Blowing up the power of a singular cardinal of uncountable cofinality.

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Abstract

A new method for blowing up the power of a singular cardinal is presented. It allows to blow up the power of a singular in the core model cardinal of uncountable cofinality. The method make a use of overlapping extenders.

1 Introduction.

The purpose of this paper is to present a method for blowing up the power of a singular cardinal which differs from those used in [1] and in [2] to deal with cofinality $\omega$. The advantage of the present technique is that it generalizes to singular cardinals of uncountable cofinality, which was open.

The main result can be stated as follows:

Theorem 1.1 Assume GCH. Let $\eta$ be a regular cardinal. Suppose that there is an increasing sequence $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ of strong cardinals with $\kappa_0 > \eta$. Let $\lambda > \bigcup_{\alpha < \eta} \kappa_\alpha$ be a regular cardinal. Then there is a cardinal preserving extension in which $\bigcup_{\alpha < \eta} \kappa_\alpha$ is a strong limit cardinal and $2^{\bigcup_{\alpha < \eta} \kappa_\alpha} = \lambda$.

If $\eta > \aleph_0$ and $\lambda > (\bigcup_{\alpha < \eta} \kappa_\alpha)^+$, then, by [4], $\mathcal{O}^*$ should exists.

A slightly weaker assumption than $\eta$—many strongs is actually used.

We assume that there is a sequence $\langle E(\alpha) \mid \alpha < \eta \rangle$ of extenders such that for every $\alpha < \eta$

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1. $E(\alpha)$ is a $(\kappa_\alpha, \lambda)$—extender, 
i.e., $j_{E(\alpha)} : V \rightarrow M_{E(\alpha)} \cong \text{Ult}(V, E(\alpha))$, $\text{crit}(j_{E(\alpha)}) = \kappa_\alpha, j_{E(\alpha)}(\kappa_\alpha) > \lambda$, 
$M_{E(\alpha)} \supseteq H_\lambda, ^{\kappa_\alpha}M_{E(\alpha)} \subseteq M_{E(\alpha)}$;

2. for every $\beta < \alpha$, $E(\beta) \triangleleft E(\alpha)$.
Note that this condition is equivalent to $\langle E(\beta) \mid \beta < \alpha \rangle \in M_{E(\alpha)}$, 
since $^{\kappa_\alpha}M_{E(\alpha)} \subseteq M_{E(\alpha)}$.

Our conjecture is that this assumption is optimal for blowing up the power of singular
in the core model cardinal of uncountable cofinality.

We will start with countable cofinality. Then a general case will be considered and finally
some generalizations will be stated.

## 2 Blowing up the power of a singular cardinal of cofinality $\omega$.

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of cardinals, $\kappa_\omega = \bigcup_{n<\omega} \kappa_n$ and $\langle E_n \mid n < \omega \rangle$ be
a sequence such that for every $n < \omega$

1. $E(n)$ is a $(\kappa_n, \lambda)$—extender, 
i.e., $j_{E(n)} : V \rightarrow M_{E(n)} \cong \text{Ult}(V, E(n))$, $\text{crit}(j_{E(n)}) = \kappa_n, j_{E(n)}(\kappa_n) > \lambda$, 
$M_{E(n)} \supseteq H_\lambda, ^{\kappa_n}M_{E(n)} \subseteq M_{E(n)}$;

2. $E(n) \triangleleft E(n + 1)$.

Denote by $P(n)$ the one element extender based Prikry forcing with $E(n)$. We would
like to combine the $P(n)$’s together. It would be a kind of Magidor product, but will involve
restrictions and reflections. Namely, if for some $n < \omega$ a non-direct extension is made in
$P(n)$, then he will restrict each $E(m), m < n$ to the corresponding member of the Prikry
sequence for $\kappa_n$ and reflect the information the condition contains about coordinates $m < n$
below $\kappa_n$.

Let us start with a simpler situation where instead of $\omega$ extenders we have only two.

### 2.1 A single extender.

Let us describe a variation of the one element extender based Prikry forcing that will be used
here. It will be very close to those of C. Merimovich [8]. A difference will be that sequences
inside conditions will be either empty or of length one only.
Let $E$ be a $(\kappa, \lambda)$--extender. We will define the sets $\mathcal{P}^*_E$ and $\mathcal{P}_E^{(1)}$ which will lead us to the definition of the forcing notion $\mathcal{P}_E$.

Let $d \subseteq \lambda \setminus \kappa$ of cardinality at most $\kappa$. Define a $\kappa$--ultrafilter $E(d)$ on $[d \times \kappa]^{<\kappa}$ as follows:

$$X \in E(d) \iff \{\langle j_E(\alpha), \alpha \rangle \mid \alpha \in d\} \in j_E(X).$$

Actually, $E(d)$ concentrates on a smaller set called $\text{OB}(d)$ in [8].

The advantage of using $E(d)$ is that once $A$ is a typical set of $E(d)$--measure one and $a \in A$, then $a$ is of the form $\langle \langle \alpha_\xi, \beta_\xi \rangle \mid \xi < \rho \rangle$, where

1. $\rho < \kappa$,
2. $\text{dom}(a) = \{\alpha_\xi \mid \xi < \rho\} \subseteq d$,
3. $\beta_\xi < \kappa$, for every $\xi < \rho$.

So, a measure one set provides an explicit connection between elements of Prikry sequences and the measures to which they belong.

We assume further that always $\langle \alpha_\xi \mid \xi < \rho \rangle$ and $\langle \beta_\xi \mid \xi < \rho \rangle$ are strictly increasing sequences of ordinals.

**Definition 2.1** Let $\mathcal{P}^*_E$ be the set of all functions $f$ such that

1. $\text{dom}(f) \subseteq \lambda \setminus \kappa$ is of cardinality at most $\kappa$,
2. $\kappa \in \text{dom}(f)$,
3. for every $\alpha \in \text{dom}(f)$, $f(\alpha)$ is either empty or a one element sequence which consists of an element of $\kappa$.

**Definition 2.2** Let $f, g \in \mathcal{P}^*_E$. Set $f \geq^* g$ iff $f \supseteq g$.

**Definition 2.3** Let $f \in \mathcal{P}^*_E$ and $\tilde{v} \in [\text{dom}(f) \times \kappa]^{<\kappa}$. Define $g = f_{(\tilde{v})} \in \mathcal{P}^*_E$ as follows:

1. $\text{dom}(g) = \text{dom}(f)$,
2. for every $\alpha \in \text{dom}(g)$,

$$g(\alpha) = \begin{cases} 
\langle \tilde{v}(\alpha) \rangle, & \text{if } \alpha \in \text{dom}(\tilde{v}) \text{ and } f(\alpha) \text{ is empty sequence;} \\
\langle \tilde{v}(\alpha) \rangle, & \text{if } \alpha \in \text{dom}(\tilde{v}), f(\alpha) \text{ is not empty and } \tilde{v}(\alpha) > f(\alpha); \\
f(\alpha), & \text{otherwise.}
\end{cases}$$
The difference from the original definition by Merimovich in [8], is that we do not keep $f(\alpha)$ if $\bar{v}(\alpha) > f(\alpha)$, but rather replace $f(\alpha)$ by $\bar{v}(\alpha)$.

Define now the pure part $\mathcal{P}_E^1$ of the main forcing $\mathcal{P}_E$.

**Definition 2.4** A pure condition $p \in \mathcal{P}_E^1$ is of the form $\langle f, A \rangle$, where

1. $f \in \mathcal{P}_E^*$,
2. $f(\kappa)$ is the empty sequence,
3. $A \in E(\text{dom}(f))$.

Define the order on $\mathcal{P}_E^1$ as follows:

**Definition 2.5** Let $p = \langle f, A \rangle, q = \langle g, B \rangle \in \mathcal{P}_E^1$. Set $p \geq^* q$ iff

1. $f \geq^* g$ in $\mathcal{P}_E^*$,
2. $A \upharpoonright \text{dom}(g) \subseteq B$.

The forcing $\mathcal{P}_E$ will be the union of $\mathcal{P}_E^1$ with

$$\{ f \in \mathcal{P}_E^* \mid f(\kappa) \neq \langle \rangle \}.$$

The direct order extension will be just the union of $\leq^*$ orders of both parts. Let us define the forcing order $\leq$ on $\mathcal{P}$. We do this by defining one element extensions of members of $\mathcal{P}_E^1$.

**Definition 2.6** Let $p = \langle f, A \rangle$ be in $\mathcal{P}_E^1$ and $\bar{v} \in A$. Define $p \bar{v} \in \mathcal{P}_E^*$ to be $f(\bar{v})$.

**Definition 2.7** Let $p = \langle f, A \rangle$ be in $\mathcal{P}_E^1$ and $g$ be in $\mathcal{P}_E^*$. Set $p \leq g$ iff there is $\bar{v} \in A$ such that $f(\bar{v}) \leq^* g$.

The next lemma follows from the definitions:

**Lemma 2.8** The forcing $\langle \mathcal{P}_E, \leq \rangle$ is equivalent to the Cohen forcing for adding $\lambda$–many Cohen subsets to $\kappa^+$.

However, more can be deduced:

**Lemma 2.9** $\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.
Proof. Let us sketch the basic argument following Merimovich presentation \cite{8}. Let \( p = \langle f^p, A^p \rangle \in P^*_E \) and \( \sigma \) be a statement of the forcing language. We would like to find a direct extension of \( p \) which decides \( \sigma \). Suppose that there is no such extension.

Proceed as in 3.12 of \cite{7}. Construct by induction an increasing chain of elementary submodels \( \langle N_\xi \mid \xi < \kappa \rangle \) of \( H_\chi \), for \( \chi \) large enough, and a sequence \( \langle f_\xi \mid \xi < \kappa \rangle \) of members of \( P^*_E \), such that

1. \( p, P_E, \sigma \in N_0 \),
2. \( N_0 \supseteq \kappa \),
3. for every \( \xi < \kappa \),
   (a) \( |N_\xi| = \kappa \),
   (b) \( \kappa > N_\xi \subseteq N_\xi \),
   (c) \( \langle f_\xi \mid \zeta < \xi \rangle \in N_\xi \),
   (d) \( f_\xi \in \bigcap \{ D' \in N_\xi \mid D' \text{ is a dense open subset of } P^*_E \text{ above } f^p \} \),
   (e) \( f^p \leq^* f_0 \),
   (f) \( f_\xi \geq^* f_\zeta \), for every \( \zeta < \xi \).

Set \( N = \bigcup_{\xi < \kappa} N_\xi \) and \( f^* = \bigcup \{ f_\xi \mid \xi < \kappa \} \).

Let \( A \subseteq [\text{dom}(f^*) \times \kappa]^{<\kappa} \) be such that

- \( A \upharpoonright \text{dom}(f^p) \subseteq A^p \),
- \( A \in E(\text{dom}(f^*)) \).

Note that \( A \subseteq N \), since \( \text{dom}(f^*) \subseteq N \), and so, \([\text{dom}(f^*) \times \kappa]^{<\kappa} \subseteq N \).

Let \( \vec{u} \in A \).

Define \( D_{\vec{u}} \) to be the set of all \( f \in P^*_E, f \geq f^p \) such that

\[ f_{\vec{u}} \parallel \sigma. \]

\footnote{Carmi Merimovich pointed out that there is no need here in elementary chain of models and it is possible to define \( N \) directly. This observation applies also to our further constructions.}
Then $D_\vec{\alpha}$ is a dense open subset of $P_E^*$ above $f''$.
It is definable with parameters in $N$, hence $D_\vec{\alpha} \in N$.
Then, $f^* \in D_\vec{\alpha}$.

Shrink now $A$ to $A^* \in E(\text{dom}(f^*))$, if necessary, such that for every $\vec{\nu}, \vec{\nu}'$ inside $A^*$ we will have

$$f^*_\vec{\nu} \models \sigma \text{ iff } f^*_\vec{\nu}' \models \sigma.$$ 

Suppose that for every $\vec{\nu} \in A^*$, $f^*_\vec{\nu} \models \sigma$.

Now, we claim that already $\langle f^*, A^* \rangle \models \sigma$.
Supose otherwise. Then there is $g \geq \langle f^*, A^* \rangle$ which forces $\neg \sigma$. Then for some $\vec{\nu} \in A^*$, $g \geq f^*_\vec{\nu}$, by Definition 2.7. But $f^* \in D_\vec{\alpha}$, hence already $f^*_\vec{\nu} \models \neg \sigma$, which is impossible by the choice of $A^*$.
Contradiction.
\hfill $\square$

### 2.2 Two extenders.

We deal now with two extenders $E(0)$ and $E(1)$.

