Silver type theorems for collapses.

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The classical theorem of Silver states that GCH cannot break for the first time over a singular cardinal of uncountable cofinality. On the other hand it is easy to obtain a situation where GCH breaks on a club below a singular cardinal $\kappa$ of an uncountable cofinality but $2^\kappa = \kappa^+$. We would like here to investigate the situation once blowing up power of singular cardinals is replaced by collapses of their successors.

1 ZFC results.

The following basic result should be well known and goes back to Silver:

**Theorem 1.1** Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:

1. $\kappa$ is a cardinal in $W$,
2. $\kappa$ changes its cofinality to $\omega_1$ in $V$ witnessed by a club $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$,
3. for every $\alpha < \omega_1$, $(\kappa^+_\alpha)^W < \kappa^+_\alpha$ (or only for stationary many $\alpha$’s),
4. $\kappa$ is a strong limit in $V$ or just it is a limit cardinal and $\kappa^{< \omega_1}_\alpha < \kappa$, for every $\alpha < \omega_1$.

Then $(\kappa^+)^W < \kappa^+$.

**Proof.** Suppose that $(\kappa^+)^W = \kappa^+$.

Fix in $W$ a sequence $\langle f_i \mid i < \kappa^+ \rangle$ of $\kappa^+$ first canonical functions in $\langle \prod_{\nu < \kappa} \nu^+, <_{j^{\text{bd}}_\kappa} \rangle$ or just any sequence of $\kappa^+$–many functions in $\prod_{\nu < \kappa} \nu^+$ increasing mod $j^{\text{bd}}_\kappa$.

Set in $V$

$g_i = f_i \upharpoonright \{ \kappa_\alpha \mid \alpha < \omega_1 \}$, for every $i < \kappa^+$. Then $\langle g_i \mid i < \kappa^+ \rangle$ is an increasing sequence
of functions in $\langle \prod_{\alpha<\omega_1} (\kappa^+_\alpha)^W, <_{J_{\kappa}^{bd}} \rangle$. By the assumption (3) we have that for every $\alpha < \omega_1$, $(\kappa^+_\alpha)^W < \kappa^+_\alpha$. Now, as in the Baumgartner-Prikry proof of the Silver Theorem (see K. Kunen [2] p.296 (H5)), it is impossible to have $\kappa^+$-many such functions. Hence $(\kappa^+)^W < \kappa^+$. □

Let us deal now with double successors.

**Theorem 1.2** Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:

1. $\kappa$ is a cardinal in $W$,
2. $2^\kappa \geq \kappa^{++}$, and moreover there is a sequence of $\kappa^{++}$-many functions in $\prod_{\nu<\kappa} \nu^{++}$ increasing mod $J_{\kappa}^{bd}$,
3. $\kappa$ changes its cofinality to $\omega_1$ in $V$ witnessed by a club $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$,
4. for every $\alpha < \omega_1$, $(\kappa^{++}_\alpha)^W < \kappa^+_\alpha$ (or only for stationary many $\alpha$’s),
5. $\kappa$ is a strong limit in $V$ or just it is a limit cardinal and $\kappa^{|\omega_1} < \kappa$, for every $\alpha < \omega_1$.

Then $(\kappa^{++})^W < \kappa^+$. 

The condition (2) allows to repeat the proof of 1.1.

Let state the following relevant result of Shelah ([3](4.9,p.304)), which says that once $(\kappa^+)^W$ changes its cofinality, then we must have $(\kappa^{++})^W < \kappa^+$ unless $\text{cof}((\kappa^+)^W) = \text{cof}(|(\kappa^+)^W|) = \text{cof}(\kappa)$.

**Proposition 1.3** Let $F$ be the $\kappa$–complete filter of co-bounded subsets of $\mathcal{P}_\kappa(\kappa^+)$, i.e. the filter generated by the sets $\{ P \in \mathcal{P}_\kappa(\kappa^+) \mid \alpha \in P \}$, $\alpha < \kappa^+$.

Then there is a sequence $\langle f_i \mid i < \kappa^{++} \rangle$ of functions such that

1. $f_i : \mathcal{P}_\kappa(\kappa^+) \to \kappa$,
2. $f_i(P) < |P|^+$, for all $P \in \mathcal{P}_\kappa(\kappa^+)$,
3. $f_i >_F f_j$, whenever $i > j$.

**Proof.** We define a sequence $\langle f_i \mid i < \kappa^{++} \rangle$ by induction.

Suppose that $\langle f_j \mid j < i \rangle$ is defined. Define $f_i$.  

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Case 1. $i = i' + 1.$
Set $f_i(P) = f_{i'}(P) + 1.$

Case 2. $i$ is a limit ordinal of cofinality $\delta < \kappa$.
Pick a cofinal in $i$ sequence $\langle i_\tau \mid \tau < \delta \rangle$. Set $f_i(P) = \bigcup_{\tau < \delta} f_{i_\tau}(P) + 1$.

