Silver type theorems for collapses.

Moti Gitik*

Abstract

Let κ be a cardinal of cofinality ω_1 witnessed by a club of cardinals $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$. We study Silver's type effects of collapsing of κ_α^+'s on κ^+. A model in which κ_α^+'s (and also κ^+) are collapsed on a stationary co-stationary set is constructed.

1 Introduction.

The classical theorem of Silver states that GCH cannot fail for the first time at a singular cardinal of uncountable cofinality. On the other hand, it is easy to obtain a situation in which GCH fails on a club below a singular cardinal κ of an uncountable cofinality but $2^\kappa = \kappa^+$.

We investigate here on which sets of cardinals $\aleph_\alpha$ below a singular of uncountable cofinality the successor $\aleph_{\alpha+1}$ can be collapsed in a forcing extension without collapsing the successor of the singular itself. Thus, blowing up powers of singular cardinals is replaced by collapses of their successors in the question above.

This paper is motivated by one of the difficult problems in large cardinal forcing in cardinal arithmetic: *is it consistent that the set of cardinals on which the GCH fails below a singular cardinal of uncountable cofinality is stationary and co-stationary in the cardinal.* This question is still open. We refer to [2] for some related partial results on it.

In the present article the following modification of the question is considered: rather than blowing up the power set of cardinals below the limit, a single cofinal set of small order type is added to their successors. The question now becomes: is it possible that in a forcing extension the set of cardinals below a singular of uncountable cofinality whose successors collapse in the extension is stationary and co-stationary. This question is provided with a positive answer by Theorem 3.1.

Section 1 is devoted to preparing the way in that it clarifies which constraints such a forcing

*The work was partially supported by Israel Science Foundation Grants No. 58/14, 1216/18. We would like to thank to the referee of the paper for his suggestions, remarks and corrections.

construction must obey. First, in Theorem 2.1 it is shown that collapsing stationarily many successors necessitates collapsing the successor of the strong limit cardinal κ. The proof follows pretty much the lines of the simplified proof of Silver's original theorem. Then the investigation is extended by considering the structure of $\mathcal{P}_\kappa(\kappa^+)$ instead of κ^+.

2 ZFC results.

The following basic result should be well known and goes back to Silver:

Theorem 2.1 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is a cardinal in W,*
2. *κ has cofinality ω_1 in V, witnessed by a club of cardinals $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$,*
3. *for every $\alpha < \omega_1, (\kappa_\alpha^+)^W < (\kappa_\alpha^+)^V$, that is, $(\kappa_\alpha^+)^W$ is no longer a cardinal in V, (or only for stationary many α's),*
4. *κ is a strong limit in V or just is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \omega_1$.*

Then $(\kappa^+)^W < \kappa^+$.

Proof. Suppose that $(\kappa^+)^W = (\kappa^+)^V$.

Fix in W a sequence $\langle f_i \mid i < \kappa^+ \rangle$ of of the first κ^+ canonical functions in $\langle \prod_{\nu<\kappa} \nu^+, <_{J_\kappa^{bd}} \rangle$ (observe that κ may be regular in W), or just any sequence of κ^+–many functions in $\prod_{\nu<\kappa} \nu^+$ increasing mod J_κ^{bd}.

Set in V, $g_i = f_i \restriction \{\kappa_\alpha \mid \alpha < \omega_1\}$, for every $i < \kappa^+$. Then $\langle g_i \mid i < \kappa^+ \rangle$ is an increasing sequence of functions in $\langle \prod_{\alpha<\omega_1} (\kappa_\alpha^+)^W, <_{J_{\omega_1}^{bd}} \rangle$. By the assumption (3) we have that for every $\alpha < \omega_1, (\kappa_\alpha^+)^W < (\kappa_\alpha^+)^V$. Now, as in the Baumgartner-Prikry proof of the Silver Theorem (see K. Kunen [4] p.296 (H5)), it is impossible to have κ^+–many such functions. Hence $(\kappa^+)^W < (\kappa^+)^V$.

□

Let us deal now with double successors.

Theorem 2.2 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is a cardinal in W,*

2. *$2^\kappa \geq \kappa^{++}$, and moreover there is a sequence of κ^{++}–many functions in $\prod_{\nu<\kappa} \nu^{++}$ increasing mod J^{bd}_κ,*

3. *κ has cofinality ω_1 in V, witnessed by a club $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$,*

4. *for every $\alpha < \omega_1, (\kappa_\alpha^{++})^W < \kappa_\alpha^+$ (or only for stationary many α's),*

5. *κ is a strong limit in V or just it is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \omega_1$.*

Then $(\kappa^{++})^W < \kappa^+$.

The condition (2) allows to repeat the proof of 2.1.

Let state the following relevant result of Shelah ([7](4.9,p.304)), which says that once $(\kappa^+)^W$ changes its cofinality, then we must have $(\kappa^{++})^W < \kappa^+$ unless
$\mathrm{cof}((\kappa^+)^W) = \mathrm{cof}(|(\kappa^+)^W|) = \mathrm{cof}(\kappa)$.

The next two statements deal with $\mathcal{P}_\kappa(\kappa^+)$ instead of κ^+. Usually, this structure is relevant if we like to change the cofinality of a successor cardinal say using the supercompact Prikry or Magidor forcing, see for example [1].

