Extenders based forcings with overlapping extenders and negations of the Shelah Weak Hypothesis.

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Dedicated to the memory of Mati Rubin.

Abstract

Extender based Prikry-Magidor forcing for overlapping extenders is introduced. As an application, models with strong forms of negations of the Shelah Weak Hypothesis for various cofinalities are constructed.

1 Introduction.

In this article we continue [3] to develop extenders based forcings with overlapping extenders. Applications to Cardinal Arithmetic and mainly to the Shelah Weak Hypothesis are presented.

We recall first some basic definitions and then state main results.

Let $\mathfrak{a}$ be a set of regular cardinals, with $|\mathfrak{a}| < \min(\mathfrak{a})$, and let J be an ideal on $\mathfrak{a}$.
If $f, g \in \prod \mathfrak{a}$, then $f <_J g$ iff $\{\nu \in \mathfrak{a} \mid f(\nu) \geq g(\nu)\} \in J$.

Definition 1.1 (S. Shelah [10]) A regular cardinal λ is called $\mathrm{tcf}(\prod \mathfrak{a}, <_J)$ iff there exists an $<_J$ −increasing sequence of functions $\langle f_\alpha \mid \alpha < \lambda\rangle$ in $\prod \mathfrak{a}$ such that for every $g \in \prod \mathfrak{a}$ there is $\alpha < \lambda$ with $g <_J f_\alpha$.

Definition 1.2 (S. Shelah [10]) Let κ be a singular cardinal.
$\mathrm{pp}(\kappa) = \sup(\{\mathrm{tcf}(\prod \mathfrak{a}, <_J) \mid \mathfrak{a} \subseteq \kappa, \mathfrak{a}$ consists of regular cardinals, $\mathfrak{a}$ is unbounded in κ,

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$|\mathfrak{a}| = \mathrm{cof}(\kappa)$, J is an ideal on $\mathfrak{a}$ which includes the ideal $J^{bd}_{\mathfrak{a}}$ of bounded subsets of $\mathfrak{a}$ and such that $\mathrm{tcf}(\prod \mathfrak{a}, <_J)$ exists $\}$.

The main theorem can be stated as follows:

Theorem 1.3 *Assume GCH. Let η be an ordinal and δ be a regular cardinal. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of strong cardinals and a cardinal $\lambda > \bigcup_{\alpha<\eta} \kappa_\alpha$. Then there is a cardinal preserving extension in which , for every $\alpha < \eta$,*

1. $\mathrm{cof}(\kappa_\alpha) = \delta$,
2. $\mathrm{pp}(\kappa_\alpha) \geq \lambda$.

The actual result, Theorem 8.3, uses a bit weaker assumptions.

The Shelah weak hypothesis (SWH) was introduced by S. Shelah in [11], see also [12]. For uncountable cofinality it states that

for every λ the set $\{\kappa < \lambda \mid \omega < \mathrm{cof}(\kappa) < \kappa, \mathrm{pp}(\kappa) \geq \lambda\}$ is finite.

It was shown in [6] that any finite size is realizable.

For countable cofinality it states that

for every λ the set $\{\kappa < \lambda \mid \omega = \mathrm{cof}(\kappa) < \kappa, \mathrm{pp}(\kappa) \geq \lambda\}$ is at most countable.

It was shown in [6] that any finite size is realizable, then in [5] that this set can be countable and, finally, in [4], that it can have size $\aleph_1$.

Strong forms of ¬SWH follow from Theorem 1.3. We are just free to pick various values of parameters η and δ there.

A detailed analysis of the cardinal arithmetic structure in the constructed models will be given in the last section of the paper.

Let us mention an additional result that is deduced using the method:

Assuming $\omega+1-$many strong cardinals, there is a cardinal preserving extension in which $\mathrm{cof}(\mathrm{pp}(\kappa)) = \omega_1$, for a singular cardinal κ of countable cofinality.

This gives a negative answer to a question of S. Shelah (Problem (ε), Analytical Guide, [10]):

Is $\mathrm{cof}(\mathrm{pp}(\kappa)) > \kappa$?

Note that $\mathrm{cof}(\mathrm{pp}(\kappa)) > \mathrm{cof}(\kappa)$ and in view of the König Theorem it is reasonable to have $\mathrm{cof}(\mathrm{pp}(\kappa)) > \kappa$.

The paper is organized as follows:

In Section 2, we deal with a situation in which an extender overlaps a measure. It is a basic block of further more involved constructions. Our hope is that main ideas and the intuition are explained here. Section 3 deals with two extenders which overlap and in Section 4 we consider an extender which overlaps many other extenders. The main forcing is defined in Section 7. In the last section the cardinal arithmetic structure of the main generic extension is studied. Applications to the Shelah Weak Hypothesis are deduced there.

2 An extender which overlaps a measure.

Assume GCH. Let $\kappa < \lambda$ be measurable cardinals. Fix a normal measure $U(\lambda)$ over λ. Suppose that E is a $(\kappa, \lambda^{++})-$extender. Let $j_E : V \to M_E \simeq \mathrm{Ult}(V, E)$ be the corresponding embedding. Then $\kappa = \mathrm{crit}(E)$ and $M_E \supseteq H_{\lambda^{++}}$. Assume that $j_E(\kappa) > \lambda^{++}$, ${}^\kappa M_E \subseteq M_E$.[1]
Let $x \in H_{\lambda^{++}}$. Denote by E_x the measure on V_κ generated by x, i.e.

$$E_x = \{X \subseteq V_\kappa \mid x \in j_E(X)\}.$$

We would like to force using E and $U(\lambda)$ in order to change the cofinality of κ to ω simultaneously blowing up its power above λ and changing the cofinality of λ to ω.
It is possible to achieve this either as in [6] using a preparation Prikry forcing below κ or as in [5] using a triangle type of construction.
Here we present another more direct method. In particular, we will construct a generic extension in which κ is a strong limit, only $\omega-$many cardinals below it change cofinality and $2^\kappa > \lambda^+$.

Assume for simplicity that $U(\lambda)$ is represented by the normal measure of E, i.e. $U(\lambda) = j_E(s)(\kappa)$ for some $s : \kappa \to V_\kappa$. Assume also that $s(\alpha)$ is a normal ultrafilter $U(\alpha)$ over λ_α, for each $\alpha < \kappa$.

Carmi Marimovich in [8, 9] have found a very elegant version of Extender Based Prikry forcing. We will use here a variation of it adapted to the present situation.

Let us state briefly few basic definitions.

Let $d \subseteq \lambda^{++} \setminus \kappa$ of cardinality at most κ. Define a $\kappa-$ultrafilter $E(d)$ on $[d \times \kappa]^{<\kappa}$ as follows:

$$X \in E(d) \Leftrightarrow \{\langle j_{E(\kappa)}(\alpha), \alpha\rangle \mid \alpha \in d\} \in j_E(X).$$

Actually, $E(d)$ concentrates on a smaller set called $\mathrm{OB}(d)$ in [9].
The advantage of using $E(d)$ is that once A is typical set of $E(d)-$measure one and $a \in A$, then a is of the form $\langle\langle \alpha_\xi, \beta_\xi\rangle \mid \xi < \rho\rangle$, where

[1] If one likes to add only λ^+-many Prikry sequences, then what is need here is $M_E \supseteq H_{\lambda^+}$ and $U(\lambda) \in M_E$.

1. $\rho < \kappa$,
2. $\mathrm{dom}(a) = \{\alpha_\xi \mid \xi < \rho\} \subseteq d$,
3. $\beta_\xi < \kappa$, for every $\xi < \rho$.

So, already a measure one set provides an explicit connection between elements of Prikry sequences and the measures to which they belong.

Note that $E(d)$ is actually equivalent to the ultrafilter E_d over V_κ defined by

$$Y \in E_d \Leftrightarrow d \in j_E(Y).$$

Thus, clearly, $E(d)$ is Rudin-Kiesler above E_d, just project to the second coordinate. For the opposite direction note $\mathrm{Ult}(V, E_d)$ is closed under $\kappa-$sequences of ordinals, hence $j"_{E_d}d$ is there. Then using it, we can define $E(d)$ easily.

Definition 2.1 Suppose now that $B \in U(\lambda), B \subseteq \lambda \setminus \kappa$ and $F : [B]^{<\omega} \to \mathcal{P}_{\kappa^+}(\lambda^{++} \setminus \kappa)$. We call F *a relevant function* iff

1. $\kappa \in F(\langle\rangle)$,
2. for every $\langle \xi_1, ..., \xi_n\rangle \in [B]^n$, $\{\xi_1, ..., \xi_n\} \subseteq F(\xi_1, ..., \xi_n)$,
3. for every $\vec{\nu}_1, \vec{\nu}_2$ in $[B]^{<\omega}$, if $\vec{\nu}_1$ extends $\vec{\nu}_2$, then $F(\vec{\nu}_1) \supseteq F(\vec{\nu}_2)$,
4. if $\vec{\nu}_1, \vec{\nu}_2$ in $[B]^{<\omega}$ are of the same length, then $\mathrm{otp}(F(\vec{\nu}_1)) = \mathrm{otp}(F(\vec{\nu}_2))$
5. if $\vec{\nu}_1, \vec{\nu}_2$ in $[B]^{<\omega}$ are of the same length, then
   $E_{F(\vec{\nu}_1)} = E_{F(\vec{\nu}_2)}$.
   Note that the number of ultrafilters over V_κ is small relatively to λ, and so this can be easily arranged on $U(\lambda)-$measure one sets.

Note the size of a relevant function is λ and so it belongs to $H_{\lambda^{++}}$. If its values are in λ^+, then the function will belong to H_{λ^+}.

Definition 2.2 Let F, G be relevant functions. Set $F \geq^* G$ iff

1. $\mathrm{dom}(F) \subseteq \mathrm{dom}(G)$,
2. for every $\langle \xi_1, ..., \xi_n\rangle \in \mathrm{dom}(F)$, $F(\xi_1, ..., \xi_n) \supseteq G(\xi_1, ..., \xi_n)$.

Consider a relevant function F and $M = \mathrm{Ult}(V, E)$ with $j = j_E : V \to M$ the corresponding elementary embedding.
Then, $F \in M$. Pick $\delta_F < \lambda^{++}$ and $\tilde{F} : \kappa \to V_\kappa$ such that $j_E(\tilde{F})(\delta_F) = F$.
We will fix this notation throughout, i.e. δ_F and $\tilde{F}$ for a given relevant function F. We fix some wellordering $\prec$ of V_κ and will use $j_E(\prec)$. We have $V_{\lambda+2} = (V_{\lambda+2})^{M_E}$, so $j_E(\prec)$ well orders $V_{\lambda+2}$.
Let $Z \subseteq \lambda^{++}$ be a set of cardinality κ. Denote by h^Z the $j_E(\prec)-$least one to one function from $|Z|$ onto Z.
A typical use of this will be as follows.
Let $\beta \in F(\langle\rangle)$. Then $\beta <_E \delta_F$. We have $|F(\langle\rangle)| \leq \kappa$. Hence there is $\alpha < |F(\langle\rangle)|$ such that $\beta = h^{F(\langle\rangle)}(\alpha)$.
Let ν be in a typical E_{δ_F} set of measure one. Suppose that ν^{nor}, its canonical projection to the (least) normal measure of E, is bigger than α.
Let $h^{\tilde{F}(\nu)(\langle\rangle)}_{\nu^{nor}}$ be $\prec$ $-$least bijection between ν^{nor} and $\tilde{F}(\nu)(\langle\rangle)$. Then $h^{\tilde{F}(\nu)(\langle\rangle)}_{\nu^{nor}}(\alpha)$ will correspond to β.

Turn now to the definition of the forcing.

Following [8, 9], define first $\mathcal{P}^*_E$.

Definition 2.3 Let $\mathcal{P}^*_E$ be the set of all functions f such that

1. $\mathrm{dom}(f) \subseteq [\lambda]^{<\omega} \times \mathcal{P}_{\kappa^+}(\lambda^{++} \setminus \kappa)$ is a relevant function, i.e., we view $\mathrm{dom}(f)$ as a partial function from $[\lambda]^{<\omega}$ into $\mathcal{P}_{\kappa^+}(\lambda^{++} \setminus \kappa)$.

2. For every $\langle \xi_1, ..., \xi_n \rangle \in \mathrm{dom}(\mathrm{dom}(f))$ (i.e., $\langle \xi_1, ..., \xi_n \rangle$ is in the domain of the relevant function $(\mathrm{dom}(f))$) and for every $\alpha \in (\mathrm{dom}(f))(\xi_1, ..., \xi_n)) \subseteq \lambda^{++} \setminus \kappa$, $(f(\xi_1, ..., \xi_n))(\alpha) = \langle (f(\xi_1, ..., \xi_n))(\alpha)_0, ..., (f(\xi_1, ..., \xi_n))(\alpha)_{k-1} \rangle$ is a finite sequence of elements of V_κ, for some $k < \omega$, which canonical projections to the normal measure form an increasing sequence and such that for every $i < k$,

   (a) $(f(\xi_1, ..., \xi_n))(\alpha)_i$ is an ordinal
   or

   (b) for some $\nu < \kappa$ and a measurable cardinal $\lambda_\nu, \nu < \lambda_\nu < \kappa$, a normal ultrafilter $U(\lambda_\nu)$ over λ_ν and a set $B_{\lambda_\nu} \in U(\lambda_\nu)$, $(f(\xi_1, ..., \xi_n))(\alpha)_i : [B_{\lambda_\nu}]^n \to \lambda_\nu^{++}$. [2]

[2] Note that further not every such λ_ν will change its cofinality. Moreover, only λ_ν's with ν's which the members of the Prikry sequence for the normal measure of the extender, i.e. for E_κ, will change their cofinality.

Require, in addition, that if α' is an other member of $(\mathrm{dom}(f))(\xi_1, ..., \xi_n))$ and some of the elements of $(f(\xi_1, ..., \xi_n))(\alpha')$ has the same ν, then the set B_{λ_ν} is also the same.

3. For every $\langle \xi_1, ..., \xi_n \rangle, \langle \xi'_1, ..., \xi'_n \rangle \in [\lambda]^n \cap \mathrm{dom}(\mathrm{dom}(f))$,
   (a) $\mathrm{otp}((\mathrm{dom}(f)(\xi_1, ..., \xi_n))) = \mathrm{otp}((\mathrm{dom}(f)(\xi'_1, ..., \xi'_n)))$,
   (b) for every $\beta < \mathrm{otp}((\mathrm{dom}(f)(\xi_1, ..., \xi_n)))$, if α, α' are the $\beta-$th members of $\mathrm{dom}(f)(\xi_1, ..., \xi_n)$ and $\mathrm{dom}(f)(\xi'_1, ..., \xi'_n)$ respectively, then $(f(\xi_1, ..., \xi_n))(\alpha) = (f(\xi'_1, ..., \xi'_n))(\alpha')$.
4. if a sequence $\vec{\xi} \in [\lambda]^m$ extends a sequence $\vec{\xi'} \in [\lambda]^n$ and $\alpha \in (\mathrm{dom}(f))(\vec{\xi'}))$ (and hence, $\alpha \in (\mathrm{dom}(f))(\vec{\xi})))$, then $f(\vec{\xi})(\alpha) = f(\vec{\xi'})(\alpha)$.

The intuition behind this definition is that $f(\langle\rangle)$ acts exactly as in [8, 9]. In addition, we have here λ that is supposed to change its cofinality to ω. So, $\langle \xi_1, ..., \xi_n \rangle \in [\lambda]^n$ is a possible initial segment of the Prikry sequence of λ and $f(\xi_1, ..., \xi_n)$ provides the correspondence between its domain, which is a subset of $\lambda^{++} \setminus \kappa$ of cardinality $\leq \kappa$, and finite sequences in V_κ, again as in [8, 9].

Definition 2.4 Let f, g be in $\mathcal{P}^*_E$. Set $f \geq^* g$ iff $\mathrm{dom}(f) \geq^* \mathrm{dom}(g)$, as relevant functions and for every $n < \omega$, $\langle \xi_1, ..., \xi_n \rangle \in \mathrm{dom}(\mathrm{dom}(f)), \alpha \in \mathrm{dom}(g)(\xi_1, ..., \xi_n)$, $f(\xi_1, ..., \xi_n)(\alpha) = g(\xi_1, ..., \xi_n)(\alpha)$.

Lemma 2.5 *$\langle \mathcal{P}^*_E, \leq^* \rangle$ is κ^+-closed.*

Proof. Let $\langle f_\gamma \mid \gamma < \kappa \rangle$ be a $\leq^*$ $-$increasing sequence of elements of $\mathcal{P}^*_E$.
For every $\gamma < \kappa$, let $\mathrm{dom}(\mathrm{dom}(f_\gamma)) = [B_\gamma]^{<\omega}$, for some $B_\gamma \in U(\lambda)$. Set $B = \bigcap_{\gamma<\kappa} B_\gamma$. Then $B \in U(\lambda)$, since $\kappa < \lambda$.
Define $f \in \mathcal{P}^*_E$. Set $\mathrm{dom}(\mathrm{dom}(f)) = [B]^{<\omega}$. Now, for every $\langle \xi_1, ..., \xi_n \rangle \in \mathrm{dom}(\mathrm{dom}(f))$, let $\mathrm{dom}(f(\xi_1, ..., \xi_n)) = \bigcup_{\gamma<\kappa} \mathrm{dom}(f_\gamma(\xi_1, ..., \xi_n))$. Finally, if $\alpha \in \mathrm{dom}(f(\xi_1, ..., \xi_n))$, then pick $\gamma < \kappa$ such that $\alpha \in \mathrm{dom}(f_\gamma(\xi_1, ..., \xi_n))$ and set $f(\xi_1, ..., \xi_n)(\alpha) = f_\gamma(\xi_1, ..., \xi_n)(\alpha)$.
Clearly, $f \in \mathcal{P}^*_E$ and $f \geq^* f_\gamma$, for every $\gamma < \kappa$.
□

Turn now to the definition of our forcing.
Define first conditions with empty stems $\mathcal{P}_0$.