Case 3. $i$ is a limit ordinal of cofinality $\delta \geq \kappa$, i.e. $\delta = \kappa$ or $\delta = \kappa^+$. 
Pick a cofinal in $i$ sequence $\langle i_\tau \mid \tau < \delta \rangle$. Set $f_i(P) = \bigcup_{\tau \in P} f_{i_\tau}(P) + 1$.

\[ \square \]

Theorem 1.4 Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:

1. $\kappa$ is an inaccessible in $W$,
2. $\kappa > (\text{cof}(\kappa))^V = \delta$ for some uncountable (in $V$) cardinal $\delta$.
3. $\kappa$ is a strong limit in $V$ or just it is a limit cardinal and for every $\xi < \kappa$, $\xi^\delta < \kappa$.
4. There exist a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in $\kappa$ (or just a stationary set) \(^1\) and a sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ such that:
   (a) $P_\alpha \in (P_\kappa(\kappa^+))^W$, for each $\alpha < \delta$,
   (b) $(|P_\alpha|^+)^W < \kappa_\alpha^+$, for each $\alpha < \delta$,
   (c) $(\kappa^+)^W = \bigcup_{\alpha < \delta} P_\alpha$,
   (d) for every $Q \in (P_\kappa(\kappa^+))^W$, there is $\alpha < \delta$ such that for every $\beta, \alpha \leq \beta < \delta$, $Q \subseteq P_\beta$.

Then $(\kappa^{++})^W < \kappa^+$.

Proof. Suppose otherwise. Then $(\kappa^{++})^W = \kappa^+$, by the assumption (b),(c) above.
Let $\langle f_i \mid i < \kappa^{++} \rangle$ be a sequence of functions in $W$ given by Proposition 1.3.
We can repeat the argument of 1.1 with slight adaptations. Thus, set in $V$
$g_i(\alpha) = f_i(P_\alpha)$, for every $\alpha < \omega_1$ and $i < (\kappa^{++})^W = \kappa^+$. Let $\nu_\alpha := (|P_\alpha|^+)^W$. By the assumption, $\nu_\alpha < \kappa_\alpha^+$. Then $\langle g_i \mid i < \kappa^+ \rangle$ is an increasing sequence of functions in $\langle \prod_{\alpha < \delta} \nu_\alpha; < J^{+} \rangle$, since for every $A \in \mathcal{F}$ we have $\{P_\alpha \mid \alpha \geq \alpha_0\} \subseteq A$, for some $\alpha_0 < \delta$. This is impossible, since $\nu_\alpha < \kappa_\alpha^+$, for every $\alpha < \delta$. Contradiction.

\[ \square \]

\(^1\)Note that if $\delta = \omega_1$, then we can just force a club into it without effecting things above.
Theorem 1.5 Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that for some inaccessible in $W$ cardinal $\kappa$ both $\kappa$ and its successor in $W$ change their cofinality to some uncountable (in $V$) cardinal $\delta$ and $\kappa$ remains a cardinal in $V$. Then the following conditions are equivalent:

1. (*) There are a clubs $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in $\kappa$ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ in $(\kappa^+)^W$ such that for every limit $\alpha < \delta$ (or just for stationary many $\alpha$'s)$^2$ the set $\{\eta_\beta \mid \beta < \alpha\}$ can be covered by a set $a_\alpha \in W$ with $(|a_\alpha|^+)^W < \kappa^+$. 

2. (**) There are a clubs $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in $\kappa$ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ in $(\kappa^+)^W$ such that for every limit $\alpha < \delta$ (or just for stationary many $\alpha$'s) the set $\{\eta_\beta \mid \beta < \alpha\}$ has an unbounded intersection with a set $b_\alpha \in W$ with $(|b_\alpha|^+)^W < \kappa^+$. 

3. There exist a club $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ in $\kappa$ and a sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ such that
   
   a. $P_\alpha \in (\mathcal{P}_\kappa(\kappa^+))^W$, for each $\alpha < \delta$, 
   b. $P_\alpha \cap \kappa = \kappa_\alpha$, for each $\alpha < \delta$, 
   c. $(|P_\alpha|^+)^W < \kappa^+$, for each $\alpha < \delta$, 
   d. $(\kappa^+)^W = \bigcup_{\alpha < \omega_1} P_\alpha$, 
   e. for every $Q \in (\mathcal{P}_\kappa(\kappa^+))^W$, there is $\alpha < \omega_1$ such that for every $\beta, \alpha \leq \beta < \omega_1$, $Q \subseteq P_\beta$.

4. There exists an increasing sequence $\langle P_\alpha \mid \alpha < \delta \rangle$ which satisfies all the requirements of the previous item.

Proof. Split the proof into lemmas.

Lemma 1.6 (*) iff (**).

