Silver type theorems for collapses.

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The classical theorem of Silver states that GCH cannot break for the first time over a singular cardinal of uncountable cofinality. On the other hand it is easy to obtain a situation where GCH breaks on a club below a singular cardinal κ of an uncountable cofinality but $2^\kappa = \kappa^+$.

We would like here to investigate the situation once blowing up power of singular cardinals is replaced by collapses of their successors.

1 ZFC results.

The following basic result should be well known and goes back to Silver:

Theorem 1.1 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is a cardinal in W,*
2. *κ changes its cofinality to ω_1 in V witnessed by a club $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$,*
3. *for every $\alpha < \omega_1, (\kappa_\alpha^+)^W < \kappa_\alpha^+$ (or only for stationary many α's),*
4. *κ is a strong limit in V or just it is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \omega_1$.*

Then $(\kappa^+)^W < \kappa^+$.

Proof. Suppose that $(\kappa^+)^W = \kappa^+$.

Fix in W a sequence $\langle f_i \mid i < \kappa^+ \rangle$ of κ^+ first canonical functions in $\langle \prod_{\nu<\kappa} \nu^+, <_{J^{bd}_\kappa} \rangle$ or just any sequence of κ^+–many functions in $\prod_{\nu<\kappa} \nu^+$ increasing mod J^{bd}_κ.

Set in V

$g_i = f_i \restriction \{\kappa_\alpha \mid \alpha < \omega_1\}$, for every $i < \kappa^+$. Then $\langle g_i \mid i < \kappa^+ \rangle$ is an increasing sequence of functions in $\langle \prod_{\alpha<\omega_1} (\kappa_\alpha^+)^W, <_{J^{bd}_{\omega_1}} \rangle$. By the assumption (3) we have that for every $\alpha < \omega_1, (\kappa_\alpha^+)^W < \kappa_\alpha^+$. Now, as in the Baumgartner-Prikry proof of the Silver Theorem (see K. Kunen [2] p.296 (H5)), it is impossible to have κ^+–many such functions. Hence $(\kappa^+)^W < \kappa^+$. □

Let us deal now with double successors.

Theorem 1.2 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is a cardinal in W,*
2. *$2^\kappa \geq \kappa^{++}$, and moreover there is a sequence of κ^{++}–many functions in $\prod_{\nu<\kappa} \nu^{++}$ increasing mod J^{bd}_κ,*
3. *κ changes its cofinality to ω_1 in V witnessed by a club $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$,*
4. *for every $\alpha < \omega_1, (\kappa_\alpha^{++})^W < \kappa_\alpha^+$ (or only for stationary many α's),*
5. *κ is a strong limit in V or just it is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \omega_1$.*

Then $(\kappa^{++})^W < \kappa^+$.

The condition (2) allows to repeat the proof of 1.1.

Let state the following relevant result of Shelah ([3](4.9,p.304)), which says that once $(\kappa^+)^W$ changes its cofinality, then we must have $(\kappa^{++})^W < \kappa^+$ unless $\mathrm{cof}((\kappa^+)^W) = \mathrm{cof}(|(\kappa^+)^W|) = \mathrm{cof}(\kappa)$.

Proposition 1.3 *Let $\mathcal{F}$ be the κ–complete filter of co-bounded subsets of $\mathcal{P}_\kappa(\kappa^+)$, i.e. the filter generated by the sets $\{P \in \mathcal{P}_\kappa(\kappa^+) \mid \alpha \in P\}$, $\alpha < \kappa^+$.*
Then there is a sequence $\langle f_i \mid i < \kappa^{++} \rangle$ of functions such that

1. *$f_i : \mathcal{P}_\kappa(\kappa^+) \to \kappa$,*
2. *$f_i(P) < |P|^+$, for all $P \in \mathcal{P}_\kappa(\kappa^+)$,*
3. *$f_i >_{\mathcal{F}} f_j$, whenever $i > j$.*

Proof. We define a sequence $\langle f_i \mid i < \kappa^{++} \rangle$ by induction.
Suppose that $\langle f_j \mid j < i \rangle$ is defined. Define f_i.

Case 1. $i = i' + 1$.
Set $f_i(P) = f_{i'}(P) + 1$.

Case 2. i is a limit ordinal of cofinality $\delta < \kappa$.
Pick a cofinal in i sequence $\langle i_\tau \mid \tau < \delta \rangle$. Set $f_i(P) = \bigcup_{\tau<\delta} f_{i_\tau}(P) + 1$.