We will deal with the forcing notion $P_{E(0),E(1)}$. The definition uses the sets constructed in previous subsection, i.e., $P^*_{E(i)}, P^i_{E(i)}, P_{E(i)}, i < 2$. In addition we will define the following: $P^*_{E(0),E(1)}, P^i_{E(0),E(1)}, P^0_{E(0),E(1)}, P^1_{E(0),E(1)}$.

**Definition 2.10** The set of pure conditions $P^i_{(E(0),E(1))}$ consists of all pairs $\langle p(0), p(1) \rangle$ such that

1. $p(0) = \langle f^0, A^0 \rangle \in P^i_{E(0)}$,
2. $p(1) = \langle f^1, A^1 \rangle \in P^i_{E(1)}$,
3. $\text{dom}(f^0) \setminus \kappa_1 \subseteq \text{dom}(f^1)$,
4. for every $\alpha \in \text{dom}(f^0) \setminus \kappa_1$, if $f^1(\alpha)$ is not the empty sequence, then for every $\vec{\nu} \in A^1$, $\alpha \in \text{dom}(\vec{\nu})$ and $\vec{\nu}(\alpha) > f^1(\alpha)$.

The intuition behind this condition is that the current value $f^1(\alpha)$ may interfere with values of one element Prikry sequences over $\kappa_0$. Namely, with the $\alpha$–th Prikry sequence over $\kappa_0$. Now, if $\vec{\nu}(\alpha) > f^1(\alpha)$, then $f^*_\vec{\nu}(\alpha) = \vec{\nu}(\alpha)$, by Definition 2.3, and so, the value $f^1(\alpha)$ just disappears.
5. For every $\gamma \in \text{dom}(f^0) \cap \kappa_1, \vec{v} \in A^1$ and $\alpha \in \text{dom}(\vec{v})$, $\vec{v}(\alpha) > \gamma$.
   Note that $|\text{dom}(f^0)| \leq \kappa_0$, so it is easy to arrange this.

6. For every $\vec{v} \in A^1$, the measures $E(0)(\text{dom}(f^0))$ and $E(0)((\text{dom}(f^0) \cap \kappa_1) \cup \{\vec{v}(\alpha) \mid \alpha \in \text{dom}(f^0) \setminus \kappa_1\})$ are basically the same in the following sense:
   
   $X \in E(0)(\text{dom}(f^0))$ iff $X^{ref} \in E(0)((\text{dom}(f^0) \cap \kappa_1) \cup \{\vec{v}(\alpha) \mid \alpha \in \text{dom}(f^0) \setminus \kappa_1\}),$
   
   where
   
   $X^{ref} = \{(\alpha, \beta) \in X \mid \alpha < \kappa_1\} \cup \{((\vec{v}(\alpha), \beta) \mid (\alpha, \beta) \in X, \alpha \geq \kappa_1\}.$
   
   Note that this property is true in the ultrapower by $E(1)$, so it holds on a set of measure one, as well.

Turn now to non-pure extensions.
First consider the situation with non-pure part over $\kappa_0$.

**Definition 2.11** The set of conditions $\mathcal{P}^{(0)}_{(E(0), E(1))}$ consists of all pairs $\langle f^0, p(1) \rangle$ such that

1. $f^0 \in \mathcal{P}^*_{E(0)},$
2. $p(1) = \langle f^1, A^1 \rangle \in \mathcal{P}_{E(1)},$
3. $\text{dom}(f^0) \setminus \kappa_1 \subseteq \text{dom}(f^1),$
4. for every $\alpha \in \text{dom}(f^0) \setminus \kappa_1$, if $f^1(\alpha)$ is not the empty sequence, then for every $\vec{v} \in A^1,$ $\alpha \in \text{dom}(\vec{v})$ and $\vec{v}(\alpha) > f^1(\alpha),$
5. for every $\gamma \in \text{dom}(f^0) \cap \kappa_1, \vec{v} \in A^1$ and $\alpha \in \text{dom}(\vec{v}),$ $\vec{v}(\alpha) > \gamma.$

Now we define conditions with a pure part over $\kappa_0$ and a non-pure over $\kappa_1$.
Assume for simplicity that there is $h_\lambda : \kappa_1 \rightarrow \kappa_1$ such that $j_{E(1)}(h_\lambda)(\kappa_1) = \lambda.$

**Definition 2.12** The set of conditions $\mathcal{P}^{(1)}_{(E(0), E(1))}$ consists of all pairs $\langle p(0), f^1 \rangle$ such that

1. $f^1 \in \mathcal{P}^*_{E(1)},$
2. $f^1(\kappa_1)$ is non-empty,
3. $p(0) \in \mathcal{P}_{E(0)\mid h_\lambda(f^1(\kappa_1))}.$ The meaning is that if the value of the Prikry sequence for the normal measure of $E(1)$ is decided, then we cut $E(0)$ to the reflection of $\lambda$ below $\kappa_1$, i.e. to $h_\lambda(f^1(\kappa_1)).$
Define now a completely non-pure part of the forcing.

**Definition 2.13** The set of conditions $\mathcal{P}^*_\langle E(0), E(1) \rangle$ consists of all pairs $\langle f^0, f^1 \rangle$ such that

1. $f^1 \in \mathcal{P}^*_{E(1)}$,
2. $f^1(\kappa_1)$ is non-empty,
3. $f^0 \in \mathcal{P}^*_{E(0)}$,
4. $f^0(\kappa_0)$ is non-empty,
5. $\text{dom}(f^0) \subseteq h_\lambda(f^1(\kappa_1))$.

The meaning is that if the value of the Prikry sequence for the normal measure of $E(1)$ is decided, then we add only $h_\lambda(f^1(\kappa_1))$ Cohen subsets to $\kappa^+_0$.

Now let us put everything together.

**Definition 2.14** $\mathcal{P}_{\langle E(0), E(1) \rangle} = \mathcal{P}^{(1)}_{\langle E(0), E(1) \rangle} \cup \mathcal{P}^{(0)}_{\langle E(0), E(1) \rangle} \cup \mathcal{P}^{(1)}_{\langle E(0), E(1) \rangle} \cup \mathcal{P}^*_\langle E(0), E(1) \rangle$.

Define the orders $\leq, \leq^*$ over $\mathcal{P}_{\langle E(0), E(1) \rangle}$.

$\leq^*$ is just the union of the orders at each of the components.

Let us give now the main definition.

**Definition 2.15** Let $p, q \in \mathcal{P}_{\langle E(0), E(1) \rangle}$. If $p, q$ are in the same component, then set $p \geq q$ iff $p \geq^* q$. Suppose that they are in different components.

Split into cases.

1. Suppose that $q \in \mathcal{P}^{(1)}_{\langle E(0), E(1) \rangle}$; i.e., in the pure part of $\mathcal{P}_{\langle E(0), E(1) \rangle}$, $p \in \mathcal{P}^{(0)}_{\langle E(0), E(1) \rangle}$; i.e., only the part of $p$ over $\kappa_1$ is a pure condition.

Let then $q = \langle \langle g^0, B_0^0 \rangle, \langle g^1, B_1^1 \rangle \rangle$, $p = \langle f^0, \langle f^1, A^1 \rangle \rangle$.

Set $p \geq q$ iff $f^0 \geq g^0, B_0^0$ in $\mathcal{P}_{E(0)}$ and $f^1 \geq^* g^1, B_1^1$ in $\mathcal{P}_{E(1)}$.

2. Suppose that $q \in \mathcal{P}^{(1)}_{\langle E(0), E(1) \rangle}$; i.e., in the part over $\kappa_0$ is pure and those over $\kappa_1$ is not pure, $p \in \mathcal{P}^*_\langle E(0), E(1) \rangle$; i.e. $p$ is a completely non-pure condition.

Let then $q = \langle \langle g^0, B^0_0 \rangle, g^1 \rangle$ and $p = \langle f^0, f^1 \rangle$.

Set $p \geq q$ iff $f^0 \geq g^0, B_0^0$ in $\mathcal{P}_{E(0)}$ and $f^1 \geq g^1$ in $\mathcal{P}_{E(1)}$. 


3. (Principal case 1.)
Suppose that $q \in \mathcal{P}^{(0)}_{(E_0),E(1)}$, i.e. in the part over $\kappa_1$ is pure and those over $\kappa_0$ is not pure, $p \in \mathcal{P}^{*}_{(E_0),E(1)}$, i.e. $p$ is a completely non-pure condition.
Let then $q = \langle g^0, (g^1, B^1) \rangle$ and $p = \langle f^0, f^1 \rangle$.
Set $p \geq q$ iff $f^1 \geq \langle g^1, B^1 \rangle$ in $\mathcal{P}_{E(1)}$ and $f^0 \geq (g^0)^{\text{ref}}$ in $\mathcal{P}^*_{E_0 | h_\lambda (f^1(\kappa_1))}$, where $(g^0)^{\text{ref}}$
the reflection of $g^0$ below $\kappa_1$ is defined as follows:

(a) $\text{dom}((g^0)^{\text{ref}}) = (\text{dom}(g^0) \cap \kappa_1) \cup \{f^1(\alpha) | \alpha \in \text{dom}(g^0) \setminus \kappa_1\}$,

(b) for every $\alpha \in \text{dom}(g^0) \cap \kappa_1 = \text{dom}(g^0) \cap \text{dom}((g^0)^{\text{ref}})$, $(g^0)^{\text{ref}}(\alpha) = g^0(\alpha)$,

(c) for every $\alpha \in \text{dom}(g^0) \setminus \kappa_1$, $(g^0)^{\text{ref}}(f^1(\alpha)) = g^0(\alpha)$.

It is crucial here that $f^1 \upharpoonright (\text{dom}(g^0) \setminus \kappa_1)$ is one to one and the values there are above $\text{rng}(g^0) \cap \kappa_1$.
This follows by conditions (4),(5) of Definitions 2.10,2.11.

4. (Principal case 2.)
Suppose that $q \in \mathcal{P}^{(1)}_{(E_0),E(1)}$, i.e., both parts are pure, $p \in \mathcal{P}^{(1)}_{(E_0),E(1)}$, i.e., only the part over $\kappa_0$ is pure.
Let then $q = \langle \langle g^0, B^0 \rangle, (g^1, B^1) \rangle$ and $p = \langle \langle f^0, A^0 \rangle, f^1 \rangle$.
Set $p \geq q$ iff $f^1 \geq \langle g^1, B^1 \rangle$ in $\mathcal{P}_{E(1)}$ and $f^0 \geq (g^0)^{\text{ref}}$ in $\mathcal{P}^*_{E_0 | h_\lambda (f^1(\kappa_1))}$, where
$(g^0, B^0)^{\text{ref}}$ the reflection of $\langle g^0, B^0 \rangle$ below $\kappa_1$ is defined as follows:

(a) $\text{dom}((g^0)^{\text{ref}}) = (\text{dom}(g^0) \cap \kappa_1) \cup \{f^1(\alpha) | \alpha \in \text{dom}(g^0) \setminus \kappa_1\}$,

(b) for every $\alpha \in \text{dom}(g^0) \cap \kappa_1 = \text{dom}(g^0) \cap \text{dom}((g^0)^{\text{ref}})$, $(g^0)^{\text{ref}}(\alpha) = g^0(\alpha)$,

(c) for every $\alpha \in \text{dom}(g^0) \setminus \kappa_1$, $(g^0)^{\text{ref}}(f^1(\alpha)) = g^0(\alpha)$.

Again, it is crucial here that $f^1 \upharpoonright (\text{dom}(g^0) \setminus \kappa_1)$ is one to one and the values there are above $\text{dom}(g^0) \cap \kappa_1$, and this follows by conditions (4),(5) of Definitions 2.10,2.11.
One more crucial observation here is that the measure $(E(0))(\text{dom}(g^0))$, to which $B^0$ belongs, reflects to basically the same measure,
It follows by (6) of Definitions 2.10.

(d) $A^0 \upharpoonright \text{dom}((g^0)^{\text{ref}}) \subseteq (B^0)^{\text{ref}}$, where $(B^0)^{\text{ref}} = \{\bar{v}^{\text{ref}} | \bar{v} \in B^0\}$ and if $\bar{v} = \langle \langle \alpha_\xi, \beta_\xi \rangle | \xi < \rho \rangle$, then
\[\bar{v}^{\text{ref}} = \langle \langle \alpha_\xi, \beta_\xi \rangle | \xi < \rho, \alpha_\xi < \kappa_1 \rangle \cap \langle \langle f^1(\alpha_\xi), \beta_\xi \rangle | \xi < \rho, \alpha_\xi \geq \kappa_1 \rangle.\]
Denote further in this subsection $\mathcal{P}_{(E(0),E(1))}$ by just $\mathcal{P}$.

