Strongly compact Magidor forcing.

Moti Gitik

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Abstract

We present a strongly compact version of the Supercompact Magidor forcing ([3]). A variation of it is used to show that the following is consistent:

$V \supseteq W$ are transitive models of ZFC+GCH with the same ordinals such that:

1. κ is an inaccessible in W,
2. κ changes its cofinality to ω_1 in V witnessed by a club $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$,
3. for every $\alpha < \omega_1, (\kappa_\alpha^{++})^W < \kappa_\alpha^+$,
4. $(\kappa^{++})^W = \kappa^+$.

1 Preliminary settings.

Assume GCH. Let κ be a $\kappa^{+4}-$supercompact cardinal and $j : V \to M$ be a witnessing embedding. Denote the normal measure over κ derived from j by U, i.e.

$$X \in U \text{ iff } \kappa \in j(X).$$

We assume that

$$\{\alpha < \kappa \mid \alpha \text{ is a } \kappa^{++} - \text{supercomact cardinal } \} \in U.$$

Let

$$i : V \to N$$

be the ultrapower embedding and

$$k : N \to M$$

be defined by $k([f]_U) = j(f)(\kappa)$. Then it is elementary and the corresponding diagram is commutative.

Pick some large enough $\chi >> \kappa$ which is a fixed point of k. We fix inside N a well-ordering

$\prec$ of V_χ such that $\prec\restriction\eta$ wellorders $\mathcal{P}(\eta)$ in order type η^+, for each cardinal $\eta < \chi$(of N). Then $k(\prec)$ does the same in M.

We use j in a Radin fashion (see [4],[1]) to define a sequence of ultrafilters

$$\langle W(\kappa,\beta) \mid \beta < \omega_1\rangle.$$

Set

$$X \in W(\kappa,0) \text{ iff } j``\kappa^{+3} \in j(X).$$

Suppose that $\beta < \omega_1$ and the sequence $\langle W(\kappa,\beta) \mid \beta' < \beta\rangle$ is defined. Set

$$X \in W(\kappa,\beta) \text{ iff } \langle j``\kappa^{+3}, \langle W(\kappa,\beta) \mid \beta' < \beta\rangle\rangle \in j(X).$$

Then each $W(\kappa,\beta)$ will be a $\kappa-$complete ultrafilter over $\mathcal{P}_\kappa(V_{\kappa+3})$. $W(\kappa,0)$ will be a normal ultrafilter over $\mathcal{P}_\kappa(\kappa^{+3})$.
We denote by

$$j_{W(\kappa,\beta)} : V \to M_{W(\kappa,\beta)}$$

the elementary embedding of $W(\kappa,\beta)$ and let

$$k_{W(\kappa,\beta)} : M_{W(\kappa,\beta)} \to M$$

be defined by setting

$$k_{W(\kappa,\beta)}([f]_{W(\kappa,\beta)}) = j(f)(\langle j``\kappa^{+3}, \langle W(\kappa,\beta') \mid \beta' < \beta\rangle\rangle).$$

Then $k_{W(\kappa,\beta)}$ is elementary and the resulting diagram is commutative. Then

$$j_{W(\kappa,\beta)}``\kappa^{+3} \in M_{W(\kappa,\beta)}$$

and, hence

$${}^{\kappa^{+3}}M_{W(\kappa,\beta)} \subseteq M_{W(\kappa,\beta)}, \text{ and } \mathrm{crit}(k_{W(\kappa,\beta)}) = (\kappa^{+5})^{M_{W(\kappa,\beta)}}.$$

In addition, if $\beta' < \beta < \omega_1$, then

$$W(\kappa,\beta') \in M_{W(\kappa,\beta)}$$

and we have an elementary embedding

$$k_{W(\kappa,\beta'),W(\kappa,\beta)} : M_{W(\kappa,\beta')} \to M_{W(\kappa,\beta)},$$

where

$$k_{W(\kappa,\beta'),W(\kappa,\beta)}([f]_{W(\kappa,\beta')}) = j_{W(\kappa,\beta)}(f)(\langle j_{W(\kappa,\beta)}{}''\kappa^{+3}, \langle W(\kappa,\beta') \mid \beta'' < \beta'\rangle\rangle).$$

Also all corresponding diagrams are commutative.