Definition 2.6 The set $\mathcal{P}_0$ consists of pairs $\langle \langle f, A, \delta \rangle, \langle \langle\rangle, B \rangle \rangle$ such that

1. $\langle\langle\rangle, B\rangle$ is a condition in the Prikry forcing with $U(\lambda)$,

2. $f \in \mathcal{P}^*_E$,

3. $\mathrm{dom}(\mathrm{dom}(f)) = [B]^{<\omega}$,

4. $\delta < \lambda^{++}$,

5. $f(\langle\rangle)(\kappa) = \langle\rangle$,

6. $f = j_E(\tilde{f})(\delta)$, where $\tilde{f} : \kappa \to V_\kappa$ is the $j_E(\prec)-$least function like this.[3]

7. $A \in E(\mathrm{dom}(f(\langle\rangle)) \cup \{\delta\})$ or, alternatively, A is an $\omega-$tree with splittings in $E(\mathrm{dom}(f(\langle\rangle)) \cup \{\delta\})$.
   It will be convenient further to view A as an element of the equivalent measure E_δ.

Define the direct extension $\leq^*$ on $\mathcal{P}_0$ in the usual fashion by extending support and by shrinking measure one sets.

Definition 2.7 Let $p = \langle\langle f, A, \delta\rangle, \langle\langle\rangle, B\rangle\rangle$,
$p' = \langle\langle f', A', \delta'\rangle, \langle\langle\rangle, B'\rangle\rangle \in \mathcal{P}_0$. Set $p >^* p'$ iff

1. $B \subseteq B'$,

2. $f \geq^* f'$,

3. $\delta' \in \mathrm{dom}(f(\langle\rangle))$,

4. $A \restriction (\mathrm{dom}(f'(\langle\rangle)) \cup \{\delta'\}) \subseteq A'$.
   In particular, this implies:

   (a) $\delta \geq_E \delta'$,

   (b) the canonical projection of A to δ' is a subtree of A', i.e. $\pi_{\delta\delta'}{}''A \subseteq A'$.

Turn now to one element extensions of members of $\mathcal{P}_0$.

Definition 2.8 Let $p = \langle\langle f, A, \delta\rangle, \langle\langle\rangle, B\rangle\rangle \in \mathcal{P}_0$ and $\eta \in A$ or at the first level of A, if we view A as a tree. Define the extension of p by η, $p^\frown\eta$ as follows.
Set $p^\frown\eta$ to be $\langle\langle\rangle, \mathrm{dom}(\mathrm{dom}((\tilde{f})(\eta)))\rangle^\frown\langle f^\frown\eta, \langle\kappa, \eta^{nor}\rangle, \langle\delta, \eta\rangle, A_{\langle\eta\rangle}\rangle, \langle\langle\rangle, B\rangle\rangle$,
where

[3] δ is essentially the maximal coordinate of the condition.

1. $\langle\langle\rangle, \mathrm{dom}(\mathrm{dom}((\tilde{f})(\eta)))\rangle$ is a condition in the Prikry forcing over $\lambda_{\eta^{nor}}$ with $U(\lambda_{\eta^{nor}})$. Note that $\mathrm{dom}(\mathrm{dom}((j_E(\tilde{f})(\delta_f))) = [B]^{<\omega}$.

2. $A_{\langle\eta\rangle}$ is the tree A above $\langle\eta\rangle$ (or $A \setminus \eta$ if working with sets),

3. $f^\frown\eta$ is defined as follows:
   its domain is identical to those of f;
   let $n < \omega$ and $\langle\xi_1, ..., \xi_n\rangle \in [B]^n$ and $\alpha \in dom(f)(\xi_1, ..., \xi_n)$.
   Set $(f^\frown\eta(\xi_1, ..., \xi_n))(\alpha) = (f(\xi_1, ..., \xi_n))(\alpha)$, unless η^{nor} is above all elements of the sequence $(f(\xi_1, ..., \xi_n))(\alpha)$. For elements of the sequence $(f(\xi_1, ..., \xi_n))(\alpha)$ which are names, we mean that η^{nor} is above the measurable (or its double successor) which defines it.
   Suppose that η^{nor} is above every element of $(f(\xi_1, ..., \xi_n))(\alpha)$.
   We include here also the case when the sequence is empty. This happens, in particular, if $\alpha = \kappa$.
   Split into two cases.

   (a) If $\alpha = \kappa$, then set $(f^\frown\eta(\xi_1, ..., \xi_n))(\alpha) = \langle\eta^{nor}\rangle$,

   (b) suppose that $\alpha \neq \kappa$. Let α be the $\beta(\alpha)-$th member of $\mathrm{dom}(f)(\xi_1, ..., \xi_n))$, for some $\beta(\alpha) < \eta^{nor}$, i.e. $h^{\mathrm{dom}(f)(\xi_1,...,\xi_n))}(\beta(\alpha)) = \alpha$.
   Consider $\mathrm{dom}(\tilde{f})(\eta) : [B(\eta)]^{<\omega} \to \mathcal{P}_{(\eta^{nor})^+}(\lambda(\eta^{nor})^{++})$.
   Define $g : [B(\eta)]^n \to \lambda(\eta^{nor})^{++}$ by setting $g(\tau_1, ..., \tau_n)$ to be the $\beta(\alpha)-$th member of $\mathrm{dom}(\tilde{f})(\eta)(\tau_1, ..., \tau_n)$, i.e. $h_{\eta^{nor}}^{\mathrm{dom}(f)(\xi_1,...,\xi_n))}(\beta(\alpha))$.
   Add then g to the sequence, i.e. let
   $(f^\frown\eta(\xi_1, ..., \xi_n))(\alpha) = (f(\xi_1, ..., \xi_n))(\alpha)^\frown g$.

An additional way to extend conditions is to extend its Prikry parts. Start first over λ.

Definition 2.9 $p = \langle\langle f, A, \delta_f\rangle, \langle\langle\rangle, B\rangle\rangle \in \mathcal{P}_0$ and $\langle\xi_1, ..., \xi_m\rangle \in [B]^m$. Define the extension of p by $\langle\xi_1, ..., \xi_m\rangle$, $p^\frown\langle\xi_1, ..., \xi_m\rangle$ as follows.
Set $p^\frown\langle\xi_1, ..., \xi_m\rangle$ to be $\langle\langle f_{\langle\xi_1,...,\xi_m\rangle} A', \delta_{f_{\langle\xi_1,...,\xi_m\rangle}}\rangle, \langle\langle\xi_1, ..., \xi_m\rangle, B \setminus \xi_m + 1\rangle\rangle$, where

1. $f_{\langle\xi_1,...,\xi_m\rangle}$ is the obvious restriction of f to extensions of the sequence $\langle\xi_1, ..., \xi_m\rangle$ inside B.

2. A' is the pre-image of A under the canonical projection $\pi_{\delta_{f_{\langle\xi_1,...,\xi_m\rangle}}, \delta_f}$ of $E_{\delta_{f_{\langle\xi_1,...,\xi_m\rangle}}}$ onto E_{δ_f}.

Note that ξ_i's need not be $\leq_E \delta_f$. Moreover, most of ξ's mod $U(\lambda)$ are not below δ_f. However, almost every E_ξ mod $U(\lambda)$ is Rudin-Keisler below E_{δ_f}, since $f = j_E(\tilde{f})(\delta_f)$.

Let us add such extensions to $\mathcal{P}_0$. Denote the result by $\mathcal{P}_1$.

Similarly define extension in Prikry parts below κ.

Definition 2.10 Let $q = \langle\langle\rangle, \mathrm{dom}((\mathrm{dom}(\tilde{f}))(\eta))^\frown\langle f, A_{\langle\eta\rangle}, \delta\rangle, \langle\langle b\rangle, B\rangle\rangle \in \mathcal{P}_1$. Suppose that $x \in [\mathrm{dom}((\mathrm{dom}(\tilde{f}))(\eta)]^{<\omega}$. Then we put
$q^\frown x = \langle\langle x\rangle, \mathrm{dom}((\mathrm{dom}(\tilde{f}))(\eta))^\frown\langle f, A_{\langle\eta\rangle}, \delta\rangle, \langle\langle b\rangle, B\rangle\rangle$ into $\mathcal{P}_1$, as well.

One element extensions of elements of $\mathcal{P}_1$ are defined in the same fashion. Denote the resulting set by $\mathcal{P}_2$. Proceed by induction and define similar $\mathcal{P}_n$, for every $n < \omega$.
Finaly set $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}_n$.

Let us start with the Prikry condition.

The next lemma is an analog of 3.12 from [8].

Lemma 2.11 *Let $p = \langle\langle f^p, A^p, \delta^p\rangle, \langle\langle\rangle, B^p\rangle\rangle \in \mathcal{P}$ and $D \subseteq \mathcal{P}$ be a dense open. Then there are $p^* \geq^* p$ and $n < \omega$ such that $p^{*\frown}\vec{\nu}^\frown\vec{t}^\frown\vec{x} \in D$,
for every $\vec{\nu} = \langle\nu_0, ..., \nu_{n-1}\rangle \in [A^{p^*}]^n, \vec{t} \in [B^{p^*}]^n$, $\vec{x} = \langle\vec{x}_k \mid k < n\rangle$ such that for every $k < n$, $\vec{x}_k \in [B_k^{p^*}]^n$, where $B_k^{p^*} = \mathrm{dom}(\tilde{f}^*)(\nu_0, ..., \nu_k)$, i.e. the set of measure one corresponding to B^{p^*} at the level k.*

Proof. Suppose otherwise.
Proceed as in 3.12 of [8]. Construct by induction an $\in$ −increasing chain of elementary submodels $\langle N_\xi \mid \xi < \kappa\rangle$ of H_χ, for χ large enough[4], and a sequence $\langle f_\xi \mid \xi < \kappa\rangle$ of members of $\mathcal{P}_E^*$, such that

1. $p, \mathcal{P}_E^*, \mathcal{P}, D \in N_0$,
2. $N_0 \supseteq \kappa$,
3. for every $\xi < \kappa$,
    (a) $|N_\xi| = \kappa$,
    (b) ${}^{\kappa>}N_\xi \subseteq N_\xi$,
    (c) $\langle f_\zeta \mid \zeta < \xi\rangle \in N_\xi$,

[4]Note that we do not require the continuity of this chain.

(d) $f_\xi \in \bigcap\{D' \in N_\xi \mid D'$ is a dense open subset of $\mathcal{P}_E$ above $f^p\}$,

(e) $f^p \leq^* f_0$,

(f) $\delta^p \in \mathrm{dom}(f_0(\langle\rangle))$,

(g) $\{\delta_{f_\zeta} \mid \zeta < \xi\} \subseteq \mathrm{dom}(f_\xi(\langle\rangle))$,

(h) $f_\xi \geq^* f_\zeta$, for every $\zeta < \xi$.

Set $N = \bigcup_{\xi<\kappa} N_\xi$ and f^* the upper bound of $\langle f_\xi \mid \xi < \kappa\rangle$.
Consider $\{\delta_{f_\xi} \mid \xi < \kappa\}$. Let δ be the least code of this set in our fixed wellordering.
Define an ultrafilter $\bar{E}$ over $[\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa]^{<\kappa}$ which is equivalent (Rudin-Keisler) to $E_{\{\kappa,\delta_{f_\xi}|\xi<\kappa\}}$ and is below E_δ (in the order $<_E$ of E).
Set $Z \in \bar{E}$ iff $\{(j_E(\kappa), \kappa), (j_E(\delta^p), \delta^p), (j_E(\delta_{f_0}), \delta_{f_0}), ..., j_E(\delta_{f_\xi}), \delta_{f_\xi}), ... \mid \xi < \kappa\} \in j_E(Z)$.
Note that $\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \subseteq N$, and so, $\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa \subseteq N$. Hence, also, $[\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa]^{<\kappa} \subseteq N$. The function $\langle \beta_\gamma \mid \gamma < \epsilon < \kappa\rangle \mapsto \langle\langle\kappa, \beta_0\rangle, \langle\delta^p, \beta_1\rangle, \langle\delta_{f_0}, \beta_2\rangle, ..., \langle\delta_{f_\xi}, \beta_{2+\xi}\rangle, ... \mid \xi < \epsilon\rangle$ witnesses the equivalence between $E_{\{\kappa,\delta_{f_\xi}|\xi<\kappa\}}$ and $\bar{E}$.

Let us define an additional ultrafilter $\bar{\bar{E}}$ in order to take care of Prikry forcings below κ. It will concentrate on $[\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa]^{<\kappa} \times \kappa$.
Set $Y \in \bar{\bar{E}}$ iff $\{(j_E(\kappa), \kappa), (j_E(\delta^p), \delta^p), (j_E(\delta_{f_0}), \delta_{f_0}), ..., j_E(\delta_{f_\xi}), \delta_{f_\xi}), ... \mid \xi < \kappa\} \in j_E(Y \restriction 1)$ and $j_E(Y)_{\langle(j_E(\kappa),\kappa),(j_E(\delta^p),\delta^p),(j_E(\delta_{f_0}),\delta_{f_0}),...,j_E(\delta_{f_\xi}),\delta_{f_\xi}),...|\xi<\kappa\rangle} \in U(\lambda)$, where
$Y \restriction 1 = \{y \mid \exists\alpha < \kappa((y, \alpha) \in Y)\}$ and if $\vec{\nu} \in Y \restriction 1$, then $Y_{\vec{\nu}} = \{\alpha < \kappa \mid (\vec{\nu}, \alpha) \in Y\}$.

We will use closely related ultrafilters $\bar{\bar{E}}_n$, for $n, 1 \leq n < \omega$ over $[\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa]^{<\kappa} \times [\kappa]^n$.[5]
Set $Y \in \bar{\bar{E}}_n$ iff $\{(j_E(\kappa), \kappa), (j_E(\delta^p), \delta^p), (j_E(\delta_{f_0}), \delta_{f_0}), ..., j_E(\delta_{f_\xi}), \delta_{f_\xi}), ... \mid \xi < \kappa\} \in j_E(Y \restriction 1)$ and $j_E(Y)_{\langle(j_E(\kappa),\kappa),(j_E(\delta^p),\delta^p),(j_E(\delta_{f_0}),\delta_{f_0}),...,j_E(\delta_{f_\xi}),\delta_{f_\xi}),...|\xi<\kappa\rangle} \in U(\lambda)^n$.

Let $A \in \bar{E}$ be a set which projection to δ^p is a subset of A^p.

For each $k < \omega$ and
$\langle\langle\eta_0, \vec{x}_0\rangle, ..., \langle\eta_{k-1}, \vec{x}_{k-1}\rangle\rangle \in [[\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa]^{<\kappa} \times [\kappa]^k]^k$
such that $\langle\eta_0, ..., \eta_{k-1}\rangle \in [A]^k$, let
$D_{\langle\langle\eta_0,\vec{x}_0\rangle,...,\langle\eta_{k-1},\vec{x}_{k-1}\rangle\rangle}$ be the set of all $f \geq^* f^p$ such that either

$$\exists T\exists B \subseteq B^p\exists C_0 \subseteq \mathrm{dom}(\mathrm{dom}((\tilde{f}^p)(\eta_0))), ..., C_{k-1} \subseteq \mathrm{dom}(\mathrm{dom}((\tilde{f}^p)(\eta_{k-1}))) \tag{1}$$

$$(\langle\langle\vec{x}_0, C_0\rangle, ..., \langle\vec{x}_{k-1}, C_{k-1}\rangle\rangle, \langle f_{\langle\eta_0,...,\eta_{k-1}\rangle}, T, \langle\langle\rangle, B\rangle\rangle \in D)$$

[5] An alternative way to proceed is to deal only with $\bar{E}$. This will be explored further in a more general situation.

or

$$(2)\forall g \geq^* f \forall T \forall B \subseteq B^p \forall C_0 \subseteq \mathrm{dom}(\mathrm{dom}((\tilde{f^p})(\eta_0))), ..., C_{k-1} \subseteq \mathrm{dom}(\mathrm{dom}((\tilde{f^p})(\eta_{k-1})))$$

$$(\langle\langle\vec{x}_0, C_0\rangle, ..., \langle\vec{x}_{k-1}, C_{k-1}\rangle\rangle, \langle\langle g_{\langle\eta_0,...,\eta_{k-1}\rangle}, T, \langle\langle\rangle, B\rangle\rangle \notin D).$$

Such defined $D_{\langle\langle\eta_0,\vec{x}_0\rangle,...,\langle\eta_{k-1},\vec{x}_{k-1}\rangle\rangle}$ is obviously dense in $\mathcal{P}^*_E$ above f^p. Then, for each $\langle\langle\eta_0,\vec{x}_0\rangle, ..., \langle\eta_{k-1},\vec{x}_{k-1}\rangle\rangle \in [[\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa]^{<\kappa} \times [\kappa]^k]^k$ with $\langle\eta_0, ..., \eta_{k-1}\rangle \in [A]^k$, $f^* \in D_{\langle\langle\eta_0,\vec{x}_0\rangle,...,\langle\eta_{k-1},\vec{x}_{k-1}\rangle\rangle}$.