Proposition 2.3 *Let $\mathcal{F}$ be the κ–complete filter of co-bounded subsets of $\mathcal{P}_\kappa(\kappa^+)$, i.e. the filter generated by the sets $\{P \in \mathcal{P}_\kappa(\kappa^+) \mid \alpha \in P\}$, $\alpha < \kappa^+$.*
Then there is a sequence $\langle f_i \mid i < \kappa^{++} \rangle$ of functions such that

1. *$f_i : \mathcal{P}_\kappa(\kappa^+) \to \kappa$,*

2. *$f_i(P) < |P|^+$, for all $P \in \mathcal{P}_\kappa(\kappa^+)$,*

3. *$f_i >_{\mathcal{F}} f_j$, whenever $i > j$.*

Proof. We define a sequence $\langle f_i \mid i < \kappa^{++} \rangle$ by induction.
Suppose that $\langle f_j \mid j < i \rangle$ is defined. Define f_i.

Case 1. $i = i' + 1$.
Set $f_i(P) = f_{i'}(P) + 1$.

Case 2. i is a limit ordinal of cofinality $\delta < \kappa$.
Pick a cofinal in i sequence $\langle i_\tau \mid \tau < \delta \rangle$. Set $f_i(P) = \bigcup_{\tau<\delta} f_{i_\tau}(P) + 1$.

Case 3. i is a limit ordinal of cofinality $\delta \geq \kappa$, i.e. $\delta = \kappa$ or $\delta = \kappa^+$.
Pick a cofinal in i sequence $\langle i_\tau \mid \tau < \delta \rangle$. Set $f_i(P) = \bigcup_{\tau \in P} f_{i_\tau}(P) + 1$.
□

Theorem 2.4 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is an inaccessible in W,*
2. *$\kappa > (\mathrm{cof}(\kappa))^V = \delta$ for some uncountable (in V) cardinal δ.*
3. *κ is a strong limit in V or just it is a limit cardinal and for every $\xi < \kappa$, $\xi^\delta < \kappa$.*
4. *There exist a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ (or just a stationary set) [1] and a sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ such that*
   (a) *$P_\alpha \in (\mathcal{P}_\kappa(\kappa^+))^W$, for each $\alpha < \delta$,*
   (b) *$(|P_\alpha|^+)^W < \kappa_\alpha^+$, for each $\alpha < \delta$,*
   (c) *$(\kappa^+)^W = \bigcup_{\alpha<\delta} P_\alpha$,*
   (d) *for every $Q \in (\mathcal{P}_\kappa(\kappa^+))^W$, there is $\alpha < \delta$ such that for every $\beta, \alpha \leq \beta < \delta$, $Q \subseteq P_\beta$.*

Then $(\kappa^{++})^W < \kappa^+$.

Proof. Suppose otherwise. Then $(\kappa^{++})^W = \kappa^+$, by the assumption (b),(c) above.
Let $\langle f_i \mid i < \kappa^{++} \rangle$ be a sequence of functions in W given by Proposition 2.3.
We can repeat the argument of 2.1 with slight adaptations. Thus, set in V
$g_i(\alpha) = f_i(P_\alpha)$, for every $\alpha < \omega_1$ and $i < (\kappa^{++})^W = \kappa^+$. Let $\nu_\alpha := (|P_\alpha|^+)^W$. By the assumption, $\nu_\alpha < \kappa_\alpha^+$. Then $\langle g_i \mid i < \kappa^+ \rangle$ is an increasing sequence of functions in $\langle \prod_{\alpha<\delta} \nu_\alpha, <_{J_\delta^{bd}} \rangle$, since for every $A \in \mathcal{F}$ we have $\{P_\alpha \mid \alpha \geq \alpha_0\} \subseteq A$, for some $\alpha_0 < \delta$. This is impossible, since $\nu_\alpha < \kappa_\alpha^+$, for every $\alpha < \delta$. Contradiction.
□

Theorem 2.5 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that for some inaccessible in W cardinal κ both κ and its successor in W change their cofinality to some uncountable (in V) cardinal δ and κ remains a cardinal in V. Then the following conditions are equivalent:*

[1] Note that if $\delta = \omega_1$, then we can just force a club into it without effecting things above.

1. *There are clubs* $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ *in* κ *and* $\langle \eta_\alpha \mid \alpha < \delta \rangle$ *in* $(\kappa^+)^W$ *such that for every limit* $\alpha < \delta$ *(or just for stationary many* α*'s)*[2] *the set* $\{\eta_\beta \mid \beta < \alpha\}$ *can be covered by a set* $a_\alpha \in W$ *with* $(|a_\alpha|^+)^W < \kappa_\alpha^+$.

2. *There are clubs* $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ *in* κ *and* $\langle \eta_\alpha \mid \alpha < \delta \rangle$ *in* $(\kappa^+)^W$ *such that for every limit* $\alpha < \delta$ *(or just for stationary many* α*'s) the set* $\{\eta_\beta \mid \beta < \alpha\}$ *has an unbounded intersection with a set* $b_\alpha \in W$ *with* $(|b_\alpha|^+)^W < \kappa_\alpha^+$.