Let us now define a sequence of $(\kappa,\kappa^{++})-$extenders $\langle E(\kappa,\beta) \mid \beta < \omega_1\rangle$.
Let $E(\kappa,0) = \langle E(\kappa,0)(a) \mid a \in [\kappa^{++}]^{<\omega}\rangle$ be the $(\kappa,\kappa^{++})-$extender derived from $W(\kappa,0)$, i.e.

$$X \in E(\kappa,0)(a) \text{ iff } a \in j_{W(\kappa,0)}(X).$$

Now,

$$\mathrm{crit}(k_{W(\kappa,0)}) = (\kappa^{+5})^{M_{W(\kappa,0)}} > (\kappa^{+4})^{M_{W(\kappa,0)}} = \kappa^{+4} \supseteq a.$$

Hence,

$$a \in j_{W(\kappa,0)}(X) \text{ iff } a \in j(X).$$

Clearly, $E(\kappa,0)$ is definable via $W(\kappa,0)$, and so, belongs to each $M_{W(\kappa,\beta)}, \beta < \omega_1$.
Denote by

$$i_{E(\kappa,0)} : V \to N_{E(\kappa,0)} \simeq \mathrm{Ult}(V, E(\kappa,0))$$

the corresponding elementary embedding.

Let $\eta_0 < \kappa^{+5}$ be the ordinal which codes (corresponds to) $W(\kappa,0)$ in M (and, so in each $M_{W(\kappa,\beta)}, 0 < \beta < \omega_1$) by $k(\prec)$.
Define $E(\kappa,1) = \langle E(\kappa,1)(a) \mid a \in [\kappa^{++} \cup \{\eta_0\}]^{<\omega}\rangle$ to be the extender derived from $W(\kappa,1)$, i.e.

$$X \in E(\kappa,1)(a) \text{ iff } a \in j_{W(\kappa,1)}(X).$$

Note that $W(\kappa,0) \in M_{W(\kappa,1)}$, hence $\eta_0 < (\kappa^{+5})^{M_{W(\kappa,1)}}$. Then,

$$\mathrm{crit}(k_{W(\kappa,1)}) = (\kappa^{+5})^{M_{W(\kappa,1)}} \supseteq a.$$

Hence,

$$a \in j_{W(\kappa,0)}(X) \text{ iff } a \in j(X).$$

Denote by

$$i_{E(\kappa,1)} : V \to N_{E(\kappa,1)} \simeq \mathrm{Ult}(V, E(\kappa,1))$$

the corresponding elementary embedding. Let $k_{E(\kappa,1)} : N_{E(\kappa,1)} \to M$ be the corresponding elementary embedding. The critical point of $k_{E(\kappa,1)}$ is $(\kappa^{+3})^{N_{E(\kappa,1)}}$. Denote by η_0^1 the pre-image of η_0 by $k_{E(\kappa,1)}$.

Let $W^1(\kappa,0)$ be the filter over $\mathcal{P}_\kappa(\kappa^{+3})$ coded by η_0^1 inside $N_{E(\kappa,1)}$. It is a normal ultrafilter in $N_{E(\kappa,1)}$, but only a $\kappa-$complete filter in V.
We have

1. $E(\kappa,0)\in N_{E(\kappa,1)}$,
2. $E(\kappa,0)=E(\kappa,1)\restriction\kappa^{++}$.

Continue by induction and define $E(\kappa,\beta)$ for every $\beta<\omega_1$. Thus suppose that $\beta<\omega_1$ and for every $\beta'<\beta$, $E(\kappa,\beta')$ is defined. Define $E(\kappa,\beta)$.

