_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}}(n_2^{\bar{k},\bar{u}}), n_2^{\bar{k},\bar{u}}) \text{ (by (2))} \\
&= \underset{\sim}{r}_{f_{\alpha_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}}(n_2^{\bar{k},\bar{u}}), f_{\alpha_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}}(n_2^{\bar{k},\bar{u}})}(0) \\
&= \underset{\sim}{s}_{\alpha_{j_2^{\bar{k},\bar{u}} l_2^{\bar{k},\bar{u}}}}(n_2^{\bar{k},\bar{u}})\text{”}.
\end{aligned}$$

So $\bar{b} \nleq \| \underset{\sim}{s}_{\alpha_{j_1^{\bar{k},\bar{u}}, l_1^{\bar{k},\bar{u}}}}(n_1^{\bar{k},\bar{u}}) \neq \underset{\sim}{s}_{\alpha_{j_2^{\bar{k},\bar{u}}, l_2^{\bar{k},\bar{u}}}}(n_2^{\bar{k},\bar{u}}) \|$, and since $\bar{b} \leq b$, we have

$$b \nleq \| \underset{\sim}{s}_{\alpha_{j_1^{\bar{k},\bar{u}}, l_1^{\bar{k},\bar{u}}}}(n_1^{\bar{k},\bar{u}}) \neq \underset{\sim}{s}_{\alpha_{j_2^{\bar{k},\bar{u}}, l_2^{\bar{k},\bar{u}}}}(n_2^{\bar{k},\bar{u}}) \| .$$

It follows that $(j_1^{\bar{k},\bar{u}}, j_2^{\bar{k},\bar{u}}, l_1^{\bar{k},\bar{u}}, l_2^{\bar{k},\bar{u}}, n_1^{\bar{k},\bar{u}}, n_2^{\bar{k},\bar{u}}) \notin \Delta$, which is a contradiction. The second part of the claim is evident and the claim follows. □

Call (j,l) appears in Δ if $(j,l) = (j_1, l_1)$ for some $(j_1, j_2, l_1, l_2, n_1, n_2) \in \Delta$. Also set

$$\Lambda = \{(j,l) : (j,l) \text{ appears in } \Delta\}.$$

Then $|\Lambda| \leq 2|\Delta|$ is finite. Let m^*, with $n^* \leq m^* < \omega$, be such that for all $n \geq m^*$ all of the values

$$f_{\alpha_1^*}(n),\ f_{\alpha_{j_1,l_1}}(n),\ f_{\alpha_{j_2,l_2}}(n),\ f_{\alpha_2^*}(n),$$

where $(j_1, l_1), (j_2, l_2) \in \Lambda$.

Claim 3.4. *There exists $p_1 \leq p$ such that for all $(j,l) \in \Lambda$*

$$p_1 \Vdash \text{“}k(\imath_{jl}) = \min\{k < \omega : \underset{\sim}{r}_{\imath_{jl}, \imath_{jl}}(k) = 1\} = m^* \text{”}.$$

Proof. Let $S_{p_1} \subseteq 2^{(\kappa \times \kappa) \times \omega}$ be defined by

$$S_{p_1} = \{y \in 2^{(\kappa \times \kappa) \times \omega} : \forall (j,l) \in \Lambda\ [(\forall n < m^*,\ y(\imath_{jl}, \imath_{jl}, n) = 0) \text{ and } y(\imath_{jl}, \imath_{jl}, m^*) = 1]\}.$$

Then $\mu_{\kappa \times \kappa}(S_{p_1}) = 2^{-|\Lambda|(m^*+1)} > 0$, so $p_1 = [S_{p_1}] \in \mathbb{R}(\kappa \times \kappa)$. Further, for all $(j,l) \in \Lambda$ and $n < m^*$, $p_1 \Vdash$ “$\underset{\sim}{r}_{\imath_{jl}, \imath_{jl}}(n) = 0$”, also $p_1 \Vdash$ “$\underset{\sim}{r}_{\imath_{jl}, \imath_{jl}}(m^*) = 1$”, thus for $(j,l) \in \Lambda$,

$$p_1 \Vdash \text{“}k(\imath_{jl}) = \min\{k < \omega : \underset{\sim}{r}_{\imath_{jl}, \imath_{jl}}(k) = 1\} = m^* \text{”},$$

as required □

Before we continue, let us make an assumption on T_b. For each $n < \omega$ let $\Phi_n = \{(\alpha_{\imath_{jl}}, m) : (j,l) \in \Lambda, m < n\} \subseteq I \times \omega$. Then for a countable subset T' of $I \times \omega$, $\{x \restriction \Phi_n : x \in T'\} = 2^{H_n}$, for all $n < \omega$. As $[T_b] = [T_b \cup T']$, so let's assume without lose of generality that $T' \subseteq T_b$.