Proof. Clearly, (*) implies (**). Let us show the opposite direction.
We fix a bijection $\pi_\xi : \kappa \leftrightarrow \xi$ in $W$, for every $\xi < (\kappa^+)^W$. 
Fix in $V$ a function $\pi : \kappa \rightarrow^{onto} (\kappa^+)^W$. Set now for every $\alpha < \delta$, $\eta_\alpha = \sup(\pi^\alpha \kappa_\alpha)$. Then, clearly, $\{\eta_\alpha \mid \alpha < \delta\}$ is a club in $(\kappa^+)^W$. Now given a sequence which witnesses (**). Without loss of generality we can assume that it is the sequence $\langle \eta_\alpha \mid \eta < \delta \rangle$ defined above. Otherwise

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$^2$If $\delta = \omega_1$, then it is basically the same, since once we have only stationary many such $\alpha$’s, then force a club into it. Everything is a the level of $\omega_1$, so this will have no effect on the cardinal arithmetic above.
just intersect two clubs.
Define an increasing continuous sequence \( \langle N_\alpha \mid \alpha < \delta \rangle \) of elementary submodels of some \( H_\chi \), with \( \chi \) big enough such that

1. \( \delta, \kappa, \langle \kappa_\alpha \mid \alpha < \delta \rangle, \langle \pi_\xi \mid \xi < (\kappa^+)^W \rangle, \pi \in N_0, \)
2. \( |N_\alpha| < \delta, \)
3. \( N_\alpha \cap \delta \) is an ordinal,
4. \( \langle N_\beta \mid \beta \leq \alpha \rangle \in N_{\alpha + 1}. \)

Denote \( N_\alpha \cap \delta \) by \( \delta_\alpha. \)
Then \( \sup(N_\alpha \cap \kappa) = \kappa_\delta \) and \( \sup(N_\alpha \cap (\kappa^+)^W) = \eta_\delta. \) Clearly, \( \delta_\alpha = \alpha \) for a club many \( \alpha \)'s.

Suppose now that for some \( \alpha < \delta \) we have \( \delta_\alpha = \alpha \) and there is a set \( X \in W \) such that

- \( (|X|)^W < \kappa_\alpha^+, \)
- \( X \cap \{ \eta_\beta \mid \beta < \alpha \} \) is unbounded in \( \eta_\alpha. \)

Note that \( \eta_\beta \in N_\alpha, \) for every \( \beta < \alpha \) and then, also, \( \pi_{\eta_\beta} \in N_\alpha. \) By elementarity, then \( \pi_{\eta_\beta} \upharpoonright (N_\alpha \cap \kappa_\alpha) : N_\alpha \cap \kappa_\alpha \leftrightarrow N_\alpha \cap \eta_\beta. \) In particular, \( \pi_{\eta_\beta} \upharpoonright \kappa_\alpha \supseteq \{ \eta_\gamma \mid \gamma < \beta \}. \)
Set
\[
Y := \{ \pi_\xi \upharpoonright \kappa_\alpha \mid \xi \in X \cap \eta_\alpha \}.
\]
Then, \( Y \in W, |Y|^W \leq \kappa_\alpha + |X|^W \), and so \( (|X|)^W < \kappa_\alpha^+. \) But, in addition, \( Y \supseteq \{ \eta_\gamma \mid \gamma < \alpha \} \), since for unboundedly many \( \beta < \alpha, \) we have \( \eta_\beta \in X \) and so, \( \pi_{\eta_\beta} \upharpoonright \kappa_\alpha \supseteq \{ \eta_\gamma \mid \gamma < \beta \}. \)
\( \square \) of the lemma.

**Lemma 1.7** (1) implies (3)

**Proof.**

Fix clubs \( \langle \kappa_\alpha \mid \alpha < \delta \rangle \) and \( \langle \eta_\alpha \mid \alpha < \delta \rangle \) witnessing (1).
Let us build first a sequence \( \langle R_\alpha \mid \alpha < \delta \rangle \) which satisfies all the requirements of (3), but probably is not increasing.

Set \( R_0 = \kappa_0 \cup ((\pi_{\eta_0} \upharpoonright \kappa_0) \setminus \kappa). \)
Let \( \alpha, 0 < \alpha < \delta \) be an ordinal. Pick \( a_\alpha \in W, a_\alpha \subseteq \eta_\alpha \) to be a cover of \( \{ \eta_\beta \mid \beta < \alpha \} \) with \( (|a_\alpha|)^W < \kappa_\alpha^+. \) Set \( R'_\alpha = \bigcup \{ \pi_\xi \upharpoonright \kappa_\alpha \mid \xi \in b_\alpha \cup \{ \eta_\alpha \} \}. \) Let \( R_\alpha = \kappa_\alpha \cup (R'_\alpha \setminus \kappa). \)
The constructed sequence satisfies trivially the requirements \(a\), \(b\) and \(c\). Let us check \(e\). \(d\) clearly follows from \(e\).

Let \(Q \in (P_\kappa(\kappa))^W\). There is \(\beta < \omega_1\) such that \(Q \subseteq \eta_\beta\). Consider \(\pi_{\eta_\beta}^{-1}Q\). It is a bounded subset of \(\kappa\). Hence there is \(\gamma < \omega_1\) such that \(\kappa_\gamma \supseteq \pi_{\eta_\beta}^{-1}Q\). So \(\pi_{\eta_\beta}^{-1} \kappa_\beta \supseteq Q\). Let \(\alpha < \omega_1\) be an ordinal above \(\beta, \gamma\). Then \(R_\delta \supseteq Q\), for every \(\delta \geq \alpha\).