Case 3. i is a limit ordinal of cofinality $\delta \geq \kappa$, i.e. $\delta = \kappa$ or $\delta = \kappa^+$.
Pick a cofinal in i sequence $\langle i_\tau \mid \tau < \delta \rangle$. Set $f_i(P) = \bigcup_{\tau \in P} f_{i_\tau}(P) + 1$.
□

Theorem 1.4 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is an inaccessible in W,*
2. *$\kappa > (\mathrm{cof}(\kappa))^V = \delta$ for some uncountable (in V) cardinal δ.*
3. *κ is a strong limit in V or just it is a limit cardinal and for every $\xi < \kappa$, $\xi^\delta < \kappa$.*
4. *There exist a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ (or just a stationary set)* [1] *and a sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ such that*
   (a) *$P_\alpha \in (\mathcal{P}_\kappa(\kappa^+))^W$, for each $\alpha < \delta$,*
   (b) *$(|P_\alpha|^+)^W < \kappa_\alpha^+$, for each $\alpha < \delta$,*
   (c) *$(\kappa^+)^W = \bigcup_{\alpha<\delta} P_\alpha$,*
   (d) *for every $Q \in (\mathcal{P}_\kappa(\kappa^+))^W$, there is $\alpha < \delta$ such that for every $\beta, \alpha \leq \beta < \delta$, $Q \subseteq P_\beta$.*

Then $(\kappa^{++})^W < \kappa^+$.

Proof. Suppose otherwise. Then $(\kappa^{++})^W = \kappa^+$, by the assumption (b),(c) above.
Let $\langle f_i \mid i < \kappa^{++} \rangle$ be a sequence of functions in W given by Proposition 1.3.
We can repeat the argument of 1.1 with slight adaptations. Thus, set in V
$g_i(\alpha) = f_i(P_\alpha)$, for every $\alpha < \omega_1$ and $i < (\kappa^{++})^W = \kappa^+$. Let $\nu_\alpha := (|P_\alpha|^+)^W$. By the assumption, $\nu_\alpha < \kappa_\alpha^+$. Then $\langle g_i \mid i < \kappa^+ \rangle$ is an increasing sequence of functions in $\langle \prod_{\alpha<\delta} \nu_\alpha, <_{J_\delta^{bd}} \rangle$, since for every $A \in \mathcal{F}$ we have $\{P_\alpha \mid \alpha \geq \alpha_0\} \subseteq A$, for some $\alpha_0 < \delta$. This is impossible, since $\nu_\alpha < \kappa_\alpha^+$, for every $\alpha < \delta$. Contradiction.
□

[1] Note that if $\delta = \omega_1$, then we can just force a club into it without effecting things above.

Theorem 1.5 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that for some inaccessible in W cardinal κ both κ and its successor in W change their cofinality to some uncountable (in V) cardinal δ and κ remains a cardinal in V. Then the following conditions are equivalent:*

1. *(*) There are a clubs $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ in $(\kappa^+)^W$ such that for every limit $\alpha < \delta$ (or just for stationary many α's)[2] the set $\{\eta_\beta \mid \beta < \alpha\}$ can be covered by a set $a_\alpha \in W$ with $(|a_\alpha|^+)^W < \kappa_\alpha^+$.*

2. *(**) There are a clubs $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ in $(\kappa^+)^W$ such that for every limit $\alpha < \delta$ (or just for stationary many α's) the set $\{\eta_\beta \mid \beta < \alpha\}$ has an unbounded intersection with a set $b_\alpha \in W$ with $(|b_\alpha|^+)^W < \kappa_\alpha^+$.*

3. *There exist a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ and a sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ such that*

   (a) $P_\alpha \in (\mathcal{P}_\kappa(\kappa^+))^W$, for each $\alpha < \delta$,

   (b) $P_\alpha \cap \kappa = \kappa_\alpha$, for each $\alpha < \delta$,

   (c) $(|P_\alpha|^+)^W < \kappa_\alpha^+$, for each $\alpha < \delta$,

   (d) $(\kappa^+)^W = \bigcup_{\alpha<\omega_1} P_\alpha$,

   (e) for every $Q \in (\mathcal{P}_\kappa(\kappa^+))^W$, there is $\alpha < \omega_1$ such that for every $\beta, \alpha \leq \beta < \omega_1$, $Q \subseteq P_\beta$.

4. *There exists an increasing sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ which satisfies all the requirements of the previous item.*

Proof. Split the proof into lemmas.

