Blowing up the power of a singular cardinal of uncountable cofinality.

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January 22, 2020

Abstract

A new method for blowing up the power of a singular cardinal is presented. It allows to blow up the power of a singular in the core model cardinal of uncountable cofinality. The method make a use of overlapping extenders.

1 Introduction.

The purpose of this paper is to present a method for blowing up the power of a singular cardinal which differs from those used in [1] and in [2] to deal with cofinality ω. The advantage of the present technique is that it generalizes to singular cardinals of uncountable cofinality, which was open.

The main result can be stated as follows:

Theorem 1.1 *Assume GCH. Let η be a regular cardinal. Suppose that there is an increasing sequence $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ of strong cardinals with $\kappa_0 > \eta$. Let $\lambda > \bigcup_{\alpha<\eta} \kappa_\alpha$ be a regular cardinal. Then there is a cardinal preserving extension in which $\bigcup_{\alpha<\eta} \kappa_\alpha$ is a strong limit cardinal and $2^{(\bigcup_{\alpha<\eta} \kappa_\alpha)} = \lambda$.*

If $\eta > \aleph_0$ and $\lambda > (\bigcup_{\alpha<\eta} \kappa_\alpha)^+$, then, by [4], $o^\P$ should exists.
A slightly weaker assumption than $\eta-$many strongs is actually used.
We assume that there is a sequence $\langle E(\alpha) \mid \alpha < \eta \rangle$ of extenders such that for every $\alpha < \eta$

*The work was partially supported by Israel Science Foundation Grant No. 58/14. We are grateful to Carmi Merimovich for reading a draft of the paper and specially for the inspiration we got from his works on extender based forcings. We like to thank to James Cummings and to Sittinon Jirattikansakul for suggested corrections. We are grateful to the referee of the paper for his long and detailed list of corrections.

1. $E(\alpha)$ is a $(\kappa_\alpha, \lambda)-$extender,
   i.e., $j_{E(\alpha)} : V \to M_{E(\alpha)} \simeq \mathrm{Ult}(V, E(\alpha))$, $crit(j_{E(\alpha)}) = \kappa_\alpha, j_{E(\alpha)}(\kappa_\alpha) > \lambda$,
   $M_{E(\alpha)} \supseteq H_\lambda, {}^{\kappa_\alpha} M_{E(\alpha)} \subseteq M_{E(\alpha)}$;

2. for every $\beta < \alpha$, $E(\beta) \triangleleft E(\alpha)$.
   Note that this condition is equivalent to $\langle E(\beta) \mid \beta < \alpha \rangle \in M_{E(\alpha)}$,
   since ${}^{\kappa_\alpha} M_{E(\alpha)} \subseteq M_{E(\alpha)}$.

Our conjecture is that this assumption is optimal for blowing up the power of singular in the core model cardinal of uncountable cofinality.

We will start with countable cofinality. Then a general case will be considered and finally some generalizations will be stated.

2 Blowing up the power of a singular cardinal of cofinality ω.

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of cardinals, $\kappa_\omega = \bigcup_{n<\omega} \kappa_n$ and $\langle E_n \mid n < \omega \rangle$ be a sequence such that for every $n < \omega$

1. $E(n)$ is a $(\kappa_n, \lambda)-$extender,
   i.e., $j_{E(n)} : V \to M_{E(n)} \simeq \mathrm{Ult}(V, E(n))$, $crit(j_{E(n)}) = \kappa_n, j_{E(n)}(\kappa_n) > \lambda$,
   $M_{E(n)} \supseteq H_\lambda, {}^{\kappa_n} M_{E(n)} \subseteq M_{E(n)}$;

2. $E(n) \triangleleft E(n+1)$.

Denote by $\mathcal{P}(n)$ the one element extender based Prikry forcing with $E(n)$. We would like to combine the $\mathcal{P}(n)$'s together. It would be a kind of Magidor product, but will involve restrictions and reflections. Namely, if for some $n < \omega$ a non-direct extension is made in $\mathcal{P}(n)$, then be will restrict each $E(m)$, $m < n$ to the corresponding member of the Prikry sequence for κ_n and reflect the information the condition contains about coordinates $m < n$ below κ_n.

Let us start with a simpler situations where instead of ω extenders we have only one or two.

2.1 A single extender.

Let us describe a variation of the one element extender based Prikry forcing that will be used here. It will be very close to those of C. Merimovich [8]. A difference will be that sequences inside conditions will be either empty or of length one only.

Let E be a $(\kappa, \lambda)-$extender. We will define the sets $\mathcal{P}^*_E$ and $\mathcal{P}^{\{\}}_E$ which will lead us to the definition of the forcing notion $\mathcal{P}_E$.

Let $d \subseteq \lambda \setminus \kappa$ of cardinality at most κ. Define a $\kappa-$ultrafilter $E(d)$ on $[d \times \kappa]^{<\kappa}$ as follows:

$$X \in E(d) \Leftrightarrow \{\langle j_E(\alpha), \alpha\rangle \mid \alpha \in d\} \in j_E(X).$$

Actually, $E(d)$ concentrates on a smaller set called $\mathrm{OB}(d)$ in [8].
The advantage of using $E(d)$ is that once A is a typical set of $E(d)-$measure one and $a \in A$, then a is of the form $\langle\langle\alpha_\xi, \beta_\xi\rangle \mid \xi < \rho\rangle$, where

1. $\rho < \kappa$,
2. $\mathrm{dom}(a) = \{\alpha_\xi \mid \xi < \rho\} \subseteq d$,
3. $\beta_\xi < \kappa$, for every $\xi < \rho$.

So, a measure one set provides an explicit connection between elements of Prikry sequences and the measures to which they belong.
We assume further that always $\langle\alpha_\xi \mid \xi < \rho\rangle$ and $\langle\beta_\xi \mid \xi < \rho\rangle$ are strictly increasing sequences of ordinals.

Definition 2.1 Let $\mathcal{P}^*_E$ be the set of all functions f such that

1. $\mathrm{dom}(f) \subseteq \lambda \setminus \kappa$ is of cardinality at most κ,
2. $\kappa \in \mathrm{dom}(f)$,
3. for every $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is either empty or a one element sequence which consists of an element of κ.

Note that $\mathcal{P}^*_E$ does not depend on E, but only on κ and λ. In particular if we replace E by another $(\kappa, \lambda)-$extender E', then $\mathcal{P}^*_E = \mathcal{P}^*_{E'}$.

Definition 2.2 Let $f, g \in \mathcal{P}^*_E$. Set $f \geq^* g$ iff $f \supseteq g$.

Definition 2.3 Let $f \in \mathcal{P}^*_E$ and $\vec{\nu} \in [\mathrm{dom}(f) \times \kappa]^{<\kappa}$. Define $g = f_{\langle \vec{\nu} \rangle} \in \mathcal{P}^*_E$ as follows:

1. $\mathrm{dom}(g) = \mathrm{dom}(f)$,
2. for every $\alpha \in \mathrm{dom}(g)$,
$$g(\alpha) = \begin{cases} \langle \vec{\nu}(\alpha) \rangle, & \text{if } \alpha \in \mathrm{dom}(\vec{\nu}) \text{ and } f(\alpha) \text{ is empty sequence;} \\ \langle \vec{\nu}(\alpha) \rangle, & \text{if } \alpha \in \mathrm{dom}(\vec{\nu}),\ f(\alpha) \text{ is not empty and } \vec{\nu}(\alpha) > f(\alpha); \\ f(\alpha), & \text{otherwise.} \end{cases}$$

The difference from the original definition by Merimovich in [8], is that we do not keep $f(\alpha)$ if $\vec{\nu}(\alpha) > f(\alpha)$, but rather replace $f(\alpha)$ by $\vec{\nu}(\alpha)$.

Define now the pure part $\mathcal{P}^{\{\}}_E$ of the main forcing $\mathcal{P}_E$.

Definition 2.4 A pure condition $p \in \mathcal{P}^{\{\}}_E$ is of the form $\langle f, A \rangle$, where

1. $f \in \mathcal{P}^*_E$,
2. $f(\kappa)$ is the empty sequence,
3. $A \in E(\mathrm{dom}(f))$.

Define the order on $\mathcal{P}^{\{\}}_E$ as follows:

Definition 2.5 Let $p = \langle f, A \rangle, q = \langle g, B \rangle \in \mathcal{P}^{\{\}}_E$. Set $p \geq^* q$ iff

1. $f \geq^* g$ in $\mathcal{P}^*_E$,
2. $A \restriction \mathrm{dom}(g) \subseteq B$.

The forcing $\mathcal{P}_E$ will be the union of $\mathcal{P}^{\{\}}_E$ with

$$\{f \in \mathcal{P}^*_E \mid f(\kappa) \neq \langle\rangle\}.$$

The direct order extension will be just the union of $\leq^*$ orders of both parts. Let us define the forcing order $\leq$ on $\mathcal{P}$. We do this by defining one element extensions of members of $\mathcal{P}^{\{\}}_E$.

Definition 2.6 Let $p = \langle f, A \rangle$ be in $\mathcal{P}^{\{\}}_E$ and $\vec{\nu} \in A$. Define $p^\frown \vec{\nu} \in \mathcal{P}^*_E$ to be $f_{\langle \vec{\nu} \rangle}$.

Definition 2.7 Let $p = \langle f, A \rangle$ be in $\mathcal{P}^{\{\}}_E$ and g be in $\mathcal{P}^*_E$. Set $p \leq g$ iff there is $\vec{\nu} \in A$ such that $f_{\langle \vec{\nu} \rangle} \leq^* g$.

The next lemma follows from the definitions:

Lemma 2.8 *The forcing $\langle \mathcal{P}_E, \leq \rangle$ is equivalent to the Cohen forcing for adding $\lambda-$many Cohen subsets to κ^+.*

However, more can be deduced:

Lemma 2.9 *$\langle \mathcal{P}_E, \leq, \leq^* \rangle$ is a Prikry type forcing notion.*

Proof. Let us sketch the basic argument following Merimovich presentation [8].
Let $p = \langle f^p, A^p \rangle \in \mathcal{P}_E^0$ and σ be a statement of the forcing language.
We would like to find a direct extension of p which decides σ. Suppose that there is no such extension.

Proceed as in 3.12 of [7]. Construct by induction an increasing chain of elementary submodels $\langle N_\xi \mid \xi < \kappa \rangle$ of H_χ, for χ large enough, and a sequence $\langle f_\xi \mid \xi < \kappa \rangle$ of members of $\mathcal{P}_E^*$, such that

1. $p, \mathcal{P}_E, \sigma \in N_0$,
2. $N_0 \supseteq \kappa$,
3. for every $\xi < \kappa$,
   (a) $|N_\xi| = \kappa$,
   (b) ${}^{\kappa>}N_\xi \subseteq N_\xi$,
   (c) $\langle f_\zeta \mid \zeta < \xi \rangle \in N_\xi$,
   (d) $f_\xi \in \bigcap \{ D' \in N_\xi \mid D'$ is a dense open subset of $\mathcal{P}_E^*$ above $f^p \}$,
   (e) $f^p \leq^* f_0$,
   (f) $f_\xi \geq^* f_\zeta$, for every $\zeta < \xi$.

Set $N = \bigcup_{\xi<\kappa} N_\xi$ and $f^* = \bigcup \{ f_\xi \mid \xi < \kappa \}$.[1]
Let $A \subseteq [\mathrm{dom}(f^*) \times \kappa]^{<\kappa}$ be such that

- $A \restriction \mathrm{dom}(f^p) \subseteq A^p$,
- $A \in E(\mathrm{dom}(f^*))$.

[1]Carmi Merimovich pointed out that there is no need here in elementary chain of models and it is possible to define N directly. This observation applies also to our further constructions.

Note that $A \subseteq N$, since $\mathrm{dom}(f^*) \subseteq N$, and so, $[\mathrm{dom}(f^*) \times \kappa]^{<\kappa} \subseteq N$.

Let $\vec{\nu} \in A$.

Define $D_{\vec{\nu}}$ to be the set of all $f \in \mathcal{P}^*_E, f \geq f^p$ such that

$$f_{\vec{\nu}} \parallel \sigma.$$

Then $D_{\vec{\nu}}$ is a dense open subset of $\mathcal{P}^*_E$ above f^p.

It is definable with parameters in N, hence $D_{\vec{\nu}} \in N$.

Then, $f^* \in D_{\vec{\nu}}$.

Shrink now A to $A^* \in E(\mathrm{dom}(f^*))$, if necessary, such that for every $\vec{\nu}, \vec{\nu}'$ inside A^* we will have

$$f^*_{\vec{\nu}} \Vdash \sigma \text{ iff } f^*_{\vec{\nu}'} \Vdash \sigma.$$

Suppose that for every $\vec{\nu} \in A^*$, $f^*_{\vec{\nu}} \Vdash \sigma$.

Now, we claim that already $\langle f^*, A^* \rangle \Vdash \sigma$.

Supose otherwise. Then there is $g \geq \langle f^*, A^* \rangle$ which forces $\neg\sigma$. Then for some $\vec{\nu} \in A^*$, $g \geq f^*_{\vec{\nu}}$, by Definition 2.7. But $f^* \in D_{\vec{\nu}}$, hence already $f^*_{\vec{\nu}} \Vdash \neg\sigma$, which is impossible by the choice of A^*.