\(\square\) of the lemma.

**Lemma 1.8** \((3)\) iff \((4)\).

**Proof.** Clearly \((4)\) implies \((3)\). Let us show the opposite direction.

Let a club \(\langle \kappa_\alpha \mid \alpha < \delta \rangle\) in \(\kappa\) and a sequence \(\langle R_\alpha \mid \alpha < \delta \rangle\) witness \((3)\).

Define an increasing subsequence \(\langle P_\alpha \mid \alpha < \delta \rangle\)

Set \(P'_0 = R_0\). By \(e\) there is \(\alpha_0\) such that for every \(\beta, \alpha_0 \leq \beta < \delta\), \(P'_\beta \subseteq R_\beta\). Set \(P'_1 = R_{\alpha_1}\). Continue by induction. Suppose that \(\nu < \delta\) and for every \(\nu' < \nu\), increasing sequences \(\langle \alpha_{\nu'} \mid \nu' < \nu\rangle\) and \(\langle P'_{\nu'} \mid \nu' < \nu\rangle\) are defined and satisfy the following:

1. \(P'_{\nu'} = R_{\alpha_{\nu'}}\),

2. for every \(\beta, \alpha_{\nu'} \leq \beta < \delta\), \(P'_{\nu'} \subseteq R_\beta\).

If \(\nu\) is a successor ordinal, then let \(\nu = \mu + 1\), for some \(\mu\). Set \(P'_\nu = R_{\alpha_{\mu}}\) and let \(\alpha_{\nu} < \delta\) be such that for every \(\beta, \alpha_{\nu} \leq \beta < \delta\), \(P'_\beta \subseteq R_\beta\).

If \(\nu\) is a limit ordinal, then let \(P'_\nu = R_{\bigcup_{\nu' < \nu} \alpha_{\nu'}}\) and define \(\alpha_{\nu}\) as in the successor case.

Finally let us define an increasing subsequence of \(\langle P'_\alpha \mid \alpha < \delta \rangle\) which satisfies the properties \(a\)-\(e\) of \((3)\).

Let \(C := \{\nu < \delta \mid \nu = \bigcup_{\nu' < \nu} \alpha_{\nu'}\}\). Clearly it is a club. Set \(P_\nu = P'_\nu\), for every \(\nu \in C\).

Then \(\langle \kappa_\alpha \mid \alpha \in C \rangle\) and \(\langle P_\alpha \mid \alpha \in C \rangle\) are as desired.

\(\square\) of the lemma.

**Lemma 1.9** \((3)\) implies \((1)\).

**Proof.** Let a club \(\langle \kappa_\alpha \mid \alpha < \delta \rangle\) in \(\kappa\) and a sequence \(\langle P_\alpha \mid \alpha < \delta \rangle\) witness \((3)\). Let \(\eta_\alpha \mid \alpha < \delta\) be a club in \((\kappa^+)^W\).

We claim that there is a club \(C \subseteq \delta\) such that for every \(\alpha \in C\), \(P_\alpha \supseteq \{\eta_\beta \mid \beta < \alpha\}\).

Suppose otherwise. Then there is a stationary \(S \subseteq \delta\) such that for every \(\alpha \in S\) there is \(\alpha' < \alpha\) with \(\eta_{\alpha'} \not\in P_\alpha\). Then there are a stationary set \(S^* \subseteq S\) and \(\alpha^* < \delta\) such that for every \(\alpha \in S^*, \eta_{\alpha^*} \not\in P_\alpha\). This is impossible by \((d)\).
Theorem 1.10 Suppose that $V \supseteq W$ are transitive models of ZFC with the same ordinals such that:

1. $\kappa$ is an inaccessible in $W$,
2. $\kappa > (\text{cof}(\kappa))^V = \delta$ for some uncountable (in $V$) cardinal $\delta > \omega_1$. Let $\langle \kappa_\alpha \mid \alpha < \delta \rangle$ be a witnessing club.
3. For every $\alpha < \delta$, $(\kappa_\alpha^\alpha)^W < \kappa_\alpha^+$ (or only for stationary many $\alpha$'s),
4. $\kappa$ is a strong limit in $V$ or just it is a limit cardinal and $\kappa_\alpha^{\omega_1} < \kappa$, for every $\alpha < \delta$.
5. There is a regular cardinal $\delta^*$, $\omega < \delta^* < \delta$ such that for every regular cardinal $\rho < \kappa$ of $W$ which became a singular of cofinality $\delta^*$ in $V$, there is a club a club sequence $\langle \rho_i \mid i < \delta^* \rangle$ in $\rho$ such that for every club $c \subseteq \delta^*$ the set $\{(\text{cof}(\rho_i))^W \mid i \in c\}$ is unbounded in $|\rho|$.