Lemma 1.6 *(*) iff (**).*

Proof. Clearly, (*) implies (**). Let us show the opposite direction.
We fix a bijection $\pi_\xi : \kappa \longleftrightarrow \xi$ in W, for every $\xi < (\kappa^+)^W$.
Fix in V a function $\pi : \kappa \longrightarrow^{onto} (\kappa^+)^W$. Set now for every $\alpha < \delta$, $\eta_\alpha = \sup(\pi``\kappa_\alpha)$. Then, clearly, $\{\eta_\alpha \mid \alpha < \delta\}$ is a club in $(\kappa^+)^W$. Now given a sequence which witnesses (**). Without loss of generality we can assume that it is the sequence $\langle \eta_\alpha \mid \eta < \delta \rangle$ defined above. Otherwise

[2] If $\delta = \omega_1$, then it is basically the same, since once we have only stationary many such α's, then force a club into it. Everything is a the level of ω_1, so this will have no effect on the cardinal arithmetic above.

just intersect two clubs.
Define an increasing continuous sequence $\langle N_\alpha \mid \alpha < \delta\rangle$ of elementary submodels of some H_χ, with χ big enough such that

1. $\delta, \kappa, \langle \kappa_\alpha \mid \alpha < \delta\rangle, \langle \pi_\xi \mid \xi < (\kappa^+)^W\rangle, \pi \in N_0$,

2. $|N_\alpha| < \delta$,

3. $N_\alpha \cap \delta$ is an ordinal,

4. $\langle N_\beta \mid \beta \leq \alpha\rangle \in N_{\alpha+1}$.

Denote $N_\alpha \cap \delta$ by δ_α.
Then $\sup(N_\alpha \cap \kappa) = \kappa_{\delta_\alpha}$ and $\sup(N_\alpha \cap (\kappa^+)^W) = \eta_{\delta_\alpha}$. Clearly, $\delta_\alpha = \alpha$ for a club many α's.
Suppose now that for some $\alpha < \delta$ we have $\delta_\alpha = \alpha$ and there is a set $X \in W$ such that

- $(|X|^+)^W < \kappa_\alpha^+$,

- $X \cap \{\eta_\beta \mid \beta < \alpha\}$ is unbounded in η_α.

Note that $\eta_\beta \in N_\alpha$, for every $\beta < \alpha$ and then, also, $\pi_{\eta_\beta} \in N_\alpha$. By elementarity, then $\pi_{\eta_\beta} \restriction (N_\alpha \cap \kappa_\alpha) : N_\alpha \cap \kappa_\alpha \longleftrightarrow N_\alpha \cap \eta_\beta$. In particular, $\pi_{\eta_\beta}``\kappa_\alpha \supseteq \{\eta_\gamma \mid \gamma < \beta\}$.
Set

$$Y := \{\pi_\zeta``\kappa_\alpha \mid \zeta \in X \cap \eta_\alpha\}.$$

Then, $Y \in W, |Y|^W \leq \kappa_\alpha + |X|^W$, and so $(|Y|^+)^W < \kappa_\alpha^+$. But, in addition, $Y \supseteq \{\eta_\gamma \mid \gamma < \alpha\}$, since for unboundedly many $\beta < \alpha$, we have $\eta_\beta \in X$ and so, $\pi_{\eta_\beta}``\kappa_\alpha \supseteq \{\eta_\gamma \mid \gamma < \beta\}$.
$\square$ of the lemma.

Lemma 1.7 *(1) implies (3)*

Proof.
Fix clubs $\langle \kappa_\alpha \mid \alpha < \delta\rangle$ and $\langle \eta_\alpha \mid \alpha < \delta\rangle$ witnessing (1).
Let us build first a sequence $\langle R_\alpha \mid \alpha < \delta\rangle$ which satisfies all the requirements of (3), but probably is not increasing.
Set $R_0 = \kappa_0 \cup ((\pi_{\eta_0}``\kappa_0) \setminus \kappa)$.
Let $\alpha, 0 < \alpha < \delta$ be an ordinal. Pick $a_\alpha \in W, a_\alpha \subseteq \eta_\alpha$ to be a cover of $\{\eta_\beta \mid \beta < \alpha\}$ with $(|a_\alpha|^+)^W < \kappa_\alpha^+$. Set $R'_\alpha = \bigcup\{\pi_\xi``\kappa_\alpha \mid \xi \in b_\alpha \cup \{\eta_\alpha\}\}$. Let $R_\alpha = \kappa_\alpha \cup (R'_\alpha \setminus \kappa)$.

The constructed sequence satisfies trivially the requirements (a),(b) and (c). Let us check (e). (d) clearly follows from (e).
Let $Q \in (\mathcal{P}_\kappa(\kappa^+))^W$. There is $\beta < \omega_1$ such that $Q \subseteq \eta_\beta$. Consider $\pi^{-1}_{\eta_\beta}$"Q. It is a bounded subset of κ. Hence there is $\gamma < \omega_1$ such that $\kappa_\gamma \supseteq \pi^{-1}_{\eta_\beta}$"$Q$. So π_{η_β}"$\kappa_\beta \supseteq Q$. Let $\alpha < \omega_1$ be an ordinal above β, γ. Then $R_\delta \supseteq Q$, for every $\delta \geq \alpha$.
$\square$ of the lemma.