Contradiction.

□

2.2 Two extenders.

We deal now with two extenders $E(0)$ and $E(1)$.

$E(0)$ is a $(\kappa_0, \lambda)-$extender, $E(1)$ is a $(\kappa_1, \lambda)-$extender, $\kappa_0 < \kappa_1$ and $E(0) \lhd E(1)$.

Assume for simplicity that there is $h_\lambda : \kappa_1 \to \kappa_1$ such that $j_{E(1)}(h_\lambda)(\kappa_1) = \lambda$ and that there is $h_{E(0)} : \kappa_1 \to V_{\kappa_1}$ such that $j_{E(1)}(h_{E(0)})(\kappa_1) = E(0)$.

Note that having a Woodin cardinal, it is possible to pick such $E(0)$ and $E(1)$ with $E(0) = j_{E(1)}(E(0)) \restriction \lambda$.

We will define the forcing notion $\mathcal{P}_{E(0),E(1)}$. The definition uses the sets constructed in previous subsection, i.e., $\mathcal{P}^*_{E(i)}, \mathcal{P}^{\{\}}_{E(i)}, \mathcal{P}_{E(i)}, i < 2$. In addition we will define the following: $\mathcal{P}^*_{E(0),E(1)}, \mathcal{P}^{\{\}}_{E(0),E(1)}, \mathcal{P}^{\{0\}}_{E(0),E(1)}$, and $\mathcal{P}^{\{1\}}_{E(0),E(1)}$.

Definition 2.10 The set of pure conditions $\mathcal{P}^{\{\}}_{\langle E(0),E(1)\rangle}$ consists of all pairs $\langle p(0), p(1)\rangle$ such that

1. $p(0) = \langle f^0, A^0 \rangle \in \mathcal{P}^{\{\}}_{E(0)}$,

2. $p(1) = \langle f^1, A^1 \rangle \in \mathcal{P}^{\{\}}_{E(1)}$,

3. $\mathrm{dom}(f^0) \setminus \kappa_1 \subseteq \mathrm{dom}(f^1)$,

4. for every $\alpha \in \mathrm{dom}(f^0) \setminus \kappa_1$, if $f^1(\alpha)$ is not the empty sequence, then for every $\vec{\nu} \in A^1$, $\alpha \in \mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\alpha) > f^1(\alpha)$.
   The intuition behind this condition is that the current value $f^1(\alpha)$ may interfere with values of one element Prikry sequences over κ_0. Namely, with the $\alpha-$th Prikry sequence over κ_0. Now, if $\vec{\nu}(\alpha) > f^1(\alpha)$, then $f^1_{\vec{\nu}}(\alpha) = \vec{\nu}(\alpha)$, by Definition 2.3, and so, the value $f^1(\alpha)$ just disappears.

5. For every $\gamma \in \mathrm{dom}(f^0) \cap \kappa_1, \vec{\nu} \in A^1$ and $\alpha \in \mathrm{dom}(\vec{\nu})$, $\vec{\nu}(\alpha) > \gamma$.
   Note that $|\mathrm{dom}(f^0)| \leq \kappa_0$, so it is easy to arrange this.

6. For every $\vec{\nu} \in A^1$, the measures $E(0)(\mathrm{dom}(f^0))$ and $h_{E(0)}(\vec{\nu}(\kappa_1))((\mathrm{dom}(f^0) \cap \kappa_1) \cup \{\vec{\nu}(\alpha) \mid \alpha \in \mathrm{dom}(f^0) \setminus \kappa_1\})$ are basically the same in the following sense:

   $X \in E(0)(\mathrm{dom}(f^0))$ iff $X^{ref} \in h_{E(0)}(\vec{\nu}(\kappa_1))((\mathrm{dom}(f^0) \cap \kappa_1) \cup \{\vec{\nu}(\alpha) \mid \alpha \in \mathrm{dom}(f^0) \setminus \kappa_1\})$,

   where
   $$X^{ref} = \{(\alpha, \beta) \in X \mid \alpha < \kappa_1\} \cup \{(\vec{\nu}(\alpha), \beta) \mid (\alpha, \beta) \in X, \alpha \geq \kappa_1\}.$$
   Note that this property is true in the ultrapower by $E(1)$, so it holds on a set of measure one, as well.

Turn now to non-pure extensions.
First consider the situation with non-pure part over κ_0.

Definition 2.11 The set of conditions $\mathcal{P}^{\{0\}}_{\langle E(0), E(1) \rangle}$ consists of all pairs $\langle f^0, p(1) \rangle$ such that

1. $f^0 \in \mathcal{P}^*_{E(0)}$,

2. $p(1) = \langle f^1, A^1 \rangle \in \mathcal{P}_{E(1)}$,

3. $\mathrm{dom}(f^0) \setminus \kappa_1 \subseteq \mathrm{dom}(f^1)$,

4. for every $\alpha \in \mathrm{dom}(f^0) \setminus \kappa_1$, if $f^1(\alpha)$ is not the empty sequence, then for every $\vec{\nu} \in A^1$, $\alpha \in \mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\alpha) > f^1(\alpha)$,

5. for every $\gamma \in \mathrm{dom}(f^0) \cap \kappa_1, \vec{\nu} \in A^1$ and $\alpha \in \mathrm{dom}(\vec{\nu})$, $\vec{\nu}(\alpha) > \gamma$.

Now we define conditions with a pure part over κ_0 and a non-pure over κ_1.

Definition 2.12 The set of conditions $\mathcal{P}^{\{1\}}_{\langle E(0),E(1)\rangle}$ consists of all pairs $\langle p(0), f^1\rangle$ such that

1. $f^1 \in \mathcal{P}^*_{E(1)}$,
2. $f^1(\kappa_1)$ is non-empty,
3. $p(0) \in \mathcal{P}_{h_{E(0)}(f^1(\kappa_1))}$. The meaning is that if the value of the Prikry sequence for the normal measure of $E(1)$ is decided, then we reflect $E(0)$ below κ_1
   to the $(\kappa_0, h_\lambda(f^1(\kappa_1))-$extender $h_{E(0)}(f^1(\kappa_1))$.

Define now a completely non-pure part of the forcing.

Definition 2.13 The set of conditions $\mathcal{P}^*_{\langle E(0),E(1)\rangle}$ consists of all pairs $\langle f^0, f^1\rangle$ such that

1. $f^1 \in \mathcal{P}^*_{E(1)}$,
2. $f^1(\kappa_1)$ is non-empty,
3. $f^0 \in \mathcal{P}^*_{E(0)}$,
4. $f^0(\kappa_0)$ is non-empty,
5. $\mathrm{dom}(f^0) \subseteq h_\lambda(f^1(\kappa_1))$.
   The meaning is that if the value of the Prikry sequence for the normal measure of $E(1)$ is decided, then we add only $h_\lambda(f^1(\kappa_1))$ Cohen subsets to κ_0^+.

Now let us put everything together.

Definition 2.14 $\mathcal{P}_{\langle E(0),E(1)\rangle} = \mathcal{P}^{\{\}}_{\langle E(0),E(1)\rangle} \cup \mathcal{P}^{\{0\}}_{\langle E(0),E(1)\rangle} \cup \mathcal{P}^{\{1\}}_{\langle E(0),E(1)\rangle} \cup \mathcal{P}^*_{\langle E(0),E(1)\rangle}$.

Define the orders $\leq, \leq^*$ over $\mathcal{P}_{\langle E(0),E(1)\rangle}$.
$\leq^*$ is just the union of the orders at each of the components.
Let us give now the main definition.

Definition 2.15 Let $p, q \in \mathcal{P}_{\langle E(0),E(1)\rangle}$. If p, q are in the same component, then set $p \geq q$ iff $p \geq^* q$. Suppose that they are in different components.
Split into cases.

1. Suppose that $q \in \mathcal{P}^{\{\}}_{\langle E(0),E(1)\rangle}$, i.e. in the pure part of $\mathcal{P}_{\langle E(0),E(1)\rangle}$, $p \in \mathcal{P}^{\{0\}}_{\langle E(0),E(1)\rangle}$, i.e., only the part of p over κ_1 is a pure condition.
   Let then $q = \langle\langle g^0, B^0\rangle, \langle g^1, B^1\rangle\rangle, p = \langle f^0, \langle f^1, A^1\rangle\rangle$.
   Set $p \geq q$ iff $f^0 \geq \langle g^0, B^0\rangle$ in $\mathcal{P}_{E(0)}$ and $\langle f^1, A^1\rangle \geq^* \langle g^1, B^1\rangle$ in $\mathcal{P}_{E(1)}$.

2. Suppose that $q \in \mathcal{P}^{\{1\}}_{\langle E(0),E(1)\rangle}$, i.e., in the part over κ_0 is pure and those over κ_1 is not pure, $p \in \mathcal{P}^*_{\langle E(0),E(1)\rangle}$, i.e. p is a completely non-pure condition.
   Let then $q = \langle\langle g^0, B^0\rangle, g^1\rangle$ and $p = \langle f^0, f^1\rangle$.
   Set $p \geq q$ iff $f^0 \geq \langle g^0, B^0\rangle$ in $\mathcal{P}_{h_{E(0)}(g^1(\kappa_1))}$ and $f^1 \geq g^1$ in $\mathcal{P}_{E(1)}$.

3. (Principal case 1.)
   Suppose that $q \in \mathcal{P}^{\{0\}}_{\langle E(0),E(1)\rangle}$, i.e. in the part over κ_1 is pure and those over κ_0 is not pure, $p \in \mathcal{P}^*_{\langle E(0),E(1)\rangle}$, i.e. p is a completely non-pure condition.
   Let then $q = \langle g^0, \langle g^1, B^1\rangle\rangle$ and $p = \langle f^0, f^1\rangle$.
   Set $p \geq q$ iff $f^1 \geq \langle g^1, B^1\rangle$ in $\mathcal{P}_{E(1)}$ and $f^0 \geq (g^0)^{ref}$ in $\mathcal{P}^*_{E(0)\restriction h_\lambda(f^1(\kappa_1))}$, where $(g^0)^{ref}$ the reflection of g^0 below κ_1 is defined as follows:

   (a) $\mathrm{dom}((g^0)^{ref}) = (\mathrm{dom}(g^0) \cap \kappa_1) \cup \{f^1(\alpha) \mid \alpha \in \mathrm{dom}(g^0) \setminus \kappa_1\}$,

   (b) for every $\alpha \in \mathrm{dom}(g^0) \cap \kappa_1 = \mathrm{dom}(g^0) \cap \mathrm{dom}((g^0)^{ref})$, $(g^0)^{ref}(\alpha) = g^0(\alpha)$,

   (c) for every $\alpha \in \mathrm{dom}(g^0) \setminus \kappa_1$, $(g^0)^{ref}(f^1(\alpha)) = g^0(\alpha)$.
   It is crucial here that $f^1 \restriction (\mathrm{dom}(g^0) \setminus \kappa_1)$ is one to one and the values there are above $\mathrm{rng}(g^0) \cap \kappa_1$.
   This follows by conditions (4),(5) of Definitions 2.10,2.11.

4. (Principal case 2.)
   Suppose that $q \in \mathcal{P}^{\{\}}_{\langle E(0),E(1)\rangle}$, i.e., both parts are pure, $p \in \mathcal{P}^{\{1\}}_{\langle E(0),E(1)\rangle}$, i.e., only the part over κ_0 is pure.
   Let then $q = \langle\langle g^0, B^0\rangle, \langle g^1, B^1\rangle\rangle$ and $p = \langle\langle f^0, A^0\rangle, f^1\rangle$.
   Set $p \geq q$ iff $f^1 \geq \langle g^1, B^1\rangle$ in $\mathcal{P}_{E(1)}$ and $\langle f^0, A^0\rangle \geq (\langle g^0, B^0\rangle)^{ref}$ in $\mathcal{P}_{h_{E(0)}(f^1(\kappa_1))}$, where $(\langle g^0, B^0\rangle)^{ref}$ the reflection of $\langle g^0, B^0\rangle$ below κ_1 is defined as follows:

   (a) $\mathrm{dom}((g^0)^{ref}) = (\mathrm{dom}(g^0) \cap \kappa_1) \cup \{f^1(\alpha) \mid \alpha \in \mathrm{dom}(g^0) \setminus \kappa_1\}$,

   (b) for every $\alpha \in \mathrm{dom}(g^0) \cap \kappa_1 = \mathrm{dom}(g^0) \cap \mathrm{dom}((g^0)^{ref})$, $(g^0)^{ref}(\alpha) = g^0(\alpha)$,

   (c) for every $\alpha \in \mathrm{dom}(g^0) \setminus \kappa_1$, $(g^0)^{ref}(f^1(\alpha)) = g^0(\alpha)$.
   Again, it is crucial here that $f^1 \restriction (\mathrm{dom}(g^0) \setminus \kappa_1)$ is one to one and the values

there are above $\mathrm{dom}(g^0) \cap \kappa_1$, and this follows by conditions (4),(5) of Definitions 2.10,2.11.