The next lemma follows from the definitions:

**Lemma 2.16** The forcing $\langle \mathcal{P}, \leq \rangle$ is equivalent to Cohen($\kappa_0^+, \eta) \times$ Cohen($\kappa_1^+, \lambda)$, for some $\eta < \kappa_1$ which depends on the choice of a non-pure condition for $\mathcal{P}_{E(1)}$.

However, as usual, more can be deduced:

**Lemma 2.17** $\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.

**Proof.** The proof is similar to those of Lemma 2.9 (and in turn to those of Merimovich [8]).

Suppose otherwise.

Let $p \in \mathcal{P}$ be a pure condition and $\sigma$ a statement of the forcing language which is undecided by pure extensions of $p$. Then $p$ is of the form $\langle \langle f_0^p, A_0^p \rangle, \langle f_1^p, A_1^p \rangle \rangle$.

Proceed as in 3.12 of [7]. Construct by induction an increasing chain of elementary submodels $\langle N_1^\gamma | \gamma < \kappa_1 \rangle$ of $H_\chi$, for $\chi$ large enough, and a sequence $\langle f_1^\gamma | \gamma < \kappa_1 \rangle$ of members of $\mathcal{P}_{E(1)}^*$, such that

1. $p, \mathcal{P}, \sigma \in N_0^1$,
2. $N_0^1 \supseteq \kappa_1$,
3. for every $\xi < \kappa_1$,
   (a) $|N_\xi^1| = \kappa_1$
   (b) $\kappa_1 > N_\xi^1 \subseteq N_\xi^1$
   (c) $\langle f_1^\gamma | \gamma < \xi \rangle \in N_\xi^1$
   (d) $f_\xi^1 \in \cap \{ D' \in N_\xi^1 | D' \text{ is a dense open subset of } \mathcal{P}_{E(1)}^* \text{ above } f_1^1 \}$
   (e) $f_1^1 \leq^* f_0^1$
   (f) $f_\xi^1 \geq^* f_\zeta^1$, for every $\zeta < \xi$.

Set $N^1 = \bigcup_{\xi < \kappa_1} N_\xi^1$ and $f^{1*} = \bigcup \{ f_\xi^1 | \xi < \kappa \}$. Let $A \subseteq [\text{dom}(f^{1*}) \times \kappa_1]^{< \kappa_1}$ be such that

- $A \upharpoonright \text{dom}(f_1^1) \subseteq A_0^1$,
- $A \in (E(1))(\text{dom}(f^{1*}))$. 

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Note that $A \subseteq N^1$, since $\text{dom}(f^{1*}) \subseteq N^1$, and so, $[\text{dom}(f^{1*}) \times \kappa_1]^{<\kappa_1} \subseteq N^1$.

Let $\bar{\nu} \in A$. Consider $\lambda^\bar{\nu}_1 := h_\lambda(\bar{\nu}(\kappa_1))$, i.e. the cardinal below $\kappa_1$ that now corresponds to $\lambda$. Suppose for simplicity that $\text{dom}(f^{p_0}) \subseteq \lambda^\bar{\nu}_1$, otherwise just reflect the part above $\kappa_1$ below as in Definition 2.15.

Consider $P_{E(0)\lambda^\bar{\nu}_1}$. Clearly, it is contained and belongs to $N^1$.

Let $\langle t_\xi : \xi < \lambda^\bar{\nu}_1 \rangle$ be an enumeration of this forcing notion in $N_1$.

Let $f \in P_{E(1)}^*, f \geq^* f^{p_1}$.

Proceed by induction on $\xi < \lambda^\bar{\nu}_1$ and define an $\leq^*$-increasing sequence $\langle f_\xi : \xi < \lambda^\bar{\nu}_1 \rangle$ of direct extensions of $f$ such that, for every $\xi < \lambda^\bar{\nu}_1$, either

1. $\langle t_\xi, (f_\xi)_{\bar{\nu}} \rangle \not\forces \sigma$,
   or
2. for every $g \geq^* (f_\xi)_{\bar{\nu}}, \langle t_\xi, g \rangle \forces \sigma$.

Let $\bar{f} = \bigcup_{\xi < \lambda^\bar{\nu}_1} f_\xi$.

Then, for every $t \in P_{E(0)\lambda^\bar{\nu}_1}$ either

1. $\langle t, \bar{f}_{\bar{\nu}} \rangle \not\forces \sigma$,
   or
2. for every $g \geq^* \bar{f}_{\bar{\nu}}, \langle t, g \rangle \forces \sigma$.

Consider now the following statement of the forcing language of $P_{E(0)\lambda^\bar{\nu}_1}$:

$$\varphi \equiv \exists t \in G(\langle t, \bar{f}_{\bar{\nu}} \rangle \forces \sigma).$$

By the Prikry condition of the forcing $P_{E(0)\lambda^\bar{\nu}_1}$ (Lemma 2.9 ), there is $t^* \geq^* (f^{p_0}, A^{p_0})$ which decides $\varphi$.

**Claim 1** $t^* \forces \varphi$.

*Proof.* Suppose otherwise. Then $t^* \not\forces \neg \varphi$. This means that whenever $t \in P_{E(0)\lambda^\bar{\nu}_1}$ and $t \geq t^*$, $\langle t, \bar{f}_{\bar{\nu}} \rangle \not\forces \sigma$.

Pick now some $\langle t, g \rangle \in P_{E(0),E(1)}, \langle t, g \rangle \geq \langle t^*, \bar{f}_{\bar{\nu}} \rangle$ which decides $\sigma$.

Then, for some $\xi < \lambda^\bar{\nu}_1, t = t_\xi$, and then, $\langle t, (f_\xi)_{\bar{\nu}} \rangle \forces \sigma$. So, $\langle t, \bar{f}_{\bar{\nu}} \rangle \not\forces \sigma$.

Contradiction.

$\square$ of the claim.
Now use again the Prikry condition of the forcing $\mathcal{P}_{E(0)|\lambda_1^\natural}$ to decide the following statement

$$\psi \equiv \exists t \in G(\langle t, \bar{f}_\varnothing \rangle \Vdash \sigma).$$

Let $t(\bar{v}, f) \triangleright^* t^*$ be a condition which decides $\psi$. If $t(\bar{v}, f) \Vdash \psi$, then $\langle t(\bar{v}, f), \bar{f}_\varnothing \rangle \Vdash \sigma$.

If $t(\bar{v}, f) \Vdash \neg \psi$, then $\langle t(\bar{v}, f), \bar{f}_\varnothing \rangle \Vdash \neg \sigma$.

Define $D_\varnothing$ to be the set of all $f \in \mathcal{P}_{E(1)^*}, f \triangleright^* f^{p1}$ such that

$$\langle t(\bar{v}, f), \bar{f}_\varnothing \rangle \Vdash \sigma.$$ 

The next claim follows now:

**Claim 2** $D_\varnothing$ is a dense open subset of $\mathcal{P}_{E(1)^*}$ above $f^{p1}$.

$D_\varnothing$ is definable with parameters in $N$, hence $D_\varnothing \in N$.

Then, $f^{1*} \in D_\varnothing$, for every $\bar{v} \in A$.

So, $\langle t(\bar{v}, f^{1*}), f^{1*}_\varnothing \rangle \Vdash \sigma$, for every $\bar{v} \in A$. Shrink $A$, if necessary, to a set $A^{1*} \in (E(1))(\text{dom}(f^{1*}))$, such that for any two $\bar{v}, \bar{v}' \in A^{1*}$ the decision is the same, say $\sigma$ is forced.

Consider now $\langle f^{1*}, A^{1*} \rangle$. It is a pure condition in $\mathcal{P}_{E(1)}$. Use the function $\bar{v} \mapsto t(\bar{v}, f^{1*})$ in order to get a pure condition in $\mathcal{P}_{E(0)}$, just use the one which this function represents in the ultrapower by $(E(1))(\text{dom}(f^{1*}))$.

Let us explain how do we naturally combine the result into a condition in $\mathcal{P}_{E(0), E(1)}$.

Let $t(\bar{v}, f^{1*}) = \langle f^{0\bar{v}}, A^{0\bar{v}} \rangle$, for every $\bar{v} \in A^{1*}$. Consider $f^{0\bar{v}}$. It is a set of at most $\kappa_0$ many pairs $(\alpha, \beta)$, where $\alpha < \lambda_1^\varnothing < \kappa_1$ and $\beta$ is either the empty sequence or an ordinal $< \kappa_0$.

Shrinking $A^{1*}$ if necessary, we can assume that there are $x$ and $\kappa^*_0 < \kappa^+_0$ such that for every $\bar{v}, \bar{v}' \in A^{1*}$ the following hold:

1. $\text{dom}(f^{0\bar{v}}) \cap \bar{v}(\kappa_1) = x$,
2. $\text{dom}(f^{0\bar{v}}) \setminus \bar{v}(\kappa_1) = \{ \gamma^\varnothing_\tau \mid \tau < \kappa^*_0 \}$ is an increasing enumeration,
3. for every $\alpha \in x$, $f^{0\bar{v}}(\alpha) = f^{0\bar{v}}(\alpha)$,
4. for every $\tau < \kappa^*_0$, $f^{0\bar{v}}(\gamma^\varnothing_\tau) = f^{0\bar{v}}(\gamma^\varnothing_\tau)$

Consider, for every $\tau < \kappa^*_0$ a function $s_\tau$ on $A^{1*}$ defined by setting $s_\tau(\bar{v}) = \gamma^\varnothing_\tau$.

Let

$$\gamma_\tau = j_{E(1)}(s_\tau)(\langle (j_{E(1)}(\alpha), \alpha) \mid \alpha \in \text{dom}(f^{1*}) \rangle).$$
Extend now $f^{1*}$ to $f^{1**}$ by adding all $\gamma_\tau, \tau < \kappa_0^*$ to its domain and setting $f^{1**}(\gamma_\tau)$ to be the empty sequence whenever $\gamma_\tau \notin \text{dom}(f^{1*})$.

Define $A^{1**} \in E(1)(\text{dom}(f^{1**}))$ as follows.

Set $\vec{\nu} \in A^{1**}$ iff

1. $\vec{\nu} \upharpoonright \text{dom}(f^{1*}) \in A^{1*},$
2. $\text{dom}(\vec{\nu}) \supseteq \{\gamma_\tau \mid \tau < \kappa_0^*\},$
3. if $\gamma_\tau \in \text{dom}(f^{1*})$ and $f^{1*}(\gamma_\tau)$ is not the empty sequence, then $\vec{\nu}(\gamma_\tau) > f^{1*}(\gamma_\tau),$
4. $\vec{\nu}(\gamma_\tau) = s_\tau(\vec{\nu} \upharpoonright \text{dom}(f^{1*})).$

For every $\vec{\nu} \in A^{1**}$, set $\langle g^{\vec{\nu}}, B^{\vec{\nu}} \rangle = \langle f^{0\vec{\nu}\upharpoonright \text{dom}(f^{1*})}, A^{0\vec{\nu}\upharpoonright \text{dom}(f^{1*})} \rangle.$

Consider the function $\vec{\nu} \mapsto \langle g^{\vec{\nu}}, B^{\vec{\nu}} \rangle$, $\vec{\nu} \in A^{1**}$. Let $\langle f^{0*}, A^{0*} \rangle$ be represented by it in the ultrapower with $E(1)$.

It follows that $\langle \langle f^{0*}, A^{0*} \rangle, \langle f^{1**}, A^{1**} \rangle \rangle$ is a pure condition in $P_{E(0), E(1)}$ which extends $p$.

The next claim completes the argument:

**Claim 3** $\langle \langle f^{0*}, A^{0*} \rangle, \langle f^{1**}, A^{1**} \rangle \rangle \Vdash \sigma$.

**Proof.** Suppose otherwise. Then there is $\langle f, g \rangle \geq \langle \langle f^{0*}, A^{0*} \rangle, \langle f^{1**}, A^{1**} \rangle \rangle$ a non-pure in both coordinates which forces $\neg \sigma$. There is $\vec{\nu} \in A^{1**} \upharpoonright \text{dom}(f^{1*})$ such that $g \geq f^{1*}_{\vec{\nu}}$. But then $f \geq t(\vec{\nu}, f^{1*})$, and so, $\langle f, f^{1*}_{\vec{\nu}} \rangle \Vdash \sigma$. Contradiction.

$\square$ of the claim.

$\Box$

### 2.3 $\omega$–many extenders.

We deal now with a sequence $\langle E(n) \mid n < \omega \rangle$, where each $E(n)$ is a $(\kappa_n, \lambda)$–extender and $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence.