3. *There exist a club* $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ *in* κ *and a sequence* $\langle P_\alpha \mid \alpha < \delta \rangle$ *such that*

   (a) $P_\alpha \in (\mathcal{P}_\kappa(\kappa^+))^W$, *for each* $\alpha < \delta$,

   (b) $P_\alpha \cap \kappa = \kappa_\alpha$, *for each* $\alpha < \delta$,

   (c) $(|P_\alpha|^+)^W < \kappa_\alpha^+$, *for each* $\alpha < \delta$,

   (d) $(\kappa^+)^W = \bigcup_{\alpha < \omega_1} P_\alpha$,

   (e) *for every* $Q \in (\mathcal{P}_\kappa(\kappa^+))^W$, *there is* $\alpha < \omega_1$ *such that for every* $\beta, \alpha \leq \beta < \omega_1$, $Q \subseteq P_\beta$.

4. *There exists an increasing sequence* $\langle P_\alpha \mid \alpha < \delta \rangle$ *which satisfies all the requirements of the previous item.*

Proof. Split the proof into lemmas.

Lemma 2.6 *Conditions (1) and (2) in 2.5 are equivalent.*

Proof. Clearly, (1) implies (2). Let us show the opposite direction.
We fix a bijection $\pi_\xi : \kappa \longleftrightarrow \xi$ in W, for every $\xi < (\kappa^+)^W$.
Fix in V a function $\pi : \kappa \longrightarrow^{onto} (\kappa^+)^W$. Set now for every $\alpha < \delta$, $\eta_\alpha = \sup(\pi``\kappa_\alpha)$. Then, clearly, $\{\eta_\alpha \mid \alpha < \delta\}$ is a club in $(\kappa^+)^W$. Now given a sequence which witnesses (2). Without loss of generality we can assume that it is the sequence $\langle \eta_\alpha \mid \eta < \delta \rangle$ defined above. Otherwise just intersect both clubs.
Define an increasing continuous sequence $\langle N_\alpha \mid \alpha < \delta \rangle$ of elementary submodels of some H_χ, with χ big enough such that

1. $\delta, \kappa, \langle \kappa_\alpha \mid \alpha < \delta \rangle, \langle \pi_\xi \mid \xi < (\kappa^+)^W \rangle, \pi \in N_0$,

[2]If $\delta = \omega_1$, then it is basically the same, since once we have only stationary many such α's, then force a club into it. Everything is a the level of ω_1, so this will have no effect on the cardinal arithmetic above.

2. $|N_\alpha| < \delta$,

3. $N_\alpha \cap \delta$ is an ordinal,

4. $\langle N_\beta \mid \beta \leq \alpha \rangle \in N_{\alpha+1}$.

Denote $N_\alpha \cap \delta$ by δ_α.
Then $\sup(N_\alpha \cap \kappa) = \kappa_{\delta_\alpha}$ and $\sup(N_\alpha \cap (\kappa^+)^W) = \eta_{\delta_\alpha}$. Clearly, $\delta_\alpha = \alpha$ for a club many α's.

Suppose now that for some $\alpha < \delta$ we have $\delta_\alpha = \alpha$ and there is a set $X \in W$ such that

- $(|X|^+)^W < \kappa_\alpha^+$,
- $X \cap \{\eta_\beta \mid \beta < \alpha\}$ is unbounded in η_α.

Note that $\eta_\beta \in N_\alpha$, for every $\beta < \alpha$ and then, also, $\pi_{\eta_\beta} \in N_\alpha$. By elementarity, then $\pi_{\eta_\beta} \restriction (N_\alpha \cap \kappa_\alpha) : N_\alpha \cap \kappa_\alpha \longleftrightarrow N_\alpha \cap \eta_\beta$. In particular, $\pi_{\eta_\beta}``\kappa_\alpha \supseteq \{\eta_\gamma \mid \gamma < \beta\}$.
Set

$$Y := \{\pi_\zeta``\kappa_\alpha \mid \zeta \in X \cap \eta_\alpha\}.$$

Then, $Y \in W, |Y|^W \leq \kappa_\alpha + |X|^W$, and so $(|Y|^+)^W < \kappa_\alpha^+$. But, in addition, $Y \supseteq \{\eta_\gamma \mid \gamma < \alpha\}$, since for unboundedly many $\beta < \alpha$, we have $\eta_\beta \in X$ and so, $\pi_{\eta_\beta}``\kappa_\alpha \supseteq \{\eta_\gamma \mid \gamma < \beta\}$.
$\square$ of the lemma.

Lemma 2.7 *(1) implies (3)*

Proof.
Fix clubs $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ witnessing (1).
Let us build first a sequence $\langle R_\alpha \mid \alpha < \delta \rangle$ which satisfies all the requirements of (3), but probably is not increasing.

Set $R_0 = \kappa_0 \cup ((\pi_{\eta_0}``\kappa_0) \setminus \kappa)$.
Let $\alpha, 0 < \alpha < \delta$ be an ordinal. Pick $a_\alpha \in W, a_\alpha \subseteq \eta_\alpha$ to be a cover of $\{\eta_\beta \mid \beta < \alpha\}$ with $(|a_\alpha|^+)^W < \kappa_\alpha^+$. Set $R'_\alpha = \bigcup \{\pi_\xi``\kappa_\alpha \mid \xi \in b_\alpha \cup \{\eta_\alpha\}\}$. Let $R_\alpha = \kappa_\alpha \cup (R'_\alpha \setminus \kappa)$.