Set

$$J = \{f_{\alpha_{jl}}(m^* + m) : (j,l) \in \Lambda \text{ and } m < \omega\} \subseteq \kappa.$$

Note that by our choice of m^*, for all m and all $(j_1, l_1), (j_2, l_2) \in \Lambda$, $f_{\alpha_{j_1 l_1}}(m^* + m) \neq f_{\alpha_{j_2 l_2}}(m^* + m)$. Set

$$\bar{S} = \{y \in S_{p_1} : \forall n < \omega, \exists x \in T_b, \forall (j,l) \in \Lambda, \forall m < n$$
$$(\ y(f_{\alpha_{jl}}(m^* + m), f_{\alpha_{jl}}(m^* + m), m) = x(\alpha_{\imath_{jl}}, m)\ )\}.$$

By the above remarks, $\bar{S}$ is well-defined. We also have $\bar{S} = \bigcap_{n<\omega} S_n$, where

$$S_n = \{y \in S_{p_1} : \exists x \in T_b, \forall (j,l) \in \Lambda, \forall m < n(\ y(f_{\alpha_{jl}}(m^*+m), f_{\alpha_{jl}}(m^*+m), m) = x(\alpha_{\imath_{jl}}, m)\ )\}.$$

Let

$$W_n = \{(f_{\alpha_{jl}}(m^* + m), f_{\alpha_{jl}}(m^* + m), m) : (j,l) \in \Lambda, m < n\}$$

and

$$\Delta_n = \{t : W_n \to 2 : \exists x \in T_b, \forall (j,l) \in \Lambda, \forall m < n, (y(f_{\alpha_{jl}}(m^*+m), f_{\alpha_{jl}}(m^*+m), m) = x(\alpha_{\imath_{jl}}, m))\}.$$

By our assumption $T' \subseteq T_b, |\Delta_n| = 2^{|W_n|}$ and hence, $\mu_{\kappa\times\kappa}(\bigcup_{t\in\Delta_n}[t]) = \sum_{t\in\Delta_n} 2^{|t|} = 2^{|W_n|}2^{-|W_n|} = 1$. We have, $S_n = S_{p_1} \cap \bigcup_{t\in\Delta_n}[t]$, so

$$\mu_{\kappa\times\kappa}(S_n) = \mu_{\kappa\times\kappa}(S_{p_1}) + \mu_{\kappa\times\kappa}(\bigcup_{t\in\Delta_n}[t]) - \mu_{\kappa\times\kappa}(S_{p_1} \cup \bigcup_{t\in\Delta_n}[t]) = \mu_{\kappa\times\kappa}(S_{p_1}).$$

It follows that $\mu_{\kappa\times\kappa}(S_{p_1} \setminus S) = \mu_{\kappa\times\kappa}(\bigcup_{n<\omega}(S_{p_1} \setminus S_n) \leq \sum_{n<\omega} \mu_{\kappa\times\kappa}(S_{p_1} \setminus S_n) = 0$, and so $\mu_{\kappa\times\kappa}(S) = \mu_{\kappa\times\kappa}(S_{p_1}) > 0$. Let $\bar{p} = [\bar{S}]$. Then $\bar{p} \in \mathbb{R}(\kappa \times \kappa)$ and $\bar{p} \leq p$.

Claim 3.5. $\bar{p} \Vdash$ "$\langle \underset{\sim}{s}_{\alpha_{jl}} : (j,l) \in \Lambda\rangle$ *extends* b".

Proof. Suppose $(j,l) \in \Lambda$ and $n < \omega$. Let $y \in \bar{S}$. Thus we can find $x \in T_b$ such that

$$\forall m < n\ (\ y(f_{\alpha_{jl}}(m^* + m), f_{\alpha_{jl}}(m^* + m), m) = x(\alpha_{\imath_{jl}}, m)\ ).$$

But then

$$\begin{aligned}\bar{p} \Vdash\ \text{“}\underset{\sim}{s}_\alpha(m) &= \underset{\sim}{s}_{\alpha_{jl}}(m)\\ &= \underset{\sim}{r}_{f_{\alpha_{jl}}(m^*+m), f_{\alpha_{jl}}(m^*+m)}(0)\\ &= y(f_{\alpha_{jl}}(m^* + m), f_{\alpha_{jl}}(m^* + m), 0)\\ &= x(\alpha_{jl}, m)\\ &= x(\alpha, m)\text{”}.\end{aligned}$$