Define the forcing notion $P_{E(n) \mid n < \omega}$. The definition will use several components. Let $P_{E(i)}^{*}, P_{E(i)}$, $i < \omega$ be as defined before. In addition we will define the following sets: $P_{\{m_1, ..., m_k\} \mid E(n) \mid n < \omega}$, where $k < \omega$ and $m_1 < ... < m_k$.

**Definition 2.18** The set of pure conditions $P_{\{E(n) \mid n < \omega\}}^{*}$ consists of all sequences $\langle p(n) \mid n < \omega \rangle$ such that for every $n < \omega$, the following hold:

1. $p(n) = \langle f^n, A^n \rangle \in P_{E(n)}$,
2. \( \text{dom}(f^n) \setminus \kappa_{n+1} \subseteq \text{dom}(f^{n+1}) \),

3. for every \( m \leq n \), for every \( \alpha \in \text{dom}(f^m) \setminus \kappa_{n+1} \), if \( f^{n+1}(\alpha) \) is not the empty sequence, then for every \( \tilde{\nu} \in A^{n+1} \), \( \alpha \in \text{dom}(\tilde{\nu}) \) and \( \tilde{\nu}(\alpha) > f^{n+1}(\alpha) \).

The idea behind this is as in the case of two extenders.

4. For every \( \tilde{\nu} \in A^{n+1} \) and \( m \leq n \), the measures \( E(m)(\text{dom}(f^m)) \) and \( E(m)((\text{dom}(f^m) \cap \kappa_{n+1}) \cup \{ \tilde{\nu}(\alpha) \mid \alpha \in \text{dom}(f^m) \setminus \kappa_{n+1} \}) \) are basically the same in the following sense:

\[
X \in E(m)(\text{dom}(f^m)) \iff X^{ref} \in E(m)((\text{dom}(f^m) \cap \kappa_{n+1}) \cup \{ \tilde{\nu}(\alpha) \mid \alpha \in \text{dom}(f^m) \setminus \kappa_{n+1} \}),
\]

where

\[
X^{ref} = \{ (\alpha, \beta) \in X \mid \alpha < \kappa_{n+1} \} \cup \{ (\tilde{\nu}(\alpha), \beta) \mid (\alpha, \beta) \in X, \alpha \geq \kappa_{n+1} \}.
\]

Note that this property is true in the ultrapower by \( E(n + 1) \), so it holds on a set of measure one, as well.

Turn now to non-pure extensions. As usual, in Magidor type iterations, non-pure extensions are allowed only at finitely many coordinates.

Start with a non-pure extension at a single coordinate and then proceed by induction.

We assume that for each \( m < \omega \) there is a function \( h^m_\lambda : \kappa_m \to \kappa_m \) such that \( j_{E(m)}(h^m_\lambda)(\kappa_m) = \lambda \).

**Definition 2.19** Let \( m < \omega \). Define the set \( \mathcal{P}^{(m)}_{(E(n)|n<\omega)} \) of conditions with only non-pure part over the coordinate \( m \). \( \mathcal{P}^{(m)}_{(E(n)|n<\omega)} \) consists of all sequences \( \langle p(n) \mid n < \omega \rangle \) such that for every \( n < \omega \), the following hold:

1. \( \langle p(n) \mid n < \omega, n \neq m \rangle \) is a pure condition in \( \mathcal{P}_{(E(n)|n<\omega, n\neq m)} \),

2. \( p(m) = f^m \in \mathcal{P}^*_E(m) \),

3. \( \text{dom}(f^m) \setminus \kappa_n \subseteq \text{dom}(f^n) \), for every \( n, m < n < \omega \),

4. for every \( n, m < n < \omega \), for every \( \alpha \in \text{dom}(f^m) \setminus \kappa_n \), if \( f^n(\alpha) \) is not the empty sequence, then for every \( \tilde{\nu} \in A^n, \alpha \in \text{dom}(\tilde{\nu}) \) and \( \tilde{\nu}(\alpha) > f^n(\alpha) \),

5. for every \( n, m < n < \omega \), for every \( \gamma \in \text{dom}(f^m) \cap \kappa_n, \tilde{\nu} \in A^n \) and \( \alpha \in \text{dom}(\tilde{\nu}), \tilde{\nu}(\alpha) > \gamma \).
6. If \( m > 0 \), then the sequence \( \langle p(n) \mid n < m \rangle \) is a condition in the pure part of \( \mathcal{P}_{\langle E(n)h_{\lambda}(f^m(\kappa_m))\rangle\mid n < m} \). The meaning is that if the value of the Prikry sequence for the normal measure of \( E(m) \) is decided, then we cut all extenders \( E(n), n < m \) to the reflection of \( \lambda \) below \( \kappa_m \), i.e. to \( h_\lambda^m(f^m(\kappa_m)) \).

Let \( m_1 < \ldots < m_k < \omega, 1 \leq k < \omega \) and suppose that \( \mathcal{P}_{\langle E(n)\rangle\mid n < \omega} \) the set of conditions with non-pure extensions over coordinates \( (m_1, \ldots, m_k) \) only, is defined.

Let \( m < \omega, m \notin \{m_1, \ldots, m_k\} \).

Define non-pure extensions at the set of coordinates \( \{m_1, \ldots, m_k\} \cup \{m\} \).

**Definition 2.20** Let \( m < \omega \). Define the set \( \mathcal{P}_{\langle E(n)\rangle\mid n < \omega} \) of conditions with only non-pure part over the coordinate \( m_1, \ldots, m_k \) and \( m \). \( \mathcal{P}_{\langle E(n)\rangle\mid n < \omega} \) consists of all sequences \( \langle p(n) \mid n < \omega \rangle \) such that for every \( n < \omega \), the following hold:

1. \( \langle p(n) \mid n < \omega, n \neq m \rangle \) is a condition in \( \mathcal{P}_{\langle E(n)\rangle\mid n < \omega, n \neq m} \),
2. \( p(m) = f^m \in \mathcal{P}_{E(m)} \),
3. If \( m > \max\{m_1, \ldots, m_k\} \), then following hold:
   
   (a) \( \text{dom}(f^m) \setminus \kappa_n \subseteq \text{dom}(f^n) \), for every \( n, m < n < \omega \),
   
   (b) for every \( n, m < n < \omega \), for every \( \alpha \in \text{dom}(f^m) \setminus \kappa_n \), if \( f^n(\alpha) \) is not the empty sequence, then for every \( \vec{v} \in A^n, \alpha \in \text{dom}(\vec{v}) \) and \( \vec{v}(\alpha) > f^n(\alpha) \),
   
   (c) for every \( n, m < n < \omega \), for every \( \gamma \in \text{dom}(f^m) \cap \kappa_n, \vec{v} \in A^n \) and \( \alpha \in \text{dom}(\vec{v}) \), \( \vec{v}(\alpha) > \gamma \).
   
   (d) If \( m > 0 \), then the sequence \( \langle p(n) \mid n < m \rangle \) is a condition in \( \mathcal{P}_{\langle E(n)\rangle\mid h_{\lambda}(f^m(\kappa_m))\rangle\mid n < m} \).

   The meaning is that if the value of the Prikry sequence for the normal measure of \( E(m) \) is decided, then we cut all extenders \( E(n), n < m \) to the reflection of \( \lambda \) below \( \kappa_m \), i.e. to \( h_\lambda^m(f^m(\kappa_m)) \).

4. If \( m \leq \max\{m_1, \ldots, m_k\} \), then let \( i^* \) be the least such that \( m \leq m_{i^*} \). We require the following:

   (a) \( \langle p(n) \mid n < m_{i^*} \rangle \in \mathcal{P}_{\langle E(n)\rangle\mid h_{\lambda}(f^{m_{i^*}}(\kappa_{m_{i^*}}))\rangle\mid n < m_{i^*}} \).
Finally set

\[ P_{E(n)|n<\omega} = \bigcup \{ P_{\{E(n)|n<\omega\}|m_k} \mid k < \omega, m_1 < \ldots < m_k < \omega \}. \]

Define the direct extension order \( \leq^* \) over \( P_{E(n)|n<\omega} \) to be the union of such orders over every \( P_{\{E(n)|n<\omega\}|m_k} \), for every \( k < \omega, m_1 < \ldots < m_k < \omega \).

Turn now to the definition of the forcing order \( \leq \) over \( P_{E(n)|n<\omega} \).

Let \( m < \omega, m \notin \{m_1, \ldots, m_k\} \). Define a one element extension at coordinate \( m \) of a condition in \( P_{\{E(n)|n<\omega\}|m_k} \).

**Definition 2.21** Let \( p \in P_{\{E(n)|n<\omega\}|m_k} \) and \( q \in P_{\{E(n)|n<\omega\}|m_k} \). Set \( p \geq q \) iff the following hold:

1. Suppose that \( m = 0 \).
   
   Then \( p(0) = f^0 \in P_{E(0)} \) and \( q(0) = (g^0, B^0) \) is a pure condition in \( P_{E(0)} \).
   
   Set \( p \geq q \) iff \( f^0 \geq (g^0, B^0) \) in \( P_{E(0)} \) and \( \langle p(n) \mid 0 < n < \omega \rangle \geq^* \langle q(n) \mid 0 < n < \omega \rangle \) in \( P_{\{E(n)|0<n<\omega\}} \).

2. Suppose that \( m > 0 \).
   
   Then \( P(m) = f^m \in P_{E(m)} \) and \( q(m) = (g^m, B^m) \) is a pure condition in \( P_{E(m)} \).
   
   Set \( p \geq q \) iff

   (a) \( f^m \geq (g^m, B^m) \) in \( P_{E(m)} \) and \( \langle p(n) \mid m < n < \omega \rangle \geq^* \langle q(n) \mid m < n < \omega \rangle \) in \( P_{\{E(n)|m<n<\omega\}} \).

   And

   (b) \( \langle p(n) \mid n < m \rangle \geq^* \langle q(n) \mid n < m \rangle^{ref} \) in \( P_{\{E(n)|n<m\}} \), where \( \langle q(n) \mid n < m \rangle^{ref} \) - the reflection of \( \langle q(n) \mid n < m \rangle \) below \( \kappa_m \) is defined as follows, where \( q(n) = (g^n, B^n) \), if \( n \notin \{m_1, \ldots, m_k\} \) and \( q(n) = (g^n) \) otherwise.

   i. Suppose first that \( n \in \{m_1, \ldots, m_k\} \).

   Then

   A. \( \text{dom}((g^n)^{ref}) = (\text{dom}(g^n) \cap \kappa_m) \cup \{ f^m(\alpha) \mid \alpha \in \text{dom}(g^n) \setminus \kappa_m \} \),

   B. for every \( \alpha \in \text{dom}(g^n) \cap \kappa_m = (g^n)^{ref} \cap \text{dom}(g^n)^{ref} \), \( (g^n)^{ref}(\alpha) = g^n(\alpha) \),

   C. for every \( \alpha \in \text{dom}(g^n) \setminus \kappa_m \), \( (g^n)^{ref}(f^m(\alpha)) = g^n(\alpha) \).

   It is crucial here that \( f^m \mid (\text{dom}(g^n) \cap \kappa_m) \) is one to one and the values there are above \( \text{rng}(g^n) \cap \kappa_m \).

   This follows by conditions (4),(5) of Definitions 2.10,2.11.
ii. Suppose now that

\[ n \not\in \{m_1, ..., m_k\}. \]

Then

A. \( \text{dom}((g^n)_{ref}) = (\text{dom}(g^n) \cap \kappa_m) \cup \{f^m(\alpha) \mid \alpha \in \text{dom}(g^n) \setminus \kappa_m\} \).

B. for every \( \alpha \in \text{dom}(g^n) \cap \kappa_m = \text{dom}(g^n) \cap (\text{dom}(g^n)_{ref}), (g^n)_{ref}(\alpha) = g^n(\alpha) \).

C. for every \( \alpha \in \text{dom}(g^n) \setminus \kappa_m \), \( (g^n)_{ref}(f^m(\alpha)) = g^n(\alpha) \).

Again, it is crucial here that \( f^m \upharpoonright (\text{dom}(g^n) \setminus \kappa_m) \) is one to one and the values there are above \( \text{dom}(g^n) \setminus \kappa_m \), and this follows by conditions (3),(4) of Definition 2.18 and (4),(5) of Definition 2.19.

One more crucial observation here is that the measure \( (E(n))(\text{dom}(g^n)) \), to which \( B^n \) belongs, reflects to basically the same measure.

It follows by (4) of Definition 2.18.