If for some $n < \omega$, for a set of $\bar{\bar{E}}^n_n-$measure one the possibility (1) occurs, then we get a contradiction to our initial assumption.

So, suppose that for every $n < \omega$, there is a set $Y_n \in \bar{\bar{E}}^n_n-$ on which the possibility (2) occurs.

Let us construct a condition in $\mathcal{P}$ which is based on f^* and Y_n's.

We will use the following observations.

Claim 1 Suppose that $Y \in \bar{\bar{E}}$. Define a function $g : Y \restriction 1 \to V_\kappa$ by setting $g(\vec{\eta}) = Y_{\vec{\eta}}$. Then $j_E(g)(\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\}) \in U(\lambda)$.

Proof. Follows by the definition of $\bar{\bar{E}}$.
□ of the claim.

Claim 2 Suppose that $Y \in \bar{\bar{E}}^2$. Let $Z = \{(z_1, z_2) \mid \exists\beta_1, \beta_2((z_1, \beta_1, z_2, \beta_2) \in Y)\}$. Then there is $Y^* \in \bar{E}$ such that $Z \supseteq [Y^*]^2$.[6]

Proof. Clear.
□ of the claim.

Claim 3 Suppose that $Y \in \bar{\bar{E}}^2$, then there is $Y' \subseteq Y, Y' \in \bar{\bar{E}}^2$ such that for some $Z \in \bar{E}$ and a function $g : Z \to V_\kappa$ the following hold:

1. $Y'_{1,3} = [Z]^2$, where $Y'_{1,3}$ is the projection of Y' to the first and third coordinate, i.e. $Y'_{1,3} = \{(z_1, z_2) \mid \exists\beta_1, \beta_2((z_1, \beta_1, z_2, \beta_2) \in Y')\}$.

[6]Here and in similar situations, given a $\kappa-$complete ultrafilter W over V_κ which is not necessarily normal, it is still possible to use a diagonal intersections with a slight correction. Namely, let $W^* \leq_{Rudin-Keisler} W$ be the normal measure over κ generated by κ and let $\pi : V_\kappa \to \kappa$ be a projection of W to W^*. Suppose that $\{A_x \mid x \in V_\kappa\} \subseteq W$. Set $A = \{y \in V_\kappa \mid \forall x \in V_\kappa(x \in V_{\pi(y)} \to y \in A_x)\}$. Then $A \in W$.
Also, by $[X]^2$ we mean in such a context the set $\{(x, y) \in V_\kappa \mid x \in V_{\pi(y)}\}$.
Clealy, if W is a normal measure, then the above coincides with the usual notions.

2. For every $(z_1, \beta_1, z_2) \in Y'_{1,2,3}$, $Y'_{(z_1,\beta_1,z_2)} = g(z_2)$, where $Y'_{1,2,3}$ is the projection of Y' to the first three coordinates.

Proof. First we pick Y^* by the previous claim. Let $(z_1, \beta_1, z_2) \in Y_{1,2,3}$ with $(z_1, z_2) \in [Y^*]^2$. We have then that $Y_{(z_1,\beta_1,z_2)} \in U(\lambda_{z_2^{nor}})$, where z_2^{nor} is the first element of z_2, i.e. one that corresponds to the normal measure E_κ. The ultrafilter $U(\lambda_{z_2^{nor}})$ is $\lambda_{z_2^{nor}}-$complete, so $Y(z_2) := \bigcap\{Y_{(z'_1,\beta'_1,z_2)} \mid (z'_1, \beta'_1, z_2) \in Y_{1,2,3}\} \in U(\lambda_{z_2^{nor}})$.
Set $g(z_2) = Y(z_2)$.
□ of the claim.

The analogues statement holds for every $n, 1 \leq n < \omega$ with a similar proof.

Claim 4 Let $n, 1 \leq n < \omega$. Suppose that $Y \in \bar{\bar{E}}^{2n}$, then there is $Y' \subseteq Y, Y' \in \bar{\bar{E}}^{2n}$ such that for some $Z \in \bar{E}$ and a function $g : Z \to V_\kappa$ the following hold:

1. $Y'_{1,3,...,2n-1} = [Z]^n$, where $Y'_{1,3,...,2n-1}$ is the projection of Y' to its odd coordinates.

2. For every $(z_1, \beta_1, z_2, ..., z_k) \in Y'_{1,2,3,...,k}$, $Y'_{(z_1,\beta_1,z_2,...,z_k)} = g(z_k)$, where $Y'_{1,2,3,...,k}$ is the projection of Y' to the first $k-$coordinates.

Now, for each $n, 1 \leq n < \omega$ we apply the claims to $Y_n \in \bar{\bar{E}}^n_n$ and find $Z_n \in \bar{E}$ and $g_n : Z_n \to V_\kappa$. Let $Z^* = \bigcap_{1 \leq n < \omega} Z_n$. Set $B_n = j_E(g_n)(\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\})$. Then each B_n is in $U(\lambda)$. Set $B^* = B^p \cap \bigcap_{1 \leq n < \omega} B_n$.

Now, based on f^*, Z^* and B^* we form a condition in $\mathcal{P}$ in the obvious fashion.
Let $p^* = \langle\langle f^{**}, A^*, \delta^*\rangle, \langle\langle\rangle, B^*\rangle\rangle \geq^* p$, where A^* is contained in the pre-image of Z^* and f^{**} extends f^*. Further let us abuse the notation and denote f^{**} still by f^*.

Claim 5 Suppose that $q \geq p^*$ and the component of q over λ still have empty sequence, i.e. it is of the form $\langle\langle\rangle, B^q\rangle$. Then $q \notin D$.

Proof. Suppose otherwise. Then, for some $k < \omega$ and
$\langle\langle \eta_0, \vec{x}_0\rangle, ..., \langle \eta_{k-1}, \vec{x}_{k-1}\rangle\rangle \in [[\{\kappa, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa\} \times \kappa]^{<\kappa} \times [\kappa]^k]^k$
with $\langle \eta_0, ..., \eta_{k-1}\rangle \in [A^*]^k$,
f^q is a $\leq^*$ $-$extension of some $f \leq^* f^*, f \in D_{\langle\langle \eta_0,\vec{x}_0\rangle,...,\langle \eta_{k-1},\vec{x}_{k-1}\rangle\rangle} \cap N$. But then, the possibility (2) must occur for f. This means that q cannot be in D. Contradiction.
□ of the claim.

Now, we repeat the process above with the empty sequence over λ replaced by any $\vec{b} \in [B^p]^{<\omega}$.

Let $p_{\langle\rangle} = p^* = \langle\langle f^*, A^*, \delta^*\rangle, \langle\langle\rangle, B^p\rangle\rangle$. Denote by $p_{\vec{b}} = \langle\langle f_{\vec{b}}, A_{\vec{b}}, \delta_{\vec{b}}\rangle, \langle\vec{b}, B_{\vec{b}}\rangle\rangle$ obtained as p^*, but with $\langle\vec{b}, B_{\vec{b}}\rangle$ instead of $\langle\langle\rangle, B\rangle$. Arrange also that if $\vec{b}$ extends $\vec{b}'$, then $\langle f_{\vec{b}}, A_{\vec{b}}, \delta_{\vec{b}}\rangle \geq^* \langle f_{\vec{b}'}, A_{\vec{b}'}, \delta_{\vec{b}'}\rangle$.

Shrink B^p if necessary such that for every $n < \omega$, if $\vec{b}, \vec{b}' \in [B]^n$, then

1. $p_{\vec{b}}, p_{\vec{b}'}$ give the same conclusion about being in D,
2. $E_{\delta_{\vec{b}}} = E_{\delta_{\vec{b}'}}$,
3. $A_{\vec{b}} = A_{\vec{b}'}$.

Denote the constant values of the measures by $E(n)$ and the corresponding sets of measures one by $A(n)$, for every $n < \omega$.

Consider $F = \langle p_{\vec{b}} \mid \vec{b} \in [B]^{<\omega}\rangle$. It is an element of $H_{\lambda^{++}}$. Pick $\tilde{F} : \kappa \to V_\kappa$ and $\delta < \lambda^{++}$ such that $j_E(\tilde{F})(\delta) = F$.

Deal with $n = 1$. Let $\xi \in B$.
Consider a simpler (but typical) case when $\delta_{\langle\xi\rangle} = \delta_\xi$ is just ξ.

We have then that for every $\xi \in B$, $\xi \in j_E(A(1))$, since $A(1) \in E_\xi$. So, $B \subseteq j_E(A(1)) \cap \lambda$. In particular, $j_E(A(1)) \cap \lambda \in U(\lambda)$.

Let $j_\delta : V \to M_\delta \simeq \mathrm{Ult}(V, E_\delta)$ be the canonical embedding and $k_\delta : M_\delta \to M_E$ be defined as follows: $k_\delta(j_\delta(g)([id]_{E_\delta})) = j_E(g)(\delta)$. By the choice of δ, $B, A(1)$ etc. are in the range of k_δ.
Denote B_δ, $A(1)_\delta$ the pre-images. Then, using the commutativity of the corresponding diagrams, $B_\delta \subseteq j_{E_\delta}(A(1)) \cap \lambda_{E_\delta}$ and $j_{E_\delta}(A(1)) \cap \lambda_{E_\delta} \in U(\lambda_\delta)$. Hence, the set

$$A^*(1) = \{\eta < \kappa \mid B_\eta \subseteq A(1) \cap \lambda_\eta \in U(\lambda_\eta)\}$$

is in E_δ.

Such defined $A^*(1)$ will take care of compatibility with members of $A(1)$. Namely, we need to combine $f_{\vec{b}}$'s into a single element of $\mathcal{P}^*_E$ and attach $\langle A^*(1), \delta\rangle$ to the result.

Let us deal now with the general case.
First we combine in the natural fashion all $f_{\vec{b}}$'s into a single element f^* of $\mathcal{P}^*_E$.
Then each $\delta_{\vec{b}}$ will become a member of $f^*(\vec{b})$. By shrinking B, if necessary, we can assume that for every $n < \omega$, there is $\beta(n) < \kappa$, such that for every $\vec{b} \in [B]^n$, $\delta_{\vec{b}}$ is $\beta(n)-$th element of $(\mathrm{dom}(f^*)(\vec{b})$, i.e. $h^{(\mathrm{dom}(f^*))(\vec{b})}(\beta(n)) = \delta_{\vec{b}}$.
Then, for every $n < \omega$, for every $\vec{b} \in [B]^n$, $A(n) = A_{\vec{b}} \in E_{\delta_{\vec{b}}}$. Now, $A(n) \in E_{\delta_{\vec{b}}}$ implies that $h^{(\mathrm{dom}(f^*))(\vec{b})}(\beta(n)) = \delta_{\vec{b}} \in j_E(A(n)) \cap \lambda^{++}$.

We have $\tilde{f}^* : \kappa \to V_\kappa$ such that $j_E(\tilde{f}^*)(\delta) = f^*$. For every $n, 1 \leq n < \omega$, define

$$A^*(n) = \{\eta < \kappa \mid \beta(n) < \eta^{nor} \wedge (\forall \vec{c} \in [B_\eta]^n (h_{\eta^{nor}}^{(\mathrm{dom}(\tilde{f}^*(\eta))(\vec{c})}(\beta(n)) \in A(n) \cap \lambda_\eta^{++} \in U(\lambda_\eta)))\}.$$

Then $A^*(n) \in E_\delta$. Set $A^* = \bigcap_{1 \leq n < \omega} A^*(n)$.

Now we form a condition in $\mathcal{P}$ in the obvious form based on f^* and A^*, i.e. $\langle\langle f^*, A^*, \delta\rangle, \langle\langle\rangle, B\rangle\rangle$.
The contradiction is derived then as in Claim 5.
□

Let us force now with the part of $\mathcal{P}_E$ over λ, i.e. with the Prikry forcing with $U(\lambda)$. Let $\vec{\lambda} = \langle \lambda(n) \mid n < \omega \rangle$ be a Prikry sequence.
Consider $\mathcal{P}_E/\vec{\lambda}$. Its members are all $p \in \mathcal{P}_E$ with $\{\lambda(n) \mid n < \omega\} \subseteq b^p \cup B^p$.

Lemma 2.12 *The forcing $\mathcal{P}_E/\vec{\lambda}$ satisfies $\kappa^{++}-$c.c. in $V[\vec{\lambda}]$.*

Proof. Let $\{p(\alpha) \mid \alpha < \kappa^{++}\} \subseteq \mathcal{P}_E/\vec{\lambda}$. Assume that all of them have the same stem and suppose for simplicity that it is just empty.
We interpret each $f^{p(\alpha)}$ according to the Prikry sequence $\langle \lambda(n) \mid n < \omega \rangle$ and, then, using a $\Delta-$system find $\alpha \neq \beta$ for which these interpretations are compatible.
Let us argue that $p(\alpha)$ and $p(\beta)$ are compatible as well.
Consider $B = B^{p(\alpha)} \cap B^{p(\beta)}$.
For every $\langle \eta_1, ..., \eta_n \rangle \in [B]^n$ (or with increasing projections to the normals in more general settings) find $C_{\langle \eta_1, ..., \eta_n \rangle} \subseteq B$ of measure one such that for every $\eta \in C_{\langle \eta_1, ..., \eta_n \rangle}$ either

1. the finite sequences of the corresponding parts of $p(\alpha)^\frown \langle \eta_1, ..., \eta_n \rangle^\frown \eta$ and $p(\beta)^\frown \langle \eta_1, ..., \eta_n \rangle^\frown \eta$ do not contradict;
   or

2. there are two finite sequences of the corresponding parts of $p(\alpha)^\frown \langle \eta_1, ..., \eta_n \rangle^\frown \eta$ and $p(\beta)^\frown \langle \eta_1, ..., \eta_n \rangle^\frown \eta$ over the same place that are different (i.e. contradict one an other).

Let $C = \Delta_{\langle \eta_1, ..., \eta_n \rangle \in [B]^n} C_{\langle \eta_1, ..., \eta_n \rangle}$. Then C has measure one.
There is $k < \omega$ such that for every $n, k \leq n < \omega, \lambda(n) \in C$. Extend both $p(\alpha)$ and $p(\beta)$ by $\langle \lambda(m) \mid m < k \rangle$. Then the condition obtained by the obvious merging of $p(\alpha)_{\langle \lambda(m) | m < k \rangle}$ and $p(\beta)_{\langle \lambda(m) | m < k \rangle}$ will be in $\mathcal{P}_E/\vec{\lambda}$ and it will witness the desired compatibility.
□

3 An extender which overlaps another extender.

It is possible to replace in the previous construction the measure $U(\lambda)$ with an extender E^1 over λ such that $E^1 \triangleleft E$. Let $\mu, \lambda^{++} \geq \mu > \lambda$ be a regular cardinal and E^1 be a $(\lambda, \mu)-$extender.

This way the following is obtained:

There is a cardinals preserving extension $V[G]$ of V such that

1. κ remains a strong limit,
2. $\mathrm{cof}(\kappa) = \mathrm{cof}(\lambda) = \omega$,
3. only $\omega-$many cardinals below κ change their cofinality and it is changed to ω,
4. no cardinal bigger than κ and different from λ changed its cofinality,
5. $\mathrm{pp}(\lambda) = \mu < 2^\kappa = \mathrm{pp}(\kappa) = \lambda^{++}$, if $\mu < \lambda^{++}$, and $\mathrm{pp}(\lambda) = \mu = 2^\kappa = \mathrm{pp}(\kappa) = \lambda^{++}$, if $\mu = \lambda^{++}$.

Suppose now that there are two extenders $E(0)$ over κ_0 and $E(1)$ over $\kappa_1 > \kappa_0$ such that $E(0) \triangleright E(1)$.
The corresponding forcing $\mathcal{P} = \mathcal{P}_{\langle E(0),E(1)\rangle}$ is defined similar to those of the previous section with obvious changes of the Prikry forcing over the larger cardinal by the extender based Prikry with $E(1)$ and its reflections below κ_0.
A typical pure condition is of the form $p = \langle\langle f^p, A^p, \delta^p\rangle, \langle g^p, B^p\rangle\rangle$, where $\langle g^p, B^p\rangle$ is now a condition in the extender based Prikry forcing with $E(1)$. As before, we will have a function $\tilde{f}^p$ which represents f^p in the ultrapower with the measure δ^p of the extender $E(0)$, but let us require here that also g^p is represented there, i.e. $(j_{E(0)}(\tilde{f}^p))(\delta^p) = \langle f^p, g^p\rangle$.

The only new point appears in the Prikry condition argument (Lemma 2.11). Let us deal with a corresponding lemma in the present situation.

Lemma 3.1 *Let $p = \langle\langle f^p, A^p, \delta^p\rangle, \langle g^p, B^p\rangle\rangle\rangle \in \mathcal{P}_{\langle E(0),E(1)\rangle}$ and $D \subseteq \mathcal{P}_{\langle E(0),E(1)\rangle}$ be a dense open. Then there are $p^* \geq^* p$ and $n < \omega$ such that $p^{*\frown}\vec{\nu}^\frown\vec{t}^\frown\vec{x} \in D$,
for every $\vec{\nu} = \langle \nu_0, ..., \nu_{n-1}\rangle \in [A^{p^*}]^n, \vec{t} \in [B^{p^*}]^n$, $\vec{x} = \langle \vec{x}_k \mid k < n\rangle$ such that for every $k < n$, $\vec{x}_k \in [B_k^{p^*}]^n$, where $B_k^{p^*} = \mathrm{dom}(\tilde{f}^*)(\nu_0, ..., \nu_k)$, i.e., the set of measure one corresponding to B^{p^*} at the level k.*

Proof. Suppose otherwise.