The result follows. □

We now consider those (j,l)'s, $j \in \{1,2\}, l < k_j$, which do not appear in Δ. Fix such a pair (j,l). Also let $n < \omega$. Then there is $(j_1,l_1) \in \Lambda$ and such that for each $m < n$, $b \nleq c(j,j_1,l,l_1,m,m)$, i.e., $b \nVdash$"$\underset{\sim}{s}_{\alpha_{j,l}}(m) \neq \underset{\sim}{s}_{\alpha_{j_1,l_1}}(m)$". So there exists $b_{jln} = [T_{jln}] \leq b$ such that $\forall m < n,\ b_{jln} \Vdash$"$\underset{\sim}{s}_{\alpha_{j,l}}(m) = \underset{\sim}{s}_{\alpha_{j_1,l_1}}(m)$".

Note that $\mu_I(T_{jln} \setminus T_b) = 0$. Since there are only countably many such tuples (j,l,n),

$$\mu_I(\bigcup_{n<\omega,(j,l)\in\Lambda} T_{jln} \setminus T_b) = 0.$$

This implies $[T_b] = [T_b \cup \bigcup_{n<\omega,(j,l)\in\Lambda} T_{jln}]$, so without loss of generality, each $T_{jln} \subseteq T_b$, where $n < \omega$ and $(j,l) \in \Lambda$. Now Claim 3.5 implies the following:

Claim 3.6. *$\bar{p} \Vdash$"$\langle \underset{\sim}{s}_\alpha : \alpha \in I\rangle$ extends b".*

$(*)$ follows, which completes the proof of Lemma 3.2. □

Theorem 3.1 follows. □

The next theorem follows immediately from Theorem 3.1 and the arguments from [1].

Theorem 3.7. (a) *Suppose that V satisfies GCH, $\kappa = \bigcup_{n<\omega} \kappa_n$ and $\bigcup_{n<\omega} o(\kappa_n) = \kappa$ (where $o(\kappa_n)$ is the Mitchell order of κ_n). Then there exists a cardinal preserving generic extension V_1 of V satisfying GCH and having the same reals as V does, so that adding $\kappa-$many random reals over V_1 produces κ^+-many random reals over V.*

(b) *Suppose V is a model of GCH. Then there is a generic extension V_1 of V satisfying GCH so that the only cardinal of V which is collapsed in V_1 is $\aleph_1$ and such that adding $\aleph_\omega-$many random reals to V_1 produces $\aleph_{\omega+1}-$many of them over V.*

(c) *Suppose V satisfies GCH. Then there is a generic extension V_1 of V satisfying GCH and having the same reals as V does, so that the only cardinals of V which are collapsed in V_1 are $\aleph_2$ and $\aleph_3$ and such that adding $\aleph_\omega-$many random reals to V_1 produces $\aleph_{\omega+1}-$many of them over V.*

(d) *Suppose that κ is a strong cardinal, $\lambda \geq \kappa$ is regular and GCH holds. Then there exists a cardinal preserving generic extension V_1 of V having the same reals as V does, so that adding $\kappa-$many random reals over V_1 produces $\lambda-$many of them over V.*

(e) *Suppose that there is a strong cardinal and GCH holds. Let $\alpha < \omega_1$. Then there is a model $V_1 \supset V$ having the same reals as V and satisfying GCH below $\aleph_\omega^{V_1}$ such that adding $\aleph_\omega^{V_1}-$many random reals to V_1 produces $\aleph_{\alpha+1}^{V_1}-$many of them over V.*

We can also use ideas of the proof of Theorem 3.1, to get the following theorem, which is an analogue of [1, Theorem 3.1] for random reals.

Theorem 3.8. *Suppose that V satisfies GCH. Then there is a cofinality preserving generic extension V_1 of V satisfying GCH so that adding a random real over V_1 produces $\aleph_1-$many random reals over V.*

4. The second general fact about adding many random reals

In this section, we prove our second general result which is an analogue of Theorem 2.1 form [2]. Then we use the result to obtain similar results as in [2] for random reals.

Theorem 4.1. *Suppose $\kappa < \lambda$ are infinite (regular or singular) cardinals, and let V_1 be an extension of V. Suppose that in V_1 :*

(a) *$\kappa < \lambda$ are still infinite cardinals.*

(b) *there exists an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals, cofinal in κ. In particular $cf(\kappa) = \omega$.*

(c) *there is an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in the product $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$.*

(d) *there is a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ into sets of size λ such that for every countable set $I \in V$ and every $\sigma < \kappa$ we have $|I \cap S_\sigma| < \aleph_0$.*

Then adding $\kappa-$many random reals over V_1 produces $\lambda-$many random reals over V.