Lemma 1.8 *(3) iff (4).*

Proof. Clearly (4) implies(3). Let us show the opposite direction.
Let a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ and a sequence $\langle R_\alpha \mid \alpha < \delta \rangle$ witness (3).
Define an increasing subsequence $\langle P'_\alpha \mid \alpha < \delta \rangle$
Set $P'_0 = R_0$. By (e) there is α_0 such that for every $\beta, \alpha_0 \leq \beta < \delta$, $P'_0 \subseteq R_\beta$. Set $P'_1 = R_{\alpha_1}$. Continue by induction. Suppose that $\nu < \delta$ and for every $\nu' < \nu$, increasing sequences $\langle \alpha_{\nu'} \mid \nu' < \nu \rangle$ and $\langle P'_{\nu'} \mid \nu' < \nu \rangle$ are defined and satisfy the following:

1. $P'_{\nu'} = R_{\alpha_{\nu'}}$,
2. for every $\beta, \alpha_{\nu'} \leq \beta < \delta$, $P'_{\nu'} \subseteq R_\beta$.

If ν is a successor ordinal, then let $\nu = \mu + 1$, for some μ. Set $P'_\nu = R_{\alpha_\mu}$ and let $\alpha_\nu < \delta$ be such that for every $\beta, \alpha_\nu \leq \beta < \delta$, $P'_\nu \subseteq R_\beta$.
If ν is a limit ordinal, then let $P'_\nu = R_{\bigcup_{\nu'<\nu} \alpha_{\nu'}}$ and define α_ν as in the successor case.

Finally let us define an increasing subsequence of $\langle P'_\alpha \mid \alpha < \delta \rangle$ which satisfies the properties (a)-(e) of (3).
Let $C := \{\nu < \delta \mid \nu = \bigcup_{\nu'<\nu} \alpha_{\nu'}\}$. Clearly it is a club. Set $P_\nu = P'_\nu$, for every $\nu \in C$.
Then $\langle \kappa_\alpha \mid \alpha \in C \rangle$ and $\langle P_\alpha \mid \alpha \in C \rangle$ are as desired.
$\square$ of the lemma.

Lemma 1.9 *(3) implies (1).*

Proof. Let a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in κ and a sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ witness (3). Let $\eta_\alpha \mid \alpha < \delta \rangle$ be a club in $(\kappa^+)^W$.
We claim that there is a club $C \subseteq \delta$ such that for every $\alpha \in C$, $P_\alpha \supseteq \{\eta_\beta \mid \beta < \alpha\}$.

Suppose otherwise. Then there is a stationary $S \subseteq \delta$ such that for every $\alpha \in S$ there is $\alpha' < \alpha$ with $\eta_{\alpha'} \notin P_\alpha$. Then there are a stationary set $S^* \subseteq S$ and $\alpha^* < \delta$ such that for every $\alpha \in S^*$, $\eta_{\alpha^*} \notin P_\alpha$. This is impossible by (d).

$\square$ of the lemma.

$\square$

Theorem 1.10 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is an inaccessible in W,*
2. *$\kappa > (\mathrm{cof}(\kappa))^V = \delta$ for some uncountable (in V) cardinal $\delta > \omega_1$. Let $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ be a witnessing club.*
3. *For every $\alpha < \delta, (\kappa_\alpha^{++})^W < \kappa_\alpha^+$ (or only for stationary many α's),*
4. *κ is a strong limit in V or just it is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \delta$.*
5. *There is a regular cardinal $\delta^*, \omega < \delta^* < \delta$ such that for every regular cardinal $\rho < \kappa$ of W which became a singular of cofinality δ^* in V, there is a club a club sequence $\langle \rho_i \mid i < \delta^* \rangle$ in ρ such that for every club $c \subseteq \delta^*$ the set $\{(\mathrm{cof}(\rho_i))^W \mid i \in c\}$ is unbounded in $|\rho|$.*

   Or
6. *Like the previous item but only for ρ's of the form $(\mathrm{cof}(\eta_\alpha))^W$ with $\alpha < \delta$ of cofinality δ^*, where $\langle \eta_\alpha \mid \alpha < \delta \rangle$ is a club in $(\kappa^+)^W$.*

Then $(\kappa^{++})^W < \kappa^+$.

Proof.