One more crucial observation here is that the measure $(E(0))(\mathrm{dom}(g^0))$, to which B^0 belongs, reflects to basically the same measure,

It follows by (6) of Definitions 2.10.

(d) $A^0 \restriction \mathrm{dom}((g^0)^{ref}) \subseteq (B^0)^{ref}$, where $(B^0)^{ref} = \{\vec{\nu}^{ref} \mid \vec{\nu} \in B^0\}$ and if $\vec{\nu} = \langle\langle \alpha_\xi, \beta_\xi\rangle \mid \xi < \rho\rangle$, then
$\vec{\nu}^{ref} = \langle\langle \alpha_\xi, \beta_\xi\rangle \mid \xi < \rho, \alpha_\xi < \kappa_1\rangle^\frown\langle\langle f^1(\alpha_\xi), \beta_\xi\rangle \mid \xi < \rho, \alpha_\xi \geq \kappa_1\rangle$.

Denote further in this subsection $\mathcal{P}_{\langle E(0),E(1)\rangle}$ by just $\mathcal{P}$.

The next lemma follows from the definitions:

Lemma 2.16 *The forcing $\langle \mathcal{P}, \leq \rangle$ is equivalent to Cohen$(\kappa_0^+, \eta) \times$ Cohen(κ_1^+, λ), for some $\eta < \kappa_1$ which depends on the choice of a non-pure condition for $\mathcal{P}_{E(1)}$.*

However, as usual, more can be deduced:

Lemma 2.17 *$\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.*

Proof. The proof is similar to those of Lemma 2.9 (and in turn to those of Merimovich [8]).

Suppose otherwise.

Let $p \in \mathcal{P}$ be a pure condition and σ a statement of the forcing language which is undecided by pure extensions of p. Then p is of the form $\langle\langle f^{p0}, A^{p0}\rangle, \langle f^{p1}, A^{p1}\rangle\rangle$.

Proceed as in 3.12 of [7]. Construct by induction an increasing chain of elementary submodels $\langle N^1_\xi \mid \xi < \kappa_1\rangle$ of H_χ, for χ large enough, and a sequence $\langle f^1_\xi \mid \xi < \kappa_1\rangle$ of members of $\mathcal{P}^*_{E(1)}$, such that

1. $p, \mathcal{P}, \sigma \in N^1_0$,
2. $N^1_0 \supseteq \kappa_1$,
3. for every $\xi < \kappa_1$,
   (a) $|N^1_\xi| = \kappa_1$,
   (b) ${}^{\kappa_1>}N^1_\xi \subseteq N^1_\xi$,
   (c) $\langle f^1_\zeta \mid \zeta < \xi\rangle \in N^1_\xi$,
   (d) $f^1_\xi \in \bigcap\{D' \in N^1_\xi \mid D'$ is a dense open subset of $\mathcal{P}^*_{E(1)}$ above $f^{p1}\}$,

(e) $f^{p1} \leq^* f^1_0$,

(f) $f^1_\xi \geq^* f^1_\zeta$, for every $\zeta < \xi$.

Set $N^1 = \bigcup_{\xi<\kappa_1} N^1_\xi$ and $f^{1*} = \bigcup\{f^1_\xi \mid \xi < \kappa\}$. Let $A \subseteq [\mathrm{dom}(f^{1*}) \times \kappa_1]^{<\kappa_1}$ be such that

- $A \restriction \mathrm{dom}(f^{p1}) \subseteq A^{p1}$,
- $A \in (E(1))(\mathrm{dom}(f^{1*}))$.

Note that $A \subseteq N^1$, since $\mathrm{dom}(f^{1*}) \subseteq N^1$, and so, $[\mathrm{dom}(f^{1*}) \times \kappa_1]^{<\kappa_1} \subseteq N^1$.

Let $\vec{\nu} \in A$. Consider $\lambda_1^{\vec{\nu}} := h_\lambda(\vec{\nu}(\kappa_1))$, i.e. the cardinal below κ_1 that now corresponds to λ. Suppose for simplicity that $\mathrm{dom}(f^{p0}) \subseteq \lambda_1^{\vec{\nu}}$, otherwise just reflect the part above κ_1 below as in Definition 2.15.
Consider $\mathcal{P}_{h_{E(0)}(\vec{\nu}(\kappa_1))}$. Clearly, it is contained and belongs to N^1.
Let $\langle t_\xi \mid \xi < \lambda_1^{\vec{\nu}} \rangle$ be an enumeration of this forcing notion in N^1.
Let $f \in \mathcal{P}^*_{E(1)}, f \geq^* f^{p1}$.
Proceed by induction on $\xi < \lambda_1^{\vec{\nu}}$ and define an $\leq^*$ $-$increasing sequence $\langle f_\xi \mid \xi < \lambda_1^{\vec{\nu}} \rangle$ of direct extensions of f such that, for every $\xi < \lambda_1^{\vec{\nu}}$, either

(1) $\langle t_\xi, (f_\xi)_{\vec{\nu}} \rangle \parallel \sigma$,
or

(2) for every $g \geq^* (f_\xi)_{\vec{\nu}}$, $\langle t_\xi, g \rangle \nparallel \sigma$.

Let $\bar{f} = \bigcup_{\xi<\lambda_1^{\vec{\nu}}} f_\xi$.
Then, for every $t \in \mathcal{P}_{h_{E(0)}(\vec{\nu}(\kappa_1))}$ either

(1) $\langle t, \bar{f}_{\vec{\nu}} \rangle \parallel \sigma$,
or

(2) for every $g \geq^* \bar{f}_{\vec{\nu}}$, $\langle t, g \rangle \nparallel \sigma$.

Consider now the following statement of the forcing language of $\mathcal{P}_{E(0) \restriction \lambda_1^{\vec{\nu}}}$:

$$\varphi \equiv \exists t \in \underset{\sim}{G}(\langle t, \bar{f}_{\vec{\nu}} \rangle \parallel \sigma).$$

By the Prikry condition of the forcing $\mathcal{P}_{h_{E(0)}(\vec{\nu}(\kappa_1))}$ (Lemma 2.9), there is $t^* \geq^* \langle f^{p0}, A^{p0} \rangle$ which decides φ.

Claim 1 $t^* \Vdash \varphi$.

Proof. Suppose otherwise. Then $t^* \Vdash \neg\varphi$. This means that whenever $t \in \mathcal{P}_{h_{E(0)}(\vec{\nu}(\kappa_1))}$ and $t \geq t^*$, $\langle t, \bar{f}_{\vec{\nu}} \rangle \not\parallel \sigma$.
Pick now some $\langle t, g \rangle \in \mathcal{P}_{E(0),E(1)}, \langle t, g \rangle \geq \langle t^*, \bar{f}_{\vec{\nu}} \rangle$ which decides σ.
Then, for some $\xi < \lambda_1^{\vec{\nu}}, t = t_\xi$, and then, $\langle t, (f_\xi)_{\vec{\nu}} \rangle \parallel \sigma$. So, $\langle t, \bar{f}_{\vec{\nu}} \rangle \parallel \sigma$.
Contradiction.
$\square$ of the claim.

Now use again the Prikry condition of the forcing $\mathcal{P}_{h_{E(0)}(\vec{\nu}(\kappa_1))}$ to decide the following statement

$$\psi \equiv \exists t \in \underset{\sim}{G}(\langle t, \bar{f}_{\vec{\nu}} \rangle \Vdash \sigma).$$

Let $t(\vec{\nu}, f) \geq^* t^*$ be a condition which decides ψ. If $t(\vec{\nu}, f) \Vdash \psi$, then $\langle t(\vec{\nu}, f), \bar{f}_{\vec{\nu}} \rangle \Vdash \sigma$. If $t(\vec{\nu}, f) \Vdash \neg\psi$, then $\langle t(\vec{\nu}, f), \bar{f}_{\vec{\nu}} \rangle \Vdash \neg\sigma$.

Define $D_{\vec{\nu}}$ to be the set of all $f \in \mathcal{P}^*_{E(1)}, f \geq^* f^{p1}$ such that

$$\langle t(\vec{\nu}, f), f_{\vec{\nu}} \rangle \parallel \sigma.$$

The next claim follows now:

Claim 2 *$D_{\vec{\nu}}$ is a dense open subset of $\mathcal{P}^*_{E(1)}$ above f^{p1}.*

$D_{\vec{\nu}}$ is definable with parameters in N, hence $D_{\vec{\nu}} \in N$.
Then, $f^{1*} \in D_{\vec{\nu}}$, for every $\vec{\nu} \in A$.
So, $\langle t(\vec{\nu}, f^{1*}), f^{1*}_{\vec{\nu}} \rangle \parallel \sigma$, for every $\vec{\nu} \in A$. Shrink A, if necessary, to a set $A^{1*} \in (E(1))(\text{dom}(f^{1*}))$, such that for any two $\vec{\nu}, \vec{\nu}' \in A^{1*}$ the decision is the same, say σ is forced.

Consider now $\langle f^{1*}, A^{1*} \rangle$. It is a pure condition in $\mathcal{P}_{E(1)}$. Use the function $\vec{\nu} \mapsto t(\vec{\nu}, f^{1*})$ in order to get a pure condition in $\mathcal{P}_{E(0)}$, just use the one which this function represents in the ultrapower by $(E(1))(\text{dom}(f^{1*}))$.
Let us explain how do we naturally combine the result into a condition in $\mathcal{P}_{E(0),E(1)}$.
Let $t(\vec{\nu}, f^{1*}) = \langle f^{0\vec{\nu}}, A^{0\vec{\nu}} \rangle$, for every $\vec{\nu} \in A^{1*}$. Consider $f^{0\vec{\nu}}$. It is a set of at most κ_0 many pairs (α, β), where $\alpha < \lambda_1^{\vec{\nu}} < \kappa_1$ and β is either the empty sequence or an ordinal $< \kappa_0$.
Shrinking A^{1*} if necessary, we can assume that there are x and $\kappa_0^* < \kappa_0^+$ such that for every $\vec{\nu}, \vec{\nu}' \in A^{1*}$ the following hold:

1. $\text{dom}(f^{0\vec{\nu}}) \cap \vec{\nu}(\kappa_1) = x$,

2. $\text{dom}(f^{0\vec{\nu}}) \setminus \vec{\nu}(\kappa_1) = \{\gamma_\tau^{\vec{\nu}} \mid \tau < \kappa_0^*\}$ is an increasing enumeration,

3. for every $\alpha \in x$, $f^{0\vec{\nu}}(\alpha) = f^{0\vec{\nu}'}(\alpha)$,

4. for every $\tau < \kappa_0^*$, $f^{0\vec{\nu}}(\gamma_\tau^{\vec{\nu}}) = f^{0\vec{\nu}'}(\gamma_\tau^{\vec{\nu}'})$

Consider, for every $\tau < \kappa_0^*$ a function s_τ on A^{1*} defined by setting $s_\tau(\vec{\nu}) = \gamma_\tau^{\vec{\nu}}$.
Let

$$\gamma_\tau = j_{E(1)}(s_\tau)(\langle (j_{E(1)}(\alpha), \alpha) \mid \alpha \in \mathrm{dom}(f^{1*})\rangle).$$

Extend now f^{1*} to f^{1**} by adding all $\gamma_\tau, \tau < \kappa_0^*$ to its domain and setting $f^{1**}(\gamma_\tau)$ to be the empty sequence whenever $\gamma_\tau \notin \mathrm{dom}(f^{1*})$.
Define $A^{1**} \in E(1)(\mathrm{dom}(f^{1**})$ as follows.
Set $\vec{\nu} \in A^{1**}$ iff

1. $\vec{\nu} \restriction \mathrm{dom}(f^{1*}) \in A^{1*}$,

2. $\mathrm{dom}(\vec{\nu}) \supseteq \{\gamma_\tau \mid \tau < \kappa_0^*\}$,

3. if $\gamma_\tau \in \mathrm{dom}(f^{1*})$ and $f^{1*}(\gamma_\tau)$ is not the empty sequence, then $\vec{\nu}(\gamma_\tau) > f^{1*}(\gamma_\tau)$,

4. $\vec{\nu}(\gamma_\tau) = s_\tau(\vec{\nu} \restriction \mathrm{dom}(f^{1*}))$.

For every $\vec{\nu} \in A^{1**}$, set $\langle g^{\vec{\nu}}, B^{\vec{\nu}}\rangle = \langle f^{0\vec{\nu}\restriction \mathrm{dom}(f^{1*})}, A^{0\vec{\nu}\restriction \mathrm{dom}(f^{1*})}\rangle$.
Consider the function $\vec{\nu} \mapsto \langle g^{\vec{\nu}}, B^{\vec{\nu}}\rangle$, $\vec{\nu} \in A^{1**}$. Let $\langle f^{0*}, A^{0*}\rangle$ be represented by it in the ultrapower with $E(1)$.

It follows that $\langle\langle f^{0*}, A^{0*}\rangle, \langle f^{1**}, A^{1**}\rangle\rangle$ is a pure condition in $\mathcal{P}_{E(0),E(1)}$ which extends p.
The next claim completes the argument:

Claim 3 $\langle\langle f^{0*}, A^{0*}\rangle, \langle f^{1**}, A^{1**}\rangle\rangle \Vdash \sigma$.