Let $\eta_{\beta'}<\kappa^{+5}$ be the ordinal which codes (corresponds) $W(\kappa,\beta')$ in M (and, so in each $M_{W(\kappa,\gamma)},\beta\leq\gamma<\omega_1$) by $k(\prec)$, for every $\beta'<\beta$. Pick $\eta'_\beta<\kappa^{+5}$ be the ordinal which codes $\langle\eta_{\beta'}\mid\beta'<\beta\rangle$. We need this η'_β in order to keep the ultrapower by the extender closed under $\omega-$sequences.
Define $E(\kappa,\beta)=\langle E(\kappa,\beta)(a)\mid a\in[\kappa^{++}\cup\{\eta_{\beta'}\mid\beta'<\beta\}\cup\{\eta'_\beta\}]^{<\omega}\rangle$ to be the extender derived from $W(\kappa,\beta)$, i.e.

$$X\in E(\kappa,\beta)(a)\text{ iff }a\in j_{W(\kappa,\beta)}(X).$$

Note that $W(\kappa,\beta')\in M_{W(\kappa,\beta)}$, for every $\beta'<\beta$. Hence $\eta'_\beta<(\kappa^{+5})^{M_{W(\kappa,\beta)}}$. Then,

$$\text{crit}(k_{W(\kappa,\beta)})=(\kappa^{+5})^{M_{W(\kappa,\beta)}}\supseteq a.$$

Hence,

$$a\in j_{W(\kappa,\beta)}(X)\text{ iff }a\in j(X).$$

Denote by

$$i_{E(\kappa,\beta)}:V\to N_{E(\kappa,1)}\simeq\text{Ult}(V,E(\kappa,1))$$

the corresponding elementary embedding. Let $k_{E(\kappa,\beta)}:N_{E(\kappa,\beta)}\to M$ be the corresponding elementary embedding. The critical point of $k_{E(\kappa,\beta)}$ is $(\kappa^{+3})^{N_{E(\kappa,\beta)}}$. Denote by $\eta^\beta_{\beta'}$ the pre-image of $\eta_{\beta'}$ by $k_{E(\kappa,\beta)}$.
Let $W^\beta(\kappa,\beta')$ be the filter over $\mathcal{P}_\kappa((V_{\kappa+3})^{N_{E(\kappa,\beta)}})$ coded by $\eta^\beta_{\beta'}$ inside $N_{E(\kappa,\beta)}$, for every $\beta'<\beta$.

$$N_{E(\kappa,\beta)}\vDash W^\beta(\kappa,\beta')\text{ is an ultrafilter with the ultrapower closed under }\kappa^{+3}-\text{sequences}.$$

However, in V, it is only a $\kappa-$complete fine filter over $\mathcal{P}_\kappa((V_{\kappa+3})^{N_{E(\kappa,\beta)}})$.
Now, for every $\beta'<\beta$, we have

1. $E(\kappa,\beta') \in N_{E(\kappa,\beta)}$,

2. $E(\kappa,\beta') = E(\kappa,\beta) \restriction \eta_{\beta'}$.
   Denote the induced elementary embedding by

$$k_{E(\kappa,\beta'),E(\kappa,\beta)} : N_{E(\kappa,\beta')} \to N_{E(\kappa,\beta)}.$$

Let $\mathcal{V}$ denotes the $\prec -$least normal ultrafilter over $\mathcal{P}_\kappa(i(\kappa^{++}))$ in N (the ultrapower by the normal measure U over κ). Denote the image of $\mathcal{V}$ in $N_{E(\kappa,\beta)}$ by $\mathcal{V}_\beta$, for every $\beta < \omega_1$. Then the $\prec -$least normal ultrafilter over $\mathcal{P}_\kappa(i_{E(\kappa,\beta)}(\kappa^{++}))$ in $N_{E(\kappa,\beta)}$.
Note that $i_{E(\kappa,\beta)}(\kappa^{++}) < \kappa^{+3}$, and so fine κ-complete ultrafilters over $\mathcal{P}_\kappa(\kappa^{+3})$ can be used in order to extend $\mathcal{V}_\beta$ to an ultrafilter. However, we do not have any specific information about functions which represent ordinals below κ^{++} in such extensions and this knowledge will be important further in order to to link things over κ with those below. So, let us deal not directly with $\mathcal{V}_\beta$'s, but rather replace them by iteration which starts with extenders $E(\kappa,\beta)$'s.