The constructed sequence satisfies trivially the requirements (a),(b) and (c). Let us check (e). (d) clearly follows from (e).
Let $Q \in (\mathcal{P}_\kappa(\kappa^+))^W$. There is $\beta < \omega_1$ such that $Q \subseteq \eta_\beta$. Consider $\pi_{\eta_\beta}^{-1}``Q$. It is a bounded subset of $\kappa$. Hence there is $\gamma < \omega_1$ such that $\kappa_\gamma \supseteq \pi_{\eta_\beta}^{-1}``Q$. So $\pi_{\eta_\beta}``\kappa_\beta \supseteq Q$. Let $\alpha < \omega_1$ be an ordinal above β, γ. Then $R_\delta \supseteq Q$, for every $\delta \geq \alpha$.
$\square$ of the lemma.

Lemma 2.8 *(3) iff (4).*

Proof. Clearly (4) implies(3). Let us show the opposite direction.
Let a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ and a sequence $\langle R_\alpha \mid \alpha < \delta \rangle$ witness (3).
Define an increasing subsequence $\langle P'_\alpha \mid \alpha < \delta \rangle$
Set $P'_0 = R_0$. By (e) there is α_0 such that for every $\beta, \alpha_0 \leq \beta < \delta$, $P'_0 \subseteq R_\beta$. Set $P'_1 = R_{\alpha_1}$. Continue by induction. Suppose that $\nu < \delta$ and for every $\nu' < \nu$, increasing sequences $\langle \alpha_{\nu'} \mid \nu' < \nu \rangle$ and $\langle P'_{\nu'} \mid \nu' < \nu \rangle$ are defined and satisfy the following:

1. $P'_{\nu'} = R_{\alpha_{\nu'}}$,
2. for every $\beta, \alpha_{\nu'} \leq \beta < \delta$, $P'_{\nu'} \subseteq R_\beta$.

If ν is a successor ordinal, then let $\nu = \mu + 1$, for some μ. Set $P'_\nu = R_{\alpha_\mu}$ and let $\alpha_\nu < \delta$ be such that for every $\beta, \alpha_\nu \leq \beta < \delta$, $P'_\nu \subseteq R_\beta$.
If ν is a limit ordinal, then let $P'_\nu = R_{\bigcup_{\nu' < \nu} \alpha_{\nu'}}$ and define α_ν as in the successor case.

Finally let us define an increasing subsequence of $\langle P'_\alpha \mid \alpha < \delta \rangle$ which satisfies the properties (a)-(e) of (3).
Let $C := \{\nu < \delta \mid \nu = \bigcup_{\nu' < \nu} \alpha_{\nu'}\}$. Clearly it is a club. Set $P_\nu = P'_\nu$, for every $\nu \in C$.
Then $\langle \kappa_\alpha \mid \alpha \in C \rangle$ and $\langle P_\alpha \mid \alpha \in C \rangle$ are as desired.
$\square$ of the lemma.

Lemma 2.9 *(3) implies (1).*

Proof. Let a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ and a sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ witness (3). Let $\eta_\alpha \mid \alpha < \delta \rangle$ be a club in $(\kappa^+)^W$.
We claim that there is a club $C \subseteq \delta$ such that for every $\alpha \in C$, $P_\alpha \supseteq \{\eta_\beta \mid \beta < \alpha\}$.

Suppose otherwise. Then there is a stationary $S \subseteq \delta$ such that for every $\alpha \in S$ there is $\alpha' < \alpha$ with $\eta_{\alpha'} \notin P_\alpha$. Then there are a stationary set $S^* \subseteq S$ and $\alpha^* < \delta$ such that for every $\alpha \in S^*$, $\eta_{\alpha^*} \notin P_\alpha$. This is impossible by (d).
$\square$ of the lemma.
$\square$

Theorem 2.10 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is an inaccessible cardinal in W,*

2. *$\kappa > (\mathrm{cof}(\kappa))^V = \delta$ for some uncountable (in V) cardinal $\delta > \omega_1$. Let $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ be a witnessing club of cardinals.*

3. *For every $\alpha < \delta, (\kappa_\alpha^{++})^W < \kappa_\alpha^+$ (or only for stationary many α's),*

4. *κ is a strong limit in V or just it is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \delta$.*

5. *There is a regular cardinal $\delta^*, \omega < \delta^* < \delta$ such that for every regular cardinal $\rho < \kappa$ of W which became a singular of cofinality δ^* in V, there is a club a club sequence $\langle \rho_i \mid i < \delta^* \rangle$ in ρ such that for every club $c \subseteq \delta^*$ the set $\{(\mathrm{cof}(\rho_i))^W \mid i \in c\}$ is unbounded in $|\rho|$.*

   Or

6. *Like the previous item but only for ρ's of the form $(\mathrm{cof}(\eta_\alpha))^W$ with $\alpha < \delta$ of cofinality δ^*, where $\langle \eta_\alpha \mid \alpha < \delta \rangle$ is a club in $(\kappa^+)^W$.*

Then $(\kappa^{++})^W < \kappa^+$.

Proof.