Let us argue that (**) of 1.4 holds.
Assume for simplicity that $\delta^* = \omega_1$.
Let $\langle N_\alpha \mid \alpha < \delta \rangle$ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ be as in 1.6. Pick $\alpha < \delta$ of cofinality ω_1 with $\delta_\alpha = \alpha$. Consider η_α. Then $\mathrm{cof}(\eta_\alpha) = \omega_1$. If $(\mathrm{cof}(\eta_\alpha))^W < \kappa_\alpha^+$, then we pick in W a club X in η_α of the order type $(\mathrm{cof}(\eta_\alpha))^W$. Then $X \cap \{\eta_\beta \mid \beta < \alpha\}$ is a club, and so, unbounded in η_α.

Suppose now that $(\mathrm{cof}(\eta_\alpha))^W \geq \kappa_\alpha^+$. Denote $(\mathrm{cof}(\eta_\alpha))^W$ by ρ. Then $\rho \leq \kappa$, since $\eta_\alpha < (\kappa^+)^W$. It is impossible to have $\rho = \kappa$, since $\mathrm{cof}(\kappa) > \omega_1 = \mathrm{cof}(\alpha) = \mathrm{cof}(\eta_\alpha) = \mathrm{cof}(\rho)$. Hence $\kappa_\alpha^+ \leq \rho < \kappa$. In particular, $|\rho| \geq \kappa_\alpha^+$.
By the assumption (5) of the theorem, there is a club a club sequence $\langle \rho_i \mid i < \omega_1 \rangle$ such that for every club $c \subseteq \omega_1$ the set $\{(\mathrm{cof}(\rho_i))^W \mid i \in c\}$ is unbounded in $|\rho|$. Let $e = \{e_\xi \mid \xi < \rho\} \in W$ be a club in η_α. Consider $d := \{\eta_\beta \mid \beta < \alpha\} \cap e$. It is a club in η_α. So there are some

$\gamma < \alpha$ and $j < \omega_1$ such that $\eta_\gamma = e_{\rho_j}$ and $(\mathrm{cof}(\rho_j))^W > \kappa_\alpha$. But this is impossible, since $\eta_\gamma \in N_\alpha$, and hence, $(\mathrm{cof}(\eta_\gamma))^W = (\mathrm{cof}(\rho_j))^W \in N_\alpha \cap \kappa \subseteq \kappa_\alpha$.

Hence, always $(\mathrm{cof}(\eta_\alpha))^W < \kappa_\alpha^+$.

So, the set $\{\eta_\alpha \mid \alpha < \delta \text{ and } \mathrm{cof}(\alpha) = \omega_1\}$ witnesses (**) and we are done.

□

Lemma 1.11 *For every $\beta < \delta$,*
$\{(\mathrm{cof}(\eta_\gamma))^W \mid \gamma < \beta\} \subseteq \kappa_\beta$.

Proof. Otherwise there is $\gamma < \beta$ such that $(\mathrm{cof}(\eta_\gamma))^W \geq \kappa_\beta$. Recall that $\kappa < \eta_\gamma < (\kappa^+)^W$. Hence, $(\mathrm{cof}(\eta_\gamma))^W \leq \kappa$. It is impossible to have $(\mathrm{cof}(\eta_\gamma))^W = \kappa$, since $\mathrm{cof}(\kappa) = \delta > |N_\gamma| \geq \mathrm{cof}(\eta_\gamma) = \mathrm{cof}((\mathrm{cof}(\eta_\gamma))^W)$. So, $(\mathrm{cof}(\eta_\gamma))^W < \kappa$. But $(\mathrm{cof}(\eta_\gamma))^W \in N_\beta$ and $\sup(N_\beta \cap \kappa) = \kappa_\beta$.

□

Lemma 1.12 *Suppose that for every $\beta < \delta$, κ_β^+ is is successor cardinal in W and ν_β is its immediate predecessor, then, for a club many $\beta < \delta$ of uncountable cofinality*
$(\mathrm{cof}(\eta_\beta))^W \geq \nu_\beta$.

Proof. Otherwise there will be stationary many β's of uncountable cofinality with $(\mathrm{cof}(\eta_\beta))^W < \nu_\beta$. Then (**) holds on this stationary set.

□

Lemma 1.13 *Suppose that for every $\beta < \delta$, κ_β^+ is a limit cardinal of W, then, for a club many $\beta < \delta$ of uncountable cofinality*
$(\mathrm{cof}(\eta_\beta))^W > \kappa_\beta^+$.

Proof. Otherwise there will be stationary many β's of uncountable cofinality with $(\mathrm{cof}(\eta_\beta))^W < \kappa_\beta^+$. Then (**) holds on this stationary set.