Let $D' \subseteq \mathcal{P}_{E(1)}$ be the set of all $\langle g, B\rangle \in \mathcal{P}_{E(1)}$ satisfying the following:
there are f^*, A^*, δ^* such that

1. $\langle\langle f^*, A^*, \delta^*\rangle, \langle g, B\rangle\rangle \in \mathcal{P}_{\langle E(0),E(1)\rangle}$,
2. $\langle\langle f^*, A^*, \delta^*\rangle, \langle g, B\rangle\rangle \geq p$,
3. for every $\vec{\eta} \in [A^*]^{<\omega}$,
   (1) $\langle\langle f^*, A^*, \delta^*\rangle, \langle g, B\rangle\rangle^\frown\vec{\eta} \in D$[7]
   or
   (2) for every $p' \geq^* \langle\langle f^*, A^*, \delta^*\rangle, \langle g, B\rangle\rangle$, $p'^\frown\vec{\eta} \notin D$.
   Moreover, the same conclusion valid for any two such $\vec{\eta}, \vec{\eta}'$ of the same length.

Let us argue that such D' is dense in $\mathcal{P}_{E(1)}$ above $\langle g^p, B^p\rangle$.

Claim 6 *D' is dense in $\mathcal{P}_{E(1)}$ above $\langle g^p, B^p\rangle$.*

Proof.
Let $\langle g, B\rangle \in \mathcal{P}_{E(1)}$.

Define an elementary submodel N, $\bar{E}, A \in \bar{E}$ as in Lemma 2.11 with obvious adjustments.

Now instead of defining a dense set in $P^*_{E(0)}$, as it was done in Lemma 2.11, we proceed by induction of the length κ_0 and build increasing $\leq^*$ $-$sequences $f_\xi, \langle g_\xi, B_\xi\rangle, \xi < \kappa_0$ with each stage inside N.

At limit stages union is taken. Let us deal with a successor stage. So, suppose that $f_\xi, \langle g_\xi, B_\xi\rangle \in N$ is constructed.
Let $\langle \eta_0, ..., \eta_{k-1}\rangle \in [A]^k$, be the $\xi-$th member of an enumeration of
$[[\{\kappa_0, \delta^p, \delta_{f_0}, ..., \delta_{f_\zeta}, ... \mid \zeta < \kappa_0\} \times \kappa_0]^{<\kappa_0}$.

Let $f = f_\xi, \langle g, B\rangle = \langle g_\xi, B_\xi\rangle$. We would like first to deal with the forcing below κ_0 which $\vec{\eta} := \langle \eta_0, ..., \eta_{k-1}\rangle$ induces and to use its Prikry property. Suppose for simplicity that $k = 1$, so we deal with η_0 and the corresponding extender based Prikry forcing over it.
There are $f_* \geq^* f, \langle g_*, B_*\rangle \geq^* \langle g, B\rangle$ such that

for every $\langle \bar{x}, \bar{C}\rangle$ in the extender based Prikry over $\kappa_1(\eta_0)$[8] there is $\langle x, C\rangle \geq \langle \bar{x}, \bar{C}\rangle$ so that either

[7] Adding $\vec{\eta}$ requires to add a lower part of the condition $\tilde{f}^*(\vec{\eta})$, where $\tilde{f}^*$ represents now $\langle f^*, g, B\rangle$ in the ultrapower.

[8] $\kappa_1(\eta_0)$ denotes the reflection of κ_1 to the level of η_0^{nor}.

$$(1) \qquad \exists T \langle\langle x, C\rangle, \langle f_{*\langle\eta_0\rangle}, T\rangle, \langle g_*, B_*\rangle\rangle \in D),$$

or

$$(2) \forall \langle x', C'\rangle \geq \langle x, C\rangle \forall f' \geq^* f_* \forall \langle g', B'\rangle \geq^* \langle g_*, B_*\rangle \forall T \langle\langle x', C'\rangle, \langle f'_{\langle\eta_0\rangle}, T\rangle, \langle g', B'\rangle\rangle \notin D).$$

Just proceed by induction and use the fact the number of possibilities for $\langle x, C\rangle$ is much less than the degree of completeness of $\leq^*$.

This defines a dense open set $D(\eta_0)$ in the extender based Prikry over η_0. So, for every $\langle x, C\rangle$ there are $\langle x_*, C_*\rangle \geq^* \langle x, C\rangle$ and m_* minimal such that for every $\vec{\rho} \in [C_*]^{m_*}$, $\langle x_*, C_*\rangle^\frown\vec{\rho}$ is in this set. By shrinking C_* if necessary, we can assume that the same conclusion about being in D or not holds. Note that if "not in D" conclusion (2) holds, then this will be true already with $\langle x_*, C_*\rangle$, and so, $m_* = 0$.

Also note that all this lower parts $\langle x, C\rangle$ are in N. Hence, there are such $f_*, \langle g_*, B_*\rangle$, in N, by elementarity.

We pick $f_{\xi+1}, g_{\xi+1}, B_{\xi+1}$ to be such a sequence inside N.

Finally, let $f^*, \langle g^*, B^*\rangle$ be the upper bound of $\langle f_\xi, \langle g_\xi, B_\xi\rangle \mid \xi < \kappa\rangle$.

Let $\vec{\eta} \in [A]^{<\omega}$. Again, assume for simplicity that $\vec{\eta}$ is just η_0.

Consider $\tilde{f}^*(\eta_0)$. Let $\langle x(\eta_0), C(\eta_0)\rangle$ be the part of $\tilde{f}^*(\eta_0)$ which is a condition in the extender based Prikry forcing over $\kappa_1(\eta_0)$. Then there are $\langle x(\eta_0)_*, C(\eta_0)_*\rangle \geq^* \langle x(\eta_0), C(\eta_0)\rangle$ and $m(\eta_0)_* < \omega$ such that for every $\vec{\rho} \in [C(\eta_0)_*]^{m_*}$, $\langle x(\eta_0)_*, C(\eta_0)_*\rangle^\frown\vec{\rho} \in D(\eta_0)$.

Now, use the function $\eta_0 \mapsto \langle x(\eta_0)_*, C(\eta_0)_*\rangle$ and extend directly $\langle f^*, \langle g^*, B^*\rangle\rangle$ to $\langle f^{**}, \langle g^{**}, B^{**}\rangle\rangle$.[9]

Now we shrink A in order to get the same conclusion (1) or (2) with $\langle f^{**}, \langle g^{**}, B^{**}\rangle\rangle$.

If it is (1), then $\langle g^{**}, B^{**}\rangle \in D'$. Suppose that it is (2).

Let us argue that then for every $\vec{\eta} \in [A^*]^{<\omega}$, for every $p' \geq^* \langle\langle f^{**}, A^*, \delta^{**}\rangle, \langle g^{**}, B^{**}\rangle\rangle$, $p'^\frown\vec{\eta} \notin D$.

Oterwise, there are $\vec{\eta} \in [A^*]^{<\omega}$ and $p' \geq^* \langle\langle f^*, A^*, \delta^*\rangle, \langle g^{**}, B^{**}\rangle\rangle$ such that $p'^\frown\vec{\eta} \in D$. But $\vec{\eta}$ was considered at a stage $\xi < \kappa$ of the construction. The existence of p' implies that already $f_{\xi+1}, \langle g_{\xi+1}, B_{\xi+1}\rangle$ forced this, which is impossible.

$\square$ of the claim.

[9] There was no need to do this in the previous section, since the basic Prikry forcing was used, and any two direct extension in this forcing are compatible. This is not the case with the extender based Prikry used here.

Now we use the Prikry condition for the extender Prikry forcing $\mathcal{P}_{E(1)}$ and find $n < \omega$ and $\langle g, B\rangle$ such that for every $\vec{b} \in [B]^n$, $\langle g, B\rangle^\frown \vec{b} \in D'$.
Next, for each $\vec{b} \in [B]^n$, let $f_{\vec{b}}, A_{\vec{b}}, \delta_{\vec{b}}$ be witnesses for $\langle g, B\rangle^\frown \vec{b} \in D'$. We put them together as in Lemma 2.11.
□

Turn now to the chain condition. The argument is similar to those of Lemma 2.12 with obvious adaptations. Let us force now with the part of $\mathcal{P}_{\langle E(0),E(1)\rangle}$ over κ_1, i.e. with the extender based Prikry forcing with $E(1)$. Let $G(1)$ be a generic subset of $\mathcal{P}_{E(1)}$.
Consider $\mathcal{P}_{E(0)}/G(1)$. Its members are all $p = \langle p(0), p(1)\rangle \in \mathcal{P}_{\langle E(0),E(1)\rangle}$ with $p(1) \in G(1)$.

Lemma 3.2 *The forcing $\mathcal{P}_{E(0)}/G(1)$ satisfies $\kappa_0^{++}-$c.c. in $V[G(1)]$.*

Proof. Let $\{p_\alpha = \langle p_\alpha(0), p_\alpha(1) \mid \alpha < \kappa_0^{++}\} \subseteq \mathcal{P}_{E(0)}/G(1)$.
Let $p_\alpha(0) = \langle f^0_\alpha, A^0_\alpha, \delta^0_\alpha\rangle$, for every $\alpha < \kappa_0^{++}$.
We interpret each f^0_α according to the Prikry sequence of the maximal coordinate of $p_\alpha(1)$ and, then, using a $\Delta-$system find $\alpha \neq \beta$ for which this interpretations are compatible.
Let us argue that p_α and p_β are compatible as well. First find $p(1) = \langle f^{p(1)}, A^{p(1)}, \delta^{p(1)}\rangle \in G(1)$ which is stronger than both $p_\alpha(1), p_\beta(1)$.
Now use the Prikry condition for the forcing $\mathcal{P}_{E(1)}$ and find $q(1) = \langle f^{q(1)}, A^{q(1)}, \delta^{q(1)}\rangle \in G(1), q(1) \geq p(1)$ such that for every $\langle \eta_1, ..., \eta_n\rangle \in [A^{q(1)}]^n$, $q(1)$ decides (in $\mathcal{P}_{E(1)}$) the following statement:

> the finite sequences of the corresponding parts of
> $p_\alpha(0)^\frown\langle \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_1), ..., \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_n)\rangle$ and $p_\beta(0)^\frown\langle \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_1), ..., \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_n)\rangle$
> do not contradict, where $\pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}$, as usual, denotes the canonical projection between the coordinates $\delta^{q(1)}, \delta^{p(1)}$ of the extender $E(1)$.

By the choice of p_α and p_β, the decision should positive, i.e.
$q(1)$ forces (in $\mathcal{P}_{E(1)}$) that:

> the finite sequences of the corresponding parts of
> $p_\alpha(0)^\frown\langle \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_1), ..., \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_n)\rangle$ and $p_\beta(0)^\frown\langle \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_1), ..., \pi^{E(1)}_{\delta^{q(1)},\delta^{p(1)}}(\eta_n)\rangle$
> do not contradict.

Then the condition obtained by the obvious merging of p_α, p_β and $q(1)$ will witness the desired compatibility.
□

4 An extender which overlaps many extenders.

The argument of the previous section applies with minor changes to the following situation:

1. $\langle \kappa_\xi \mid \xi < \rho \rangle$ is an increasing sequence of measurable cardinals of a length $\rho < \kappa_0$;
2. for every $\xi < \rho$, $E(\xi)$ is an extender over κ_ξ;
3. for every $\xi, \rho > \xi > 0$, the length of $E(\xi)$ is at most $\kappa_{\xi+1}$;
4. the length of $E(0)$ is $(\bigcup_{\xi<\rho} \kappa_\xi)^+$;
   In particular, $E(0)$ overlaps each $E(\xi), \xi > 0$.
5. $\langle E(\xi) \mid 0 < \xi < \rho \rangle \in \mathrm{Ult}(V, E(0))$.

The corresponding forcing $\mathcal{P} = \mathcal{P}_{\langle E(\xi) | \xi < \rho \rangle}$ is defined similar to those of the previous sections. For every $\xi, 0 < \xi < \rho$, the extender $E(\xi)$ is used only up to the first element of the Prikry sequence of the normal measure of $E(\xi + 1)$.
A typical pure condition is of the form $p = \langle \langle f^p, A^p, \delta^p \rangle, \langle \langle g^p_\xi, B^p_\xi \rangle \mid 0 < \xi < \rho \rangle \rangle$, where $\langle g^p_\xi, B^p_\xi \rangle$ is now a condition in the extender based Prikry forcing with $E(\xi)$. As before, we will have a function $\tilde{f}^p$ which represents f^p in the ultrapower with the measure δ^p of the extender $E(0)$, but let us require here that also $\langle g^p_\xi \mid 0 < \xi < \rho \rangle$ is represented there, i.e. $(j_{E(0)}(\tilde{f}^p))(\delta^p) = \langle f^p, \langle g^p_\xi \mid 0 < \xi < \rho \rangle \rangle$.

The proof of the Prikry condition repeats the argument of Lemma 3.1. At the final stage after showing that D' is dense open in the forcing $\mathcal{P}_{\langle E(\xi) | 0 < \xi < \rho \rangle}$, i.e. the forcing with all extenders $E(\xi), 0 < \xi < \rho$, we use the fact that $\mathcal{P}_{\langle E(\xi) | 0 < \xi < \rho \rangle}$ has the Prikry property. In order to entre D' it is enough to find finitely many places $\xi_1, ..., \xi_k$ and $n < \omega$ such that any choice of $n-$elements from each set of measure one for all the places $\xi_1, ..., \xi_k$, the resulting extension will be in D'.

Under the same lines it is possible to deal with more general situations. What is crucial here is that the overlapped part satisfies the Prikry property and its direct extension ordering is closed enough.

5 A partial overlapping.

Suppose that $\kappa < \lambda$, there are two normal measures $U(\lambda, 0) \lhd U(\lambda, 1)$ over λ and a $(\kappa, \lambda)-$extender E over κ which ultrapower is closed under $\kappa-$sequences.

Consider $j_{U(\lambda,1)} : V \to M_{U(\lambda,1)} \simeq \mathrm{Ult}(V, U(\lambda,1))$. Use the embedding $j_{U(\lambda,1)}$ to stretch the extender E. Thus, we consider $j_{U(\lambda,1)}(E)$. It is a $(\kappa, j_{U(\lambda,1)}(\lambda))-$extender over κ in $M_{U(\lambda,1)}$ which ultrapower is closed under $\kappa-$sequences.
Consider $E^1 := j_{U(\lambda,1)}(E) \restriction (\lambda^{++})^{M_{U(\lambda,1)}}$. Then, in V, it is extender over κ of the length $(\lambda^{++})^{M_{U(\lambda,1)}}$, with ultrapower M_{E^1} closed under $\kappa-$sequences, $H_{\lambda^+} \subseteq M_{E^1}$ and $U(\lambda, 0) \in M_{E^1}$. Clearly, $E^1 \lhd U(\lambda, 1)$.

We would like to describe the forcing that does the following:

1. it picks a one element Prikry sequence λ_{10} for $U(\lambda, 1)$;
2. adds a full Prikry sequence to λ_{10} using the reflection of $U(\lambda, 0)$;
3. adds $\lambda_{10}^{++}-$many Prikry sequences to κ using E^1.

The forcing of the first section will be used here as a part of the present forcing notion. Note that $M_{U(\lambda,1)}$ is closed under $\lambda-$sequences of its elements, and so, $\mathrm{cof}((\lambda^{++})^{M_{U(\lambda,1)}}) = \lambda^+$ and every relevant function (defined as in Section 1 using E^1 and $U(\lambda, 0)$) will be in $M_{U(\lambda,1)}$.

Definition 5.1 $\mathcal{P}_{E^1,U(\lambda,0),U(\lambda,1)}$ consists of triples $\langle\langle f, A, \delta\rangle, \langle b, B\rangle, \langle\langle\rangle, C\rangle\rangle$ such that

1. $\langle\langle f, A, \delta\rangle, \langle b, B\rangle\rangle$ is as in the forcing of Section 1,
   only with λ^{++} replaced by $(\lambda^{++})^{M_{U(\lambda,1)}}$,
2. $\langle\langle\rangle, C\rangle$ is a condition in the Prikry forcing with $U(\lambda, 1)$ with $\min(C) > \max(b)$.

The orders are defined in the obvious fashion. If an element ν from C is picked, then we use the functions $h_\delta : \lambda \to \lambda, h_f : \lambda \to V_\lambda$ which represent δ , f mod $U(\lambda, 1)$, and push $\langle\langle f, A, \delta\rangle, \langle b, B\rangle\rangle$ down to ν replacing it by $\langle\langle h_f(\nu), A, h_\delta(\nu)\rangle, \langle b, B \cap \nu\rangle\rangle$.
The continuation is as in Section 1.

An other way of partial overlapping was considered in [3]. A typical situation there is that we have, for example, $\kappa_0 < \kappa_1 < \lambda$ and two extenders $E(0), E(1)$ such

- $E(0)$ is a $(\kappa_0, \lambda)-$extender,
- $E(1)$ is a $(\kappa_1, \lambda)-$extender,
- $E(0) \lhd E(1)$.

A forcing $\mathcal{P}_{\langle E(0),E(1)\rangle}$ that involves both of them is defined in a way that if a non-pure extension is made using $E(1)$, then the part of the forcing with $E(0)$ is restricted to $E(0) \restriction \eta$, for some $\eta < \kappa_1$.
Further we will use this type of constructions.

6 Uncountable cofinality.

Use the combination of C. Merimovich forcing [9] with Sections 1,2.