Let us argue that (2) of 2.4 holds.
Assume for simplicity that $\delta^* = \omega_1$.
Let $\langle N_\alpha \mid \alpha < \delta \rangle$ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ be as in 2.6. Pick $\alpha < \delta$ of cofinality ω_1 with $\delta_\alpha = \alpha$. Consider η_α. Then $\mathrm{cof}(\eta_\alpha) = \omega_1$. If $(\mathrm{cof}(\eta_\alpha))^W < \kappa_\alpha^+$, then we pick in W a club X in η_α of the order type $(\mathrm{cof}(\eta_\alpha))^W$. Then $X \cap \{\eta_\beta \mid \beta < \alpha\}$ is a club, and so, unbounded in η_α.

Suppose now that $(\mathrm{cof}(\eta_\alpha))^W \geq \kappa_\alpha^+$. Denote $(\mathrm{cof}(\eta_\alpha))^W$ by ρ. Then $\rho \leq \kappa$, since $\eta_\alpha < (\kappa^+)^W$. It is impossible to have $\rho = \kappa$, since $\mathrm{cof}(\kappa) > \omega_1 = \mathrm{cof}(\alpha) = \mathrm{cof}(\eta_\alpha) = \mathrm{cof}(\rho)$. Hence $\kappa_\alpha^+ \leq \rho < \kappa$. In particular, $|\rho| \geq \kappa_\alpha^+$.
By the assumption (5) of the theorem, there is a club a club sequence $\langle \rho_i \mid i < \omega_1 \rangle$ such that for every club $c \subseteq \omega_1$ the set $\{(\mathrm{cof}(\rho_i))^W \mid i \in c\}$ is unbounded in $|\rho|$. Let $e = \{e_\xi \mid \xi < \rho\} \in W$ be a club in η_α. Consider $d := \{\eta_\beta \mid \beta < \alpha\} \cap e$. It is a club in η_α. So there are some $\gamma < \alpha$ and $j < \omega_1$ such that $\eta_\gamma = e_{\rho_j}$ and $(\mathrm{cof}(\rho_j))^W > \kappa_\alpha$. But this is impossible, since $\eta_\gamma \in N_\alpha$, and hence, $(\mathrm{cof}(\eta_\gamma))^W = (\mathrm{cof}(\rho_j))^W \in N_\alpha \cap \kappa \subseteq \kappa_\alpha$.

Hence, always $(\mathrm{cof}(\eta_\alpha))^W < \kappa_\alpha^+$.
So, the set $\{\eta_\alpha \mid \alpha < \delta \text{ and } \mathrm{cof}(\alpha) = \omega_1\}$ witnesses (2) and we are done.
□

Lemma 2.11 *For every $\beta < \delta$,*
$\{(\mathrm{cof}(\eta_\gamma))^W \mid \gamma < \beta\} \subseteq \kappa_\beta$.

Proof. Otherwise there is $\gamma < \beta$ such that $(\mathrm{cof}(\eta_\gamma))^W \geq \kappa_\beta$. Recall that $\kappa < \eta_\gamma < (\kappa^+)^W$. Hence, $(\mathrm{cof}(\eta_\gamma))^W \leq \kappa$. It is impossible to have $(\mathrm{cof}(\eta_\gamma))^W = \kappa$, since
$\mathrm{cof}(\kappa) = \delta > |N_\gamma| \geq \mathrm{cof}(\eta_\gamma) = \mathrm{cof}((\mathrm{cof}(\eta_\gamma))^W)$. So, $(\mathrm{cof}(\eta_\gamma))^W < \kappa$. But $(\mathrm{cof}(\eta_\gamma))^W \in N_\beta$ and $\sup(N_\beta \cap \kappa) = \kappa_\beta$.
□

Lemma 2.12 *Suppose that for every $\beta < \delta$, κ_β^+ is is successor cardinal in W and ν_β is its immediate predecessor, then, for a club many $\beta < \delta$ of uncountable cofinality*
$(\mathrm{cof}(\eta_\beta))^W \geq \nu_\beta$.

Proof. Otherwise there will be stationary many β's of uncountable cofinality
with $(\mathrm{cof}(\eta_\beta))^W < \nu_\beta$. Then (2) holds on this stationary set.
□

Lemma 2.13 *Suppose that for every $\beta < \delta$, κ_β^+ is a limit cardinal of W, then, for a club many $\beta < \delta$ of uncountable cofinality*
$(\mathrm{cof}(\eta_\beta))^W > \kappa_\beta^+$.

Proof. Otherwise there will be stationary many β's of uncountable cofinality with
$(\mathrm{cof}(\eta_\beta))^W < \kappa_\beta^+$. Then (2) holds on this stationary set.
□

Theorem 2.14 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is an inaccessible in W,*
2. *$\kappa > (\mathrm{cof}(\kappa))^V = \delta$ for some uncountable (in V) cardinal $\delta > \omega_1$. Let $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ be a witnessing club.*
3. *For every $\alpha < \delta, (\kappa_\alpha^{++})^W < \kappa_\alpha^+$ (or only for stationary many α's),*
4. *κ is a strong limit in V or just it is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \delta$.*

Assume that $(\kappa^{++})^W \geq \kappa^+$.
Then there is an increasing and unbounded sequence $\langle \rho_\alpha \mid \alpha < \delta \rangle$ in κ such that

- *Each ρ_α is a regular cardinal in W,*

- *for every limit* α, $\mathrm{cof}(\rho_\alpha) = \mathrm{cof}(\alpha)$,
- *for every limit* α *of uncountable cofinality,* $\rho_\alpha \geq |\rho_\alpha| > \kappa_\alpha \geq \sup(\{\rho_\beta \mid \beta < \alpha\})$,
- *for every limit* α *of uncountable cofinality, there is a club* c_α *in* ρ_α *such that for every* $\tau \in c_\alpha$ *we have* $(\mathrm{cof}(\tau))^W \in \{\rho_\beta \mid \beta < \alpha\}$.