Proof. Force to add $\kappa-$many random reals over V_1. Let us write them as $\langle r_{i,\sigma} : i, \sigma < \kappa \rangle$. Also in V, split κ into $\kappa-$blocks $B_\sigma, \sigma < \kappa$, each of size κ, and let $\langle f_\alpha : \alpha < \lambda \rangle \in V_1$ be an increasing (mod finite) sequence in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$. Let $\alpha < \lambda$. We define a real s_α as follows. Pick $\sigma < \kappa$ such that $\alpha \in S_\sigma$. Let $k_\alpha = min\{k < \omega : r_{\sigma,\sigma}(k)\} = 1$ and set

$$\forall n < \omega,\ s_\alpha(n) = r_{f_\alpha(n+k_\alpha),\sigma}(0).$$

The following lemma completes the proof.

Lemma 4.2. *$\langle s_\alpha : \alpha < \lambda\rangle$ is a sequence of $\lambda-$many random reals over V.*

Proof. First note that $\langle r_{i,\sigma} : i, \sigma < \kappa\rangle$ is $\mathbb{R}(\kappa \times \kappa)$-generic over V_1. By Lemma 2.3, it suffices to show that for any countable set $I \subseteq \lambda$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I\rangle$ is $\mathbb{R}(I)$-generic over V. Thus it suffices to prove the following

(∗) For every $p \in \mathbb{R}(\kappa \times \kappa)$ and every open dense subset $D \in V$ of $\mathbb{R}(I)$, there is $\bar{p} \leq p$ such that $\bar{p} \Vdash \ulcorner\langle \underset{\sim}{s}_\alpha : \alpha \in I\rangle$ extends some element of $D\urcorner$.

Let p and D be as above and for simplicity suppose that $p = 1_{\mathbb{R}(\kappa\times\kappa)} = [2^{\kappa\times\kappa\times\omega}]$. Let $b = [T_b] \in D$, where $T_b \subseteq 2^{I\times\omega}$. As I is countable, we can find $\{\sigma_j : j < \bar{\omega} \leq \omega\} \subseteq \lambda$ such that

$$I = I \cap \bigcup_{\sigma<\lambda} S_\sigma = \bigcup_{j<\bar{\omega}} (I \cap S_{\sigma_j}),$$

and each $I \cap S_{\sigma_j}$ is non-empty. By (d), each $I \cap S_{\sigma_j}$ is finite, say

$$I \cap S_{\sigma_j} = \{\alpha_{j,0}, \dots, \alpha_{j,k_j-1}\}.$$

For every $j_1, j_2 < \bar{\omega}$, $l_1 < k_{j_1}, l_2 < k_{j_2}$ and $n_1, n_2 < \omega$ set

$$c(j_1, j_2, l_1, l_2, n_1, n_2) = \| \underset{\sim}{s}_{\alpha_{j_1,l_1}}(n_1) \neq \underset{\sim}{s}_{\alpha_{j_2,l_2}}(n_2) \| .$$

The following can be proved as in Claim 3.3.

Claim 4.3. *The set $\Delta = \{(j_1, j_2, l_1, l_2, n_1, n_2) : b \leq c(j_1, j_2, l_1, l_2, n_1, n_2)\}$ is finite. Also $(j_1, j_2, l_1, l_2, n_1, n_2) \in \Delta$ implies $(j_2, j_1, l_2, l_1, n_2, n_1) \in \Delta$.*

Let $\Lambda = \{j < \bar{\omega} :$ there exists $(j_1, j_2, l_1, l_2, n_1, n_2) \in \Delta$ with $j = j_1\}$. Then Λ is finite. For each $j \in \Lambda$, by (c), we can find $n^*_j < \omega$ such that for all $n \geq n^*_j$ and $\alpha^*_1 < \alpha^*_2$ in $I \cap S_{\sigma_j}$ we have $f_{\alpha^*_1}(n) < f_{\alpha^*_2}(n)$.

Let

$$S' = [\{x \in 2^{\kappa\times\kappa\times\omega} : \forall j \in \Lambda(\forall n < n^*_j, x(\sigma_j, \sigma_j, n) = 0 \text{ and } x(\sigma_j, \sigma_j, n^*_j) = 1)\}]$$

Then $\mu_{\kappa\times\kappa}(S') = 2^{-|\Lambda|(n^*_j+1)} > 0$, and so $p' = [S'] \in \mathbb{R}(\kappa \times \kappa)$. Also, for each $j \in \Lambda$ and $l < k_j$, $p' \Vdash \ulcorner k_{\alpha_{jl}} = n^*_j\urcorner$. Let

$$\bar{S} = \{y \in S' : \forall n < \omega \exists x \in T_b, \forall j \in \Lambda \forall l < k_j \forall m < n \;(\; y(f_{\alpha_{jl}}(n^*_j + m), \sigma_j, 0) = x(\alpha_{jl}, m) \;)\}.$$

By our choice of n_j^* there are no collisions and the above definition is well-defined. Also, by the same arguments as before, $\mu_{\kappa\times\kappa}(\bar{S}) = \mu_{\kappa\times\kappa}(S') > 0$.