7 The main forcing.

We introduce here forcing notions that will be applied to the Shelah Weak Hypothesis in the next section.

7.1 Basic settings.

Let η, δ be arbitrary ordinals.
Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals, with $\eta, \delta < \kappa_0$, and λ be a cardinal $> \bigcup_{\alpha<\eta} \kappa_\alpha$.

Suppose that $\langle E(\alpha, \beta) \mid \alpha < \eta, \beta < \delta \rangle$ be a sequence such that for every $\alpha, \alpha' < \eta$, $\beta, \beta' < \delta$,

1. $E(\alpha, \beta)$ is a $(\kappa_\alpha, \lambda)-$extender, i.e.,
   (a) $E(\alpha, \beta)$ is an extender over κ_α,
   (b) $H_\lambda \subseteq M_{E(\alpha,\beta)}$, where $j_{E(\alpha,\beta)} : V \to M_{E(\alpha,\beta)} \simeq \mathrm{Ult}(V, E(\alpha, \beta))$,
   (c) $crit(j_{E(\alpha,\beta)}) = \kappa_\alpha$,
   (d) $j_{E(\alpha,\beta)}(\kappa_\alpha) > \lambda$,
   (e) ${}^{\kappa_\alpha} M_{E(\alpha,\beta)} \subseteq M_{E(\alpha,\beta)}$.
2. $E(\alpha, \beta) \rhd E(\alpha', \beta')$, if $\beta > \beta'$ or ($\beta = \beta'$ and $\alpha > \alpha'$).

Assume in addition that for every $\alpha < \eta, \beta < \delta$,

- there is a function $h^\lambda_{E(\alpha,\beta)} : \kappa_\alpha \to \kappa_\alpha$ such that $j_{E(\alpha,\beta)}(h^\lambda_{E(\alpha,\beta)})(\kappa_\alpha) = \lambda$, i.e. λ is represented already by the least normal measure of the extender $E(\alpha, \beta)$.
- For every $\alpha' < \eta, \beta' < \delta$, so that $\beta > \beta'$ or ($\beta = \beta'$ and $\alpha > \alpha'$), there is a function $h^{E(\alpha',\beta')}_{E(\alpha,\beta)} : \kappa_\alpha \to V_{\kappa_\alpha}$ such that $j_{E(\alpha,\beta)}(h^{E(\alpha',\beta')}_{E(\alpha,\beta)})(\kappa_\alpha) = E(\alpha', \beta')$, i.e. $E(\alpha', \beta')$ is represented already by the least normal measure of the extender $E(\alpha, \beta)$.
  Note that due to the closure of the ultapowers, this implies that all the sequence
  $$\langle E(\alpha', \beta') \mid \alpha' < \eta, \beta' < \delta, (\beta > \beta' \text{ or } (\beta = \beta' \text{ and } \alpha > \alpha')) \rangle$$
  is represented like this.

The forcing $\mathcal{P}_{\langle E(\alpha,\beta)|\alpha<\eta,\beta<\delta\rangle}$ defined below will change the cofinality of each $\kappa_\alpha, \alpha < \eta$ to δ (assuming that δ is a regular cardinal) and will make $\mathrm{pp}(\kappa_\alpha) \geq \lambda$.
For example if $\eta = \omega_1$ and $\delta = \omega$, then we intend to change cofinality of all κ_α's to ω and to make $\mathrm{pp}(\kappa_\alpha) \geq \lambda$.

Let us deal first with simpler situations.

Further let us denote $\langle E(\alpha,\beta) \mid \alpha < \eta\rangle$ by $\vec{E}(\beta)$, for every $\beta < \delta$.

7.2 The first level forcing.

Suppose that $\delta = 1$. Then we have a $\lhd$ −increasing sequence of extenders $\langle E(\alpha, 0) \mid \alpha < \eta\rangle$. Use the forcing $\mathcal{P}_{\langle E(\alpha,0)|\alpha<\eta\rangle}$ of [3].

7.3 Two levels forcing.

Let us define the forcing for the first two levels.
Assume that $\eta = \omega$ and $\delta = 2$.
So, we deal with extenders $E(n, k), n < \omega, k < 2$.

Recall that $E(0, 1) \rhd E(n, 0)$ and $E(n + 1, 1) \rhd E(n, 1)$, for every $n < \omega$.

We will use a setting similar to those of the first section. Instead of the Prikry forcing there - the forcing of the first level will be used. In addition, instead of the single extender based forcing there - all extenders $E(n, 1)$'s will come into the play here. The interaction between the $E(n, 1)$'s will be organized similar to the one between the $E(n, 0)$'s in 7.2 or in [3].

We will need to deal with names of ordinals in the first level forcing $\mathcal{P}_{\vec{E}(0)}$.
Define relevant functions in the present context.
Here, as in [3], we will use only one element sequences with each extender involved.

Definition 7.1 Let $n < \omega$. A function F is called a $n-$*relevant function* iff

1. There is $p \in \mathcal{P}_{\vec{E}(0)}$ such that the domain of F is the set of all finite products of sets of measure one of p, i.e.,

$$\mathrm{dom}(F) = \bigcup\{\prod_{k\in s} A^p_k \mid s \subseteq \omega, |s| < \aleph_0, \min(s) > n\}.$$

2. $F : \mathrm{dom}(F) \to \mathcal{P}_{\kappa_n^+}(\lambda)$.

3. $\kappa_n \in F(\langle\rangle)$,

4. For every $\vec{\nu}_1, \vec{\nu}_2$ in $\mathrm{dom}(F)$,
   if $\vec{\nu}_1$ extends $\vec{\nu}_2$, then $F(\vec{\nu}_1) \supseteq F(\vec{\nu}_2)$.

5. If $\vec{\nu}_1, \vec{\nu}_2$ in $\prod_{k\in s}(A^p_k)$ for some $s \subseteq \omega, |s| < \aleph_0, \min(s) > n$, then $\mathrm{otp}(F(\vec{\nu}_1)) = \mathrm{otp}(F(\vec{\nu}_2))$.

6. If $\vec{\nu}_1, \vec{\nu}_2$ in $\prod_{k\in s} A^p_k$ for some $s \subseteq \omega, |s| < \aleph_0, \min(s) > n$,
   then $E(n,1)_{F(\vec{\nu}_1)} = E(n,1)_{F(\vec{\nu}_2)}$.
   Note that the number of ultrafilters over V_{κ_n} is small relatively to κ_{n+1}, and so this can be easily arranged on measure one sets.

7. If $\vec{\nu} \in \prod_{k\in s} A^p_k$, for some $s \subseteq \omega, |s| < \aleph_0, \min(s) > n$,
   and $\nu \in A^p_m$, for some $m < \omega$, is above $\max(\vec{\nu})$, then
   $$F(\vec{\nu}^\frown \nu) \cap \pi_{mc(p_m),\kappa_m}(\nu) = F(\vec{\nu}) \cap \kappa_m,$$
   where p_m is the $m-$th component of p.

The intuition behind is that we like to use extender based forcing with $E(n,1)$. The forcings over smaller κ_m's, $m < n$, satisfy small chain condition. So we can cover names of ordinals depending on them by sets in V of size κ_n. What remains then is the part of $\mathcal{P}_{\vec{E}(0)}$ above κ_n. There $n-$relevant functions come into the play.

Definition 7.2 Let F, G be $n-$relevant functions. Set $F \geq^* G$ iff

1. $\mathrm{dom}(F) \subseteq \mathrm{dom}(G)$,

2. for every $\langle \xi_1, ..., \xi_n \rangle \in \mathrm{dom}(F)$, $F(\xi_1, ..., \xi_n) \supseteq G(\xi_1, ..., \xi_n)$.

Define now the set $\mathcal{P}^*_{E(n,1)}$. Again we will use here only one element sequences. Another difference from Merimovich [9], will be that there is a function f for every measure of a condition, even for measures over the same cardinal.

Definition 7.3 Let $\mathcal{P}^*_{E(n,1)}$ be the set of all functions f such that

1. $\mathrm{dom}(f)$ is an $n-$relevant function,

2. for every $\langle \xi_1, ..., \xi_n \rangle \in \mathrm{dom}(\mathrm{dom}(f))$ and for every $\alpha \in (\mathrm{dom}(f))(\xi_1, ..., \xi_n))$,
   $(f(\xi_1, ..., \xi_n))(\alpha)$ is either the empty or one element sequence.
   Require that if $(f(\xi_1, ..., \xi_n))(\alpha)$ is not the empty sequence, then

(a) $(f(\xi_1,...,\xi_n))(\alpha)$ is an ordinal
or

(b) for some $\nu < \kappa_n$ and a reflection $\mathcal{P}_\nu$ of $\mathcal{P}_{\vec{E}(0)} \setminus n+1$ to ν, $(f(\xi_1,...,\xi_n))(\alpha)$ is a relevant function corresponding to $\mathcal{P}_\nu$. [10]

3. For every $\langle \xi_1,...,\xi_n\rangle, \langle \xi'_1,...,\xi'_n\rangle \in \mathrm{dom}(\mathrm{dom}(f))$, such that for every $i, 1 \leq i \leq n$, ξ_i and ξ'_i are from the same place,

   (a) $\mathrm{otp}((\mathrm{dom}(f)(\xi_1,...,\xi_n))) = \mathrm{otp}((\mathrm{dom}(f)(\xi'_1,...,\xi'_n)))$,

   (b) for every $\beta < \mathrm{otp}((\mathrm{dom}(f)(\xi_1,...,\xi_n)))$, if α, α' are the $\beta-$th members of $\mathrm{dom}(f)(\xi_1,...,\xi_n)$ and $\mathrm{dom}(f)(\xi'_1,...,\xi'_n)$ respectively, then $(f(\xi_1,...,\xi_n))(\alpha) = (f(\xi'_1,...,\xi'_n))(\alpha')$.

4. If a sequence $\vec{\xi'}$ extends a sequence $\vec{\xi}$ and $\alpha \in (\mathrm{dom}(f))(\vec{\xi})$ (and hence, $\alpha \in (\mathrm{dom}(f))(\vec{\xi'})$)), then $f(\vec{\xi})(\alpha) = f(\vec{\xi'})(\alpha)$.

Definition 7.4 Let f, g be in $\mathcal{P}^*_{E(n,1)}$. Set $f \geq^* g$ iff $\mathrm{dom}(f) \geq^* \mathrm{dom}(g)$, as relevant functions and for every $n < \omega, \langle \xi_1,...,\xi_n\rangle \in \mathrm{dom}(\mathrm{dom}(f)), \alpha \in \mathrm{dom}(g)(\xi_1,...,\xi_n)$, $f(\xi_1,...,\xi_n)(\alpha) = g(\xi_1,...,\xi_n)(\alpha)$.

The proof of the next lemma is as in Section 1.

Lemma 7.5 *$\langle \mathcal{P}^*_{E(n,1)}, \leq^* \rangle$ is κ_n^+-closed.*

Turn now to the definition of our forcing.
Define first conditions with empty stems.

Definition 7.6 The set $\mathcal{P}_{\vec{E}(0),\vec{E}(1)}$ consists of all sequences $p = \langle \vec{p}(0), \langle\langle f^n, A^n, \delta^n\rangle \mid n < \omega\rangle\rangle$ such that

1. $\vec{p}(0) \in \mathcal{P}_{\vec{E}(0)}$,

2. for every $n < \omega$,

   (a) $f^n \in \mathcal{P}^*_{E(n,1)}$,

   (b) $\delta^n < \lambda$,

   (c) $f^n = j_{E(n,1)}(\tilde{f^n})(\delta^n)$, where $\tilde{f^n} : \kappa_n \to V_{\kappa_n}$ is the least function like this.

[10] Note that further not at every such ν the forcing $\mathcal{P}_\nu$ will be used. Moreover, only for ν's which the members of the Prikry sequence for the normal measure of the extender, i.e. for $E(n,1)_\kappa$, $\mathcal{P}_\nu$ will be used.

(d) $A^n \in E(n,1)(\mathrm{dom}(f^n)(\langle\rangle) \cup \{\delta^n\})$. It will be convenient to view A^n as an element of the equivalent measure $E(n,1)_{\delta^n}$,

(e) for every $m, n \le m < \omega$, $\mathrm{rng}(\mathrm{dom}(f^n)) \setminus \kappa_m \subseteq \mathrm{rng}(\mathrm{dom}(f^m))$,

(f) $\mathrm{rng}(\mathrm{dom}(f^n)) \supseteq$ all the ordinals mentioned in $\vec{p}(0) \restriction n$.

In the next definition we define the direct extension $\le^*$ in the usual fashion:
$p \ge^* p'$ iff p is obtained from p' by extending support and by shrinking measure one sets.

Definition 7.7 Let $p = \langle \vec{p}(0), \langle\langle f^n, A^n, \delta^n\rangle \mid n < \omega\rangle\rangle$,
$p' = \langle \vec{p}'(0), \langle\langle f'^n, A'^n, \delta'^n\rangle \mid n < \omega\rangle\rangle \in \mathcal{P}_{\vec{E}(0),\vec{E}(1)}$. Set $p >^* p'$ iff

1. $\vec{p}(0) \ge^*_{\mathcal{P}_{\vec{E}(0)}} \vec{p}'(0)$,
2. for every $n < \omega$,
   (a) $f^n \ge^*_{\mathcal{P}^*_{E(n,1)}} f'^n$,
   (b) $\delta'^n \in \mathrm{dom}(f^n)(\langle\rangle)$,
   (c) $A^n \restriction (\mathrm{dom}(f'(\langle\rangle)) \cup \{\delta'^n\}) \subseteq A'^n$.
   In particular, this implies:
      i. $\delta^n \ge_{E(n,1)} \delta'^n$,
      ii. the canonical projection of A^n to δ' is a subset of A'^n, i.e., $\pi^{E(n,1)}_{\delta\delta'}{}'' A^n \subseteq A'^n$.

Let $p = \langle \vec{p}(0), \langle\langle f^n, A^n, \delta^n\rangle \mid n < \omega\rangle\rangle$ be a pure condition in $\mathcal{P}_{\vec{E}(0),\vec{E}(1)}$, i.e., the one in which the measures were not used yet. Suppose that $\bar{n} < \omega$ and $\eta \in A^{\bar{n}}$.
Define $p^\frown\eta$ the one-element extension of p by η. The definition is under the usual lines and as in [3], only we will move the part $\vec{p}(0) \restriction \bar{n}$, down to $\kappa(\eta)$, and the part $\langle\langle f^{n'}, A^{n'}, \delta^{n'}\rangle \mid n' < \bar{n}\rangle$ will be restricted and transferred the reflection of $\langle \kappa_m \mid \bar{n} \le m < \omega\rangle$ over $\kappa(\eta)$, where $\kappa(\eta) < \kappa_{\bar{n}}$ denotes the cardinal which correspond to $\kappa_{\bar{n}}$ via the connection to the normal measure of $E(\bar{n},1)$. We will replace $\vec{p}(0) \setminus \bar{n}+1$ over $\kappa(\eta)$ by the value of the representing it function over η. Still, $\vec{p}(0) \setminus \bar{n}+1$ will be kept above as well.
Let us give a formal definition.

Definition 7.8 Set $p^\frown\eta$ to be
$\langle\langle\rangle, \mathrm{dom}(\mathrm{dom}(\tilde{f}^{\bar{n}}(\eta)))\rangle^\frown \vec{p}(0) \setminus \bar{n}+1^\frown\langle f^{\bar{n}\frown}\eta \cup \{\langle \kappa_{\bar{n}}, \eta^{nor}\rangle, \langle \delta^{\bar{n}}, \eta\rangle\}\rangle^\frown$
$\langle\langle f^n, A^n, \delta^n\rangle \mid \bar{n} < n < \omega\rangle\rangle$,
where

1. $\langle\langle\rangle, \mathrm{dom}(\mathrm{dom}(\tilde{f}^{\bar{n}}(\eta)))\rangle$ is a condition in the forcing $\mathcal{P}_{\vec{E}(0),\vec{E}(1)\restriction\bar{n}}$ over the level of η^{nor} with the reflections of the extenders in $\vec{E}(0)$ and $\vec{E}(1)\restriction\bar{n}$. The extender $E(\bar{n},0)$ reflects as in the Merimovich extender based Magidor forcing [9].

2. $f^{\bar{n}\frown}\eta$ is defined as follows:
its domain is identical to the domain of $f^{\bar{n}}$;
let $n<\omega$ and $\langle\xi_1,...,\xi_n\rangle\in[\mathrm{dom}(\mathrm{dom}(f^{\bar{n}}))]^n$ and $\alpha\in\mathrm{dom}(f^{\bar{n}})(\xi_1,...,\xi_n)$.
Set $(f^{\bar{n}\frown}\eta(\xi_1,...,\xi_n))(\alpha)=(f^{\bar{n}}(\xi_1,...,\xi_n))(\alpha)$, unless η^{nor} is above all elements of the sequence $(f^{\bar{n}}(\xi_1,...,\xi_n))(\alpha)$. For elements of the sequence $(f^{\bar{n}}(\xi_1,...,\xi_n))(\alpha)$ which are names, we mean that η^{nor} is above the measurable (or its double successor) which defines it.
Suppose that η^{nor} is above every element of $(f^{\bar{n}}(\xi_1,...,\xi_n))(\alpha)$.
We include here also the case when the sequence is empty. This happens, in particular, if $\alpha=\kappa$.
Split into two cases.