Proof. Suppose otherwise. Then there is $\langle f, g\rangle \geq \langle\langle f^{0*}, A^{0*}\rangle, \langle f^{1**}, A^{1**}\rangle\rangle$ a non-pure in both coordinates which forces $\neg\sigma$. There is $\vec{\nu} \in A^{1**} \restriction \mathrm{dom}(f^{1*})$ such that $g \geq^* f^{1*}_{\vec{\nu}}$. But then $f \geq t(\vec{\nu}, f^{1*})$, and so, $\langle f, f^{1*}_{\vec{\nu}}\rangle \Vdash \sigma$. Contradiction.
□ of the claim.
□

2.3 $\omega-$many extenders.

We deal now with a $\triangleleft-$increasing sequence $\vec{E} = \langle E(n) \mid n < \omega\rangle$, where each $E(n)$ is a $(\kappa_n, \lambda)-$extender and $\langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence.
Assume for simplicity that for every $m < \omega$ there is $h^m_\lambda : \kappa_m \to \kappa_m$ such that

$j_{E(m)}(h^m_\lambda)(\kappa_m) = \lambda$ and that there is $h^m_{\vec{E}\restriction m} : \kappa_m \to V_{\kappa_m}$ such that $j_{E(m)}(h^m_{\vec{E}\restriction m})(\kappa_m) = \vec{E} \restriction m$. Note that having a Woodin cardinal, it is possible to pick such $\vec{E}$ so that
$E(n) = j_{E(m)}(E(n)) \restriction \lambda$, for every $n < m < \omega$.

Define the forcing notion $\mathcal{P}_{\langle E(n) | n<\omega\rangle}$. The definition will use several components. Let $\mathcal{P}^*_{E(i)}, \mathcal{P}_{E(i)}, i < \omega$ be as defined before. In addition we will define the following sets: $\mathcal{P}^{\{m_1,...m_k\}}_{\langle E(n)|n<\omega\rangle}$, where $k < \omega$ and $m_1 < ... < m_k$.

Definition 2.18 The set of pure conditions $\mathcal{P}^{\{\}}_{\langle E(n)|n<\omega\rangle}$ consists of all sequences $\langle p(n) \mid n < \omega\rangle$ such that for every $n < \omega$, the following hold:

1. $p(n) = \langle f^n, A^n\rangle \in \mathcal{P}_{E(n)}$,
2. $\mathrm{dom}(f^n) \setminus \kappa_{n+1} \subseteq \mathrm{dom}(f^{n+1})$,
3. for every $m \le n$, for every $\alpha \in \mathrm{dom}(f^m) \setminus \kappa_{n+1}$, if $f^{n+1}(\alpha)$ is not the empty sequence, then for every $\vec{\nu} \in A^{n+1}$, $\alpha \in \mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\alpha) > f^{n+1}(\alpha)$.
   The idea behind this is as in the case of two extenders.
4. For every $\vec{\nu} \in A^{n+1}$ and $m \le n$, the measures $E(m)(\mathrm{dom}(f^m))$ and $(h^{n+1}_{\vec{E}\restriction n+1}(\vec{\nu}(\kappa_{n+1}))(m))((\mathrm{dom}(f^m) \cap \kappa_{n+1}) \cup \{\vec{\nu}(\alpha) \mid \alpha \in \mathrm{dom}(f^m) \setminus \kappa_{n+1}\})$ are basically the same in the following sense:

$$X \in E(m)(\mathrm{dom}(f^m)) \text{ iff}$$

$$X^{ref} \in (h^{n+1}_{\vec{E}\restriction n+1}(\vec{\nu}(\kappa_{n+1}))(m))((\mathrm{dom}(f^m) \cap \kappa_{n+1}) \cup \{\vec{\nu}(\alpha) \mid \alpha \in \mathrm{dom}(f^m) \setminus \kappa_{n+1}\}),$$

   where

$$X^{ref} = \{(\alpha, \beta) \in X \mid \alpha < \kappa_{n+1}\} \cup \{(\vec{\nu}(\alpha), \beta) \mid (\alpha, \beta) \in X, \alpha \ge \kappa_{n+1}\}.$$

   Note that this property is true in the ultrapower by $E(n + 1)$, so it holds on a set of measure one, as well.

Turn now to non-pure extensions. As usual, in Magidor type iterations, non-pure extensions are allowed only at finitely many coordinates.

Start with a non-pure extension at a single coordinate and then proceed by induction.

Definition 2.19 Let $m < \omega$. Define the set $\mathcal{P}^{\{m\}}_{\langle E(n)|n<\omega\rangle}$ of conditions with only non-pure part over the coordinate m. $\mathcal{P}^{(m)}_{\langle E(n)|n<\omega\rangle}$ consists of all sequences $\langle p(n) \mid n < \omega\rangle$ such that for every $n < \omega$, the following hold:

1. $\langle p(n) \mid n<\omega, n\neq m\rangle$ is a pure condition in $\mathcal{P}_{\langle E(n)|n<\omega,n\neq m\rangle}$,

2. $p(m)=f^m\in\mathcal{P}^*_{E(m)}$,

3. $\mathrm{dom}(f^m)\setminus\kappa_n\subseteq\mathrm{dom}(f^n)$, for every $n, m<n<\omega$,

4. for every $n, m<n<\omega$, for every $\alpha\in\mathrm{dom}(f^m)\setminus\kappa_n$, if $f^n(\alpha)$ is not the empty sequence, then for every $\vec{\nu}\in A^n$, $\alpha\in\mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\alpha)>f^n(\alpha)$,

5. for every $n, m<n<\omega$, for every $\gamma\in\mathrm{dom}(f^m)\cap\kappa_n$,$\vec{\nu}\in A^n$ and $\alpha\in\mathrm{dom}(\vec{\nu})$, $\vec{\nu}(\alpha)>\gamma$.

6. If $m>0$, then the sequence $\langle p(n)\mid n<m\rangle$ is a condition in the pure part of $\mathcal{P}_{h^m_{\vec{E}\restriction m}(f^m(\kappa_m))}$. The meaning is that if the value of the Prikry sequence for the normal measure of $E(m)$ is decided, then we reflect all extenders $E(n), n<m$ below κ_m to the corresponding $(\kappa_n, h^m_\lambda(f^m(\kappa_m)))-$extenders.

Let $m_1<...<m_k<\omega, 1\leq k<\omega$ and suppose that $\mathcal{P}^{\{m_1,...,m_k\}}_{\langle E(n)|n<\omega\rangle}$ the set of conditions with non-pure extensions over coordinates $(m_1,...,m_k)$ only, is defined.
Let $m<\omega, m\notin\{m_1,...,m_k\}$.
Define non-pure extensions at the set of coordinates $\{m_1,...,m_k\}\cup\{m\}$.

Definition 2.20 Let $m<\omega$. Define the set $\mathcal{P}^{\{m_1,...,m_k\}\cup\{m\}}_{\langle E(n)|n<\omega\rangle}$ of conditions with only non-pure part over the coordinate $m_1,...,m_k$ and m. $\mathcal{P}^{\{m_1,...,m_k\}\cup\{m\}}_{\langle E(n)|n<\omega\rangle}$ consists of all sequences $\langle p(n)\mid n<\omega\rangle$ such that for every $n<\omega$, the following hold:

1. $\langle p(n)\mid n<\omega, n\neq m\rangle$ is a condition in $\mathcal{P}^{\{m_1,...,m_k\}}_{\langle E(n)|n<\omega,n\neq m\rangle}$,

2. $p(m)=f^m\in\mathcal{P}^*_{E(m)}$.

3. If $m>\max\{m_1,...,m_k\}$, then following hold:

   (a) $\mathrm{dom}(f^m)\setminus\kappa_n\subseteq\mathrm{dom}(f^n)$, for every $n, m<n<\omega$,

   (b) for every $n, m<n<\omega$, for every $\alpha\in\mathrm{dom}(f^m)\setminus\kappa_n$, if $f^n(\alpha)$ is not the empty sequence, then for every $\vec{\nu}\in A^n$, $\alpha\in\mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\alpha)>f^n(\alpha)$,

   (c) for every $n, m<n<\omega$, for every $\gamma\in\mathrm{dom}(f^m)\cap\kappa_n$,$\vec{\nu}\in A^n$ and $\alpha\in\mathrm{dom}(\vec{\nu})$, $\vec{\nu}(\alpha)>\gamma$.

(d) If $m > 0$, then the sequence $\langle p(n) \mid n < m\rangle$ is a condition in $\mathcal{P}^{\{m_1,...,m_k\}}_{h^m_{\vec{E}\restriction m}(f^m(\kappa_m))}$.
The meaning is that if the value of the Prikry sequence for the normal measure of $E(m)$ is decided, then we reflect all extenders $E(n), n < m$ below κ_m to the corresponding $(\kappa_n, h^m_\lambda(f^m(\kappa_m)))-$extenders.

4. If $m \leq \max\{m_1, ..., m_k\},$, then let i^* be the least such that $m \leq m_i$. We require the following:

   (a) $\langle p(n) \mid n < m_{i^*}\rangle \in \mathcal{P}^{\{m_1,...,m_{i^*-1},m\}}_{h^{m_{i^*}}_{\vec{E}\restriction m_{i^*}}(f^{m_{i^*}}(\kappa_{m_{i^*}})}$.

Finally set

$$\mathcal{P}_{\langle E(n)|n<\omega\rangle} = \bigcup\{\mathcal{P}^{\{m_1,...,m_k\}}_{\langle E(n)|n<\omega\rangle} \mid k < \omega, m_1 < ... < m_k < \omega\}.$$

Define the direct extension order $\leq^*$ over $\mathcal{P}_{\langle E(n)|n<\omega\rangle}$ to be the union of such orders over every $\mathcal{P}^{\{m_1,...,m_k\}}_{\langle E(n)|n<\omega\rangle}$, for every $k < \omega, m_1 < ... < m_k < \omega$.

Turn now to the definition of the forcing order $\leq$ over $\mathcal{P}_{\langle E(n)|n<\omega\rangle}$.

Let $m < \omega, m \notin \{m_1, ..., m_k\}$. Define a one element extension at coordinate m of a condition in $\mathcal{P}^{\{m_1,...,m_k\}}_{\langle E(n)|n<\omega\rangle}$.

Definition 2.21 Let $p \in \mathcal{P}^{\{m_1,...,m_k\}\cup\{m\}}_{\langle E(n)|n<\omega\rangle}$ and $q \in \mathcal{P}^{\{m_1,...,m_k\}}_{\langle E(n)|n<\omega\rangle}$. Set $p \geq q$ iff the following hold:

1. Suppose that $m = 0$.
   Then $p(0) = f^0 \in \mathcal{P}^*_{E(0)}$ and $q(0) = \langle g^0, B^0\rangle$ is a pure condition in $\mathcal{P}_{E(0)}$.
   Set $p \geq q$ iff $f^0 \geq \langle g^0, B^0\rangle$ in $\mathcal{P}_{E(0)}$ and $\langle p(n) \mid 0 < n < \omega\rangle \geq^* \langle q(n) \mid 0 < n < \omega\rangle$ in $\mathcal{P}_{\langle E(n)|0<n<\omega\rangle}$.

2. Suppose that $m > 0$.
   Then $p(m) = f^m \in \mathcal{P}^*_{E(m)}$ and $q(m) = \langle g^m, B^m\rangle$ is a pure condition in $\mathcal{P}_{E(m)}$.
   Set $p \geq q$ iff

   (a) $f^m \geq \langle g^m, B^m\rangle$ in $\mathcal{P}_{E(m)}$ and $\langle p(n) \mid m < n < \omega\rangle \geq^* \langle q(n) \mid m < n < \omega\rangle$ in $\mathcal{P}_{\langle E(n)|m<n<\omega\rangle}$.
   And

(b) $\langle p(n) \mid n < m\rangle \geq^* \langle q(n) \mid n < m\rangle^{ref}$ in $\mathcal{P}_{h^m_{\vec{E}\restriction m}(f^m(\kappa_m))}$, where $\langle q(n) \mid n < m\rangle^{ref}$ - the reflection of $\langle q(n) \mid n < m\rangle$ below κ_m is defined as follows, where $q(n) = \langle g^n, B^n\rangle$, if $n \not\in \{m_1, ..., m_k\}$ and $q(n) = \langle g^n\rangle$ otherwise.

i. Suppose first that $n \in \{m_1, ..., m_k\}$.
Then

A. $\mathrm{dom}((g^n)^{ref}) = (\mathrm{dom}(g^n) \cap \kappa_m) \cup \{f^m(\alpha) \mid \alpha \in \mathrm{dom}(g^n) \setminus \kappa_m\}$,

B. for every $\alpha \in \mathrm{dom}(g^n) \cap \kappa_m = \mathrm{dom}(g^n) \cap \mathrm{dom}((g^n)^{ref})$, $(g^n)^{ref}(\alpha) = g^n(\alpha)$,

C. for every $\alpha \in \mathrm{dom}(g^n) \setminus \kappa_m$, $(g^n)^{ref}(f^m(\alpha)) = g^n(\alpha)$.
It is crucial here that $f^m \restriction (\mathrm{dom}(g^n) \setminus \kappa_m)$ is one to one and the values there are above $\mathrm{rng}(g^n) \cap \kappa_m$.
This follows by conditions (4),(5) of Definitions 2.10,2.11.

ii. Suppose now that $n \not\in \{m_1, ..., m_k\}$.
Then

A. $\mathrm{dom}((g^n)^{ref}) = (\mathrm{dom}(g^n) \cap \kappa_m) \cup \{f^m(\alpha) \mid \alpha \in \mathrm{dom}(g^n) \setminus \kappa_m\}$,

B. for every $\alpha \in \mathrm{dom}(g^n) \cap \kappa_m = \mathrm{dom}(g^n) \cap \mathrm{dom}((g^n)^{ref})$, $(g^n)^{ref}(\alpha) = g^n(\alpha)$,

C. for every $\alpha \in \mathrm{dom}(g^n) \setminus \kappa_m$, $(g^n)^{ref}(f^m(\alpha)) = g^n(\alpha)$.
Again, it is crucial here that $f^m \restriction (\mathrm{dom}(g^n) \setminus \kappa_m)$ is one to one and the values there are above $\mathrm{dom}(g^n) \cap \kappa_m$, and this follows by conditions (3),(4) of Definition 2.18 and (4),(5) of Definition 2.19.
One more crucial observation here is that the measure $(E(n))(\mathrm{dom}(g^n)$, to which B^n belongs, reflects to basically the same measure,
It follows by (4) of Definition 2.18.