D. \( A^n \upharpoonright \text{dom}((g^n)_{ref}) \subseteq \{ (\alpha, \beta) \mid (\alpha, \beta) \in B^n, \alpha < \kappa_m \} \cup \{ (f^m(\alpha), \beta) \mid (\alpha, \beta) \in B^n, \alpha \geq \kappa_m \} \).

Denote further in this subsection \( P_{(E(n)|n<\omega)} \) by just \( P \).

The next lemma follows from the definitions:

Lemma 2.22 For every \( m < \omega \), the forcing \( \langle P_{(E(n)|n<m)}, \leq \rangle \) is equivalent to the product of Cohen forcings \( \text{Cohen}(\kappa_n^+, \eta_n) \)'s, for some \( \eta_n < \kappa_{n+1} \)'s which depend on the choice of a non-pure condition for \( P_{E(n+1)} \).

Lemma 2.23 For every \( m < \omega \), the forcing \( \langle P_{(E(n)|m \leq n < \omega)}, \leq^* \rangle \) is \( \kappa_m \)-closed.

Lemma 2.24 The forcing \( \langle P, \leq \rangle \) satisfies \( \kappa^{++}_m \)-c.c.

Proof. Use the standard \( \Delta \)-system argument.

\[ \square \]

Lemma 2.25 \( \langle P, \leq, \leq^* \rangle \) is a Prikry type forcing notion.

Proof. The proof is similar to those of Lemmas 2.9, 2.17 (and in turn to those of Merimovich [8]).

Assume that for every \( m < \omega \), \( \langle P_{(E(n)|n<m)}, \leq, \leq^* \rangle \) is a Prikry type forcing notion.

Suppose that \( \langle P, \leq, \leq^* \rangle \) does not have the Prikry property.

Let \( p \in P \) be a pure condition and \( \sigma \) a statement of the forcing language which is undecided by pure extensions of \( p \). Then \( p \) is of the form \( \{ (f^m, A^m) \mid n < \omega \} \).
Proceed by induction on $m < \omega$ and define an $\leq^*$ increasing sequence $\langle p_m \mid m < \omega \rangle$ of direct extensions of $p$.

Let $p_0$ be $p$. Assume that for every $n < m$, $p_n$ is defined. Define $p_m$.

At stage $m$ we deal with the coordinate $m$ of the condition.

Construct by induction an increasing chain of elementary submodels $\langle N^m_\xi \mid \xi < \kappa_m \rangle$ of $H_\chi$, for $\chi$ large enough, and a sequence $\langle f_\xi \mid \xi < \kappa_m \rangle$ of members of $\mathcal{P}^*_E(m)$, such that

1. $p, p_{m-1}, \mathcal{P}, \sigma \in N^m_0$,
2. $N^m_0 \supseteq \kappa_m$,
3. for every $\xi < \kappa_m$,
   (a) $|N^m_\xi| = \kappa_m$,
   (b) $\kappa_m > N^m_\xi \subseteq N^m_\xi$,
   (c) $\langle (f^m_\xi, r^m_\xi) \mid \zeta < \xi \rangle \in N^m_\xi$,
   (d) $\langle f^m_\xi, r^m_\xi \rangle \in \bigcap \{D' \in N^m_\xi \mid |D'\text{ is a dense open subset of } \mathcal{P}^*_E(m) \times \langle \mathcal{P}_E(n) \mid m < n < \omega \rangle, \leq^* \rangle \}
   $\hspace{1cm}$\text{above } \langle f^{m-1}_\xi, (p_{m-1}(n) \mid m < n < \omega) \rangle$, 
   (e) $f^{m-1}_\xi \leq^* f^m_0$, $\langle p_{m-1}(n) \mid m < n < \omega \rangle \leq^* r^m_0$,
   (f) $f^m_\xi \supseteq^* f^m_0$, $r^m_\xi \leq^* r^m_0$, for every $\zeta < \xi$.

Set $N^m = \bigcup_{\xi < \kappa_m} N^m_\xi$ and $f^{m*} = \bigcup \{f^m_\xi \mid \xi < \kappa_m \}$. Pick $p^m_\xi$, to be $\leq^*$ stronger than every $r^m_\xi, \xi < \kappa_m$. Let $A \subseteq [\text{dom}(f^{m*}) \times \kappa_m]^{<\kappa_m}$ be such that

- $A \upharpoonright \text{dom}(f^{m*}) \subseteq A^{pm}$,
- $A \in (E(m))(\text{dom}(f^{m*}))$.

Note that $A \subseteq N^m$, since $\text{dom}(f^{m*}) \subseteq N^m$, and so, $[\text{dom}(f^{m*}) \times \kappa_m]^{<\kappa_m} \subseteq N^m$.

Let $\bar{\nu} \in A$. Consider $\lambda^\bar{\nu}_m := h^\bar{\nu}_m(\bar{\nu}(\kappa_m))$, i.e. the cardinal below $\kappa_m$ that now corresponds to $\lambda$. Suppose for simplicity that $\text{dom}(f^{m*}) \subseteq \lambda^\bar{\nu}_m$, for every $n < m$, otherwise just reflect the part above $\kappa_m$ below as in Definition 2.21.

Consider $\mathcal{P}_{(E(n)|\lambda^\bar{\nu}_m[n<m)$. Clearly, it is contained and belongs to $N^m$.

Let $\langle t_\xi \mid \xi < \lambda^\bar{\nu}_m \rangle$ be an enumeration of this forcing notion in $N^m$.

Let $f \in \mathcal{P}^*_E(m)$, $f \supseteq^* f^{m*}$.

Proceed by induction on $\xi < \lambda^\bar{\nu}_m$. Define an $\leq^*$ increasing sequence $\langle f_\xi \mid \xi < \lambda^\bar{\nu}_m \rangle$ of direct
extensions of $f$ and an $\leq^*$-increasing sequence $\langle p^m_\xi \mid \xi < \lambda^\varphi_m \rangle$ of direct extensions of $\langle p_{m-1}(n) \mid m < n < \omega \rangle$

such that, for every $\xi < \lambda^\varphi_m$, either

(1) $(t_\xi, (f_\xi)_\varphi, p^m_\varphi) \parallel \sigma$, \\
or

(2) for every $q \geq^* (\langle f_\xi \rangle_\varphi, p^m_\varphi)$, $(t_\xi, q) \not\parallel \sigma$.

Let $\tilde{f} = \bigcup_{\xi < \lambda^\varphi_1} f_\xi$ and $\tilde{p}^{>m}$ be a direct extension of $\langle p^m_\xi \mid \xi < \lambda^\varphi_1 \rangle$.

Then, for every $t \in P_{\langle E(n) \mid \lambda^\varphi_1(n) < m \rangle}$ either

(1) $(t, \tilde{f}_\varphi, \tilde{p}^{>m}) \parallel \sigma$, \\
or

(2) for every $q \geq^* (\tilde{f}_\varphi, \tilde{p}^{>m})$, $(t, q) \not\parallel \sigma$.

Consider now the following statement of the forcing language of $P_{\langle E(n) \mid \lambda^\varphi_1(n) < m \rangle}$:

$$\varphi \equiv \exists t \in G((t, \tilde{f}_\varphi, \tilde{p}^{>m}) \parallel \sigma).$$

By the Prikry condition of the forcing $P_{\langle E(n) \mid \lambda^\varphi_1(n) < m \rangle}$, there is $t^* \geq^* (p_{m-1}(n) \mid n < m)$ which decides $\varphi$.

If $t^* \models \neg \varphi$, then set $t(\tilde{f}, f) = t^*$.

If $t^* \models \varphi$, then use again the Prikry condition of the forcing $P_{\langle E(n) \mid \lambda^\varphi_1(n) < m \rangle}$ to decide the following statement

$$\psi \equiv \exists t \in G((t, \tilde{f}_\varphi, \tilde{p}^{>m}) \models \sigma).$$

Let $t(\tilde{f}, f) \geq^* t^*$ be a condition which decides $\psi$.

**Claim 4** Let $t \geq t(\tilde{f}, f)$ in $P_{\langle E(n) \mid \lambda^\varphi_1(n) < m \rangle}$, $(g, q) \geq^* (\tilde{f}_\varphi, \tilde{p}^{>m})$ in $P_{\langle E(n) \mid n < m \rangle}$.

Suppose that $(t, g, q) \models \sigma$ (or $(t, g, q) \models \neg \sigma$),

then already $(t(\tilde{f}, f), \tilde{f}_\varphi, \tilde{p}^{>m}) \models \sigma$ (or $(t(\tilde{f}, f), \tilde{p}^{>m}) \models \neg \sigma$).

**Proof.** Let $t \geq t(\tilde{f}, f)$ in $P_{\langle E(n) \mid \lambda^\varphi_1(n) < m \rangle}$, $(g, q) \geq^* (\tilde{f}_\varphi, \tilde{p}^{>m})$ in $P_{\langle E(n) \mid n < m \rangle}$.

Suppose that $(t, g, q) \models \sigma$.

Then, for some $\xi < \lambda^\varphi_1$, $t = t_\xi$, and then, $(t, (f_\xi)_\varphi, p^m_\varphi) \parallel \sigma$. So, $(t, \tilde{f}_\varphi, \tilde{p}^{>m}) \parallel \sigma$. 

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Then $t^* \models \varphi$. Hence, $\langle t(\tilde{\nu}, f), \tilde{f}_\nu, \tilde{p}_\nu^{>m} \rangle \models \sigma$.

Define $D_\varphi$ to be the set of all $\langle f, p_\nu^{>m} \rangle \in P^*_E(m) \times P_{(E(n))\lambda^m | n < \omega}; f \geq^* f^{p_m-1m}$, $p^{>m}_\nu \geq^* p^{>m}_{m-1}$, such that either

(1) $\langle t(\tilde{\nu}, f), \tilde{f}_\nu, \tilde{p}_\nu^{>m} \rangle \models \sigma$

or

(2) for every $t \geq t(\tilde{\nu}, f)$ in $P_{(E(n))\lambda^m | n < \omega}$, for every $\langle g, q \rangle \geq^* \langle f, p^{>m}_\nu \rangle$ in $P_{(E(n))\lambda^m | n < \omega}$,

$\langle t, g, q \rangle \models \sigma$.

The next claim follows now from the previous one:

**Claim 5** $D_\varphi$ is a dense open subset of $P^*_E(m) \times (P_{(E(n))\lambda^m | n < \omega}; \leq^*)$ above

$\langle f^{p_m-1m}, \{p_m-1(n) \mid m < n < \omega\} \rangle$.

$D_\varphi$ is definable with parameters in $N^m$, hence $D_\varphi \in N^m$.

Then, $\langle f^{m*}, p^{>m}_{m*} \rangle \in D_\varphi$, for every $\tilde{\nu} \in A$.

So, for every $\tilde{\nu} \in A$ we have either

(3) $\langle t(\tilde{\nu}, f^{m*}), \tilde{f}_\nu, \tilde{p}_\nu^{>m} \rangle \models \sigma$

or

(4) for every $t \geq t(\tilde{\nu}, f^{m*})$ in $P_{(E(n))\lambda^m | n < \omega}$, for every $\langle g, q \rangle \geq^* \langle f, p^{>m}_\nu \rangle$ in $P_{(E(n))\lambda^m | n < \omega}$,

$\langle t, g, q \rangle \models \sigma$.

Shrink $A$, if necessary, to a set $A^{m*} \in (E(m))(\text{dom}(f^{m*}))$, such that for any two $\tilde{\nu}, \tilde{\nu}' \in A^{m*}$ the decision is the same.

Consider now $\langle f^{m*}, A^{m*} \rangle$ it is a pure condition in $P_{E(m)}$. Use the function $\tilde{\nu} \mapsto t(\tilde{\nu}, f^{m*})$ in order to get a pure condition in $P_{(E(n))\lambda^m | n < \omega}$, just use the one this function represents in the ultrapower by $(E(m))(\text{dom}(f^{m*}))$. Denote it by $\langle \langle f^{m*}, A^{m*} \rangle \mid n < m \rangle$.

Let us explain how do we naturally combine the result into a condition in $P_{(E(n))\lambda^m | n < \omega}$.

Let $t(\tilde{\nu}, f^{m*}) = \langle \langle f^{n\tilde{\nu}}, A^{n\tilde{\nu}} \rangle \mid n < m \rangle$, for every $\tilde{\nu} \in A^{m*}$. Consider $f^{n\tilde{\nu}}, n < m$. It is a set of at most $\kappa_n$ many pairs $(\alpha, \beta)$, where $\alpha < \lambda^m_n < \kappa_m$ and $\beta$ is either the empty sequence or an ordinal $< \kappa_n$.