   (a) If $\alpha=\kappa$, then set $(f^{\bar{n}\frown}\eta(\xi_1,...,\xi_n))(\alpha)=\langle\eta^{nor}\rangle$,

   (b) suppose that $\alpha\neq\kappa$. Let α be the $\beta(\alpha)-$th member of $\mathrm{dom}(f^{\bar{n}})(\xi_1,...,\xi_n))$, for some $\beta(\alpha)<\eta^{nor}$, i.e. $h^{\mathrm{dom}(f)(\xi_1,...,\xi_n))}(\beta(\alpha))=\alpha$.
   Consider $\mathrm{dom}(\tilde{f}^{\bar{n}})(\eta)$. Define a function g by setting $g(\tau_1,...,\tau_n)$ to be the $\beta(\alpha)-$th member of $\mathrm{dom}(\tilde{f}^{\bar{n}})(\eta)(\tau_1,...,\tau_n)$, i.e. $h_{\eta^{nor}}^{\mathrm{dom}(\tilde{f}^{\bar{n}})(\xi_1,...,\xi_n))}(\beta(\alpha))$.
   Add then g to the sequence, i.e. let
   $(f^{\bar{n}\frown}\eta(\xi_1,...,\xi_n))(\alpha)=(f^{\bar{n}}(\xi_1,...,\xi_n))(\alpha)^\frown g$.

7.4 General case.

Let $\alpha\leq\delta$ and suppose that for every $\alpha'<\alpha$ the forcing $\mathcal{P}_{\langle\vec{E}(\beta)|\beta<\alpha'\rangle}$ is defined.
Define $\mathcal{P}_{\langle\vec{E}(\beta)|\beta<\alpha\rangle}$. We proceed as in 7.3.

Assume for simplicity that $\delta=\omega_1$ and $\eta=\omega$.

Definition 7.9 The pure part of $\mathcal{P}_{\langle\vec{E}(\beta)|\beta<\alpha\rangle}$ consists of sequences $p=\langle\vec{p}(\beta)\mid\beta<\alpha\rangle$ such that for every $\alpha'<\alpha$, $p\restriction\alpha'=\langle\vec{p}(\beta)\mid\beta<\alpha'\rangle$ is in the pure part of $\mathcal{P}_{\langle\vec{E}(\beta)|\beta<\alpha'\rangle}$.
The above defines $\mathcal{P}_{\langle\vec{E}(\beta)|\beta<\alpha\rangle}$ for a limit α.
Deal with a successor case. Suppose that $\alpha=\alpha'+1$.
Then, as in 7.3, $p=\langle\vec{p}(\beta)\mid\beta<\alpha'\rangle^\frown\langle\langle f^n,A^n,\delta^n\rangle\mid n<\omega\rangle$, where

1. $\langle\vec{p}(\beta)\mid\beta<\alpha'\rangle\in\mathcal{P}_{\langle\vec{E}(\beta)|\beta<\alpha'\rangle}$,

2. for every $n < \omega$,

   (a) $f^n \in \mathcal{P}^*_{E(n,\alpha')}$,

   (b) $\delta^n < \lambda$,

   (c) $f^n = j_{E(n,\alpha')}(\tilde{f}^n)(\delta^n)$, where $\tilde{f}^n : \kappa_n \to V_{\kappa_n}$ is the least function like this.

   (d) $A^n \in E(n,\alpha')(\mathrm{dom}(f^n)(\langle\rangle) \cup \{\delta^n\})$. It will be convenient to view A^n as an element of the equivalent measure $E(n,\alpha')_{\delta^n}$,

   (e) for every $m, n \leq m < \omega$, $\mathrm{rng}(\mathrm{dom}(f^n)) \setminus \kappa_m \subseteq \mathrm{rng}(\mathrm{dom}(f^m))$,

   (f) $\mathrm{rng}(\mathrm{dom}(f^n)) \supseteq$ all the ordinals mentioned in $\vec{p}(\beta) \restriction n, \beta < \alpha'$.

In the next two definitions we define the direct extension $\leq^*$ in the usual fashion: $p \geq^* p'$ iff p is obtained from p' by extending support and by shrinking measure one sets.

First deal with a limit α.

Definition 7.10 Let $p = \langle \vec{p}(\beta) \mid \beta < \alpha\rangle$,$p' = \langle \vec{p'}(\beta) \mid \beta < \alpha\rangle \in \mathcal{P}_{\langle \vec{E}(\beta) \mid \beta < \alpha\rangle}$.
Set $p >^* p'$ iff for every $\alpha' < \alpha$, $p \restriction \alpha' \geq^*_{\mathcal{P}_{\langle \vec{E}(\beta)\mid\beta<\alpha'\rangle}} p' \restriction \alpha'$.

Now let us turn to a successor case. Let $\alpha = \alpha' + 1$.

Definition 7.11 Let $p = \langle \vec{p}(\beta) \mid \beta < \alpha'\rangle^\frown \langle\langle f^n, A^n, \delta^n\rangle \mid n < \omega\rangle$,
$p' = \langle \vec{p'}(\beta) \mid \beta < \alpha'\rangle^\frown \langle\langle f'^n, A'^n, \delta'^n\rangle \mid n < \omega\rangle \in \mathcal{P}_{\langle \vec{E}(\beta)\mid\beta<\alpha\rangle}$. Set $p >^* p'$ iff

1. $\langle \vec{p}(\beta) \mid \beta < \alpha'\rangle \geq^*_{\mathcal{P}_{\langle \vec{E}(\beta)\mid\beta<\alpha'\rangle}} \langle \vec{p'}(\beta) \mid \beta < \alpha'\rangle$,

2. for every $n < \omega$,

   (a) $f^n \geq^*_{\mathcal{P}^*_{E(n,\alpha')}} f'^n$,

   (b) $\delta'^n \in \mathrm{dom}(f^n)(\langle\rangle)$,

   (c) $A^n \restriction (\mathrm{dom}(f'(\langle\rangle)) \cup \{\delta'^n\}) \subseteq A'^n$.
   In particular, this implies:

      i. $\delta^n \geq_{E(n,\alpha')} \delta'^n$,

      ii. the canonical projection of A^n to δ' is a subset of A'^n, i.e., $\pi^{E(n,\alpha')}_{\delta\delta'}{}'' A^n \subseteq A'^n$.

Let $p = \langle\langle \vec{p}(\beta) \mid \beta < \alpha'\rangle, \langle\langle f^n, A^n, \delta^n\rangle \mid n < \omega\rangle\rangle$ be a pure condition in $\mathcal{P}_{\langle \vec{E}(\beta)\mid\beta<\alpha'+1\rangle}$.
Suppose that $\bar{n} < \omega$ and $\eta \in A^{\bar{n}}$.
Define $p^\frown\eta$ the one-element extension of p by η. The definition is under the usual lines, only

we will move the part $\langle \vec{p}(\beta) \mid \beta < \alpha' \rangle \restriction \bar{n}$, down to $\kappa(\eta)$, and the part $\langle \langle f^{n'}, A^{n'}, \delta^{n'} \rangle \mid n' < \bar{n} \rangle$ will be restricted and transferred the reflection of $\langle \kappa_m \mid \bar{n} \leq m < \omega \rangle$ over $\kappa(\eta)$, where $\kappa(\eta) < \kappa_{\bar{n}}$ denotes the cardinal which correspond to $\kappa_{\bar{n}}$ via the connection to the normal measure of $E(\bar{n}, \alpha')$. We will replace $\langle \vec{p}(\beta) \mid \beta < \alpha' \rangle \setminus \bar{n}+1$ by the value of the function representing it over η. Still, $\langle \vec{p}(\beta) \mid \beta < \alpha' \rangle \setminus \bar{n}+1$ will be kept above as well.
Let us give a formal definition.

Definition 7.12 Set $p^\frown \eta$ to be
$\langle \langle \rangle, \mathrm{dom}(\mathrm{dom}((\tilde{f^{\bar{n}}})(\eta))) \rangle^\frown \langle \vec{p}(\beta) \mid \beta < \alpha' \rangle \setminus \bar{n}+1^\frown$
$\langle f^{\bar{n}\frown}\eta \cup \{\langle \kappa_{\bar{n}}, \eta^{nor} \rangle, \langle \delta^{\bar{n}}, \eta \rangle\} \rangle^\frown \langle \langle f^n, A^n, \delta^n \rangle \mid \bar{n} < n < \omega \rangle \rangle$,
where

1. $\langle \langle \rangle, \mathrm{dom}(\mathrm{dom}((\tilde{f^{\bar{n}}})(\eta))) \rangle$ is a condition in the forcing $\mathcal{P}_{\langle \vec{E}(\beta) \mid \beta < \alpha' \rangle, \vec{E}(\alpha') \restriction \bar{n}}$ over the level of η^{nor} with the reflections of the extenders in $\langle \vec{E}(\beta) \mid \beta < \alpha' \rangle$ and $\vec{E}(\alpha') \restriction \bar{n}$.
   The extenders $E(\bar{n}, \beta), \beta < \alpha'$ reflect as in the Merimovich extender based Magidor forcing [9].

2. $f^{\bar{n}\frown}\eta$ is defined as follows:
   its domain is identical to the domain of $f^{\bar{n}}$;
   let $n < \omega$ and $\langle \xi_1, ..., \xi_n \rangle \in [\mathrm{dom}(\mathrm{dom}(f^{\bar{n}})]^n$ and $\alpha \in \mathrm{dom}(f^{\bar{n}})(\xi_1, ..., \xi_n)$.
   Set $(f^{\bar{n}\frown}\eta(\xi_1, ..., \xi_n))(\alpha) = (f^{\bar{n}}(\xi_1, ..., \xi_n))(\alpha)$, unless η^{nor} is above all elements of the sequence $(f^{\bar{n}}(\xi_1, ..., \xi_n))(\alpha)$. For elements of the sequence $(f^{\bar{n}}(\xi_1, ..., \xi_n))(\alpha)$ which are names, we mean that η^{nor} is above the measurable (or its double successor) which defines it.
   Suppose that η^{nor} is above every element of $(f^{\bar{n}}(\xi_1, ..., \xi_n))(\alpha)$.
   We include here also the case when the sequence is empty. This happens, in particular, if $\alpha = \kappa$.
   Split into two cases.

   (a) If $\alpha = \kappa$, then set $(f^{\bar{n}\frown}\eta(\xi_1, ..., \xi_n))(\alpha) = \langle \eta^{nor} \rangle$,

   (b) suppose that $\alpha \neq \kappa$. Let α be the $\beta(\alpha)-$th member of $\mathrm{dom}(f^{\bar{n}})(\xi_1, ..., \xi_n))$, for some $\beta(\alpha) < \eta^{nor}$, i.e. $h^{\mathrm{dom}(f)(\xi_1, ..., \xi_n))}(\beta(\alpha)) = \alpha$.
   Consider $\mathrm{dom}(\tilde{f^{\bar{n}}})(\eta)$. Define a function g by setting $g(\tau_1, ..., \tau_n)$ to be the $\beta(\alpha)-$th member of $\mathrm{dom}(\tilde{f^{\bar{n}}})(\eta)(\tau_1, ..., \tau_n)$, i.e. $h_{\eta^{nor}}^{\mathrm{dom}(\tilde{f^{\bar{n}}})(\xi_1, ..., \xi_n))}(\beta(\alpha))$.
   Add then g to the sequence, i.e. let
   $(f^{\bar{n}\frown}\eta(\xi_1, ..., \xi_n))(\alpha) = (f^{\bar{n}}(\xi_1, ..., \xi_n))(\alpha)^\frown g$.

A one element extension for a limit $\alpha \leq \omega_1$ is defined similarly by reducing to a smaller α' below α.

Define non-pure members of $\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle}$ by making finitely many one step extensions.

Our next task will be to show that for each $\alpha \leq \omega_1$ the forcing satisfies the Prikry condition, and if $n < \omega$, then $\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle} \restriction \kappa_n$ satisfies $\kappa_n^{++}-$c.c. in $V^{(\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle})_{>\kappa_n}}$.

Lemma 7.13 *Let $\alpha \leq \omega_1$ and $n < \omega$. Then $(\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle})_{>\kappa_n}$ the part of $\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle}$ above κ_n is closed under the pure extensions ordering $\leq^*$ up to the first cardinal above κ_n which changes its cofinality (i.e. can be made arbitrary large below κ_{n+1}).*

Proof. Suppose that $p \in (\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle})_{>\kappa_n}$ is pure condition and $\delta < \kappa_{n+1}$. Shrink sets of measures one in order to be above δ. Let p' denotes such extension of p. Then, above p', we will have $\delta-$closure of $\langle (\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle})_{>\kappa_n}, \leq^* \rangle$.

□

The proof of the following lemma will appear after Lemma 7.16.

Lemma 7.14 *Let $\alpha \leq \omega_1$. Then the forcing $(\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle})_{>\kappa_n}$ satisfies the Prikry condition.*

Lemma 7.15 *The forcing $\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle}$ satisfies $\kappa_\omega^{++}-$c.c., for every $\alpha \leq \omega_1$.*

Proof. Repeats the standard argument for extender based forcings.

□

Lemma 7.16 *Let $\alpha \leq \omega_1$ and $n < \omega$. Then the restriction of $\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle}$ to the first n cardinals $\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle} \restriction \kappa_n$ satisfies $\kappa_n^{++}-$c.c. inside $V^{(\mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle})_{>\kappa_n}}$.*

Proof. Repeat the argument of Lemma 3.2 with obvious adjustments.

□

Proof of Lemma 7.14.

Suppose otherwise.

We proceed by induction on the length of sequences of extenders or its order type, simultaneously over different cardinals.

Let $\alpha \leq \omega_1$.

Case 1. *$n < \omega$, $\mathcal{P} = \mathcal{P}_{\langle \vec{E}(\beta)|\beta<\alpha\rangle^\frown\langle E(m,\alpha)|m\leq n\rangle}$, p is a pure condition in $\mathcal{P}$ and $D \subseteq \mathcal{P}$ is a dense open.*

The argument is parallel to those of Lemma 3.1.

Consider a subset D' of $\mathcal{P}(> n) := \mathcal{P}_{\langle E(m,\beta)|m>n,\beta<\alpha\rangle}$ which consists of all $r(> n) \in \mathcal{P}(> n)$ so that there are f^*, A^*, δ^* such that

1. $\langle\langle f^*, A^*, \delta^*\rangle, r(> n)\rangle \in \mathcal{P}$,
2. $\langle\langle f^*, A^*, \delta^*\rangle, r(> n)\rangle \geq p$,
3. for every $\vec{\eta} \in [A^*]^{<\omega}$,
   (1) $\langle\langle f^*, A^*, \delta^*\rangle, r(> n)\rangle^\frown\vec{\eta} \in D$[11]
   or
   (2) for every $p' \geq^* \langle\langle f^*, A^*, \delta^*\rangle, r(> n)\rangle$, $p'^\frown\vec{\eta} \notin D$.
   Moreover, the same conclusion valid for any two such $\vec{\eta}, \vec{\eta}'$ of the same length.

Let us argue that such D' is dense in $\mathcal{P}(> n)$ above $r^p(> n)$, where $r^p(> n)$ denotes the relevant part of p.

Claim 7 *D' is dense in $\mathcal{P}(> n)$ above $r^p(> n)$.*

Proof.
Let $s \in \mathcal{P}(> n), s \geq r^p(> n)$.

Define an elementary submodel $N, \bar{E}, A \in \bar{E}$ as in Lemma 2.11 with obvious adjustments. Namely, construct by induction an $\in -$increasing chain of elementary submodels $\langle N_\xi \mid \xi < \kappa_n\rangle$ of H_χ, for χ large enough, and a sequence $\langle f_\xi \mid \xi < \kappa\rangle$ of members of $\mathcal{P}^*_{E(n,\gamma)}$, such that

1. $p, s, \mathcal{P}, D, D' \in N_0$,
2. $N_0 \supseteq \kappa_n$,
3. for every $\xi < \kappa_n$,
   (a) $|N_\xi| = \kappa_n$,
   (b) ${}^{\kappa_n>}N_\xi \subseteq N_\xi$,
   (c) $\langle f_\zeta \mid \zeta < \xi\rangle \in N_\xi$,
   (d) $f_\xi \in \bigcap\{D'' \in N_\xi \mid D''$ is a dense open subset of $\mathcal{P}$ above $f^p\}$,
   (e) $f^p \leq^* f_0$,

[11] Adding $\vec{\eta}$ requires to add a lower part of the condition $\tilde{f}^*(\vec{\eta})$, where $\tilde{f}^*$ represents now $\langle f^*, r(> n)\rangle$ in the ultrapower.

(f) $\delta^p \in \mathrm{dom}(f_0(\langle\rangle))$,

(g) $\{\delta_{f_\zeta} \mid \zeta < \xi\} \subseteq \mathrm{dom}(f_\xi(\langle\rangle))$,

(h) $f_\xi \geq^* f_\zeta$, for every $\zeta < \xi$.