Or

6. Like the previous item but only for $\rho$'s of the form $(\text{cof}(\eta_\alpha))^W$ with $\alpha < \delta$ of cofinality $\delta^*$, where $\langle \eta_\alpha \mid \alpha < \delta \rangle$ is a club in $\langle \kappa^+ \rangle^W$.

Then $(\kappa^\alpha^\alpha)^W < \kappa^+$.

Proof.

Let us argue that $(\ast \ast)$ of 1.4 holds.

Assume for simplicity that $\delta^* = \omega_1$.

Let $\langle N_\alpha \mid \alpha < \delta \rangle$ and $\langle \eta_\alpha \mid \alpha < \delta \rangle$ be as in 1.6. Pick $\alpha < \delta$ of cofinality $\omega_1$ with $\delta_\alpha = \alpha$. Consider $\eta_\alpha$. Then $\text{cof}(\eta_\alpha) = \omega_1$. If $(\text{cof}(\eta_\alpha))^W < \kappa_\alpha^+$, then we pick in $W$ a club $X$ in $\eta_\alpha$ of the order type $(\text{cof}(\eta_\alpha))^W$. Then $X \cap \{\eta_\beta \mid \beta < \alpha\}$ is a club, and so, unbounded in $\eta_\alpha$.

Suppose now that $(\text{cof}(\eta_\alpha))^W \geq \kappa_\alpha^+$. Denote $(\text{cof}(\eta_\alpha))^W$ by $\rho$. Then $\rho \leq \kappa$, since $\eta_\alpha < (\kappa^+)^W$. It is impossible to have $\rho = \kappa$, since $\text{cof}(\kappa) > \omega_1 = \text{cof}(\alpha) = \text{cof}(\eta_\alpha) = \text{cof}(\rho)$. Hence $\kappa_\alpha^+ \leq \rho < \kappa$. In particular, $|\rho| \geq \kappa_\alpha^+$.

By the assumption (5) of the theorem, there is a club a club sequence $\langle \rho_i \mid i < \omega_1 \rangle$ such that for every club $c \subseteq \omega_1$ the set $\{(\text{cof}(\rho_i))^W \mid i \in c\}$ is unbounded in $|\rho|$. Let $e = \{e_\xi \mid \xi < \rho\} \in W$ be a club in $\eta_\alpha$. Consider $d := \{\eta_\beta \mid \beta < \alpha\} \cap e$. It is a club in $\eta_\alpha$. So there are some
\( \gamma < \alpha \) and \( j < \omega_1 \) such that \( \eta_\gamma = e_{\rho_j} \) and \((\text{cof}(\rho_j))^W > \kappa_\alpha \). But this is impossible, since \( \eta_\gamma \in N_\alpha \), and hence, \((\text{cof}(\eta_\gamma))^W = (\text{cof}(\rho_j))^W \in N_\alpha \cap \kappa \subseteq \kappa_\alpha \).

Hence, always \((\text{cof}(\eta_\alpha))^W < \kappa_\alpha^+ \).

So, the set \( \{ \eta_\alpha : \alpha < \delta \text{ and cof}(\alpha) = \omega_1 \} \) witnesses \((**)\) and we are done.

\[ \Box \]

**Lemma 1.11** For every \( \beta < \delta \),
\[
\{ (\text{cof}(\eta_\gamma))^W : \gamma < \beta \} \subseteq \kappa_\beta.
\]

**Proof.** Otherwise there is \( \gamma < \beta \) such that \((\text{cof}(\eta_\gamma))^W \geq \kappa_\beta \). Recall that \( \kappa < \eta_\gamma < (\kappa^+)^W \).

Hence, \((\text{cof}(\eta_\gamma))^W \leq \kappa \). It is impossible to have \((\text{cof}(\eta_\gamma))^W = \kappa \), since \( \text{cof}(\kappa) = \delta > |N_\gamma| \geq \text{cof}(\eta_\gamma) = \text{cof}((\text{cof}(\eta_\gamma))^W) \). So, \((\text{cof}(\eta_\gamma))^W < \kappa \). But \((\text{cof}(\eta_\gamma))^W \in N_\beta \) and \( \sup(N_\beta \cap \kappa) = \kappa_\beta \).

\[ \Box \]

**Lemma 1.12** Suppose that for every \( \beta < \delta \), \( \kappa_\beta^+ \) is is successor cardinal in \( W \) and \( \nu_\beta \) is its immediate predecessor, then, for a club many \( \beta < \delta \) of uncountable cofinality
\[
(\text{cof}(\eta_\beta))^W \geq \nu_\beta.
\]

**Proof.** Otherwise there will be stationary many \( \beta \)'s of uncountable cofinality with \((\text{cof}(\eta_\beta))^W < \nu_\beta \). Then \((**)\) holds on this stationary set.

\[ \Box \]

**Lemma 1.13** Suppose that for every \( \beta < \delta \), \( \kappa_\beta^+ \) is a limit cardinal of \( W \), then, for a club many \( \beta < \delta \) of uncountable cofinality
\[
(\text{cof}(\eta_\beta))^W > \kappa_\beta^+.
\]