D. $A^n \restriction \mathrm{dom}((g^n)^{ref}) \subseteq \{(\alpha, \beta) \mid (\alpha, \beta) \in B^n, \alpha < \kappa_m\} \cup \{(f^m(\alpha), \beta) \mid (\alpha, \beta) \in B^n, \alpha \geq \kappa_m\}$.

Denote further in this subsection $\mathcal{P}_{\langle E(n)|n<\omega\rangle}$ by just $\mathcal{P}$.

The next lemma follows from the definitions:

Lemma 2.22 *For every $m < \omega$, the forcing $\langle \mathcal{P}_{\langle E(n)|n<m\rangle}, \leq \rangle$ is equivalent to the product of Cohen forcings Cohen(κ_n^+, η_n)'s, for some $\eta_n < \kappa_{n+1}$'s which depend on the choice of a non-pure condition for $\mathcal{P}_{E(n+1)}$.*

Lemma 2.23 *For every $m < \omega$, the forcing $\langle \mathcal{P}_{\langle E(n)|m\leq n<\omega\rangle}, \leq^* \rangle$ is κ_m−closed.*

Lemma 2.24 *The forcing $\langle \mathcal{P}, \leq \rangle$ satisfies κ_ω^{++}−c.c.*

Proof. Use the standard $\Delta-$system argument.
$\square$

Lemma 2.25 *$\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.*

Proof. The proof is similar to those of Lemmas 2.9, 2.17 (and in turn to those of Merimovich [8]).

Assume that for every $m < \omega$, $\langle \mathcal{P}_{\langle E(n)|n<m\rangle}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.

Suppose that $\langle \mathcal{P}, \leq, \leq^* \rangle$ does not have the Prikry property.
Let $p \in \mathcal{P}$ be a pure condition and σ a statement of the forcing language which is undecided by pure extensions of p. Then p is of the form $\langle\langle f^{pn}, A^{pn}\rangle \mid n < \omega\rangle$.

Proceed by induction on $m < \omega$ and define an $\leq^*-$increasing sequence $\langle p_m \mid m < \omega\rangle$ of direct extensions of p.
Let p_{-1} be p. Assume that for every $n < m$, p_n is defined. Define p_m.
At stage m we deal with the coordinate m of the condition.

Construct by induction an increasing chain of elementary submodels $\langle N^m_\xi \mid \xi < \kappa_m\rangle$ of H_χ, for χ large enough, and a sequence $\langle f_\xi \mid \xi < \kappa_m\rangle$ of members of $\mathcal{P}^*_{E(m)}$, such that

1. $p, p_{m-1}, \mathcal{P}, \sigma \in N^m_0$,
2. $N^m_0 \supseteq \kappa_m$,
3. for every $\xi < \kappa_m$,
   (a) $|N^m_\xi| = \kappa_m$,
   (b) ${}^{\kappa_m>}N^m_\xi \subseteq N^m_\xi$,
   (c) $\langle\langle f^m_\zeta, r^m_\zeta\rangle \mid \zeta < \xi\rangle \in N^m_\xi$,
   (d) $\langle f^m_\xi, r^m_\xi\rangle \in \bigcap\{D' \in N^m_\xi \mid D'$ is a dense open subset of $\mathcal{P}^*_{E(m)} \times \langle \mathcal{P}_{\langle E(n)|m<n<\omega\rangle}, \leq^* \rangle$ above $\langle f^{p_{m-1}m}, \langle p_{m-1}(n) \mid m < n < \omega\rangle\rangle$,
   (e) $f^{p_{m-1}m} \leq^* f^m_0$, $\langle p_{m-1}(n) \mid m < n < \omega\rangle \leq^* r^m_0$,
   (f) $f^m_\xi \geq^* f^m_\zeta$, $r^m_\xi \leq^* r^m_\zeta$, for every $\zeta < \xi$.

Set $N^m = \bigcup_{\xi<\kappa_m} N^m_\xi$ and $f^{m*} = \bigcup\{f^m_\xi \mid \xi < \kappa_m\}$. Pick $p^{>m}_{f^{m*}}$ to be $\leq^*-$stronger than every $r^m_\xi, \xi < \kappa_m$. Let $A \subseteq [\mathrm{dom}(f^{m*}) \times \kappa_m]^{<\kappa_m}$ be such that

- $A \restriction \mathrm{dom}(f^{pm}) \subseteq A^{pm}$,

- $A \in (E(m))(\mathrm{dom}(f^{m*}))$.

Note that $A \subseteq N^m$, since $\mathrm{dom}(f^{m*}) \subseteq N^m$, and so, $[\mathrm{dom}(f^{m*}) \times \kappa_m]^{<\kappa_m} \subseteq N^m$.

Let $\vec{\nu} \in A$. Consider $\lambda_m^{\vec{\nu}} := h_\lambda^m(\vec{\nu}(\kappa_m))$, i.e. the cardinal below κ_m that now corresponds to λ. Suppose for simplicity that $\mathrm{dom}(f^{pn}) \subseteq \lambda_m^{\vec{\nu}}$, for every $n < m$, otherwise just reflect the part above κ_m below as in Definition 2.21.
Consider $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$. Clearly, it is contained and belongs to N^m.
Let $\langle t_\xi \mid \xi < \lambda_m^{\vec{\nu}} \rangle$ be an enumeration of this forcing notion in N^m.
Let $f \in \mathcal{P}^*_{E(m)}, f \geq^* f^{pm}$.
Proceed by induction on $\xi < \lambda_m^{\vec{\nu}}$. Define an $\leq^*$ −increasing sequence $\langle f_\xi \mid \xi < \lambda_m^{\vec{\nu}} \rangle$ of direct extensions of f and an $\leq^*$ −increasing sequence $\langle p_\xi^{>m} \mid \xi < \lambda_m^{\vec{\nu}} \rangle$ of direct extensions of $\langle p_{m-1}(n) \mid m < n < \omega \rangle$

such that, for every $\xi < \lambda_m^{\vec{\nu}}$, either

(1) $\langle t_\xi, (f_\xi)_{\vec{\nu}}, p_\xi^{>m} \rangle \parallel \sigma$,
or

(2) for every $q \geq^* \langle (f_\xi)_{\vec{\nu}}, p_\xi^{>m} \rangle$, $\langle t_\xi, q \rangle \nparallel \sigma$.

Let $\bar{f} = \bigcup_{\xi < \lambda_1^{\vec{\nu}}} f_\xi$ and $\bar{p}^{>m}$ be a direct extension of $\langle p_\xi^{>m} \mid \xi < \lambda_1^{\vec{\nu}} \rangle$.
Then, for every $t \in \mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$ either

(1) $\langle t, \bar{f}_{\vec{\nu}}, \bar{p}^{>m} \rangle \parallel \sigma$,
or

(2) for every $q \geq^* \langle \bar{f}_{\vec{\nu}}, \bar{p}^{>m} \rangle$, $\langle t, q \rangle \nparallel \sigma$.

Consider now the following statement of the forcing language of $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$:

$$\varphi \equiv \exists t \in \underset{\sim}{G}(\langle t, \bar{f}_{\vec{\nu}}, \bar{p}^{>m} \rangle \parallel \sigma).$$

By the Prikry condition of the forcing $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$, there is $t^* \geq^* \langle p_{m-1}(n) \mid n < m \rangle$ which decides φ.

If $t^* \Vdash \neg\varphi$, then set $t(\vec{\nu}, f) = t^*$.
If $t^* \Vdash \varphi$, then use again the Prikry condition of the forcing $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$ to decide the following statement

$$\psi \equiv \exists t \in \underset{\sim}{G}(\langle t, \bar{f}_{\vec{\nu}}, \bar{p}^{>m} \rangle \Vdash \sigma).$$

Let $t(\vec{\nu}, f) \geq^* t^*$ be a condition which decides ψ.

Claim 4 Let $t \geq t(\vec{\nu}, f)$ in $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$, $\langle g, q\rangle \geq^* \langle \bar{f}_{\vec{\nu}}, \bar{p}^{>m}\rangle$ in $\mathcal{P}_{\langle E(n)|m\leq n<\omega\rangle}$.
Suppose that $\langle t, g, q\rangle \Vdash \sigma$ (or $\langle t, g, q\rangle \Vdash \neg\sigma$),
then already $\langle t(\vec{\nu}, f), \bar{f}_{\vec{\nu}}, \bar{p}^{>m}\rangle \Vdash \sigma$ (or $\langle t(\vec{\nu}, \bar{f}_{\vec{\nu}}, \bar{p}^{>m}\rangle \Vdash \neg\sigma$).

Proof. Let $t \geq t(\vec{\nu}, f)$ in $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$, $\langle g, q\rangle \geq^* \langle \bar{f}_{\vec{\nu}}, \bar{p}^{>m}\rangle$ in $\mathcal{P}_{\langle E(n)|m\leq n<\omega\rangle}$.
Suppose that $\langle t, g, q\rangle \Vdash \sigma$.
Then, for some $\xi < \lambda_1^{\vec{\nu}}, t = t_\xi$, and then, $\langle t, (f_\xi)_{\vec{\nu}}, p_\xi^{>m}\rangle \parallel \sigma$. So, $\langle t, \bar{f}_{\vec{\nu}}, \bar{p}^{>m}\rangle \parallel \sigma$.
Then $t^* \Vdash \varphi$. Hence, $\langle t(\vec{\nu}, f), \bar{f}_{\vec{\nu}}, \bar{p}^{>m}\rangle \Vdash \sigma$.
$\square$ of the claim.

Define $D_{\vec{\nu}}$ to be the set of all $\langle f, p_f^{>m}\rangle \in \mathcal{P}^*_{E(m)} \times \mathcal{P}_{\langle E(n)|m<n<\omega\rangle}$, $f \geq^* f^{p_{m-1}m}$, $p_f^{>m} \geq^* p_{m-1}^{>m}$, such that either

(1) $\langle t(\vec{\nu}, f), f_{\vec{\nu}}, p_f^{>m}\rangle \parallel \sigma$
or

(2) for every $t \geq t(\vec{\nu}, f)$ in $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$, for every $\langle g, q\rangle \geq^* \langle f_{\vec{\nu}}, p_f^{>m}\rangle$ in $\mathcal{P}_{\langle E(n)|m\leq n<\omega\rangle}$, $\langle t, g, q\rangle \not\parallel \sigma$.

The next claim follows now from the previous one:

Claim 5 *$D_{\vec{\nu}}$ is a dense open subset of $\mathcal{P}^*_{E(m)} \times \langle \mathcal{P}_{\langle E(n)|m<n<\omega\rangle}, \leq^* \rangle$ above $\langle f^{p_{m-1}m}, \langle p_{m-1}(n) \mid m < n < \omega\rangle\rangle$.*

$D_{\vec{\nu}}$ is definable with parameters in N^m, hence $D_{\vec{\nu}} \in N^m$.
Then, $\langle f^{m*}, p_{f^{m*}}^{>m}\rangle \in D_{\vec{\nu}}$, for every $\vec{\nu} \in A$.
So, for every $\vec{\nu} \in A$ we have either

(3) $\langle t(\vec{\nu}, f^{m*}), f_{\vec{\nu}}^{m*}, p_{f^{m*}}^{>m}\rangle \parallel \sigma$
or

(4) for every $t \geq t(\vec{\nu}, f^{m*})$ in $\mathcal{P}_{h^m_{\vec{E}\restriction m}(\vec{\nu}(\kappa_m))}$, for every $\langle g, q\rangle \geq^* \langle f_{\vec{\nu}}^{m*}, p_{f^{m*}}^{>m}\rangle$ in $\mathcal{P}_{\langle E(n)|m\leq n<\omega\rangle}$, $\langle t, g, q\rangle \not\parallel \sigma$.