Set $N = \bigcup_{\xi<\kappa_n} N_\xi$ and f^* the upper bound of $\langle f_\xi \mid \xi < \kappa_n\rangle$.
Consider $\{\delta_{f_\xi} \mid \xi < \kappa_n\}$. Let δ be the least code of this set in our fixed wellordering.
Define an ultrafilter $\bar{E}$ over $[\{\kappa_n, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa_n\} \times \kappa_n]^{<\kappa_n}$ which is equivalent (Rudin-Keisler) to $E(n,\alpha)_{\{\kappa_n, \delta_{f_\xi} | \xi<\kappa_n\}}$ and is below $E(n,\alpha)_\delta$ (in the order $<_E$ (n,α) of $E(n,\alpha)$).
Set $Z \in \bar{E}$ iff $\{(j_{E(n,\gamma)}(\kappa_n), \kappa_n), (j_{E(n,\gamma)}(\delta^p), \delta^p), (j_{E(n,\gamma)}(\delta_{f_0}), \delta_{f_0}), ..., j_{E(n,\gamma)}(\delta_{f_\xi}), \delta_{f_\xi}), ... \mid \xi < \kappa\} \in j_{E(n,\gamma)}(Z)$.
Note that $\{\kappa_n, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa_n\} \subseteq N$, and so, $\{\kappa_n, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa_n\} \times \kappa_n \subseteq N$. Hence, also, $[\{\kappa_n, \delta^p, \delta_{f_0}, ..., \delta_{f_\xi}, ... \mid \xi < \kappa_n\} \times \kappa_n]^{<\kappa_n} \subseteq N$. The function $\langle \beta_\rho \mid \rho < \epsilon < \kappa_n\rangle \mapsto \langle\langle \kappa_n, \beta_0\rangle, \langle \delta^p, \beta_1\rangle, \langle \delta_{f_0}, \beta_2\rangle, ..., \langle \delta_{f_\xi}, \beta_{2+\xi}\rangle, ... \mid \xi < \epsilon\rangle$ witnesses the equivalence between $E(n,\gamma)_{\{\kappa_n, \delta_{f_\xi} | \xi<\kappa\}}$ and $\bar{E}$.

Let $A \in \bar{E}$ be a set which projection to δ^p is a subset of A^p.
Let $k < \omega$ and $\langle \eta_0, ..., \eta_{k-1}\rangle \in [A]^k$.
Consider $r := p^\frown\langle \eta_0, ..., \eta_{k-1}\rangle$. Then r can be written as $r_{<n}{}^\frown r_n{}^\frown r_{>n}$, where $r_{<n}$ is the part of r below κ_n and, in particular it includes the reflection of $E(m,\beta)$'s which are $\lhd E(n,\alpha)$, r_n is a part over κ_n and $r_{>n}$ the part above κ_n.

We proceed by induction of the length κ_n and build increasing $\leq^*-$sequences $\langle f_\xi, r(>n)_\xi\rangle = \langle\langle g_\xi^{m\beta}, B_\xi^{m\beta}\rangle \mid m > n, \beta < \alpha\rangle \mid \xi < \kappa_0\rangle$ with each stage inside N and $s \leq^* r(>n)_0$.

At limit stages union is taken. Let us deal with a successor stage. So, suppose that $\langle f_\xi, r(>n)_\xi\rangle = \langle\langle g_\xi^{m\beta}, B_\xi^{m\beta}\rangle \mid m > n, \beta < \alpha\rangle \in N$ is constructed.
Let $\langle \eta_0, ..., \eta_{k-1}\rangle \in [A]^k$, be the $\xi-$th member of an enumeration of $[[\{\kappa_0, \delta^p, \delta_{f_0}, ..., \delta_{f_\zeta}, ... \mid \zeta < \kappa_0\} \times \kappa_0]^{<\kappa_0}$.

Let $f = f_\xi, r(>n) = r(>n)_\xi = \langle\langle g_\xi^{m\beta}, B_\xi^{m\beta}\rangle \mid m > n, \beta < \alpha\rangle$. We would like first to deal with the forcing below κ_n which $\vec{\eta} := \langle \eta_0, ..., \eta_{k-1}\rangle$ induces and to use its Prikry property. Suppose for simplicity that $k = 1$, so we deal with η_0 and the corresponding forcing over it, i.e. the reflection of $\mathcal{P}(<n) = \mathcal{P}_{\langle E(m,\beta)|m<n,\beta<\alpha\rangle^\frown\langle E(n,\beta)|\beta<\alpha\rangle}$. Denote such reflection by $\mathcal{P}(<n, \eta_0)$.

There are $f_* \geq^* f, r(>n)_* \geq^* r(>n)$ such that

for every $\bar{x}$ in $\mathcal{P}(<n, \eta_0)$

there is $x \geq \bar{x}$ so that either

$$(1) \qquad \exists T\langle x, \langle f_{*\langle\eta_0\rangle}, T\rangle, r(>n)_*\rangle \in D),$$

or

$$(2) \forall x' \geq x \forall f' \geq^* f_* \forall r(>n)' \geq^* r(>n)_* \forall T\langle x', \langle f'_{\langle\eta_0\rangle}, T\rangle, r(>n)'\rangle \notin D).$$

Just proceed by induction and use the fact the number of possibilities for x is much less than the degree of completeness of $\leq^*$.
This defines a dense open set $D(\eta_0)$ in the forcing $\mathcal{P}(<n, \eta_0)$ over η_0. Apply the induction. So, for every x there are $x_* \geq^* x$, finitely many coordinates in x_* with $C_1, .., C_k$ sets of measure one at this coordinates and $m_* < \omega$ minimal such that for every
$\vec{\rho}_1 \in [C_1]^{m_*}, ..., \vec{\rho}_k \in [C_k]^{m_*}$, $x_* {}^\frown \langle \vec{\rho}_1, ..., \vec{\rho}_k\rangle \in D(\eta_0)$.
By shrinking $C_1, ..., C_k$ if necessary, we can assume that the same conclusion about being in D or not holds. Note that if "not in D" conclusion (2) holds, then this will be true already with x_* itself, and so, $m_* = 0$.
Also note that all this lower parts x are in N. Hence, there are such $f_*, r(>n)_*$ in N, by elementarity.
We pick $f_{\xi+1}, r(>n)_{\xi+1}$ to be such a sequence inside N.

Finally, let $f^*, r(>n)^*$ be the upper bound of $\langle f_\xi, r(>n)_\xi \mid \xi < \kappa\rangle$.

Let $\vec{\eta} \in [A]^{<\omega}$. Again, assume for simplicity that $\vec{\eta}$ is just η_0.
Consider $\tilde{f}^*(\eta_0)$. Let $x(\eta_0)$ be the part of $\tilde{f}^*(\eta_0)$ which is a condition in $\mathcal{P}(<n, \eta_0)$.
Then there are $x(\eta_0)_* \geq^* x(\eta_0)$ finitely many coordinates in x_* with $C_1, .., C_k$ sets of measure one at this coordinates and $m_* < \omega$ minimal such that for every
$\vec{\rho}_1 \in [C_1]^{m_*}, ..., \vec{\rho}_k \in [C_k]^{m_*}$, $x_* {}^\frown \langle \vec{\rho}_1, ..., \vec{\rho}_k\rangle \in D(\eta_0)$.
Now, use the function $\eta_0 \mapsto x(\eta_0)_*$ and extend directly
$\langle f^*, r(>n)^*\rangle$ to $\langle f^{**}, r(>n)^{**}\rangle$.

Now we shrink A in order to get the same conclusion (1) or (2) with $\langle f^{**}, r(>n)^{**}\rangle$.
If it is (1), then $r(>n)^{**} \in D'$. Suppose that it is (2).
Let us argue that then for every $\vec{\eta} \in [A^*]^{<\omega}$, for every $p' \geq^* \langle\langle f^{**}, A^*, \delta^{**}\rangle, r(>n)^{**}\rangle$, $p' {}^\frown \vec{\eta} \notin D$.
Oterwise, there are $\vec{\eta} \in [A^*]^{<\omega}$ and $p' \geq^* \langle\langle f^*, A^*, \delta^*\rangle, r(>n)^{**}\rangle$ such that $p' {}^\frown \vec{\eta} \in D$. But $\vec{\eta}$ was considered at a stage $\xi < \kappa$ of the construction. The existence of p' implies that already $f_{\xi+1}, r(>n)_{\xi+1}$ forced this, which is impossible.
$\square$ of the claim.

Now, we apply the induction and use the Prikry condition for $\mathcal{P}(> n)$. There will be $r(> n) \geq^* r^p(> n)$, finitely many coordinates in $r(> n)$ with $C_1, .., C_k$ sets of measure one at this coordinates and $m < \omega$ such that for every $\vec{\rho}_1 \in [C_1]^m, ..., \vec{\rho}_k \in [C_k]^m$,
$r(> n)^\frown \langle \vec{\rho}_1, ..., \vec{\rho}_k \rangle \in D'$.
Next, for every such sequences $\vec{\rho}_1 \in [C_1]^m, ..., \vec{\rho}_k \in [C_k]^m$ let $f_{\langle \vec{\rho}_1, ..., \vec{\rho}_k \rangle}, A_{\langle \vec{\rho}_1, ..., \vec{\rho}_k \rangle}, \delta_{\langle \vec{\rho}_1, ..., \vec{\rho}_k \rangle}$ be witnesses for $r(> n)^\frown \langle \vec{\rho}_1, ..., \vec{\rho}_k \rangle \in D'$. We put them together as in Lemma 2.11.
$\square$ of the successor case.

Case 2. $\mathcal{P} = \mathcal{P}_{\langle E(m,\beta) | m < \omega, \beta \leq \alpha \rangle}$, *p is a pure condition in* $\mathcal{P}$ *and* $D \subseteq \mathcal{P}$ *is a dense open.*

Now we do not have the last n. However, the argument of the previous successor case still can be applied, but rather inductively going through all n's.

Define by induction an $\leq^*$ $-$increasing sequence $\langle p_n \mid n < \omega \rangle$ of direct extensions of p.

Let $n < \omega$. Suppose that for every $m < n$, p_m was defined. Set $\bar{p} = p_{n-1}$, if $n > 0$, and $\bar{p} = p$, if $n = 0$. We need to define $p_n \geq^* \bar{p}$.

Set $\mathcal{P}(> n) = \mathcal{P}_{\langle E(m,\beta) | m > n, \beta \leq \alpha \rangle}$.
By the choice of the extenders, $E(m, \beta) \lhd E(n, \alpha)$, for every $m < \omega, \beta < \alpha$ and for every $m < n, \beta = \alpha$. We are specially interested in $\langle E(m, \beta) \mid n < m < \omega, \beta < \alpha \rangle$, i.e. the extenders which $E(n, \alpha)$ overlaps.
Recall also that if a non-direct extension was made over the coordinate $(n + 1, \alpha)$ (i.e. the one that corresponds to $E(n + 1, \alpha)$), then the coordinates $\langle (m, \beta) \mid n < m, \beta < \alpha \rangle$ reflect down below κ_{n+1} and $E(n, \alpha)$ is restricted below κ_{n+1} accordingly.

Let $\langle f^{\bar{p},(n+1,\alpha)}, A^{\bar{p},(n+1,\alpha)}, \delta^{\bar{p},(n+1,\alpha)} \rangle$ be the coordinate $(n + 1, \alpha)$ of $\bar{p}$. Pick some $\tau \in A^{\bar{p},(n+1,\alpha)}$. Consider the part of $\tilde{f}^{\bar{p},(n+1,\alpha)}(\tau)$ that represents the reflection of
$\bar{p} \restriction \{(\beta, m) \mid n < m, \beta < \alpha\}$ to the level of τ^{nor}. Denote it by $\bar{p}(> n, \tau)$. Denote the reflection of the forcing $\mathcal{P}_{\langle E(m,\beta) | n < m, \beta < \alpha \rangle}$ by $\mathcal{P}(> n, \tau)$. Then $\bar{p}(> n, \tau) \in \mathcal{P}(> n, \tau)$.

Let $r(> n) \in \mathcal{P}(> n), r(> n) \geq^* (\bar{p}^\frown \tau) \restriction \mathcal{P}(> n)$. Consider a subset $D'(r(> n), \tau)$ of $\mathcal{P}(> n, \tau)$ which consists of all $r(> n, \tau) \in \mathcal{P}(> n, \tau), r(> n, \tau) \geq \bar{p}(> n, \tau)$ so that there are f^*, A^*, δ^* such that

1. $\langle \langle f^*, A^*, \delta^* \rangle, r(> n, \tau), r(> n) \rangle \in \mathcal{P}_{\langle \vec{E}(\beta) | \beta < \alpha \rangle}$,

2. $\langle \langle f^*, A^*, \delta^* \rangle, r(> n, \tau), r(> n) \rangle \geq \bar{p}$,

3. for every $\vec{\eta} \in [A^*]^{<\omega}$,
   (1) $\quad \langle \langle f^*, A^*, \delta^* \rangle, r(> n, \tau), r(> n) \rangle^\frown \vec{\eta} \in D$
   or

(2) for every $p' \geq^* \langle\langle f^*, A^*, \delta^*\rangle, r(> n, \tau), r(> n)\rangle$, $p'^\frown \vec{\eta} \notin D$.

Moreover, the same conclusion valid for any two such $\vec{\eta}, \vec{\eta}'$ of the same length.

Claim 8 *There exists $r(> n)^* \in \mathcal{P}(> n), r(> n)^* \geq^* \bar{p} \restriction \mathcal{P}(> n)$ such that $D'(r(> n)^*, \tau)$ is dense in $\mathcal{P}(> n, \tau)$ above $\bar{p}(> n, \tau)$.*

Proof. We use the fact that $|\mathcal{P}(> n, \tau)|$ is much less than κ_{n+1}, which is a degree of completeness of the forcing $\langle \mathcal{P}(> n), \leq^* \rangle$. Enumerate $\mathcal{P}(> n, \tau)$ and build a $\leq^*$ −increasing sequence conditions there each responsible for a particular member of $\mathcal{P}(> n, \tau)$.

Let $s \in \mathcal{P}(> n, \tau)$ be above $\bar{p}(> n, \tau)$ and $r(> n)^{<s} \in \mathcal{P}(> n)$ is an upper bound of $\leq^*$ −increasing sequence of conditions in $\mathcal{P}(> n)$ corresponding predecessors of s in a fixed enumeration of $\mathcal{P}(> n, \tau)$.

Proceed as in Claim 7. Only make the following change - at successor stages of the inductive construction there pick in addition to $f_* \geq^* f, r(> n, \tau)_* \geq^* r(> n, \tau)$ in $\mathcal{P}(> n, \tau)$ also $r(> n)_* \geq^* r(> n)^{<s}$ in $\mathcal{P}(> n)$ such that

for every $\bar{x}$ in $\mathcal{P}(< n, \tau, \eta_0)$

there is $x \geq \bar{x}$ so that either

$$(1) \qquad \exists T \langle x, \langle f_{*\langle\eta_0\rangle}, T\rangle, r(> n, \tau)_*, r(> n)_*\rangle \in D),$$

or

$$(2) \forall x' \geq x \forall f' \geq^* f_* \forall r(> n, \tau)' \geq^* r(> n, \tau)_* \forall r(> n)' \geq^* r(> n)_* \forall T$$
$$\langle x', \langle f'_{\langle\eta_0\rangle}, T\rangle, r(> n, \tau)', r(> n)'\rangle \notin D).$$

This process will produce $r(> n)^s \geq^* r(> n)^{<s}$ (in $\mathcal{P}(> n)$).

At the final stage of the argument we will take $r(> n)^*$ to be an upper bound of all such $r(> n)^s$'s.

$\square$ of the claim.

Next step would be to do the above for different τ's. Note that the set of measure one of the coordinate $(n + 1, \alpha)$ of $r(> n)^*$, given by the claim, may move to a different measure (or a different place) from those of $\bar{p}$.

In order to deal with this, let us pick an elementary submodel M defined as N of Claim 7 but with κ_n replaced by κ_{n+1}. Define $\bar{E}_M$ and a set $A_M \in \bar{E}_M$ accordingly. Now, we pick τ's from A_M and preform each stage of the construction inside M.

The rest of the argument is straightforward.

Case 3. *α is a limit ordinal, $n < \omega$, $\mathcal{P} = \mathcal{P}_{\langle E(m,\beta)|m \leq n, \beta < \alpha\rangle}$, p is a pure condition in $\mathcal{P}$ and $D \subseteq \mathcal{P}$ is a dense open.*

If $n = 0$, then this just an extender based Prikry-Magidor forcing. It satisfies the Prikry condition by [9].
If $n > 1$, then we combine Lemma 4.11 of Merimovich [9] with the arguments of Case 1.
A new point here is that given a coordinate (n, β), for some $\beta < \alpha$, extenders $E(m, \gamma), m < n, \beta < \gamma < \alpha$ will overlap $E(n, \beta)$ and such extenders are over cardinals $\kappa_m < \kappa_n$.
However, going through possible non-direct extensions over a coordinate (n, β) we can accumulate the information in the components f of coordinates $(m, \gamma), m < n, \beta < \gamma < \alpha$, as it was done before.

Case 4. *α is a limit ordinal, $\mathcal{P} = \mathcal{P}_{\langle E(m,\beta)|m < \omega, \beta < \alpha\rangle}$, p is a pure condition in $\mathcal{P}$ and $D \subseteq \mathcal{P}$ is a dense open.*

Here we just combine the arguments of Cases 2 and 3 in a straightforward fashion.

8 Applications to the Shelah Weak Hypothesis.

The Shelah weak hypothesis was introduced by S. Shelah in [11], see also [12].
For uncountable cofinality it states that

for every λ the set $\{\kappa < \lambda \mid \omega < \mathrm{cof}(\kappa) < \kappa, \mathrm{pp}(\kappa) \geq \lambda\}$ is finite.

It was shown in [6] that any finite size is realizable.

For countable cofinality it states that

for every λ the set $\{\kappa < \lambda \mid \omega = \mathrm{cof}(\kappa) < \kappa, \mathrm{pp}(\kappa) \geq \lambda\}$ is at most countable.

It was shown in [6] that any finite size is realizable, then in [5] that this set can be countable and, finally, in [4], that it can have size $\aleph_1$.