Proof. Let $\rho_\alpha = (\mathrm{cof}(\eta_\alpha))^W$.
Suppose that α has an uncountable cofinality. Then, by 2.13, $\rho_\alpha \geq |\rho_\alpha| \geq \kappa_\alpha^+$, and by 2.11, $\{\rho_\beta \mid \beta < \alpha\} \subseteq \kappa_\alpha$.
Fix some increasing continuous function $\varphi_\alpha : \rho_\alpha \to \eta_\alpha$ in W with $\mathrm{ran}(\varphi_\alpha)$ unbounded in η_α. Set
$c_\alpha := \{\varphi_\alpha^{-1}(\eta_\beta) \mid \beta < \alpha \text{ limit and } \eta_\beta \text{ is a limit point of } \mathrm{ran}(\varphi_\alpha)\}$.
Let $\tau \in c_\alpha$. Then $\tau = \varphi_\alpha^{-1}(\eta_\beta)$ for a limit $\beta < \alpha$ and η_β is a limit point of $\mathrm{ran}(\varphi_\alpha)$. Now the continuity of φ_α implies that $(\mathrm{cof}(\tau))^W = (\mathrm{cof}(\eta_\beta))^W$ which is ρ_β.
□

3 A forcing construction.

Let κ be a singular cardinal of cofinality ω_1 and let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be a closed cofinal in κ sequence. Our aim here will be to show that it is possible to collapse successors of κ_α's on a stationary set of $\alpha < \omega_1$ and to preserve successors of κ_α's on its complement. By the results of the previous section, then, necessary, the successor of the supremum will be collapsed.
As it was mentioned in the introduction, the parallel question for blowing up powers of κ_α's remains open. The main difference is that in the present context we will start with a GCH model, and after the collapses GCH will be preserved. In the context of powers of singular cardinals, as a result the Singular Cardinal Hypotheses must break down. Consequently, in this type of situation, Shelah's deep Cardinal Arithmetic analyses apply. We refer [2] for some negative results.

Theorem 3.1 *Assume GCH. Suppose that* κ *is a* $\kappa^{+3}-$*supercompact cardinal. Let* S *be a stationary co-stationary subset of* ω_1. *Then there are generic extension* $V^* \subseteq V^{**}$ *such that*

1. $\mathcal{P}(\omega_1)^V = \mathcal{P}(\omega_1)^{V^*} = \mathcal{P}(\omega_1)^{V^{**}}$,
2. κ *changes its cofinality to* ω_1 *in* V^*, *and so in* V^{**},

3. *In V^* there is a closed and unbounded sequence $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ of cardinals in κ such that*
$$S = \{\alpha < \omega_1 \mid (\kappa_\alpha^+)^V < (\kappa_\alpha^+)^{V^*} = (\kappa_\alpha^+)^{V^{**}}\}$$
and
$$\omega_1 \setminus S = \{\alpha < \omega_1 \mid (\kappa_\alpha^+)^V = (\kappa_\alpha^+)^{V^*} < (\kappa_\alpha^+)^{V^{**}}\}.^3$$

Proof. Assume that in the ground model we have GCH, κ is a $\kappa^{+3}-$supercompact cardinal and S is a subset of ω_1.

Fix a coherent sequence
$$\vec{W} = \langle W(\alpha, \beta) \mid \alpha \in \mathrm{dom}(\vec{W}), \beta < o^{\vec{W}}(\alpha) \rangle$$
such that

1. $\kappa = \max(\mathrm{dom}(\vec{W}),$
2. $o^{\vec{W}}(\kappa) = \omega_1,$
3. for every $\alpha \in \mathrm{dom}(\vec{W}), \beta < o^{\vec{W}}(\alpha)$, $W(\alpha, \beta)$ is a normal ultrafilter over $\mathcal{P}_\alpha(\alpha^{++})$,
4. $\vec{W} \restriction (\alpha, \beta) = j_{W(\alpha,\beta)}(f)(\alpha)$, for some $f : \alpha \to V$.

Consider the Levy collapse $Col(\kappa, \kappa^+)$. Let $p \in Col(\kappa, \kappa^+)$. Set
$$\mathcal{F}_p = \{D \subseteq Col(\kappa, \kappa^+) \mid D \text{ is a dense open above } p\}.$$
Then $\mathcal{F}_p$ is a $\kappa-$complete filter over a set of cardinality κ^+, for every $p \in Col(\kappa, \kappa^+)$. It is also fine in a sense that for every $\eta < \kappa^+$,
$$\{q \in Col(\kappa, \kappa^+) \mid \eta \in \mathrm{ran}(q)\} \in \mathcal{F}_p.$$

Let $j : V \to M$ be an elementary embedding with critical point κ and ${}^{\kappa^{++}}M \subseteq M$. For every $p \in Col(\kappa, \kappa^+)$, pick $\tilde{p} \in \bigcap j''\mathcal{F}_p$.[4] So, $\tilde{p} \in (Col(j(\kappa), j(\kappa^+)))^M$. Set
$$\tilde{F}_p = \{X \subseteq Col(\kappa, \kappa^+) \mid \tilde{p} \in j(X)\}.$$

[3]Note that by the Weak Covering Lemma W. Mitchell, E. Schimmerling, J. Steel [5], R. Jensen and J. Steel [3] there must be an inner model with a Woodin cardinal, since successors of singular cardinals are collapsed here.