Shrink A, if necessary, to a set $A^{m*} \in (E(m))(\text{dom}(f^{m*}))$, such that for any two $\vec{\nu}, \vec{\nu}' \in A^{m*}$ the decision is the same.

Consider now $\langle f^{m*}, A^{m*}\rangle$ it is a pure condition in $\mathcal{P}_{E(m)}$. Use the function $\vec{\nu} \mapsto t(\vec{\nu}, f^{m*})$ in order to get a pure condition in $\mathcal{P}_{\langle E(n)|n<m\rangle}$, just use the one this function represents in the ultrapower by $(E(m))(\text{dom}(f^{m*}))$. Denote it by $\langle\langle f^{n*}, A^{n*}\rangle \mid n < m\rangle$.

Let us explain how do we naturally combine the result into a condition in $\mathcal{P}_{\langle E(n)|n<\omega\rangle}$.
Let $t(\vec{\nu}, f^{m*}) = \langle\langle f^{n\vec{\nu}}, A^{n\vec{\nu}}\rangle \mid n < m\rangle$, for every $\vec{\nu} \in A^{m*}$. Consider $f^{n\vec{\nu}}, n < m$. It is a set of at most κ_n many pairs (α, β), where $\alpha < \lambda^{\vec{\nu}}_m < \kappa_m$ and β is either the empty sequence or an ordinal $< \kappa_n$.
Shrinking A^{m*} if necessary, we can assume that there are $\langle x_n \mid n < m\rangle$ and $\kappa^*_n < \kappa^+_n, n < m$ such that for every $\vec{\nu}, \vec{\nu}' \in A^{m*}$, for every $n < m$, the following hold:

1. $\mathrm{dom}(f^{n\vec{\nu}}) \cap \vec{\nu}(\kappa_m) = x_n$,
2. $\mathrm{dom}(f^{n\vec{\nu}}) \setminus \vec{\nu}(\kappa_m) = \{\gamma^{\vec{\nu}}_{\tau n} \mid \tau < \kappa^*_n\}$ is an increasing enumeration,
3. for every $\alpha \in x_n$, $f^{n\vec{\nu}}(\alpha) = f^{n\vec{\nu}'}(\alpha)$,
4. for every $\tau < \kappa^*_n$, $f^{n\vec{\nu}}(\gamma^{\vec{\nu}}_{\tau n}) = f^{n\vec{\nu}'}(\gamma^{\vec{\nu}'}_{\tau n})$

Consider, for every $n < m$ and $\tau < \kappa^*_n$ a function $s_{\tau n}$ on A^{m*} defined by setting $s_{\tau n}(\vec{\nu}) = \gamma^{\vec{\nu}}_{\tau n}$.
Let

$$\gamma_{\tau n} = j_{E(m)}(s_{\tau n})(\langle (j_{E(m)}(\alpha), \alpha) \mid \alpha \in \mathrm{dom}(f^{m*})\rangle).$$

Extend now f^{m*} to f^{m**} by adding all $\gamma_{\tau n}, \tau < \kappa^*_n, n < m$ to its domain and setting $f^{m**}(\gamma_{\tau n})$ to be the empty sequence whenever $\gamma_{\tau n} \notin \mathrm{dom}(f^{m*})$.
Define $A^{m**} \in E(m)(\mathrm{dom}(f^{m**})$ as follows.
Set $\vec{\nu} \in A^{m**}$ iff

1. $\vec{\nu} \restriction \mathrm{dom}(f^{m*}) \in A^{m*}$,
2. $\mathrm{dom}(\vec{\nu}) \supseteq \{\gamma_{\tau n} \mid \tau < \kappa^*_n, n < m\}$,
3. if $\gamma_{\tau n} \in \mathrm{dom}(f^{m*})$ and $f^{m*}(\gamma_{\tau n})$ is not the empty sequence, then $\vec{\nu}(\gamma_{\tau n}) > f^{m*}(\gamma_{\tau n})$, for every $n < m$,
4. $\vec{\nu}(\gamma_{\tau n}) = s_{\tau n}(\vec{\nu} \restriction \mathrm{dom}(f^{m*}))$, for every $n < m$.

For every $\vec{\nu} \in A^{m**}, n < m$, set $\langle g^{n\vec{\nu}}, B^{n\vec{\nu}}\rangle = \langle f^{n\vec{\nu}\restriction\mathrm{dom}(f^{m*})}, A^{n\vec{\nu}\restriction\mathrm{dom}(f^{m*})}\rangle$.
Consider the function $\vec{\nu} \mapsto \langle\langle g^{n\vec{\nu}}, B^{n\vec{\nu}}\rangle \mid n < m\rangle$, $\vec{\nu} \in A^{m**}$. Let $\langle\langle f^{n*}, A^{n*}\rangle \mid n < m\rangle$ be represented by it in the ultrapower with $E(m)$.

It follows that $\langle\langle\langle f^{n*}, A^{n*}\rangle \mid n < m\rangle, \langle f^{m**}, A^{m**}\rangle\rangle$ is a pure condition in $\mathcal{P}_{\langle E(n)|n\leq m\rangle}$ which extends $p_{m-1} \restriction \mathcal{P}_{\langle E(n)|n\leq m\rangle}$.
Extend purely $p^{>m}_{f^{m*}}$ in the obvious fashion to a condition $p^{>m}_{f^{m**}}$ in $\mathcal{P}_{\langle E(n)|m<n<\omega\rangle}$ such that

$\langle\langle\langle f^{n*}, A^{n*}\rangle \mid n < m\rangle, \langle f^{m**}, A^{m**}\rangle, p^{>m}_{f^{m**}}\rangle$ is a pure condition in $\mathcal{P}_{\langle E(n)|n<\omega\rangle}$. Then it extends p_{m-1}.
Set p_m to be $\langle\langle\langle f^{n*}, A^{n*}\rangle \mid n < m\rangle, \langle f^{m**}, A^{m**}\rangle, p^{>m}_{f^{m**}}\rangle$.

This completes the recursive construction of $\langle p_m \mid m < \omega\rangle$. Let $p_* \geq p_m$, for every $m < \omega$.

The next claim completes the argument:

Claim 6 $p_* \parallel \sigma$.

Proof. Suppose otherwise. Pick then $q \geq p_*$ to be a condition which decides σ and such that its last coordinate at which a non-direct extension was made is as small as possible.
Let $q \Vdash \sigma$ and this coordinate is some $m < \omega$.
Then there is $\vec{\nu} \in A^{p_*}(m)$ such that $q(m) \geq^* f^{p_*}(m)_{\vec{\nu}}$ in $\mathcal{P}^*_{E(m)}$. In addition, $q^{>m} \geq^* p^{>m}_*$ in $\mathcal{P}_{\langle E(n)|m<n<\omega\rangle}$, by the choice of m.

But, then condition (4) above cannot hold. Hence (3) is true, which means, that

$$\langle t(\vec{\nu}, f^{m*}), f^{m*}_{\vec{\nu}}, p^{>m}_{f^{m*}}\rangle \Vdash \sigma.$$

Then the same holds for every $\vec{\nu}' \in A^{p_*}(m)$. So, already $p_* \Vdash \sigma$.
Contradiction.
$\square$ of the claim.
$\square$

It follows now that the forcing $\langle \mathcal{P}, \leq\rangle$ preserves all the cardinals except maybe κ^+_ω. Using the arguments of the previous lemma it is possible to show (and we will show this later) that κ^+_ω is preserved as well.

Let G be a generic subset of $\langle \mathcal{P}, \leq\rangle$.

Lemma 2.26 *κ_ω remains a strong limit cardinal in $V[G]$.*

Proof. Given $p \in \mathcal{P}$ and $m < \omega$. Suppose that $p(m)$ is non-pure. Then $p(m)(\kappa_m)$ is defined, and hence also the reflection $h^m_\lambda(p(m)(\kappa_m))$ of λ below κ_m. By the definition of the forcing, then the part $\mathcal{P}_{\langle E(n)|n<m\rangle}$ above p will act as $\mathcal{P}_{\langle E(n)\restriction h^m_\lambda(p(m)(\kappa_m))|n<m\rangle}$. In particular, $2^{\kappa_n} \leq h^m_\lambda(p(m)(\kappa_m)) < \kappa_m$. The upper part of the forcing, i.e. $\mathcal{P}_{\langle E(n)|m\leq n<\omega\rangle}$, does not add new bounded subsets to κ_m.
So we are done.
$\square$

Lemma 2.27 *$(\kappa^+_\omega)^V$ remains a cardinal in $V[G]$.*

Let us state first the following:

Lemma 2.28 *Let $p \in \mathcal{P}$ and $\underset{\sim}{\zeta}$ be a $\langle \mathcal{P}, \leq \rangle -$name of an ordinal or just $p \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{\zeta}$ is an ordinal.*
Then there are $p^* \geq^* p$ and $n_1 < \ldots < n_k$, for some $k < \omega$, such that

1. *for every $i, 1 \leq i \leq k$, $p^*(n_i) = \langle f^{p^*}_{n_i}, A^{p^*}_{n_i} \rangle$,*
2. *for every $\vec{\nu}_1 \in A^{p^*}_{n_1}, \ldots, \vec{\nu}_k \in A^{p^*}_{n_k}$,*
   $p^* \frown \vec{\nu}_1 \ldots \frown \vec{\nu}_k$ decides $\underset{\sim}{\zeta}$.

The proof of this lemma repeats the proof of the Prikry condition of the forcing.
Proof of 2.27. Suppose otherwise. Then there is $\mu < \kappa_\omega$ such that, in $V[G]$, $\mathrm{cof}((\kappa_\omega^+)^V) = \mu$. Back in V, let $\langle \underset{\sim}{\zeta}_\tau \mid \tau < \mu \rangle$ be a name of a witnessing sequence.
Pick $\bar{n} < \omega$ with $\kappa_{\bar{n}} > \mu$. Let $p \in \mathcal{P}$ be such that $p(\bar{n}) \in \mathcal{P}^*_{E(\bar{n}}$, i.e. its $\bar{n}-$th coordinate is non-pure. Then above p the part $\mathcal{P}_{E(n) | n < \bar{n} \rangle}$ reflects down to $\mathcal{P}_{h^{\bar{n}}_{\vec{E} \restriction \bar{n}}(p(\bar{n})(\kappa_{\bar{n}}))}$, and so has cardinality below $\kappa_{\bar{n}}$.

Construct a sequence $\langle p_\tau \mid \tau < \mu \rangle$ of $\leq^* -$extensions of p such that, for every $\tau < \mu$,

1. p_τ satisfies the conclusion of Lemma 2.28 for $\underset{\sim}{\zeta}_\tau$,
2. $\langle p_\tau(n) \mid \bar{n} \leq n < \omega \rangle \leq^* \langle p_{\tau'}(n) \mid \bar{n} \leq n < \omega \rangle$ in the forcing $\mathcal{P}_{\langle E(n) | \bar{n} \leq n < \omega \rangle}$, for every $\tau < \tau' < \mu$.

Let $s \geq^* \langle p_\tau(n) \mid \bar{n} \leq n < \omega \rangle$ in the forcing $\mathcal{P}_{\langle E(n) | \bar{n} \leq n < \omega \rangle}$, for every $\tau < \mu$. Set $r = p \restriction \bar{n} \frown s$. Then, for every $\tau < \mu$, there is $\xi_\tau < \kappa_\omega^+$ such that

$$r \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{\zeta}_\tau < \xi_\tau,$$

since by the choice of p_τ, the number of possibilities for $\underset{\sim}{\zeta}_\tau$ has cardinality $< \kappa_\omega$.
Set $\xi = \bigcup_{\tau < \mu} \xi_\tau < \kappa_\omega^+$.

$$r \Vdash_{\langle \mathcal{P}, \leq \rangle} \langle \underset{\sim}{\zeta}_\tau \mid \tau < \mu \rangle \text{ is bounded by } \xi.$$

Contradiction.
□

Given $p \in \mathcal{P}$. Denote by $\mathrm{np}(p)$ the set of all coordinates n of p such that $p(n) \in \mathcal{P}^*_{E(n)}$, i.e. a non-pure extension was made at the coordinate n.

For each $\beta \in [\kappa_\omega, \lambda)$ we define in $V[G]$ a function $t_\beta : \omega \to \kappa_\omega$ as follows.
For every $n < \omega$, find $p \in G$ such that $n \in \mathrm{np}(p)$ and if $n_1 < \ldots < n_k$ is the increasing enumeration of $\mathrm{np}(p) \setminus n$ (i.e. $n = n_1$), then the following hold:

1. $\beta \in \mathrm{dom}(p(n_k))$.
   Set $\beta_k = \beta$.

2. For every $i, 1 \leq i \leq k-1$, $\beta_i \in \mathrm{dom}(p(n_i))$,
   where $\beta_i = p(n_{i+1})(\beta_{i+1})$.

Set $t_\beta(n) = p(n)(\beta_1)$.