□

Theorem 1.14 *Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:*

1. *κ is an inaccessible in W,*
2. *$\kappa > (\mathrm{cof}(\kappa))^V = \delta$ for some uncountable (in V) cardinal $\delta > \omega_1$. Let $\langle \kappa_\alpha \mid \alpha < \delta\rangle$ be a witnessing club.*

3. *For every* $\alpha < \delta, (\kappa_\alpha^{++})^W < \kappa_\alpha^+$ *(or only for stationary many* α*'s),*

4. κ *is a strong limit in* V *or just it is a limit cardinal and* $\kappa_\alpha^{\omega_1} < \kappa$*, for every* $\alpha < \delta$*.*

Assume that $(\kappa^{++})^W \geq \kappa^+$*.*
Then there is an increasing unbounded in κ *sequence* $\langle \rho_\alpha \mid \alpha < \delta \rangle$ *such that*

- ρ_α *is a regular cardinal in* W*,*
- *for every limit* α*,* $\mathrm{cof}(\rho_\alpha) = \mathrm{cof}(\alpha)$*,*
- *for every limit* α *of uncountable cofinality,* $\rho_\alpha \geq |\rho_\alpha| > \kappa_\alpha \geq \sup(\{\rho_\beta \mid \beta < \alpha\})$*,*
- *for every limit* α *of uncountable cofinality, there is a club* c_α *in* ρ_α *such that for every* $\tau \in c_\alpha$ *we have* $(\mathrm{cof}(\tau))^W \in \{\rho_\beta \mid \beta < \alpha\}$*.*

Proof. Just take $\rho_\alpha = (\mathrm{cof}(\eta_\alpha))^W$.
Suppose that α has an uncountable cofinality. Then, by 1.13, $\rho_\alpha \geq |\rho_\alpha| \geq \kappa_\alpha^+$, and by 1.11, $\{\rho_\beta \mid \beta < \alpha\} \subseteq \kappa_\alpha$.
Fix some increasing continuous function $\varphi_\alpha : \rho_\alpha \to \eta_\alpha$ in W with $\mathrm{ran}(\varphi_\alpha)$ unbounded in η_α.
Set
$c_\alpha := \{\varphi_\alpha^{-1}(\eta_\beta) \mid \beta < \alpha$ limit and η_β is a limit point of $\mathrm{ran}(\varphi_\alpha)\}$.
Let $\tau \in c_\alpha$. Then $\tau = \varphi_\alpha^{-1}(\eta_\beta)$ for a limit $\beta < \alpha$ and η_β is a limit point of $\mathrm{ran}(\varphi_\alpha)$. Now the continuity of φ_α implies that $(\mathrm{cof}(\tau))^W = (\mathrm{cof}(\eta_\beta))^W$ which is ρ_β.
□

2 A forcing construction.

We would like to show the following:

Theorem 2.1 *Suppose that* κ *is a* $\kappa^{+3}-$*supercompact cardinal. Let* S *be subset of* ω_1*. Then there are generic extensions* $V^* \subseteq V^{**}$ *such that*

1. κ *changes its cofinality to* ω_1 *in* V^{**}*,*

2. *there is a closed unbounded in* κ *sequence* $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ *of cardinals in* V^{**} *such that*

$$S = \{\alpha < \omega_1 \mid (\kappa_\alpha^+)^{V^*} = (\kappa_\alpha^+)^{V^{**}}\}$$

and

$$\omega_1 \setminus S = \{\alpha < \omega_1 \mid (\kappa_\alpha^+)^{V^*} < (\kappa_\alpha^+)^{V^{**}}\}.$$

Let us describe the construction. Assume GCH, κ is a $\kappa^{+3}-$supercompact cardinal and S is a subset of ω_1.[3]

Fix a coherent sequence

$$\vec{W} = \langle W(\alpha, \beta) \mid \alpha \in \text{dom}(\vec{W}), \beta < o^{\vec{W}}(\alpha)\rangle$$

such that

1. $\kappa = \max(\text{dom}(\vec{W})$,
2. $o^{\vec{W}}(\kappa) = \omega_1$,
3. for every $\alpha \in \text{dom}(\vec{W}), \beta < o^{\vec{W}}(\alpha)$, $W(\alpha, \beta)$ is a normal ultrafilter over $\mathcal{P}_\alpha(\alpha^{++})$,
4. $\vec{W} \restriction (\alpha, \beta) = j_{W(\alpha,\beta)}(f)(\alpha)$, for some $f : \alpha \to V$.