Shrinking $A^{m*}$ if necessary, we can assume that there are $\langle x_n \mid n < m \rangle$ and $\kappa^+_n < \kappa^+_n, n < m$ such that for every $\tilde{\nu}, \tilde{\nu}' \in A^{m*}$, for every $n < m$, the following hold:

1. $\text{dom}(f^{n\tilde{\nu}}) \cap \tilde{\nu}(\kappa_m) = x_n$,
2. \( \text{dom}(f_{\vec{n}}^m) \setminus \vec{\nu}(\kappa_m) = \{ \gamma_{\tau n}^\vec{\nu} \mid \tau < \kappa_n^* \} \) is an increasing enumeration,

3. for every \( \alpha \in x_n \), \( f_{\vec{n}}^\vec{\nu}(\alpha) = f_{\vec{\nu}}^\vec{\nu}(\alpha) \),

4. for every \( \tau < \kappa_n^* \), \( f_{\vec{n}}^\vec{\nu}(\gamma_{\tau n}^\vec{\nu}) = f_{\vec{\nu}}^\vec{\nu}(\gamma_{\tau n}^\vec{\nu}) \)

Consider, for every \( n < m \) and \( \tau < \kappa_n^* \) a function \( s_{\tau n} \) on \( A^{m*} \) defined by setting \( s_{\tau n}(\vec{\nu}) = \gamma_{\tau n}^\vec{\nu} \).

Let

\[
\gamma_{\tau n} = j_{E(m)}(s_{\tau n})(\langle (j_{E(m)}(\alpha), \alpha) \mid \alpha \in \text{dom}(f^{m*}) \rangle).
\]

Extend now \( f^{m*} \) to \( f^{m**} \) by adding all \( \gamma_{\tau n}, \tau < \kappa_n^* \), \( n < m \) to its domain and setting \( f^{m**}(\gamma_{\tau n}) \) to be the empty sequence whenever \( \gamma_{\tau n} \notin \text{dom}(f^{m*}) \).

Define \( A^{m**} \in E(m)(\text{dom}(f^{m**})) \) as follows.

Set \( \vec{\nu} \in A^{m**} \) iff

1. \( \vec{\nu} \upharpoonright \text{dom}(f^{m*}) \in A^{m*} \),
2. \( \text{dom}(\vec{\nu}) \supseteq \{ \gamma_{\tau n} \mid \tau < \kappa_n^*, n < m \} \),
3. if \( \gamma_{\tau n} \in \text{dom}(f^{m*}) \) and \( f^{m*}(\gamma_{\tau n}) \) is not the empty sequence, then \( \vec{\nu}(\gamma_{\tau n}) > f^{m*}(\gamma_{\tau n}) \),
   for every \( n < m \),
4. \( \vec{\nu}(\gamma_{\tau n}) = s_{\tau n}(\vec{\nu} \upharpoonright \text{dom}(f^{m*})) \), for every \( n < m \).

For every \( \vec{\nu} \in A^{m**}, n < m \), set

\[
\langle (g_{\vec{\nu}}^m, B_{\vec{\nu}}^m) = \langle f_{\vec{\nu}}^m \text{dom}(f^{m*}), A_{\vec{\nu}}^m \text{dom}(f^{m*}) \rangle.
\]

Consider the function \( \vec{\nu} \mapsto \langle (g_{\vec{\nu}}^m, B_{\vec{\nu}}^m) \mid n < m \rangle, \vec{\nu} \in A^{m**} \). Let \( \langle (f^{m*}, A^{m*}) \mid n < m \rangle \) be represented by it in the ultrapower with \( E(m) \).

It follows that \( \langle (f^{m*}, A^{m*}) \mid n < m \rangle, \langle f^{m**}, A^{m**} \rangle \rangle \) is a pure condition in \( P_{(E(n))|n \leq m} \) which extends \( p_m \upharpoonright P_{(E(n))|n \leq m} \).

Extend purely \( p_{f_{m*}^{m*}}^m \) in the obvious fashion to a condition \( p_{f_{m*}^{m**}}^m \) in \( P_{(E(n))|m < n < \omega} \) such that \( \langle (f^{m*}, A^{m*}) \mid n < m \rangle, \langle f^{m**}, A^{m**} \rangle, p_{f_{m*}^{m**}}^m \rangle \) is a pure condition in \( P_{(E(n))|n < \omega} \). Then it extends \( p_{m-1} \).

Set \( p_m \) to be \( \langle (f^{m*}, A^{m*}) \mid n < m \rangle, \langle f^{m**}, A^{m**} \rangle, p_{f_{m*}^{m**}}^m \rangle \).

This completes the recursive construction of \( \langle p_m \mid m < \omega \rangle \). Let \( p_* \geq p_m \), for every \( m < \omega \).

The next claim completes the argument:

**Claim 6** \( p_* \parallel \sigma \).
Proof. Suppose otherwise. Pick then $q \geq p_*$ to be a condition which decides $\sigma$ and such that its last coordinate at which a non-direct extension was made is as small as possible.

Let $q \Vdash \sigma$ and this coordinate is some $m < \omega$.

Then there is $\vec{v} \in A^{\nu}(m)$ such that $q(m) \geq f^{\nu}(m)_{\vec{v}}$ in $\mathcal{P}^{\nu}_{E(m)}$. In addition, $q^m \geq p^m_*$ in $\mathcal{P}_{E(n)|m<n<\omega}$, by the choice of $m$.

But, then condition (4) above cannot hold. Hence (3) is true, which means, that

$$\langle t(\vec{v}, f^{m*}, f_{\vec{v}}^{m*}, p^m_*) \rangle \Vdash \sigma.$$ 

Then the same holds for every $\vec{v}' \in A^{\nu}(m)$. So, already $p_* \Vdash \sigma$.

Contradiction.

Let us state first the following:

**Lemma 2.28** Let $p \in \mathcal{P}$ and $\zeta$ be a $\langle \mathcal{P}, \leq \rangle$–name of an ordinal or just $p \Vdash_{\langle \mathcal{P}, \leq \rangle} \zeta$ is an ordinal.

Then there are $p^* \geq^p p$ and $n_1 < \ldots < n_k$, for some $k < \omega$, such that

1. for every $i, 1 \leq i \leq k$, $p^*(n_i) = (f^{n_i}_{n_i}, A^{n_i}_{n_i})$, 

2. 

**Lemma 2.26** $\kappa_\omega$ remains a strong limit cardinal in $V[G]$. 

**Proof.** Given $p \in \mathcal{P}$ and $m < \omega$. Suppose that $p(m)$ is non-pure. Then $p(m)(\kappa_m)$ is defined, and hence also the reflection $h^{\kappa_m}_\kappa(p(m)(\kappa_m))$ of $\lambda$ below $\kappa_m$. By the definition of the forcing, then the part $\mathcal{P}_{E(n)|n<m}$ above $p$ will act as $\mathcal{P}_{E(n)|h^{\kappa_m}_\kappa(p(m)(\kappa_m))|n<m}$. In particular, $2^{\kappa_m} \leq h^{\kappa_m}_\kappa(p(m)(\kappa_m)) < \kappa_m$. The upper part of the forcing, i.e. $\mathcal{P}_{E(n)|m\leq n<\omega}$, does not add new bounded subsets to $\kappa_m$.

So we are done.

**Lemma 2.27** $(\kappa_\omega^+)^V$ remains a cardinal in $V[G]$. 

Let $G$ be a generic subset of $\langle \mathcal{P}, \leq \rangle$. 

**Proof.** Given $p \in \mathcal{P}$ and $m < \omega$. Suppose that $p(m)$ is non-pure. Then $p(m)(\kappa_m)$ is defined, and hence also the reflection $h^{\kappa_m}_\kappa(p(m)(\kappa_m))$ of $\lambda$ below $\kappa_m$. By the definition of the forcing, then the part $\mathcal{P}_{E(n)|n<m}$ above $p$ will act as $\mathcal{P}_{E(n)|h^{\kappa_m}_\kappa(p(m)(\kappa_m))|n<m}$. In particular, $2^{\kappa_m} \leq h^{\kappa_m}_\kappa(p(m)(\kappa_m)) < \kappa_m$. The upper part of the forcing, i.e. $\mathcal{P}_{E(n)|m\leq n<\omega}$, does not add new bounded subsets to $\kappa_m$.

So we are done.

**Lemma 2.28** 

Let us state first the following:
2. for every $\vec{v}_1 \in A^p_{n_1}, \ldots, \vec{v}_k \in A^p_{n_k}$, 
\[ p^* \vdash \vec{v}_1 \ldots \vec{v}_k \text{ decides } \zeta. \]

The proof of this lemma repeats the proof of the Prikry condition of the forcing.

**Proof of 2.27.** Suppose otherwise. Then there is $\mu < \kappa_\omega$ such that, in $V[G]$, $\operatorname{cof}(\kappa_\omega^+) = \mu$.

Back in $V$, let $\langle \zeta_{\tau} \mid \tau < \mu \rangle$ be a name of a witnessing sequence.

Pick $\bar{n} < \omega$ with $\kappa_{\bar{n}} > \mu$. Let $p \in \mathcal{P}$ be such that $p(\bar{n}) \in \mathcal{P}^*_{E(\bar{n})}$, i.e. its $\bar{n}$–th coordinate is non-pure. Then above $p$ the part $\mathcal{P}_{E(n)\mid n < \bar{n}}$ reflects down to $\mathcal{P}_{E(n)\mid n < \bar{n}}$, and so has cardinality below $\kappa_{\bar{n}}$.

Construct a sequence $\langle p_\tau \mid \tau < \mu \rangle$ of $\leq^*$ –extensions of $p$ such that, for every $\tau < \mu$,

1. $p_\tau$ satisfies the conclusion of Lemma 2.28 for $\zeta_{\tau}$,

2. $\langle p_\tau(n) \mid \bar{n} \leq n < \omega \rangle \leq^* \langle p_{\tau'}(n) \mid \bar{n} \leq n < \omega \rangle$ in the forcing $\mathcal{P}_{E(n)\mid n < \omega}$, for every $\tau < \tau' < \mu$.

Let $s \geq^* \langle p_\tau(n) \mid \bar{n} \leq n < \omega \rangle$ in the forcing $\mathcal{P}_{E(n)\mid \bar{n} \leq n < \omega}$, for every $\tau < \mu$. Set $r = p \upharpoonright \bar{n} \sim s$. Then, for every $\tau < \mu$, there is $\xi_\tau < \kappa_\omega^+$ such that

\[ r \models_{\mathcal{P}^*_{E(n)}} \zeta_{\tau} < \xi_\tau, \]

since by the choice of $p_\tau$, the number of possibilities for $\zeta_{\tau}$ has cardinality $< \kappa_\omega$.

Set $\xi = \bigcup_{\tau < \mu} \xi_\tau < \kappa_\omega^+$.

\[ r \models_{\mathcal{P}^*_{E(n)}} \langle \zeta_{\tau} \mid \tau < \mu \rangle \text{ is bounded by } \xi. \]

Contradiction.

Given $p \in \mathcal{P}$. Denote by $\operatorname{np}(p)$ the set of all coordinates $n$ of $p$ such that $p(n) \in \mathcal{P}^*_{E(n)}$, i.e. a non-pure extension was made at the coordinate $n$.

For each $\beta \in [\kappa_\omega, \lambda)$ we define in $V[G]$ a function $t_\beta : \omega \to \kappa_\omega$ as follows.

For every $n < \omega$, find $p \in G$ such that $n \in \operatorname{np}(p)$ and if $n_1 < \ldots < n_k$ is the increasing enumeration of $\operatorname{np}(p) \setminus n$ (i.e. $n = n_1$), then the following hold:

1. $\beta \in \operatorname{dom}(p(n_k))$.
   Set $\beta_k = \beta$.

2. For every $i, 1 \leq i \leq k - 1$, $\beta_i \in \operatorname{dom}(p(n_i))$,
   where $\beta_i = p(n_{i+1})(\beta_{i+1})$. 

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Set $t_\beta(n) = p(n)(\beta_1)$.

**Lemma 2.29** In $V[G]$, if $\beta, \gamma \in [\kappa, \lambda)$ and $\beta < \gamma$, then there is $n^* < \omega$ such that for every $n, n^* \leq n < \omega$, $t_\beta(n) < t_\gamma(n)$.