[4]In some fixed in advance well ordering.

Then $\tilde{F}_p$ is a $\kappa-$complete ultrafilter which extends $\mathcal{F}_p$.
Note that $\mathcal{F}_p$ is a filter on $\mathcal{P}_\kappa(\kappa \times \kappa^+)$, hence $\tilde{F}_p$ is an ultrafilter there.
Now find, in M, some (least) $\eta < j(\kappa^+)$ which codes $\langle \tilde{p} \mid p \in Col(\kappa, \kappa^+)\rangle$.
Define a $\kappa-$complete ultrafilter $\tilde{W}$ over $\mathcal{P}_\kappa(\kappa^+) \times \kappa^+$ as follows:

$$X \in \tilde{W} \text{ iff } \langle j''\kappa^+, \eta\rangle \in j(X).$$

For every $p \in Col(\kappa, \kappa^+)$, fix a projection $\pi_p : \mathcal{P}_\kappa(\kappa^+) \times \kappa^+ \to Col(\kappa, \kappa^+)$ of $\tilde{W}$ onto $\tilde{F}_p$.

Now use the coherent sequence $\vec{W}$ to define in the obvious fashion (just using the elementary embedding $j_{W(\alpha,\beta)}$ of $W(\alpha, \beta)$ instead of j above) a new coherent sequence $\vec{\tilde{W}}$ where each $\tilde{W}(\alpha, \beta)$ is an $\alpha-$complete ultrafilter over $\mathcal{P}_\alpha(\alpha^+) \times \alpha^+$ defined from $W(\alpha, \beta)$ as above. Note that $\tilde{W} \restriction (\alpha, \beta)$ will belong already to the ultrapower by $\tilde{W}(\alpha, \beta) \restriction P_\alpha(\alpha^+) = W(\alpha, \beta) \restriction P_\alpha(\alpha^+)$. Thus, $\tilde{W} \restriction (\alpha, \beta)$ belongs to the ultrapower by $W(\alpha, \beta)$, by coherency. By the condition (4) above it will be in the ultrapower by $W(\alpha, \beta) \restriction P_\alpha(\alpha^+)$, since this ultrapower is closed under κ^+-sequences.

Force the supercompact Magidor forcing (see M. Magidor [6] or [1]) with $\vec{\tilde{W}}$.[5]

Denote by V^{**} a resulting generic extension.
Let $\langle\langle P_\nu, \eta_\nu\rangle \mid \nu < \omega_1\rangle$ be the generic sequence. Then $\langle P_\nu \mid \nu < \omega_1\rangle$ be the supercompact Magidor sequence. Denote $P_\nu \cap \kappa$ by κ_ν. If $\nu' < \nu < \omega_1$, then $\langle P_{\nu'}, \eta_{\nu'}\rangle \sqsubset \langle P_\nu, \eta_\nu\rangle$. In particular, $\eta_{\nu'} \in P_\nu$. Also, $\eta_{\nu'}$ codes elements of $Col(\kappa_\nu, P_\nu)$.[6]

For every $\nu \in S$ fix a cofinal sequence $\langle \nu_n \mid n < \omega\rangle$.
Let $\nu \in S$. Consider $\langle \eta_{\nu_n} \mid n < \omega\rangle$. Denote by $\langle t^i_{\nu,n} \mid i < \kappa^+_{\nu_n}\rangle$ the sequence of members of $Col(\kappa_{\nu_{n+1}}, P_{\nu_{n+1}})$ codded by η_{ν_n}.
Let $tr_\nu : P_\nu \longleftrightarrow \kappa^+_\nu$ be the transitive collapse of P_ν.
Consider a set

$$Z_\nu := \{tr_\nu'' t^i_{\nu,n} \mid n < \omega, i < \kappa^+_{\nu_n}\}.$$

It is a subset of $Col(\kappa_\nu, \kappa^+_\nu)$. Define a partial order $\leq_\nu$ on Z_ν as follows:

$$tr_\nu'' t^i_{\nu,n} \leq_\nu tr_\nu'' t^j_{\nu,m} \text{ iff } n \leq m \text{ and } tr_\nu'' t^i_{\nu,n} \leq_{Col(\kappa_\nu,\kappa^+_\nu)} tr_\nu'' t^j_{\nu,m}.$$

Set G_ν to be the set of all unions of all $<_\nu -$increasing $\omega-$sequences of elements of Z_ν.

Lemma 3.2 *There is $g \in G_\nu$ which is generic for $Col(\kappa_\nu, \kappa^+_\nu)$ over V.*

[5]Set here $\langle Q, \xi\rangle \sqsubset \langle P, \eta\rangle$ iff $Q \cup \{\xi\} \subseteq P$ and $|Q| < P \cap \kappa$.
[6]Note that $\eta_{\nu'}$ need not code only members of $Col(\kappa_{\nu'}, P_{\nu'})$, or even of $Col(\kappa_{\nu'}, P_\nu)$.