Lemma 2.29 *In $V[G]$, if $\beta, \gamma \in [\kappa_\omega, \lambda)$ and $\beta < \gamma$, then there is $n^* < \omega$ such that for every $n, n^* \leq n < \omega$, $t_\beta(n) < t_\gamma(n)$.*

Proof. Work in V. Let $p \in \mathcal{P}$ be any condition and $\beta, \gamma \in [\kappa_\omega, \lambda), \beta < \gamma$.
Let n^* be a coordinate above $\mathrm{np}(p)$. Then $p(n) = \langle f_n^p, A_n^p \rangle$, for every $n, n^* \leq n < \omega$.
Extend p to p^* by adding β, γ to all $\mathrm{dom}(f_n^p)$ with $n^* \leq n < \omega$.
Now, by the definition of the order on $\mathcal{P}$, for every $n, n^* \leq n < \omega$ and every $q \geq p^*$ such that q defines $t_\beta(n)$ and $t_\gamma(n)$, we will have $t_\beta(n) < t_\gamma(n)$.
So,

$$p^* \Vdash (\forall n)(n^* \leq n < \omega \rightarrow \underset{\sim}{t}_\beta(n) < \underset{\sim}{t}_\gamma(n)).$$

□

It is possible to say a bit more. Namely, let in $V[G]$, for every $n < \omega$, λ_n be the reflection of λ below κ_n, i.e. for some $p \in G$ with $p(n) = f_n^p$, $\lambda_n = h_\lambda^n(f_n^p(\kappa_n))$. Then the following holds:

Lemma 2.30 *The sequence $\langle t_\beta \mid \beta \in [\kappa_\omega, \lambda) \rangle$ is a scale in $\langle \prod_{n<\omega} \lambda_n, <_{J^{bd}} \rangle$.*

3 Arbitrary cofinality.

Let η be any ordinal. We generalize the construction of the previous section to sequences of extenders of the length η. Generalization is straightforward. Let us repeat just the main points.

So, we deal now with a $\triangleleft$−increasing sequence $\vec{E} = \langle E(\alpha) \mid \alpha < \eta \rangle$, where each $E(\alpha)$ is a (κ_α, λ)−extender and $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ is an increasing sequence with $\eta < \kappa_0$.
Assume for simplicity that for every $\alpha < \eta$ there is $h^\alpha_\lambda : \kappa_\alpha \to \kappa_\alpha$ such that $j_{E(\alpha)}(h^\alpha_\lambda)(\kappa_\alpha) = \lambda$ and that there is $h^\alpha_{\vec{E}\restriction\alpha} : \kappa_\alpha \to V_{\kappa_\alpha}$ such that $j_{E(\alpha)}(h^\alpha_{\vec{E}\restriction\alpha})(\kappa_\alpha) = \vec{E} \restriction \alpha$.
Note that having a Woodin cardinal, it is possible to pick such $\vec{E}$ so that $E(\beta) = j_{E(\alpha)}(E(\beta)) \restriction \lambda$, for every $\beta < \alpha < \eta$.

Let $\mathcal{P}^*_{E(i)}, \mathcal{P}_{E(i)}, i < \eta$ be as defined before.
Define components $\mathcal{P}^{\{\beta_1,\ldots,\beta_k\}}_{\langle E(\alpha)|\alpha<\eta\rangle}, k < \omega, \beta_1 < \ldots < \beta_k < \eta$ of the main forcing $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$.

Definition 3.1 The set of pure conditions $\mathcal{P}^{\{\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$ consists of all sequences $\langle p(\alpha) \mid \alpha < \eta \rangle$ such that for every $\alpha < \eta$, the following hold:

1. $p(\alpha) = \langle f^\alpha, A^\alpha \rangle \in \mathcal{P}_{E(\alpha)}$,

2. for every $\beta < \alpha$, $\mathrm{dom}(f^\beta) \setminus \kappa_\alpha \subseteq \mathrm{dom}(f^\alpha)$,

3. for every $\beta < \alpha$, for every $\xi \in \mathrm{dom}(f^\beta) \setminus \kappa_\alpha$, if $f^\alpha(\xi)$ is not the empty sequence, then for every $\vec{\nu} \in A^\alpha$, $\xi \in \mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\xi) > f^\alpha(\xi)$.
   The idea behind is as in the case of two extenders.

4. For every $\beta < \alpha$ and $\vec{\nu} \in A^\alpha$, the measures $E(\beta)(\mathrm{dom}(f^\beta))$ and $(h^\alpha_{\vec{E}\restriction\alpha}(\vec{\nu}(\kappa_\alpha))(\beta))((\mathrm{dom}(f^\beta) \cap \kappa_\alpha) \cup \{\vec{\nu}(\xi) \mid \xi \in \mathrm{dom}(f^\beta) \setminus \kappa_\alpha\})$ are basically the same in the following sense:
$$X \in E(\beta)(\mathrm{dom}(f^\beta)) \text{ iff}$$
$$X^{ref} \in (h^\alpha_{\vec{E}\restriction\alpha}(\vec{\nu}(\kappa_\alpha))(\beta))((\mathrm{dom}(f^\beta) \cap \kappa_\alpha) \cup \{\vec{\nu}(\xi) \mid \xi \in \mathrm{dom}(f^\beta) \setminus \kappa_\alpha\}),$$
   where
$$X^{ref} = \{(\xi, \beta) \in X \mid \xi < \kappa_\alpha\} \cup \{(\vec{\nu}(\xi), \beta) \mid (\xi, \beta) \in X, \xi \geq \kappa_\alpha\}.$$
   Note that this property is true in the ultrapower by $E(\alpha)$, so it holds on a set of measure one, as well.

Turn now to non-pure extensions. As usual, in Magidor type of iterations, non-pure extensions are allowed only at finitely many coordinates.

Start with a non-pure extension at a single coordinate and then proceed by induction.

Definition 3.2 Let $\beta < \eta$. Define the set $\mathcal{P}^{\{\beta\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$ of conditions with only non-pure part over the coordinate β. $\mathcal{P}^{(\beta)}_{\langle E(\alpha)|\alpha<\eta\rangle}$ consists of all sequences $\langle p(\alpha) \mid \alpha < \eta\rangle$ such that for every $\alpha < \eta$, the following hold:

1. $\langle p(\alpha) \mid \alpha < \eta, \alpha \neq \beta\rangle$ is a pure condition in $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta,\alpha\neq\beta\rangle}$,
2. $p(\beta) = f^\beta \in \mathcal{P}^*_{E(\beta)}$,
3. $\mathrm{dom}(f^\beta) \setminus \kappa_\alpha \subseteq \mathrm{dom}(f^\alpha)$, for every $\alpha, \beta < \alpha < \eta$,
4. for every $\alpha, \beta < \alpha < \eta$, for every $\xi \in \mathrm{dom}(f^\beta) \setminus \kappa_\alpha$, if $f^\alpha(\xi)$ is not the empty sequence, then for every $\vec{\nu} \in A^\alpha$, $\xi \in \mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\xi) > f^\alpha(\xi)$,
5. for every $\alpha, \beta < \alpha < \eta$, for every $\gamma \in \mathrm{dom}(f^\beta) \cap \kappa_\alpha$,$\vec{\nu} \in A^\alpha$ and $\xi \in \mathrm{dom}(\vec{\nu})$, $\vec{\nu}(\xi) > \gamma$.
6. If $\beta > 0$, then the sequence $\langle p(\alpha) \mid \alpha < \beta\rangle$ will be a condition in the pure part of $\mathcal{P}_{h^\beta_{\vec{E}\upharpoonright\beta}(f^\beta(\kappa_\beta))}$. The meaning is that if the value of the Prikry sequence for the normal measure of $E(\beta)$ is decided, then we reflect all extenders $E(\alpha), \alpha < \beta$ below κ_β, i.e. to the corresponding $(\kappa_\alpha, h^\beta_\lambda(f^\beta(\kappa_\beta)))-$extenders.

Let $\beta_1 < ... < \beta_k < \eta, 1 \leq k < \omega$ and suppose that $\mathcal{P}^{\{\beta_1,...,\beta_k\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$ the set of conditions with non-pure extensions over coordinates $(\beta_1, ..., \beta_k)$ only, is defined.
Let $\beta < \eta, \beta \notin \{\beta_1, ..., \beta_k\}$.
Define non-pure extensions at the set of coordinates $\{\beta_1, ..., \beta_k\} \cup \{\beta\}$.

Definition 3.3 Let $\beta < \eta$. Define the set $\mathcal{P}^{\{\beta_1,...,\beta_k\}\cup\{\beta\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$ of conditions with only non-pure part over the coordinate $\beta_1, ..., \beta_k$ and β. $\mathcal{P}^{\{\beta_1,...,\beta_k\}\cup\{\beta\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$ consists of all sequences $\langle p(\alpha) \mid \alpha < \eta\rangle$ such that for every $\alpha < \eta$, the following hold:

1. $\langle p(\alpha) \mid \alpha < \eta, \alpha \neq \beta\rangle$ is a condition in $\mathcal{P}^{\{\beta_1,...,\beta_k\}}_{\langle E(\alpha)|\alpha<\eta,\alpha\neq\beta\rangle}$,
2. $p(\beta) = f^\beta \in \mathcal{P}^*_{E(\beta)}$.
3. If $\beta > \max\{\beta_1, ..., \beta_k\}$, then following hold:
   (a) $\mathrm{dom}(f^\beta) \setminus \kappa_\alpha \subseteq \mathrm{dom}(f^\alpha)$, for every $\alpha, \beta < \alpha < \eta$,

(b) for every $\alpha, \beta < \alpha < \eta$, for every $\xi \in \mathrm{dom}(f^\beta) \setminus \kappa_\alpha$, if $f^\alpha(\xi)$ is not the empty sequence, then for every $\vec{\nu} \in A^\alpha$, $\xi \in \mathrm{dom}(\vec{\nu})$ and $\vec{\nu}(\xi) > f^\alpha(\xi)$,

(c) for every $\alpha, \beta < \alpha < \eta$, for every $\gamma \in \mathrm{dom}(f^\beta) \cap \kappa_\alpha, \vec{\nu} \in A^\alpha$ and $\xi \in \mathrm{dom}(\vec{\nu})$, $\vec{\nu}(\xi) > \gamma$.

(d) If $\beta > 0$, then the sequence $\langle p(\alpha) \mid \alpha < \beta \rangle$ is a condition in $\mathcal{P}^{\{\beta_1,...,\beta_k\}}_{h^\beta_{\vec{E}\restriction\beta}(f^\beta(\kappa_\beta))}$.
The meaning is that if the value of the Prikry sequence for the normal measure of $E(\beta)$ is decided, then we reflect all extenders $E(\alpha), \alpha < \beta$ below κ_β, i.e. to the corresponding $(\kappa_\alpha, h^\beta_\lambda(f^\beta(\kappa_\beta)))-$extenders.

4. If $\beta < \max\{\beta_1, ..., \beta_k\}$, then let i^* be minimal such that $\beta < \beta_{i^*}$. Then the following hold:

(a) $\langle p(\alpha) \mid \alpha < \beta_{i^*} \rangle \in \mathcal{P}^{\{\beta_1,...,\beta_{i^*-1},\beta\}}_{h^{\beta_{i^*}}_{\vec{E}\restriction\beta_{i^*}(f^{\beta_{i^*}}(\kappa_{\beta_{i^*}}))}}$.

Finally set

$$\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle} = \bigcup\{\mathcal{P}^{\{\beta_1,...,\beta_k\}}_{\langle E(\alpha)|\alpha<\eta\rangle} \mid k < \omega, \beta_1 < ... < \beta_k < \eta\}.$$

Define the direct extension order $\leq^*$ over $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$ to be the union of such order over every $\mathcal{P}^{\{\beta_1,...,\beta_k\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$, for every $k < \omega, \beta_1 < ... < \beta_k < \eta$.

Turn now to the definition of the forcing order $\leq$ over $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$.

Let $\beta < \eta, \beta \notin \{\beta_1, ..., \beta_k\}$. Define a one element extension at coordinate β of a condition in $\mathcal{P}^{\{\beta_1,...,\beta_k\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$.

Definition 3.4 Let $p \in \mathcal{P}^{\{\beta_1,...,\beta_k\}\cup\{\beta\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$ and $q \in \mathcal{P}^{\{\beta_1,...,\beta_k\}}_{\langle E(\alpha)|\alpha<\eta\rangle}$. Set $p \geq q$ iff the following hold:

1. Suppose that $\beta = 0$.
Then $p(0) = f^0 \in \mathcal{P}^*_{E(0)}$ and $q(0) = \langle g^0, B^0 \rangle$ is a pure condition in $\mathcal{P}_{E(0)}$.
Set $p \geq q$ iff $f^0 \geq \langle g^0, B^0 \rangle$ in $\mathcal{P}_{E(0)}$ and $\langle p(\alpha) \mid 0 < \alpha < \eta \rangle \geq^* \langle q(\alpha) \mid 0 < \alpha < \eta \rangle$ in $\mathcal{P}_{\langle E(\alpha)|0<\alpha<\eta\rangle}$.