**Proof.** Work in $V$. Let $p \in \mathcal{P}$ be any condition and $\beta, \gamma \in [\kappa, \lambda), \beta < \gamma$.
Let $n^*$ be a coordinate above $\forall \beta(p)$. Then $p(n) = \langle f^*_n, A^*_n \rangle$, for every $n, n^* \leq n < \omega$.
Extend $p$ to $p^*$ by adding $\beta, \gamma$ to all dom($f^*_n$) with $n^* \leq n < \omega$.
Now, by the definition of the order on $\mathcal{P}$, for every $n, n^* \leq n < \omega$ and every $q \geq p^*$ such that $q$ defines $t_\beta(n)$ and $t_\gamma(n)$, we will have $t_\beta(n) < t_\gamma(n)$.
So,

$$p^* \models (\forall n)(n^* \leq n < \omega \rightarrow t_\beta(n) < t_\gamma(n)).$$

$\square$

It is possible to say a bit more. Namely, let in $V[G]$, for every $n < \omega$, $\lambda_n$ be the reflection of $\lambda$ below $\kappa_n$, i.e. for some $p \in G$ with $p(n) = f^*_n$, $\lambda_n = h^*_\alpha(f^*_n(\kappa_n))$. Then the following holds:

**Lemma 2.30** The sequence $\langle t_\beta \mid \beta \in [\kappa, \lambda) \rangle$ is a scale in $\langle \prod_{n<\omega} \lambda_n, <_{Jbd} \rangle$. 

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3 Arbitrary cofinality.

Let $\eta$ be any ordinal. We generalize the construction of the previous section to sequences of extenders of the length $\eta$. Generalization is straightforward. Let us repeat just the main points.

So, we deal now with a sequence $\langle E(\alpha) \mid \alpha < \eta \rangle$, where each $E(\alpha)$ is a $(\kappa_\alpha, \lambda)$—extender and $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ is an increasing sequence with $\eta < \kappa_0$.

Let $\mathcal{P}_{E(i)}^{(\alpha)}, \mathcal{P}_{E(i)}, i < \eta$ be as defined before.

Define components $\mathcal{P}_{(E(\alpha)_{\alpha < \eta})}^{(\beta_1, \ldots, \beta_k)}, k < \omega, \beta_1 < \ldots < \beta_k < \eta$ of the main forcing $\mathcal{P}_{(E(\alpha)_{\alpha < \eta})}$.

**Definition 3.1** The set of pure conditions $\mathcal{P}_{(E(\alpha)_{\alpha < \eta})}^{(\alpha)}$ consists of all sequences $\langle p(\alpha) \mid \alpha < \eta \rangle$ such that for every $\alpha < \eta$, the following hold:

1. $p(\alpha) = \langle f^\alpha, A^\alpha \rangle \in \mathcal{P}_{E(\alpha)}$,

2. for every $\beta < \alpha$, $\text{dom}(f^\beta) \setminus \kappa_\alpha \subseteq \text{dom}(f^\alpha)$,

3. for every $\beta < \alpha$, for every $\xi \in \text{dom}(f^\beta) \setminus \kappa_\alpha$, if $f^\alpha(\xi)$ is not the empty sequence, then for every $\vec{\nu} \in A^\alpha, \xi \in \text{dom}(\vec{\nu})$ and $\vec{\nu}(\xi) > f^\alpha(\xi)$.

The idea behind is as in the case of two extenders.

4. For every $\beta < \alpha$ and $\vec{\nu} \in A^\alpha$, the measures $E(\beta)(\text{dom}(f^\beta))$ and $E(\beta)(((\text{dom}(f^\beta) \cap \kappa_\alpha) \cup \{\vec{\nu}(\xi) \mid \xi \in \text{dom}(f^\beta) \setminus \kappa_\alpha\})$ are basically the same in the following sense:

$$X \in E(\beta)(\text{dom}(f^\beta)) \iff X^{\text{ref}} \in E(\beta)((\text{dom}(f^\beta) \cap \kappa_\alpha) \cup \{\vec{\nu}(\xi) \mid \xi \in \text{dom}(f^\beta) \setminus \kappa_\alpha\}),$$

where

$$X^{\text{ref}} = \{((\xi, \beta) \in X \mid \xi < \kappa_\alpha) \cup \{(\vec{\nu}(\xi), \beta) \mid (\vec{\nu}(\xi), \beta) \in X, \xi \geq \kappa_\alpha\}.$$

Note that this property is true in the ultrapower by $E(\alpha)$, so it holds on a set of measure one, as well.

Turn now to non-pure extensions. As usual, in Magidor type of iterations, non-pure extensions are allowed only at finitely many coordinates.

Start with a non-pure extension at a single coordinate and then proceed by induction.

We assume that for each $\alpha < \eta$ there is a function $h^\alpha_\lambda : \kappa_\alpha \to \kappa_\alpha$ such that $j_{E(\alpha)}(h^\alpha_\lambda)(\kappa_\alpha) = \lambda$. 

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Definition 3.2 Let \( \beta < \eta \). Define the set \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)}^{(\beta)} \) of conditions with only non-pure part over the coordinate \( \beta \). \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)}^{(\beta)} \) consists of all sequences \( \langle p(\alpha) | \alpha < \eta \rangle \) such that for every \( \alpha < \eta \), the following hold:

1. \( \langle p(\alpha) | \alpha < \eta, \alpha \neq \beta \rangle \) is a condition in \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)} \),
2. \( p(\beta) = f^{\beta} \in \mathcal{P}_{E(\beta)}^{*} \),
3. \( \text{dom}(f^{\beta}) \setminus \kappa_\alpha \subseteq \text{dom}(f^\alpha) \), for every \( \alpha, \beta < \alpha < \eta \),
4. for every \( \alpha, \beta < \alpha < \eta \), for every \( \xi \in \text{dom}(f^{\beta}) \setminus \kappa_\alpha \), if \( f^\alpha(\xi) \) is not the empty sequence, then for every \( \bar{\nu} \in A^\alpha, \xi \in \text{dom}(\bar{\nu}) \) and \( \bar{\nu}(\xi) > f^\alpha(\xi) \),
5. for every \( \alpha, \beta < \alpha < \eta \), for every \( \gamma \in \text{dom}(f^{\beta}) \cap \kappa_\alpha, \bar{\nu} \in A^\alpha \) and \( \xi \in \text{dom}(\bar{\nu}) \), \( \bar{\nu}(\xi) > \gamma \).
6. If \( \beta > 0 \), then the sequence \( \langle p(\alpha) | \alpha < \beta \rangle \) will be a condition in the pure part of \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)}^{(\beta_1, ..., \beta_k, \beta)} \). The meaning is that if the value of the Prikry sequence for the normal measure of \( E(\beta) \) is decided, then we cut all extenders \( E(\alpha), \alpha < \beta \) to the reflection of \( \lambda \) below \( \kappa_\beta \), i.e. to \( h^{(\beta_1, ..., \beta_k)}_\lambda(\kappa_\beta) \).

Let \( \beta_1 < ... < \beta_k < \eta, 1 \leq k < \omega \) and suppose that \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)}^{(\beta_1, ..., \beta_k)} \) the set of conditions with non-pure extensions over coordinates \( (\beta_1, ..., \beta_k) \) only, is defined.

Let \( \beta < \eta, \beta \notin \{\beta_1, ..., \beta_k\} \).

Define non-pure extensions at the set of coordinates \( \{\beta_1, ..., \beta_k\} \cup \{\beta\} \).

Definition 3.3 Let \( \beta < \eta \). Define the set \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)}^{(\beta_1, ..., \beta_k, \beta)} \) of conditions with only non-pure part over the coordinate \( \beta_1, ..., \beta_k \) and \( \beta \). \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)}^{(\beta_1, ..., \beta_k, \beta)} \) consists of all sequences \( \langle p(\alpha) | \alpha < \eta \rangle \) such that for every \( \alpha < \eta \), the following hold:

1. \( \langle p(\alpha) | \alpha < \eta, \alpha \neq \beta \rangle \) is a condition in \( \mathcal{P}_{(E(\alpha) | \alpha < \eta)}^{(\beta_1, ..., \beta_k)} \),
2. \( p(\beta) = f^{\beta} \in \mathcal{P}_{E(\beta)}^{*} \),
3. If \( \beta > \max\{\beta_1, ..., \beta_k\} \), then following hold:
   (a) \( \text{dom}(f^{\beta}) \setminus \kappa_\alpha \subseteq \text{dom}(f^\alpha) \), for every \( \alpha, \beta < \alpha < \eta \),
   (b) for every \( \alpha, \beta < \alpha < \eta \), for every \( \xi \in \text{dom}(f^{\beta}) \setminus \kappa_\alpha \), if \( f^\alpha(\xi) \) is not the empty sequence, then for every \( \bar{\nu} \in A^\alpha, \xi \in \text{dom}(\bar{\nu}) \) and \( \bar{\nu}(\xi) > f^\alpha(\xi) \),
(c) for every $\alpha, \beta < \alpha < \eta$, for every $\gamma \in \text{dom}(f^\beta) \cap \kappa_\alpha, \bar{v} \in A^\alpha$ and $\xi \in \text{dom}(\bar{v})$, $\bar{v}(\xi) > \gamma$.

(d) If $\beta > 0$, then the sequence $\langle p(\alpha) \mid \alpha < \beta \rangle$ is a condition in $\mathcal{P}_{(E(\alpha)\lfloor h^\alpha_\lambda((f^\beta(\kappa_\beta)))\mid \alpha < \beta)}$.

The meaning is that if the value of the Prikry sequence for the normal measure of $E(\beta)$ is decided, then we cut all extenders $E(\alpha), \alpha < \beta$ to the reflection of $\lambda$ below $\kappa_\beta$, i.e. to $h^\beta_\lambda((f^\beta(\kappa_\beta)))$.

4. If $\beta < \max\{\beta_1, \ldots, \beta_k\}$, then let $i^*$ be minimal such that $\beta < \beta_{i^*}$. Then the following hold:

(a) $\langle p(\alpha) \mid \alpha < \beta_{i^*} \rangle \in \mathcal{P}_{(E(\alpha)\lfloor h^\alpha_\lambda((f^\beta(\kappa_{\beta_{i^*}})))\mid \alpha < \beta_{i^*})}$.

Finally set

$$\mathcal{P}_{(E(\alpha)\mid \alpha < \eta)} = \bigcup \{ \mathcal{P}_{(E(\alpha)\mid \alpha < \eta)} \mid k < \omega, \beta_1 < \ldots < \beta_k < \eta \}.$$  

Define the direct extension order $\leq^*$ over $\mathcal{P}_{(E(\alpha)\mid \alpha < \eta)}$ to be the union of such order over every $\mathcal{P}_{(E(\alpha)\mid \alpha < \eta)}$ for every $k < \omega, \beta_1 < \ldots < \beta_k < \eta$.

Turn now to the definition of the forcing order $\leq$ over $\mathcal{P}_{(E(\alpha)\mid \alpha < \eta)}$.

Let $\beta < \eta, \beta \notin \{\beta_1, \ldots, \beta_k\}$. Define a one element extension at coordinate $\beta$ of a condition in $\mathcal{P}_{(E(\alpha)\mid \alpha < \eta)}$.

**Definition 3.4** Let $p \in \mathcal{P}_{\{\beta_1, \ldots, \beta_k\} \cup \{\beta\}}$ and $q \in \mathcal{P}_{\{\beta_1, \ldots, \beta_k\}}$. Set $p \geq q$ iff the following hold:

1. Suppose that $\beta = 0$.
   
   Then $p(0) = f^0 \in \mathcal{P}_{(E(0)\mid 0 < \alpha < \eta)}$ and $q(0) = \langle g^0, B^0 \rangle$ is a pure condition in $\mathcal{P}_{(E(0)\mid 0 < \alpha < \eta)}$.
   
   Set $p \geq q$ iff $f^0 \geq \langle g^0, B^0 \rangle$ in $\mathcal{P}_{(E(0)\mid 0 < \alpha < \eta)}$ and $\langle p(\alpha) \mid 0 < \alpha < \eta \rangle \geq^* \langle q(\alpha) \mid 0 < \alpha < \eta \rangle$ in $\mathcal{P}_{(E(\alpha)\mid 0 < \alpha < \eta)}$.

2. Suppose that $\beta > 0$.
   
   Then $p(\beta) = f^\beta \in \mathcal{P}_{(E(\beta)\mid \beta < \alpha < \eta)}$ and $q(\beta) = \langle g^\beta, B^\beta \rangle$ is a pure condition in $\mathcal{P}_{(E(\beta)\mid \beta < \alpha < \eta)}$.
   
   Set $p \geq q$ iff

   (a) $f^\beta \geq \langle g^\beta, B^\beta \rangle$ in $\mathcal{P}_{(E(\beta)\mid \beta < \alpha < \eta)}$ and $\langle p(\alpha) \mid \beta < \alpha < \eta \rangle \geq^* \langle q(\alpha) \mid \beta < \alpha < \eta \rangle$ in $\mathcal{P}_{(E(\alpha)\mid \beta < \alpha < \eta)}$.