Consider the Levy collapse $Col(\kappa, \kappa^+)$. Let $p \in Col(\kappa, \kappa^+)$. Set

$$\mathcal{F}_p = \{D \subseteq Col(\kappa, \kappa^+) \mid D \text{ is a dense open above } p\}.$$

Then $\mathcal{F}_p$ is a $\kappa-$complete filter over a set of cardinality κ^+, for every $p \in Col(\kappa, \kappa^+)$. It is also fine in a sense that for every $\eta < \kappa^+$,

$$\{q \in Col(\kappa, \kappa^+) \mid \eta \in \text{ran}(q)\} \in \mathcal{F}_p.$$

Let $j : V \to M$ be an elementary embedding with κ a critical point and ${}^{\kappa^{++}}M \subseteq M$. For every $p \in Col(\kappa, \kappa^+)$, pick $\tilde{p} \in \bigcap j''\mathcal{F}_p$.[4] So, $\tilde{p} \in (Col(j(\kappa), j(\kappa^+)))^M$. Set

$$\tilde{F}_p = \{X \subseteq Col(\kappa, \kappa^+) \mid \tilde{p} \in j(X)\}.$$

Then $\tilde{F}_p$ is a $\kappa-$complete ultrafilter which extends $\mathcal{F}_p$.
Note that $\mathcal{F}_p$ is a filter on $\mathcal{P}_\kappa(\kappa \times \kappa^+)$, hence $\tilde{F}_p$ is an ultrafilter there.
Now find, in M, some (least) $\eta < j(\kappa^+)$ which codes $\langle \tilde{p} \mid p \in Col(\kappa, \kappa^+)\rangle$.
Define a $\kappa-$complete ultrafilter $\tilde{W}$ over $\mathcal{P}_\kappa(\kappa^+) \times \kappa^+$ as follows:

$$X \in \tilde{W} \text{ iff } \langle j''\kappa^+, \eta\rangle \in j(X).$$

For every $p \in Col(\kappa, \kappa^+)$, fix a projection $\pi_p : \mathcal{P}_\kappa(\kappa^+) \times \kappa^+ \to Col(\kappa, \kappa^+)$ of $\tilde{W}$ onto $\tilde{F}_p$.

[3]The interesting case is when S and its compliment are both stationary.
[4]In some fixed in advance well ordering.

Now use the coherent sequence $\vec{W}$ to define in the obvious fashion a new coherent sequence $\vec{\tilde{W}}$ where each $\tilde{W}(\alpha,\beta)$ is an $\alpha-$complete ultrafilter over $\mathcal{P}_\alpha(\alpha^+)\times\alpha^+$ defined from $W(\alpha,\beta)$ as above.
Note that $\tilde{W}\upharpoonright(\alpha,\beta)$ will belong already to the ultrapower by $\tilde{W}(\alpha,\beta)\upharpoonright P_\alpha(\alpha^+)=W(\alpha,\beta)\upharpoonright P_\alpha(\alpha^+)$. Thus, $\tilde{W}\upharpoonright(\alpha,\beta)$ belongs to the ultrapower by $W(\alpha,\beta)$, by coherency. By the condition (4) above it will be in the ultrapower by $W(\alpha,\beta)\upharpoonright P_\alpha(\alpha^+)$, since this ultrapower is closed under κ^+-sequences.

Force the supercompact Magidor forcing with $\vec{\tilde{W}}$.[5]

Denote by V^{**} a resulting generic extension.
Let $\langle\langle P_\nu,\eta_\nu\rangle\mid\nu<\omega_1\rangle$ be the generic sequence. Then $\langle P_\nu\mid\nu<\omega_1\rangle$ be the supercompact Magidor sequence. Denote $P_\nu\cap\kappa$ by κ_ν. If $\nu'<\nu<\omega_1$, then $\langle P_{\nu'},\eta_{\nu'}\rangle\sqsubset\langle P_\nu,\eta_\nu\rangle$. In particular, $\eta_{\nu'}\in P_\nu$. Also, $\eta_{\nu'}$ codes elements of $Col(\kappa_\nu,P_\nu)$.[6]

For every $\nu\in S$ fix a cofinal sequence $\langle\nu_n\mid n<\omega\rangle$.
Let $\nu\in S$. Consider $\langle\eta_{\nu_n}\mid n<\omega\rangle$. Denote by $\langle t^i_{\nu,n}\mid i<\kappa^+_{\nu_n}\rangle$ the sequence of members of $Col(\kappa_{\nu_{n+1}},P_{\nu_{n+1}})$ codded by η_{ν_n}.
Let $tr_\nu:P_\nu\longleftrightarrow\kappa^+_\nu$ be the transitive collapse of P_ν.
Consider a set

$$Z_\nu:=\{tr_\nu{}''t^i_{\nu,n}\mid n<\omega,i<\kappa^+_{\nu_n}\}.$$

It is a subset of $Col(\kappa_\nu,\kappa^+_\nu)$. Define a partial order $\leq_\nu$ on Z_ν as follows:

$$tr_\nu{}''t^i_{\nu,n}\leq_\nu tr_\nu{}''t^j_{\nu,m}\text{ iff }n\leq m\text{ and }tr_\nu{}''t^i_{\nu,n}\leq_{Col(\kappa_\nu,\kappa^+_\nu)}tr_\nu{}''t^j_{\nu,m}.$$

Set G_ν to be the set of all unions of all $<_\nu-$increasing $\omega-$sequences of elements of Z_ν.