2. Suppose that $\beta > 0$.
Then $p(\beta) = f^\beta \in \mathcal{P}^*_{E(\beta)}$ and $q(\beta) = \langle g^\beta, B^\beta \rangle$ is a pure condition in $\mathcal{P}_{E(\beta)}$.
Set $p \geq q$ iff

(a) $f^\beta \geq \langle g^\beta, B^\beta\rangle$ in $\mathcal{P}_{E(\beta)}$ and $\langle p(\alpha) \mid \beta < \alpha < \eta\rangle \geq^* \langle q(\alpha) \mid \beta < \alpha < \eta\rangle$ in $\mathcal{P}_{\langle E(\alpha)|\beta<\alpha<\eta\rangle}$.
And

(b) $\langle p(\alpha) \mid \alpha < \beta\rangle \geq^* \langle q(\alpha) \mid \alpha < \beta\rangle^{ref}$ in $\mathcal{P}_{h^\beta_{\vec{E}\restriction\beta}(f^\beta(\kappa_\beta))}$, where
$\langle q(\alpha) \mid \alpha < \beta\rangle^{ref}$ - the reflection of $\langle q(\alpha) \mid \alpha < \beta\rangle$ below κ_β is defined as follows, where $q(\alpha) = \langle g^\alpha, B^\alpha\rangle$, if $\alpha \notin \{\beta_1, ..., \beta_k\}$ and $q(\alpha) = \langle g^\alpha\rangle$ otherwise.

i. Suppose first that $\alpha \in \{\beta_1, ..., \beta_k\}$.
Then

A. $\mathrm{dom}((g^\alpha)^{ref}) = (\mathrm{dom}(g^\alpha) \cap \kappa_\beta) \cup \{f^\beta(\xi) \mid \xi \in \mathrm{dom}(g^\alpha) \setminus \kappa_\beta\}$,

B. for every $\xi \in \mathrm{dom}(g^\alpha) \cap \kappa_\beta = \mathrm{dom}(g^\alpha) \cap \mathrm{dom}((g^\alpha)^{ref})$, $(g^\alpha)^{ref}(\xi) = g^\alpha(\xi)$,

C. for every $\xi \in \mathrm{dom}(g^\alpha) \setminus \kappa_\beta$, $(g^\alpha)^{ref}(f^\beta(\xi)) = g^\alpha(\xi)$.
It is crucial here that $f^\beta \restriction (\mathrm{dom}(g^\alpha) \setminus \kappa_\beta)$ is one to one and the values there are above $\mathrm{rng}(g^\alpha) \cap \kappa_\beta$.
This follows by conditions (4),(5) of Definitions 2.10,2.11.

ii. Suppose now that $\alpha \notin \{\beta_1, ..., \beta_k\}$.
Then

A. $\mathrm{dom}((g^\alpha)^{ref}) = (\mathrm{dom}(g^\alpha) \cap \kappa_\beta) \cup \{f^\beta(\xi) \mid \xi \in \mathrm{dom}(g^\alpha) \setminus \kappa_\beta\}$,

B. for every $\xi \in \mathrm{dom}(g^\alpha) \cap \kappa_\beta = \mathrm{dom}(g^\alpha) \cap \mathrm{dom}((g^\alpha)^{ref})$, $(g^\alpha)^{ref}(\xi) = g^\alpha(\xi)$,

C. for every $\xi \in \mathrm{dom}(g^\alpha) \setminus \kappa_\beta$, $(g^\alpha)^{ref}(f^\beta(\xi)) = g^\alpha(\xi)$.
Again, it is crucial here that $f^\beta \restriction (\mathrm{dom}(g^\alpha) \setminus \kappa_\beta)$ is one to one and the values there are above $\mathrm{dom}(g^\alpha) \cap \kappa_\beta$, and this follows by conditions (3),(4) of Definition 3.1 and (4),(5) of Definition 3.2.
One more crucial observation here is that the measure $(E(\alpha))(\mathrm{dom}(g^\alpha)$, to which B^α belongs, reflects to basically the same measure,
It follows by (4) of Definition 3.1.

D. $A^\alpha \restriction \mathrm{dom}((g^\alpha)^{ref}) \subseteq \{(\xi, \zeta) \mid (\xi, \zeta) \in B^\alpha, \xi < \kappa_\beta\} \cup \{(f^\beta(\xi), \zeta) \mid (\xi, \zeta) \in B^\alpha, \xi \geq \kappa_\beta\}$.

Denote further in this subsection $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$ by just $\mathcal{P}$.
The next lemma follows from the definitions:

Lemma 3.5 *For every $\beta < \eta$ and $p \in \mathcal{P}$ with $p(\beta) \in \mathcal{P}^*_{E(\beta)}$ (i.e. non-pure on the coordinate β), the part $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\beta\rangle}, \leq \rangle$ of $\mathcal{P}$ above p has cardinality $h^\beta_\lambda(p(\beta))(\kappa_\beta) < \kappa_\beta$.*

Lemma 3.6 *For every $\beta < \eta$, the forcing $\langle \mathcal{P}_{\langle E(\alpha)|\beta\leq\alpha<\eta\rangle}, \leq^* \rangle$ is $\kappa_\beta-$closed.*

Lemma 3.7 *The forcing $\langle \mathcal{P}, \leq \rangle$ satisfies $\kappa_\eta^{++}-$c.c.*

Lemma 3.8 *$\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.*

Proof. The proof proceeds by induction on the length of the sequence of extenders,i.e. on η. The argument repeats those of Lemma 2.25.
□

Denote for every limit $\alpha, 0 < \alpha \leq \eta$, $\bigcup_{\gamma<\alpha} \kappa_\gamma$ by $\bar{\kappa}_\alpha$.

It follows, by the previous lemmas, that the forcing $\langle \mathcal{P}, \leq \rangle$ preserves all the cardinals, except maybe $\bar{\kappa}_\alpha^+, 0 < \alpha \leq \eta$ a limit ordinal. Using the arguments of the previous lemma we will show that all such cardinals are preserved as well.

Let G be a generic subset of $\langle \mathcal{P}, \leq \rangle$.

Lemma 3.9 *For every limit ordinal $\mu, 0 < \mu \leq \eta$, $\bar{\kappa}_\mu$ remains a strong limit cardinal in $V[G]$.*

Proof. Given $p \in \mathcal{P}$ and $\beta < \eta$. Suppose that $p(\beta)$ is non-pure. Then $p(\beta)(\kappa_\beta)$ is defined, and hence also the reflection $h_\lambda^\beta(p(\beta)(\kappa_\beta))$ of λ below κ_β. By the definition of the forcing, then the part $\mathcal{P}_{\langle E(\alpha)|\alpha<\beta\rangle}$ above p will act as $\mathcal{P}_{h^\beta_{\vec{E}\restriction\beta}(p(\beta)(\kappa_\beta))}$. In particular, $2^{\kappa_\alpha} \leq h_\lambda^\beta(p(\beta)(\kappa_\beta)) < \kappa_\beta$. The upper part of the forcing, i.e. $\mathcal{P}_{\langle E(\alpha)|\beta\leq\alpha<\eta\rangle}$, does not add new bounded subsets to κ_β. So we are done.
□

As in the case $\eta = \omega$, the next lemma is just a variation of the Prikry condition of the forcing.

Lemma 3.10 *Let $p \in \mathcal{P}$ and $\underset{\sim}{\zeta}$ be a $\langle \mathcal{P}, \leq \rangle-$name of an ordinal or just*
$p \Vdash_{\langle \mathcal{P},\leq\rangle} \underset{\sim}{\zeta}$ is an ordinal.
Then there are $p^* \geq^* p$ and $\alpha_1 < ... < \alpha_k < \eta$, for some $k < \omega$, such that

1. *for every $i, 1 \leq i \leq k$, $p^*(\alpha_i) = \langle f_{\alpha_i}^{p^*}, A_{\alpha_i}^{p^*} \rangle$,*
2. *for every $\vec{\nu}_1 \in A_{\alpha_1}^{p^*}, ..., \vec{\nu}_k \in A_{\alpha_k}^{p^*}$,*
   $p^{*\frown}\vec{\nu}_1 ...^\frown \vec{\nu}_k$ decides $\underset{\sim}{\zeta}$.

Lemma 3.11 *For every limit ordinal $\mu, 0 < \mu \leq \eta$, $(\bar{\kappa}_\mu^+)^V$ remains a cardinal in $V[G]$.*

The proof of this lemma repeats those of Lemma 2.27.

Given $p \in \mathcal{P}$. Denote by np(p) the set of all coordinates α of p such that $p(\alpha) \in \mathcal{P}^*_{E(\alpha)}$, i.e. a non-pure extension was made at the coordinate α.

Assume that η is a limit ordinal.

For each $\tau \in [\bar{\kappa}_\eta, \lambda)$ we define in $V[G]$ a function $t_\tau : \eta \to \bar{\kappa}_\eta$ as follows.
For every $\alpha < \eta$, find $p \in G$ such that $\alpha \in$np(p) and if $\alpha_1 < ... < \alpha_k$ is the increasing enumeration of np$(p) \setminus \alpha$ (i.e. $\alpha = \alpha_1$), then the following hold:

1. $\tau \in \mathrm{dom}(p(\alpha_k))$.
   Set $\tau_k = \tau$.

2. For every $i, 1 \leq i \leq k-1$, $\tau_i \in \mathrm{dom}(p(\alpha_i))$,
   where $\tau_i = p(\alpha_{i+1})(\tau_{i+1})$.

Set $t_\tau(\alpha) = p(\alpha)(\tau_1)$.

Lemma 3.12 *In $V[G]$, if $\tau, \rho \in [\bar{\kappa}_\eta, \lambda)$ and $\tau < \rho$, then there is $\alpha^* < \eta$ such that for every $\alpha, \alpha^* \leq \alpha < \eta$, $t_\tau(\alpha) < t_\rho(\alpha)$.*

Proof. Work in V. Let $p \in \mathcal{P}$ be any condition and $\tau, \rho \in [\bar{\kappa}_\eta, \lambda)$,$\tau < \rho$.
Let α^* be a coordinate above np(p). Then $p(\alpha) = \langle f^p_\alpha, A^p_\alpha \rangle$, for every $\alpha, \alpha^* \leq \alpha < \eta$.
Extend p to p^* by adding τ, ρ to all $\mathrm{dom}(f^p_\alpha)$ with $\alpha^* \leq \alpha < \eta$.
Now, by the definition of the order on $\mathcal{P}$, for every $\alpha, \alpha^* \leq \alpha < \eta$ and every $q \geq p^*$ such that q defines $t_\tau(\alpha)$ and $t_\rho(\alpha)$, we will have $t_\tau(\alpha) < t_\rho(\alpha)$.
So,

$$p^* \Vdash (\forall \alpha)(\alpha^* \leq \alpha < \eta \to \underset{\sim}{t}_\tau(\alpha) < \underset{\sim}{t}_\rho(\alpha)).$$

□

It is possible to say a bit more. Namely, let in $V[G]$, for every $\alpha < \eta$, λ_α be the reflection of λ below κ_α, i.e. for some $p \in G$ with $p(\alpha) = f^p_\alpha$, $\lambda_\alpha = h^\alpha_\lambda(f^p_\alpha(\kappa_\alpha))$. Then the following holds:

Lemma 3.13 *The sequence $\langle t_\tau \mid \tau \in [\bar{\kappa}_\eta, \lambda) \rangle$ is a scale in $\langle \prod_{\alpha<\eta} \lambda_\alpha, <_{J^{bd}} \rangle$.*

In particular, we obtain the following:

Corollary 3.14 *It is possible to blow up the power of a singular in the core model[2] cardinal of arbitrary cofinality in a cardinal preserving extension.*

[2]Core model with strong cardinals, but below $o-$hand grenade. It was defined and studied by Ralf Schindler in [10]

4 One generalization.

In the previous section we assumed that $\eta < \kappa_0$ in order to blow up the power of a singular cardinal of cofinality η.
Let us now take η to be an inaccessible cardinal.
Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be now an increasing sequence with limit η and each $E(\alpha)$, for $\alpha < \eta$, be a $(\kappa_\alpha, \eta)-$extender.
Assume that η is the least inaccessible limit of κ_α's.

We proceed as in the previous section and define $\langle \mathcal{P}_{\langle E(\alpha) | \alpha < \eta \rangle}, \leq, \leq^* \rangle$. It shares the properties of the forcing of the previous section.
Let G be a generic subset of $\langle \mathcal{P}_{\langle E(\alpha) | \alpha < \eta \rangle}, \leq \rangle$.
Denote $\bigcup_{\beta < \alpha} \kappa_\beta$ by $\bar{\kappa}_\alpha$, for every $\alpha < \eta$. Then the following holds:

Theorem 4.1 *$V[G]$ is a cofinality preserving extension of V such that for every $\alpha < \eta$, $\bar{\kappa}_\alpha$ is a strong limit singular cardinal with $2^{\bar{\kappa}_\alpha} > \bar{\kappa}_\alpha^+$.*
In addition η remains inaccessible.

By passing to $V[G]_\eta$ we obtain the following:

Corollary 4.2 *It is possible to blow up the power of a proper class club of singular cardinals in the core model in a cofinality preserving extension.*

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