**Proof.** Otherwise there will be stationary many \( \beta \)'s of uncountable cofinality with \((\text{cof}(\eta_\beta))^W < \kappa_\beta^+ \). Then \((**)\) holds on this stationary set.

\[ \Box \]

**Theorem 1.14** Suppose that \( V \supseteq W \) are transitive models of ZFC with the same ordinals such that:

1. \( \kappa \) is an inaccessible in \( W \),

2. \( \kappa > (\text{cof}(\kappa))^V = \delta \) for some uncountable (in \( V \)) cardinal \( \delta > \omega_1 \). Let \( \langle \kappa_\alpha : \alpha < \delta \rangle \) be a witnessing club.
3. For every $\alpha < \delta$, $(\kappa_\alpha^{++})^W < \kappa_\alpha^+$ (or only for stationary many $\alpha$'s),

4. $\kappa$ is a strong limit in $V$ or just it is a limit cardinal and $\kappa_\omega^{=1} < \kappa$, for every $\alpha < \delta$.

Assume that $(\kappa^{++})^W \geq \kappa^+$.

Then there is an increasing unbounded in $\kappa$ sequence $\langle \rho_\alpha \mid \alpha < \delta \rangle$ such that

- $\rho_\alpha$ is a regular cardinal in $W$,
- for every limit $\alpha$, $\text{cof}(\rho_\alpha) = \text{cof}(\alpha)$,
- for every limit $\alpha$ of uncountable cofinality, $\rho_\alpha \geq |\rho_\alpha| \geq \kappa_\alpha \geq \sup(\{\rho_\beta \mid \beta < \alpha\})$,
- for every limit $\alpha$ of uncountable cofinality, there is a club $c_\alpha$ in $\rho_\alpha$ such that for every $\tau \in c_\alpha$ we have $(\text{cof}(\tau))^W \in \{\rho_\beta \mid \beta < \alpha\}$.

Proof. Just take $\rho_\alpha = (\text{cof}(\eta_\alpha))^W$.

Suppose that $\alpha$ has an uncountable cofinality. Then, by 1.13, $\rho_\alpha \geq |\rho_\alpha| \geq \kappa_\alpha^+$, and by 1.11, $\{\rho_\beta \mid \beta < \alpha\} \subseteq \kappa_\alpha$.

Fix some increasing continuous function $\varphi_\alpha : \rho_\alpha \to \eta_\alpha$ in $W$ with $\text{ran}(\varphi_\alpha)$ unbounded in $\eta_\alpha$.

Set $c_\alpha := \{\varphi_\alpha^{-1}(\eta_\beta) \mid \beta < \alpha \text{ limit and } \eta_\beta \text{ is a limit point of } \text{ran}(\varphi_\alpha)\}$.

Let $\tau \in c_\alpha$. Then $\tau = \varphi_\alpha^{-1}(\eta_\beta)$ for a limit $\beta < \alpha$ and $\eta_\beta$ is a limit point of $\text{ran}(\varphi_\alpha)$. Now the continuity of $\varphi_\alpha$ implies that $(\text{cof}(\tau))^W = (\text{cof}(\eta_\beta))^W$ which is $\rho_\beta$.

$\square$

2 A forcing construction.

We would like to show the following:

**Theorem 2.1** Suppose that $\kappa$ is a $\kappa^{+3}$–supercompact cardinal. Let $S$ be subset of $\omega_1$. Then there are generic extensions $V^* \subseteq V^{**}$ such that

1. $\kappa$ changes its cofinality to $\omega_1$ in $V^{**}$,
2. there is a closed unbounded in $\kappa$ sequence $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ of cardinals in $V^{**}$ such that $S = \{\alpha < \omega_1 \mid (\kappa_\alpha^{+})^{V^*} = (\kappa_\alpha^{+})^{V^{**}}\}$ and $\omega_1 \setminus S = \{\alpha < \omega_1 \mid (\kappa_\alpha^{+})^{V^*} < (\kappa_\alpha^{+})^{V^{**}}\}$. 

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Let us describe the construction. Assume GCH, $\kappa$ is a $\kappa^{+3}$-supercompact cardinal and $S$ is a subset of $\omega_1$.  

Fix a coherent sequence 

$$\vec{W} = \langle W(\alpha, \beta) \mid \alpha \in \text{dom}(\vec{W}), \beta < o^{\vec{W}}(\alpha) \rangle$$

such that

1. $\kappa = \text{max}(\text{dom}(\vec{W}))$,
2. $o^{\vec{W}}(\kappa) = \omega_1$,
3. for every $\alpha \in \text{dom}(\vec{W}), \beta < o^{\vec{W}}(\alpha), W(\alpha, \beta)$ is a normal ultrafilter over $\mathcal{P}_\alpha(\alpha^{++})$,
4. $\vec{W} \upharpoonright (\alpha, \beta) = j_{W(\alpha, \beta)}(f)(\alpha)$, for some $f : \alpha \to V$.