Lemma 2.2 *There is $g\in G_\nu$ which is generic for $Col(\kappa_\nu,\kappa^+_\nu)$ over V.*

Proof. Work in V^{**}. Define a function g as follows. Start with $tr_\nu{}''t^0_{\nu,0}$. Pick $i_1<\kappa^+_{\nu_1}$ such that $t^{i_1}_{\nu,1}$ comes from the ultrafilter $\tilde{F}_{t^0_{\nu,0}}$ over $Col(\kappa,\kappa^+)$.
Continue by induction. Suppose that t^n_{ν,i_n} is defined. Pick $i_{n+1}<\kappa^+_{\nu_n}$ such that $t^{i_{n+1}}_{\nu,n+1}$ comes from the ultrafilter $\tilde{F}_{t^{i_n}_{\nu,n}}$ over $Col(\kappa,\kappa^+)$.
Finally set

$$g=\bigcup_{n<\omega}tr_\nu{}''t^{i_n}_{\nu,n}.$$

We claim that g is as desired.

[5]Set here $\langle Q,\xi\rangle\sqsubset\langle P,\eta\rangle$ iff $Q\cup\{\xi\}\subseteq P$ and $|Q|<P\cap\kappa$.
[6]Note that $\eta_{\nu'}$ need not code only members of $Col(\kappa_{\nu'},P_{\nu'})$, or even of $Col(\kappa_{\nu'},P_\nu)$.

Work in V above a condition which already decides κ_ν. Suppose for simplicity that none of $\kappa_{\nu_n}, n < \omega$ is decided yet. Let D be a dense open subset of $Col(\kappa_\nu, \kappa_\nu^+)$. Intersect the measure one set of $\tilde{F}_{\emptyset^0}$ with D. The resulting condition will force

$$\underset{\sim}{g} \text{ extends a member of } \check{D}.$$

□

The next lemma follows from the definition of G_ν.

Lemma 2.3 *For every $n_0 < \omega$, $G_\nu \in V[\langle tr_\nu{}'' P_{\nu_n} \mid n_0 < n < \omega \rangle]$.*

Set $V^* = V[\langle G_\nu \mid \nu \in S \rangle]$.

Let now $\rho \in \omega_1 \setminus S$. We need to argue that $(\kappa_\rho^+)^V = (\kappa_\rho^+)^{V^*}$. By Lemma 2.3, it follows that

$$V[\langle G_\nu \mid \nu \in S \setminus \rho \rangle] \subseteq V[\langle \langle P_\tau, \eta_\tau \rangle \mid \rho < \tau < \omega_1 \rangle],$$

i.e. the extension of V by the same forcing but which only starts above κ_ρ. Such extension does not add new bounded subsets to κ_ρ^+ and below. Hence, it is enough to deal with the forcing up to κ_ρ.

Let us split the argument into two cases.

Case 1. ρ is a limit point of $\rho \in \omega_1 \setminus S$.

Let then $\langle \rho_k \mid k < \omega \rangle$ be a cofinal sequence consisting of elements of $\omega_1 \setminus S$. Assume for simplicity that $\rho_0 = 0$.

For every $\nu \in S \cap \rho$ find the least $k(\nu)$ such that $\nu < \rho_{k(\nu)}$. Let n_ν be the least $n < \omega$ such that

$nu_n > \rho_{k(\nu)-1}$, if $k(\nu) \geq 1$ and 0 otherwise.

Consider

$$V^\rho := V[\langle \kappa_\tau \mid \tau < \rho \rangle, \langle \langle \langle tr_\nu{}'' P_{\nu_n}, tr_\nu{}'' \eta_{\nu_n} \rangle \mid n_\nu \leq n < \omega \rangle \mid \nu \in S \cap \rho \rangle].$$

Then

$$V[\langle G_\nu \mid \nu \in S \cap \rho \rangle] \subseteq V^\rho.$$

Lemma 2.4 *V^ρ is a generic extension of V by a Prikry type forcing which satisfies $\kappa_\rho^+ - c.c.$*

Case 2. ρ is not a limit point of $\rho \in \omega_1 \setminus S$.

The treatment of this case is similar and even a bit simpler than the previous one.

References

[1] . Gitik, Prikry type forcings, Handbook of Set Theory

[2] K. Kunen, Set Theory.

[3] S. Shelah, Cardinal arithmetic