Let $\bar{p} = [\bar{S}]$. Then $\bar{p} \in \mathbb{R}(\kappa \times \kappa)$ is well-defined and for all $\alpha = \alpha_{jl} \in I$, where $j \in \Lambda$ and $l < k_j$, and all $y \in S_{\bar{p}}$ we can find $x \in T_b$ such that for $m < n$

$$\begin{aligned} \bar{p} \Vdash \text{``} \underset{\sim}{s}_\alpha(m) &= \underset{\sim}{s}_{\alpha_{jl}}(m) \\ &= \underset{\sim}{r}_{f_{\alpha_{jl}}(n_j^*+m),\sigma_j}(0) \\ &= y(f_{\alpha_{jl}}(n_j^*+m), \sigma_j, 0) \\ &= x(\alpha_{jl}, m) \\ &= x(\alpha, m)\text{''}. \end{aligned}$$

This implies

$$\bar{p} \Vdash \ulcorner \langle \underset{\sim}{s}_{\alpha_{jl}} : j \in \Lambda, l < k_j \rangle \text{ extends } b \urcorner.$$

Now, as in the proof of Claim 3.6, we have the following:

Claim 4.4. $\bar{p} \Vdash \ulcorner \langle \underset{\sim}{s}_\alpha : \alpha \in I \rangle$ *extends* $b \urcorner$.

$(*)$ follows and we are done. □

The theorem follows. □

The following theorem follows from Theorem 4.1 and the arguments from [2].

Theorem 4.5. (a) *Suppose that GCH holds in V, κ is a cardinal of countable cofinality and there are $\kappa-$many measurable cardinals below κ. Then there is a cardinal preserving not adding a real extension V_1 of V such that adding $\kappa-$many random reals over V_1 produces κ^+-many random reals over V.*

(b) *Suppose that $V_1 \supseteq V$ are such that:*

(1) *V_1 and V have the same cardinals and reals,*

(2) *$\kappa < \lambda$ are infinite cardinals of V_1,*

(3) *there is no splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of λ in V_1 as in Theorem 3.1(d).*

Then adding $\kappa-$many random reals over V_1 cannot produce $\lambda-$many random reals over V.

(c) *The following are equiconsistent:*

(1) *There exists a pair* $(V_1, V_2), V_1 \subseteq V_2$ *of models of set theory with the same cardinals and reals and a cardinal* κ *of cofinality* ω *(in* V_2*) such that adding* $\kappa-$*many random reals over* V_2 *adds more than* $\kappa-$*many random reals over* V_1.

(2) *There exists a cardinal* δ *which is a limit of* $\delta-$*many measurable cardinals.*

(d) *Suppose that* $V_1 \supseteq V$ *are such that* V_1 *and* V *have the same cardinals and reals and* $\aleph_\delta$ *is less than the first fixed point of the* $\aleph-$*function. Then adding* $\aleph_\delta-$*many random reals over* V_1 *cannot produce* $\aleph_{\delta+1}-$*many random reals over* V.

(e) *Suppose* GCH *holds and there exists a cardinal* κ *which is of cofinality* ω *and is a limit of* $\kappa-$*many measurable cardinals. Then there is pair* (V_1, V_2) *of models of* $ZFC, V_1 \subseteq V_2$ *such that:*

(1) V_1 *and* V_2 *have the same cardinals and reals.*

(2) κ *is the first fixed point of the* $\aleph-$*function in* V_1 *(and hence in* V_2*).*

(3) *Adding* $\kappa-$*many random reals over* V_2 *adds* κ^+-*many random reals over* V_1.

References

[1] Gitik, Moti; Golshani, Mohammad, Adding a lot of Cohen reals by adding a few. I. Trans. Amer. Math. Soc. 367 (2015), no. 1, 209-229.

[2] Gitik, Moti; Golshani, Mohammad, Adding a lot of Cohen reals by adding a few. II. Fund. Math. 231 (2015), no. 3, 209-224.

[3] Kunen, Kenneth, Set theory. Studies in Logic (London), 34. College Publications, London, 2011. viii+401 pp. ISBN: 978-1-84890-050-9

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