Consider the Levy collapse $\text{Col}(\kappa, \kappa^+)$. Let $p \in \text{Col}(\kappa, \kappa^+)$. Set 

$$\mathcal{F}_p = \{ D \subseteq \text{Col}(\kappa, \kappa^+) \mid D \text{ is a dense open above } p \}.$$

Then $\mathcal{F}_p$ is a $\kappa$-complete filter over a set of cardinality $\kappa^+$, for every $p \in \text{Col}(\kappa, \kappa^+)$. It is also fine in a sense that for every $\eta < \kappa^+$,

$$\{ q \in \text{Col}(\kappa, \kappa^+) \mid \eta \in \text{ran}(q) \} \in \mathcal{F}_p.$$

Let $j : V \to M$ be an elementary embedding with $\kappa$ a critical point and $\kappa^+ M \subseteq M$. For every $p \in \text{Col}(\kappa, \kappa^+)$, pick $\tilde{p} \in \bigcap j'' \mathcal{F}_p$. So, $\tilde{p} \in (\text{Col}(j(\kappa), j(\kappa^+)))^M$. Set 

$$\tilde{\mathcal{F}}_p = \{ X \subseteq \text{Col}(\kappa, \kappa^+) \mid \tilde{p} \in j(X) \}.$$

Then $\tilde{\mathcal{F}}_p$ is a $\kappa$-complete ultrafilter which extends $\mathcal{F}_p$.

Note that $\mathcal{F}_p$ is a filter on $\mathcal{P}_\kappa(\kappa \times \kappa^+)$, hence $\tilde{\mathcal{F}}_p$ is an ultrafilter there.

Now find, in $M$, some (least) $\eta < j(\kappa^+)$ which codes $\langle \tilde{p} \mid p \in \text{Col}(\kappa, \kappa^+) \rangle$.

Define a $\kappa$-complete ultrafilter $\hat{W}$ over $\mathcal{P}_\kappa(\kappa^+) \times \kappa^+$ as follows:

$$X \in \hat{W} \text{ iff } \langle j''^{\kappa^+}, \eta \rangle \in j(X).$$

For every $p \in \text{Col}(\kappa, \kappa^+)$, fix a projection $\pi_p : \mathcal{P}_\kappa(\kappa^+) \times \kappa^+ \to \text{Col}(\kappa, \kappa^+)$ of $\hat{W}$ onto $\tilde{\mathcal{F}}_p$.

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3 The interesting case is when $S$ and its compliment are both stationary.

4 In some fixed in advance well ordering.
Now use the coherent sequence \( \tilde{W} \) to define in the obvious fashion a new coherent sequence \( \tilde{W} \) where each \( \tilde{W}(\alpha, \beta) \) is an \( \alpha \)-complete ultrafilter over \( \mathcal{P}_\alpha(\alpha^+) \times \alpha^+ \) defined from \( W(\alpha, \beta) \) as above.

Note that \( \tilde{W} \upharpoonright (\alpha, \beta) \) will belong already to the ultrapower by \( \tilde{W}(\alpha, \beta) \upharpoonright \mathcal{P}_\alpha(\alpha^+) = W(\alpha, \beta) \upharpoonright \mathcal{P}_\alpha(\alpha^+) \). Thus, \( \tilde{W} \upharpoonright (\alpha, \beta) \) belongs to the ultrapower by \( W(\alpha, \beta) \), by coherency. By the condition (4) above it will be in the ultrapower by \( W(\alpha, \beta) \upharpoonright \mathcal{P}_\alpha(\alpha^+) \), since this ultrapower is closed under \( \kappa^+ \)-sequences.

Force the supercompact Magidor forcing with \( \tilde{W} \).\(^5\)

Denote by \( V^{**} \) a resulting generic extension.

Let \( \{ \{ P_\nu, \eta_\nu \} \mid \nu < \omega_1 \} \) be the generic sequence. Then \( \{ P_\nu \mid \nu < \omega_1 \} \) be the supercompact Magidor sequence. Denote \( P_\nu \cap \kappa \) by \( \kappa_\nu \). If \( \nu' < \nu < \omega_1 \), then \( \langle P_\nu', \eta_\nu' \rangle \subseteq \langle P_\nu, \eta_\nu \rangle \). In particular, \( \eta_\nu \in P_\nu \). Also, \( \eta_\nu \) codes elements of \( Col(\kappa_\nu, P_\nu) \).\(^6\)

For every \( \nu \in S \) fix a cofinal sequence \( \langle \nu_n \mid n < \omega \rangle \).

Let \( \nu \in S \). Consider \( \langle \eta_{n_\nu} \mid n < \omega \rangle \). Denote by \( \langle t_{\nu,n}^i \mid i < \kappa_{n_\nu}^+ \rangle \) the sequence of members of \( Col(\kappa_{n_\nu}+1, P_{n_\nu+1}) \) codd by \( \eta_{n_\nu} \).

Let \( tr_\nu : P_\nu \rightarrow \kappa_\nu^+ \) be the transitive collapse of \( P_\nu \).