Proof. Work in V^{**}. Define a function g as follows. Start with $tr_\nu{}''t^0_{\nu,0}$. Pick $i_1 < \kappa^+_{\nu_1}$ such that $t^{i_1}_{\nu,1}$ comes from the ultrafilter $\tilde{F}_{t^0_{\nu,0}}$ over $Col(\kappa,\kappa^+)$.
Continue by induction. Suppose that t^n_{ν,i_n} is defined. Pick $i_{n+1} < \kappa^+_{\nu_n}$ such that $t^{i_{n+1}}_{\nu,n+1}$ comes from the ultrafilter $\tilde{F}_{t^{i_n}_{\nu,n}}$ over $Col(\kappa,\kappa^+)$.
Finally set

$$g = \bigcup_{n<\omega} tr_\nu{}''t^{i_n}_{\nu,n}.$$

We claim that g is as desired.
Work in V above a condition which already decides κ_ν. Suppose for simplicity that none of $\kappa_{\nu_n}, n < \omega$ is decided yet. Let D be a dense open subset of $Col(\kappa_\nu, \kappa^+_\nu)$. Intersect the measure one set of $\tilde{F}_{\emptyset^0}$ with D. The resulting condition will force

$$\underset{\sim}{g} \text{ extends a member of } \check{D}.$$

□

The next lemma follows from the definition of G_ν.

Lemma 3.3 *For every $n_0 < \omega$, $G_\nu \in V[\langle tr_\nu{}''P_{\nu_n} \mid n_0 < n < \omega\rangle]$.*

Set $V^* = V[\langle G_\nu \mid \nu \in S\rangle]$.

Let now $\rho \in \omega_1 \setminus S$. We need to argue that $(\kappa^+_\rho)^V = (\kappa^+_\rho)^{V^*}$. By Lemma 3.3, it follows that

$$V[\langle G_\nu \mid \nu \in S \setminus \rho\rangle] \subseteq V[\langle\langle P_\tau, \eta_\tau\rangle \mid \rho < \tau < \omega_1\rangle],$$ [7]

i.e. the extension of V by the same forcing but which only starts above κ_ρ. Such extension does not add new bounded subsets to κ^+_ρ and below. Hence, it is enough to deal with the forcing up to κ_ρ.

Let us split the argument into two cases.

Case 1. ρ is a limit point of $\rho \in \omega_1 \setminus S$.

Let then $\langle \rho_k \mid k < \omega\rangle$ be a cofinal sequence consisting of elements of $\omega_1 \setminus S$. Assume for simplicity that $\rho_0 = 0$.

[7] Here is the point of using $\mathcal{F}_p$'s and then, $Col(\kappa_\nu, \kappa^+_\nu)$. This allows the splitting below ρ and above it. Note also that if we try to replace collapses by the Cohen forcing in order to obtain GCH on S and its negation on $\omega_1 \setminus S$, then the corresponding forcing with $\mathcal{F}_p$, for $\rho \in S$ will collapse κ^+_ρ, and so, GCH will be returned back.

For every $\nu \in S \cap \rho$ find the least $k(\nu)$ such that $\nu < \rho_{k(\nu)}$. Let n_ν be the least $n < \omega$ such that $\nu_n > \rho_{k(\nu)-1}$, if $k(\nu) \geq 1$ and 0 otherwise.[8]

Consider

$$V^\rho := V[\langle \kappa_\tau \mid \tau < \rho \rangle, \langle \langle \langle tr_\nu{}'' P_{\nu_n}, tr_\nu{}'' \eta_{\nu_n} \rangle \mid n_\nu \leq n < \omega \rangle \mid \nu \in S \cap \rho \rangle].$$

Then

$$V[\langle G_\nu \mid \nu \in S \cap \rho \rangle] \subseteq V^\rho.$$

Lemma 3.4 *V^ρ is a generic extension of V by a Prikry type forcing which satisfies $\kappa_\rho^+ -c.c.$*

Case 2. ρ is not a limit point of $\rho \in \omega_1 \setminus S$.

The treatment of this case is similar and even a bit simpler than the previous one.

□

References

[1] M. Gitik, Prikry type forcings, Handbook of Set Theory, Matthew Foreman and Akihiro Kanamori (Editors), Springer Verlag, 2010

[2] M. Gitik, R. Schindler and S. Shelah, PCF theory and Woodin cardinals. In Logic Colloquium '02, Vol. 27, Assoc. Symbol. Logic, La Jolla, CA, 2006, pp. 172–205.

[3] R. Jensen and J. Steel, K without the measurable, The Journal of Symbolic Logic, vol 78, Issue 3, 2013, pp. 708–734.

[4] K. Kunen, Set Theory, North-Holland, 1980.

[5] W. Mitchell, E. Schimmerling and J. Steel, The covering lemma up to a Woodin cardinal, Annals of Pure and Applied Logic Volume 84, Issue 2, 21 March 1997, Pages 219-255.

[6] M. Magidor, Singular cardinal problem I, Israel Journal of Mathematics volume 28, 1977, pp. 1–31.

[7] S. Shelah. Cardinal arithmetic, in 'Oxford Logic Guides' 29, The Clarendon Press, *Oxford University Press.*, New York, 1994.

[8] Note that countability of ρ allows to define such disjoint intervals. This may break down for uncountable ρ's, i.e., if we try to replace here ω_1 by ω_2.