Here we would like to analyze the cardinal arithmetic of the forcing extension by $\mathcal{P} = \mathcal{P}_{\langle E(\alpha,\beta)|\alpha < \eta, \beta < \delta\rangle}$ and argue that the set $\{\kappa_\alpha \mid \alpha < \eta\}$ witnesses the failure of the hypothesis.

Let G be a generic subset of $\mathcal{P}$.
By Lemmas 7.13, 7.14, 7.15,7.16, all cardinals are preserved in $V[G]$. Clearly, each κ_α changes its cofinality to δ, provided δ is a regular cardinal.
We would like to argue that, in $V[G]$, $\mathrm{pp}(\kappa_\alpha) \geq \lambda$, for every $\alpha < \eta$.

Given $p \in \mathcal{P}$. Denote by $\mathrm{np}(p)$ the set of all coordinates α of p such that for some $\beta < \delta, p(\alpha, \beta) \in \mathcal{P}^*_{E(\alpha,\beta)}$, i.e. a non-pure extension was made at the coordinate α. Denote the largest (for α) β like this by β_α.

Given $\ell < \omega$, denote by $\mathrm{np}(p)^{\geq \ell}$ the set of all coordinates α in $\mathrm{np}(p)$ such that $\beta_\alpha \geq \ell$.
The meaning behind is that if $\alpha \in \mathrm{np}(p)^{\geq \ell}$, then, for every $\alpha' < \alpha$ and every $\beta \leq \ell$ the extender $E(\alpha', \beta)$ over the coordinate α' shrinks to $E(\alpha', \beta \restriction h_\lambda^{\alpha,\ell}(p(\alpha,\ell)(\kappa_\alpha))$,
where $h_\lambda^{\alpha,\ell} : \kappa_\alpha \to \kappa_\alpha$ is the fixed function such that $j_{E(\alpha,\ell)}(h_\lambda^{\alpha,\ell})(\kappa_\alpha) = \lambda$.

Assume that η is a limit ordinal.

Set $\bar{\kappa}_\alpha = \bigcup_{\beta<\alpha} \kappa_\beta$, for every $\alpha \leq \eta$.

Fix $\alpha < \eta$.

Let $\tau \in [\bar{\kappa}_\eta, \lambda)$. Define in $V[G]$ a function $t_\tau : \delta \to \kappa_\alpha$ as follows.
For every $\ell < \delta$, find $p \in G$ such that $\alpha \in \mathrm{np}(p)^{\geq \ell}$ and if $\alpha_1 < ... < \alpha_k$ is the increasing enumeration of $\mathrm{np}(p)^{\geq \ell} \setminus \alpha$ (i.e. $\alpha = \alpha_1$), then the following hold:

1. $\tau \in \mathrm{dom}(p(\alpha_k, \ell))$.
   Set $\tau_k = \tau$.

2. For every $i, 1 \leq i \leq k-1$, $\tau_i \in \mathrm{dom}(p(\alpha_i, \ell))$,
   where $\tau_i = p(\alpha_{i+1}, \ell)(\tau_{i+1})$.

Set $t_\tau(\ell) = p(\alpha, \ell)(\tau_1)$.

Lemma 8.1 *In $V[G]$, if $\tau, \rho \in [\bar{\kappa}_\eta, \lambda)$ and $\tau < \rho$, then there is $\ell^* < \omega$ such that for every $\ell, \ell^* \leq \ell < \delta$, $t_\tau(\ell) < t_\rho(\ell)$.*

Proof. Work in V. Let $p \in \mathcal{P}$ be any condition and $\tau, \rho \in [\bar{\kappa}_\eta, \lambda)$,$\tau < \rho$.
Let ℓ^* be a coordinate above every β_θ, with $\theta \in \mathrm{np}(p)$. Then $p(\gamma, \ell) = \langle f^p_{\gamma,\ell}, A^p_{\gamma,\ell} \rangle$ is unrestricted condition, i.e. with $A^p_{\gamma,\ell} \in (E(\gamma,\ell))(\mathrm{dom}(f^p_{\gamma,\ell}))$, for every $\gamma < \eta$ and for every $\ell, \ell^* \leq \ell < \delta$.
Extend p to p^* by adding τ, ρ to all $\mathrm{dom}(f^p_{\gamma,\ell})$ with $\alpha \leq \gamma < \eta$ and $\ell^* \leq \ell < \delta$.
Now, by the definition of the order on $\mathcal{P}$, for every $\ell^* \leq \ell < \delta$ and every $q \geq p^*$ such that q defines $t_\tau(\ell)$ and $t_\rho(\ell)$, we will have $t_\tau(\ell) < t_\rho(\ell)$.
So,

$$p^* \Vdash (\forall \ell)(\ell^* \leq \ell < \omega \to \underset{\sim}{t}_\tau(\ell) < \underset{\sim}{t}_\rho(\ell)).$$

□

It is possible to say a bit more. Namely, let in $V[G]$, for every $\ell < \omega$, $\lambda_{\alpha,\ell}$ be the reflection of λ below κ_α using, $E(\kappa_\alpha, \ell)$ and the representing function $h_\lambda^{\alpha,\ell}$, i.e. for some $p \in G$ with $p(\alpha,\ell) = f^p_{\alpha,\ell}$, $\lambda_{\alpha,\ell} = h_\lambda^{\alpha,\ell}(f^p_{\alpha,\ell}(\kappa_\alpha))$. Then the following holds:

Lemma 8.2 *The sequence* $\langle t_\tau \mid \tau \in [\bar\kappa_\eta, \lambda)\rangle$ *is a scale in* $\langle \prod_{\ell<\delta} \lambda_{\alpha,\ell}, <_{J^{bd}_\delta} \rangle$.

The desired result follows now:

Theorem 8.3 *In* $V[G]$*, for every* $\alpha < \eta$, $\mathrm{pp}(\kappa_\alpha) \geq \lambda$.
The meaning is that it is possible to arrange arbitrary long failures of the Shelah Weak Hypothesis in any given cofinality.

It turns out that in $V[G]$, $\mathrm{pp}(\kappa) > \kappa^+$ for many κ's different from κ_α's, as will be shown below. However, if δ is a regular cardinal above, say $\aleph_0$ and $\mathrm{cof}(\eta) > \aleph_0$, then there will be no cardinals of countable cofinality below λ with pp above λ.

Proposition 8.4 *Let in* $V[G]$, $\kappa < \kappa_{\bar\eta}$ *be a singular cardinal of cofinality* $\delta' < \delta$*. Then* $\mathrm{pp}(\kappa) < \lambda$.

Proof. We have $\kappa < \bar\kappa_\eta = \bigcup_{\alpha<\eta} \kappa_\alpha$.
Pick $\bar\alpha < \eta$ to be the least such that $\kappa \leq \kappa_{\bar\alpha}$.
Note that in $V[G]$, $\mathrm{cof}(\kappa_{\bar\alpha}) = \delta > \delta' = \mathrm{cof}(\kappa)$.
Hence $\kappa < \kappa_{\bar\alpha}$.
Let $a \subseteq \kappa$ be an unbounded subset of κ which consists of regular cardinals and $|a| = \delta'$.
The next claim will complete the argument.

Claim 9 $\max(\mathrm{pcf}(a)) < \kappa_{\bar\alpha}$.

Proof. The point is that by the Prikry condition arguments, a depends only on $\delta'-$many extenders.
Namely, there are sets $b \subseteq \bar\alpha$ and $c \subseteq \delta$ of cardinality at most δ', such that a is reconstructible from Magidor sequences for extenders $\langle E(\alpha, \beta) \mid \alpha \in b, \beta \in c\rangle$.
Pick some $\tau, \sup(c) < \tau < \delta$.
Then, by the definition of the order $\leq$ on $\mathcal{P}$, the element of the Magidor sequence for the normal measure of $E(\bar\alpha, \tau)$ will bound $\mathrm{pcf}(a)$, since all the extenders $\langle E(\alpha, \beta) \mid \alpha \in b, \beta \in c\rangle$ relevant to a, will be restricted below this element.
□ of the claim.
□

Corollary 8.5 *Suppose that* δ *is a regular cardinal and* $\mathrm{cof}(\eta) \geq \delta$*. Then in* $V[G]$*, for every regular* $\delta' < \delta$*, for every cardinal* $\kappa < \lambda$ *of cofinality* δ', $\mathrm{pp}(\kappa) < \lambda$.

Proof. Let κ be a such cardinal. Then it changed cofinality in $V[G]$ or it is a limit of cardinals that changed their cofinality. In particular $\kappa < \bar{\kappa}_\eta = \bigcup_{\alpha<\eta} \kappa_\alpha$.
Now the previous proposition applies.
□

Assume that δ is a regular cardinal.
We denote $\bar{\kappa}_\alpha = \bigcup_{\beta<\alpha} \kappa_\beta$, for every limit $\alpha \leq \eta$.
Let us clarify the situation with $\langle \bar{\kappa}_\alpha \mid \alpha \leq \eta \text{ a limit ordinal } \rangle$.

Proposition 8.6 *Let $\alpha < \eta$ be a limit ordinal of cofinality less than δ. Then, in $V[G]$, $\mathrm{pp}(\bar{\kappa}_\alpha) = \kappa_\alpha$.*

Proof. The argument uses a refinement of 8.4.
Namely, by Claim 9 (of Proposition 8.4), we have $\mathrm{pp}(\bar{\kappa}_\alpha) \leq \kappa_\alpha$.
On the other hand, there is an increasing sequence $\langle \kappa_{\alpha_i} \mid i < \mathrm{cof}(\alpha)\rangle$ unbounded in $\bar{\kappa}_\alpha$.
Each κ_{α_i} is a cardinal that changed its cofinality to δ in $V[G]$.
Also κ_α changed its cofinality to δ in $V[G]$.
Denote the corresponding generic Magidor sequences for the normal measure of the extenders over κ_{α_i}'s by $\langle \kappa_{\alpha_i\beta} \mid \beta < \delta\rangle$ and for κ_α by $\langle \kappa_{\alpha\beta} \mid \beta < \delta\rangle$.
Fix some $\beta^* < \delta$. Let as argue that there is an unbounded $a \subseteq \bar{\kappa}_\alpha$ which consists of regular cardinals, $|a| = \mathrm{cof}(\alpha)$ such that $\max(\mathrm{pcf}(a)) > \kappa_{\alpha\beta^*}$.
Take a to be the set $\{\lambda_{\alpha_i\beta^*} \mid i < \mathrm{cof}(\alpha)\}$, where, as in Lemma 8.2, $\lambda_{\alpha_i\beta^*}$ is the reflection of λ below κ_{α_i}, using $E(\kappa_{\alpha_i}, \beta^*)$ and the representing function $h_\lambda^{\alpha_i,\beta^*}$, i.e. for some $p \in G$ with $p(\alpha_i, \beta^*) = f^p_{\alpha_i,\beta^*}$, $\lambda_{\alpha_i\beta^*} = h_\lambda^{\alpha_i,\beta^*}(f^p_{\alpha_i,\beta^*}(\kappa_{\alpha_i}))$.
The analysis of scales similar to those of Lemmas 8.1 and 8.2 shows that $\mathrm{tcf}(\prod a, <_{J^{bd}_a}) = \lambda_{\alpha\beta^*} > \kappa_{\alpha\beta^*}$, where $\lambda_{\alpha\beta^*}$ is the reflection of λ to the level β^* using $E(\kappa_\alpha, \beta^*)$.
This gives the desired conclusion.
□

Remark 8.7 Note that if one allows subsets a of κ_α of cardinality δ, then λ is reached, i.e., $\mathrm{pp}_\delta(\bar{\kappa}_\alpha) = \lambda$. The argument repeats those of Lemmas 8.1 and 8.2, only take $\mathfrak{a} = \bigcup_{i<\mathrm{cof}(\alpha)} \mathfrak{a}_i$, where $\mathfrak{a}_i = \{\lambda_{\alpha_i\beta} \mid \beta < \delta$, for every $i < \mathrm{cof}(\alpha)\}$. Instead of $J^{bd}_{\mathfrak{a}}$ use the ideal

$$J = \{x \subseteq \mathfrak{a} \mid \exists \varepsilon_0 < \mathrm{cof}(\alpha) \forall \varepsilon \in [\varepsilon_0, \mathrm{cof}(\alpha))(x \cap \mathfrak{a}_\varepsilon \in J^{bd}_{\mathfrak{a}_\varepsilon})\}.$$

□

The proposition can be used to give a negative answer to the following question of S. Shelah (Problem (ε), Analytical Guide, [10]):
Is $\mathrm{cof}(\mathrm{pp}(\kappa)) > \kappa$?
$\mathrm{cof}(\mathrm{pp}(\kappa)) > \mathrm{cof}(\kappa)$ and in view of the König Theorem it is reasonable to have
$\mathrm{cof}(\mathrm{pp}(\kappa)) > \kappa$.
However Proposition 8.6 implies the following:

Corollary 8.8 *Let $\alpha < \eta$ be a limit ordinal of cofinality less than δ.*
Then, in $V[G]$, $\mathrm{cof}(\mathrm{pp}(\bar{\kappa}_\alpha)) = \delta < \bar{\kappa}_\alpha$.

Proof. Just note that in $V[G]$, $\mathrm{cof}(\kappa_\alpha) = \delta$ and $\delta < \kappa_0 < \bar{\kappa}_\alpha$.
□

In particular, taking $\delta = \omega_1, \eta > \omega, \alpha = \omega$, we obtain the following:

Corollary 8.9 *In $V[G]$, there is a cardinal κ of cofinality ω such that $\mathrm{cof}(\mathrm{pp}(\kappa)) = \omega_1$.*

Proposition 8.10 *Let $\alpha \leq \eta$ be a limit ordinal of cofinality $\geq \delta$.*
Then, in $V[G]$, $\mathrm{pp}(\bar{\kappa}_\alpha) \geq \lambda$.
In particular, if $\eta = \aleph_9$, then this produces a set of cardinality $\aleph_9$ which violates the Shelah Weak Hypothesis for every cofinality $\leq \aleph_8$.

Proof. Fix a cofinal in α sequence $\langle \alpha_\varepsilon \mid \varepsilon < \mathrm{cof}(\alpha) \rangle$.
Set $\mathfrak{a}_\varepsilon = \{\lambda_{\alpha_\varepsilon, \ell} \mid \ell < \delta\}$, for every $\varepsilon < \mathrm{cof}(\alpha)$, where, as above, $\lambda_{\alpha_\varepsilon,\ell}$ is the reflection of λ below $\kappa_{\alpha_\varepsilon}$ using, $E(\kappa_{\alpha_\varepsilon}, \ell)$ and the representing function $h_\lambda^{\alpha_\varepsilon,\ell}$, i.e. for some $p \in G$ with $p(\alpha_\varepsilon, \ell) = f^p_{\alpha_\varepsilon,\ell}$, $\lambda_{\alpha_\varepsilon,\ell} = h_\lambda^{\alpha_\varepsilon,\ell}(f^p_{\alpha_\varepsilon,\ell}(\kappa_{\alpha_\varepsilon}))$. By Lemma 8.2, $\mathrm{tcf}(\prod \mathfrak{a}_\varepsilon, <_{J^{bd}_{\mathfrak{a}_\varepsilon}}) = \lambda$.
Set $\mathfrak{a} = \bigcup_{\varepsilon < \mathrm{cof}(\alpha)} \mathfrak{a}_\varepsilon$.
Define the ideal J to be the set

$$\{x \subseteq \mathfrak{a} \mid \exists \varepsilon_0 < \mathrm{cof}(\alpha) \forall \varepsilon \in [\varepsilon_0, \mathrm{cof}(\alpha))(x \cap \mathfrak{a}_\varepsilon \in J^{bd}_{\mathfrak{a}_\varepsilon})\}.$$

Now, as in Lemma 8.2, $\mathrm{tcf}(\prod \mathfrak{a}, <_J) = \lambda$.
□

Note that the ideals involved in computing $\mathrm{pp}(\kappa)$'s here are $\delta-$additive. S. Shelah pointed out that by [13], Theorem 1.1, we cannot hope to get more additivity.

Let us examine what exactly happen in the present construction in this respect.

Proposition 8.11 *Let $\alpha < \eta$ be a limit ordinal of cofinality $> \delta$. Let $\mathfrak{b}$ be a set of regular cardinals of cardinality $\mathrm{cof}(\alpha)$ unbounded in $\bar{\kappa}_\alpha$. Suppose that $J \supseteq J^{bd}_{\mathfrak{b}}$ is a δ^+-complete*

ideal on $\mathfrak{b}$.
Then, in $V[G]$, *if* $\operatorname{tcf}(\prod \mathfrak{b}, <_J)$ *exists, then* $\operatorname{tcf}(\prod \mathfrak{b}, <_J) < \kappa_\alpha$.

Proof. Split $\mathfrak{b}$ into sets $\langle \mathfrak{b}_\xi \mid \xi < \delta \rangle$, where $\mathfrak{b}_\xi$ consists of all members of $\mathfrak{b}$ which are not members of the Prikry-Magidor sequences of G, or are the members of such sequences of order $< \xi$. Then, as in Proposition 8.4, $\max(\operatorname{pcf}(\mathfrak{b}_\xi)) < \kappa_\alpha$, for every $\xi < \delta$.
Now, if $\operatorname{tcf}(\prod \mathfrak{b}, <_J) \geq \kappa_\alpha$, then $\mathfrak{b}_\xi \in J$, for every $\xi < \delta$. However, the δ^+-completeness of J implies then that $\mathfrak{b} = \bigcup_{\xi < \operatorname{cof}(\alpha)} \mathfrak{b}_\xi \in J$, which is impossible.
Contradiction.
□

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