Consider a set

\[
Z_\nu := \{ tr_\nu^n t_{\nu,n}^i \mid n < \omega, i < \kappa_{n_\nu}^+ \}.
\]

It is a subset of \( Col(\kappa_\nu, \kappa_\nu^+) \). Define a partial order \( \leq_\nu \) on \( Z_\nu \) as follows:

\[
tr_\nu^n t_{\nu,n}^i \leq_\nu tr_\nu^n t_{\nu,m}^j
\]

iff \( n \leq m \) and \( tr_\nu^n t_{\nu,n}^i \leq_{Col(\kappa_\nu, \kappa_\nu^+)} tr_\nu^n t_{\nu,m}^j \).

Set \( G_\nu \) to be the set of all unions of all \( <_\nu \) -increasing \( \omega \)-sequences of elements of \( Z_\nu \).

**Lemma 2.2** There is \( g \in G_\nu \) which is generic for \( Col(\kappa_\nu, \kappa_\nu^+) \) over \( V \).

**Proof.** Work in \( V^{**} \). Define a function \( g \) as follows. Start with \( tr_\nu^n t_{\nu,0}^0 \). Pick \( i_1 < \kappa_{n_\nu}^+ \) such that \( t_{\nu,1}^{i_1} \) comes from the ultrafilter \( \tilde{F}_{\nu,0} \) over \( Col(\kappa_\nu, \kappa_\nu^+) \).

Continue by induction. Suppose that \( t_{\nu,n}^i \) is defined. Pick \( i_{n+1} < \kappa_{n_\nu}^+ \) such that \( t_{\nu,n+1}^{i_{n+1}} \) comes from the ultrafilter \( \tilde{F}_{\nu,n} \) over \( Col(\kappa_\nu, \kappa_\nu^+) \).

Finally set

\[
g = \bigcup_{n < \omega} tr_\nu^n t_{\nu,n}^i.
\]

We claim that \( g \) is as desired.

\(^5\)Set here \( \langle Q, \xi \rangle \subseteq \langle P, \eta \rangle \) iff \( Q \cup \{ \xi \} \subseteq P \) and \( |Q| < P \cap \kappa \).

\(^6\)Note that \( \eta_\nu \) need not code only members of \( Col(\kappa_\nu, P_\nu) \), or even of \( Col(\kappa_\nu, P_\nu) \).
Work in $V$ above a condition which already decides $\kappa_\nu$. Suppose for simplicity that none of $\kappa_{\nu_n}, n < \omega$ is decided yet. Let $D$ be a dense open subset of $Col(\kappa_\nu, \kappa_\nu^+)$. Intersect the measure one set of $\tilde{F}_0$ with $D$. The resulting condition will force

$$g$$ extends a member of $\tilde{D}$.

□

The next lemma follows from the definition of $G_\nu$.

**Lemma 2.3** For every $n_0 < \omega$, $G_\nu \in V[\langle tr_\nu''P_{\nu_n} \mid n_0 < n < \omega \rangle]$.

Set $V^* = V[\langle G_\nu \mid \nu \in S \rangle]$.

Let now $\rho \in \omega_1 \setminus S$. We need to argue that $(\kappa_\rho^+)^V = (\kappa_\rho^+)^{V^*}$. By Lemma 2.3, it follows that

$$V[\langle G_\nu \mid \nu \in S \setminus \rho \rangle] \subseteq V[\langle \langle P_\tau, \eta_\tau \rangle \mid \rho < \tau < \omega_1 \rangle],$$

i.e. the extension of $V$ by the same forcing but which only starts above $\kappa_\rho$. Such extension does not add new bounded subsets to $\kappa_\rho^+$ and below. Hence, it is enough to deal with the forcing up to $\kappa_\rho$.

Let us split the argument into two cases.

**Case 1.** $\rho$ is a limit point of $\rho \in \omega_1 \setminus S$.

Let then $\langle \rho_k \mid k < \omega \rangle$ be a cofinal sequence consisting of elements of $\omega_1 \setminus S$. Assume for simplicity that $\rho_0 = 0$.

For every $\nu \in S \cap \rho$ find the least $k(\nu)$ such that $\nu < \rho_{k(\nu)}$. Let $n_\nu$ be the least $n < \omega$ such that $nu_n > \rho_{k(\nu)-1}$, if $k(\nu) \geq 1$ and 0 otherwise.

Consider

$$V^\rho := V[\langle \kappa_\tau \mid \tau < \rho \rangle, \langle \langle tr_\nu''P_{\nu_n}, tr_\nu''\eta_{\nu_n} \rangle \mid n_\nu \leq n < \omega \rangle, \nu \in S \cap \rho].$$

Then

$$V[\langle G_\nu \mid \nu \in S \cap \rho \rangle] \subseteq V^\rho.$$

**Lemma 2.4** $V^\rho$ is a generic extension of $V$ by a Prikry type forcing which satisfies $\kappa_\rho^+ - \text{c.c.}$

**Case 2.** $\rho$ is not a limit point of $\rho \in \omega_1 \setminus S$.

The treatment of this case is similar and even a bit simpler than the previous one.
References

[1] Gitik, Prikry type forcings, Handbook of Set Theory