   And
(b) \( \langle p(\alpha) \mid \alpha < \beta \rangle \geq^* \langle q(\alpha) \mid \alpha < \beta \rangle^{\text{ref}} \) in \( \mathcal{P}_{(E(\alpha)[h^a_\beta(f^\beta(\kappa_\beta))]\alpha < \beta)} \), where
\( \langle q(\alpha) \mid \alpha < \beta \rangle^{\text{ref}} \) - the reflection of \( \langle q(\alpha) \mid \alpha < \beta \rangle \) below \( \kappa_\beta \) is defined as follows, where \( q(\alpha) = \langle g^a, B^\alpha \rangle \), if \( \alpha \not\in \{\beta_1, ..., \beta_k\} \) and \( q(\alpha) = \langle g^a \rangle \) otherwise.

i. Suppose first that \( \alpha \in \{\beta_1, ..., \beta_k\} \).

Then
A. \( \text{dom}((g^a)^{\text{ref}}) = (\text{dom}(g^a) \cap \kappa_\beta) \cup \{f^\beta(\xi) \mid \xi \in \text{dom}(g^a) \setminus \kappa_\beta\} \),
B. for every \( \xi \in \text{dom}(g^a) \cap \kappa_\beta = \text{dom}(g^a) \cap \text{dom}((g^a)^{\text{ref}}) \), \( (g^a)^{\text{ref}}(\xi) = g^a(\xi) \).
C. for every \( \xi \in \text{dom}(g^a) \setminus \kappa_\beta \), \( (g^a)^{\text{ref}}(f^\beta(\xi)) = g^a(\xi) \).

It is crucial here that \( f^\beta \upharpoonright (\text{dom}(g^a) \setminus \kappa_\beta) \) is one to one and the values there are above \( \text{rng}(g^a) \cap \kappa_\beta \).

This follows by conditions (4),(5) of Definitions 2.10,2.11.

ii. Suppose now that \( \alpha \not\in \{\beta_1, ..., \beta_k\} \).

Then
A. \( \text{dom}((g^a)^{\text{ref}}) = (\text{dom}(g^a) \cap \kappa_\beta) \cup \{f^\beta(\xi) \mid \xi \in \text{dom}(g^a) \setminus \kappa_\beta\} \),
B. for every \( \xi \in \text{dom}(g^a) \cap \kappa_\beta = \text{dom}(g^a) \cap \text{dom}((g^a)^{\text{ref}}) \), \( (g^a)^{\text{ref}}(\xi) = g^a(\xi) \).
C. for every \( \xi \in \text{dom}(g^a) \setminus \kappa_\beta \), \( (g^a)^{\text{ref}}(f^\beta(\xi)) = g^a(\xi) \).

Again, it is crucial here that \( f^\beta \upharpoonright (\text{dom}(g^a) \setminus \kappa_\beta) \) is one to one and the values there are above \( \text{dom}(g^a) \cap \kappa_\beta \), and this follows by conditions (3),(4) of Definition 3.1 and (4),(5) of Definition 3.2.

One more crucial observation here is that the measure \( (E(\alpha))(\text{dom}(g^a)) \), to which \( B^a \) belongs, reflects to basically the same measure,

It follows by (4) of Definition 3.1.

D. \( A^a \upharpoonright \text{dom}((g^a)^{\text{ref}}) \subseteq \{(\xi, \zeta) \mid (\xi, \zeta) \in B^a, \xi < \kappa_\beta \} \cup \{(f^\beta(\xi), \zeta) \mid (\xi, \zeta) \in B^a, \xi \geq \kappa_\beta \} \).

Denote further in this subsection \( \mathcal{P}_{(E(\alpha)[\alpha < \eta])} \) by just \( \mathcal{P} \).

The next lemma follows from the definitions:

**Lemma 3.5** For every \( \beta < \eta \) and \( p \in \mathcal{P} \) with \( p(\beta) \in \mathcal{P}^*_E(\beta) \) (i.e. non-pure on the coordinate \( \beta \)), the part \( \langle \mathcal{P}(E(\alpha)[\alpha < \beta]), \leq \rangle \) of \( \mathcal{P} \) above \( p \) has cardinality \( h^a_\beta(p(\beta))(\kappa_\beta) < \kappa_\beta \).

**Lemma 3.6** For every \( \beta < \eta \), the forcing \( \langle \mathcal{P}_{(E(\alpha)[\beta < \alpha < \eta]), \leq^*} \rangle \) is \( \kappa_\beta \)-closed.

**Lemma 3.7** The forcing \( \langle \mathcal{P}, \leq \rangle \) satisfies \( \kappa_\eta^{++} - \text{c.c.} \).
Lemma 3.8 $\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.

Proof. The proof proceeds by induction on the length of the sequence of extenders, i.e. on $\eta$. The argument repeats those of Lemma 2.25.

Denote for every limit $\alpha, 0 < \alpha \leq \eta$, $\bigcup_{\gamma < \alpha} \kappa_\gamma$ by $\bar{\kappa}_\alpha$.

It follows, by the previous lemmas, that the forcing $\langle \mathcal{P}, \leq \rangle$ preserves all the cardinals, except maybe $\bar{\kappa}_\alpha^+, 0 < \alpha \leq \eta$ a limit ordinal. Using the arguments of the previous lemma we will show that all such cardinals are preserved as well.

Let $G$ be a generic subset of $\langle \mathcal{P}, \leq \rangle$.

Lemma 3.9 For every limit ordinal $\mu, 0 < \mu \leq \eta$, $\bar{\kappa}_\mu$ remains a strong limit cardinal in $V[G]$.

Proof. Given $p \in \mathcal{P}$ and $\beta < \eta$. Suppose that $p(\beta)$ is non-pure. Then $p(\beta)(\kappa_\beta)$ is defined, and hence also the reflection $h^\beta_\lambda (p(\beta)(\kappa_\beta))$ of $\lambda$ below $\kappa_\beta$. By the definition of the forcing, then the part $\mathcal{P}_{\langle E(\alpha) | \alpha < \beta \rangle}$ above $p$ will act as $\mathcal{P}_{\langle E(\alpha) | h^\beta_\lambda (p(\beta)(\kappa_\beta)) | \alpha < \beta \rangle}$. In particular, $2^{\bar{\kappa}_\alpha} \leq h^\beta_\lambda (p(\beta)(\kappa_\beta)) < \kappa_\beta$. The upper part of the forcing, i.e. $\mathcal{P}_{\langle E(\alpha) | \beta \leq \alpha < \eta \rangle}$, does not add new bounded subsets to $\kappa_\beta$.

So we are done.

As in the case $\eta = \omega$, the next lemma is just a variation of the Prikry condition of the forcing.

Lemma 3.10 Let $p \in \mathcal{P}$ and $\zeta$ be a $\langle \mathcal{P}, \leq \rangle$--name of an ordinal or just $p \Vdash_{\langle \mathcal{P}, \leq \rangle} \dot{\zeta}$ is an ordinal.

Then there are $p^* \geq^* p$ and $\alpha_1 < \ldots < \alpha_k < \eta$, for some $k < \omega$, such that

1. for every $i, 1 \leq i \leq k$, $p^*(\alpha_i) = (f^{p^*}_{\alpha_i}, A^{p^*}_{\alpha_i})$,

2. for every $\vec{\nu}_1 \in A^{p^*}_{\alpha_1}, \ldots, \vec{\nu}_k \in A^{p^*}_{\alpha_k}$,

$p^* \vdash \neg \nu_1 \ldots \neg \nu_k$ decides $\zeta$.

Lemma 3.11 For every limit ordinal $\mu, 0 < \mu \leq \eta$, $(\bar{\kappa}_\mu^+)^V$ remains a cardinal in $V[G]$.

The proof of this lemma repeats those of Lemma 2.27.

Given $p \in \mathcal{P}$. Denote by $n_p(p)$ the set of all coordinates $\alpha$ of $p$ such that $p(\alpha) \in \mathcal{P}^{*}_{E(\alpha)}$, i.e. a non-pure extension was made at the coordinate $\alpha$. 29
Assume that $\eta$ is a limit ordinal.

For each $\tau \in [\kappa_\eta, \lambda)$ we define in $V[G]$ a function $t_\tau : \eta \to \kappa_\eta$ as follows.

For every $\alpha < \eta$, find $p \in G$ such that $\alpha \in \text{np}(p)$ and if $\alpha_1 < \ldots < \alpha_k$ is the increasing enumeration of $\text{np}(p) \setminus \alpha$ (i.e. $\alpha = \alpha_1$), then the following hold:

1. $\tau \in \text{dom}(p(\alpha_k))$.
   Set $\tau_k = \tau$.

2. For every $i, 1 \leq i \leq k - 1$, $\tau_i \in \text{dom}(p(\alpha_i))$
   where $\tau_i = p(\alpha_{i+1})(\tau_{i+1})$.

Set $t_\tau(\alpha) = p(\alpha)(\tau_1)$.

**Lemma 3.12** In $V[G]$, if $\tau, \rho \in [\kappa_\eta, \lambda)$ and $\tau < \rho$, then there is $\alpha^* < \eta$ such that for every $\alpha, \alpha^* \leq \alpha < \eta$, $t_\tau(\alpha) < t_\rho(\alpha)$.

*Proof.* Work in $V$. Let $p \in P$ be any condition and $\tau, \rho \in [\kappa_\eta, \lambda)$, $\tau < \rho$.

Let $\alpha^*$ be a coordinate above $\text{np}(p)$. Then $p(\alpha) = (f^{p}_\alpha, A^{p}_\alpha)$, for every $\alpha, \alpha^* \leq \alpha < \eta$.

Extend $p$ to $p^*$ by adding $\tau, \rho$ to all $\text{dom}(f^{p}_\alpha)$ with $\alpha^* \leq \alpha < \eta$.

Now, by the definition of the order on $P$, for every $\alpha, \alpha^* \leq \alpha < \eta$ and every $q \geq p^*$ such that $q$ defines $t_\tau(\alpha)$ and $t_\rho(\alpha)$, we will have $t_\tau(\alpha) < t_\rho(\alpha)$.

So,

$$p^* \models (\forall \alpha)(\alpha^* \leq \alpha < \eta \rightarrow t_\tau(\alpha) < t_\rho(\alpha)).$$

$\square$

It is possible to say a bit more. Namely, let in $V[G]$, for every $\alpha < \eta$, $\lambda_\alpha$ be the reflection of $\lambda$ below $\kappa_\alpha$, i.e. for some $p \in G$ with $p(\alpha) = f^{p}_\alpha$, $\lambda_\alpha = h^{p}_\chi(f^{p}_\alpha(\kappa_\alpha))$. Then the following holds:

**Lemma 3.13** The sequence $\langle t_\tau \mid \tau \in [\kappa_\eta, \lambda) \rangle$ is a scale in $\prod_{\alpha < \eta} \lambda_\alpha, <_{J^{\alpha*}}$.

In particular, we obtain the following:

**Corollary 3.14** It is possible to blow up the power of a singular in the core model\(^2\) cardinal of arbitrary cofinality in a cardinal preserving extension.

\(^2\)Core model with strong cardinals, but below $o$-hand grenade. It was defined and studied by Ralf Schindler in [10]
4 One generalization.

In the previous section we assumed that $\eta < \kappa_0$ in order to blow up the power of a singular cardinal of cofinality $\eta$.

Let us now take $\eta$ to be an inaccessible cardinal.

Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be now an increasing sequence with limit $\eta$ and each $E(\alpha)$, for $\alpha < \eta$, be a $(\kappa_\alpha, \eta)$-extender.

Assume that $\eta$ is the least inaccessible limit of $\kappa_\alpha$'s.

We proceed as in the previous section and define $\langle \mathcal{P}(E(\alpha)) \rangle_{\alpha < \eta}$, for every $\alpha < \eta$. Then the following holds:

**Theorem 4.1** $V[G]$ is a cofinality preserving extension of $V$ such that for every $\alpha < \eta$, $\bar{\kappa}_\alpha$ is a strong limit singular cardinal with $2^{\bar{\kappa}_\alpha} > \bar{\kappa}_\alpha^+$. In addition $\eta$ remains inaccessible.

By passing to $V[G]_{\eta}$ we obtain the following:

**Corollary 4.2** It is possible to blow up the power of a proper class club of singular cardinals in the core model in a cofinality preserving extension.
References


[2] M. Gitik, Short extenders forcings,