Let $\beta < \omega_1$. Work inside $N_{E(\kappa,\beta+1)}$. We have there the extender $E(\kappa,\beta)$ and $\mathcal{V}_{\beta+1}$ which is a normal ultrafilter over $\mathcal{P}_\kappa(i_{E(\kappa,\beta+1)}(\kappa^{++}))$. Denote by

$$j_{\mathcal{V}_{\beta+1}} : N_{E(\kappa,\beta+1)} \to M_{\mathcal{V}_{\beta+1}}$$

the corresponding elementary embedding.
Define

$$E(\kappa,\beta) \circ \mathcal{V}_{\beta+1}.$$

It will be the iterated ultrapower first by $\mathcal{V}_{\beta+1}$ and then by $E(\kappa,\beta)$. [1]
We use Cohen functions from $\mathcal{P}_\kappa(\kappa^{++})$ to κ in order to link the generator $j_{\mathcal{V}_{\beta+1}}{}''\kappa^{++}$ of $\mathcal{V}_{\beta+1}$ with the generators of $E(\kappa,\beta)$. [2]
Then, $E(\kappa,\beta) \circ \mathcal{V}_{\beta+1}$ is a fine $\kappa-$complete ultrafilter over $\mathcal{P}_\kappa(i_{E(\kappa,\beta+1)}(\kappa^{++}))$ in $N_{E(\kappa,\beta+1)}$. Let P be an element of its typical set of measure one. Then, $P \cap \kappa$ is an inaccessible (even a measurable) cardinal, but the projection of P to the normal measure over κ is not anymore $P \cap \kappa$, but rather an ordinal (cardinal) inside $P \cap \kappa$.

Let now $\beta + 1 < \gamma < \omega_1$. Turn to $N_{E(\kappa,\gamma)}$. We have the extenders $E(\kappa,\beta), E(\kappa,\beta+1)$ inside. So,

$$E(\kappa,\beta) \circ \mathcal{V}_{\beta+1} \in N_{E(\kappa,\gamma)}.$$

[1] Note that the resulting ultrapower will be the same if we change the order, i.e. first apply $E(\kappa,\beta)$ and then the image of $\mathcal{V}_{\beta+1}$.

[2] Assume that we forced such functions initially and now only use them changing some values.

We use $W^\gamma(\kappa,\beta)$ to extend $E(\kappa,\beta)\circ\mathcal{V}_{\beta+1}$ to a fine κ-complete ultrafilter over $\mathcal{P}_\kappa(i_{E(\kappa,\beta+1)}(\kappa^{++}))$ inside $N_{E(\kappa,\gamma)}$.

Let

$$j_{W^\gamma(\kappa,\beta)} : N_{E(\kappa,\gamma)} \to M_{W^\gamma(\kappa,\beta)} \simeq \mathrm{Ult}(N_{E(\kappa,\gamma)}, W^\gamma(\kappa,\beta))$$

be the ultrapower embedding. Then

$$N_{E(\kappa,\gamma)} \models M_{W^\gamma(\kappa,\beta)} \text{ is closed under } \kappa^{+3}-\text{sequences of its elements }.$$

In particular,

$$j_{W^\gamma(\kappa,\beta)}{}''\mathcal{V}_{\beta+1} \in M_{W^\gamma(\kappa,\beta)}$$

and it is a $j_{W^\gamma(\kappa,\beta)}(\kappa)-$complete filter there. Pick the least (in $\prec$)

$$Q \in \bigcap j_{W^\gamma(\kappa,\beta)}{}''\mathcal{V}_{\beta+1}.$$

Define an embedding

$$\sigma : \mathrm{Ult}(N_{E(\kappa,\beta+1)}, E(\kappa,\beta)\circ\mathcal{V}_{\beta+1}) \to M_{W^\gamma(\kappa,\beta)}$$

as follows

$$\sigma([f]_{E(\kappa,\beta)(a)\circ\mathcal{V}_{\beta+1})}) = j_{W^\gamma(\kappa,\beta)}(a,Q).$$

It is not elementary, since $N_{E(\kappa,\beta+1)} \subset N_{E(\kappa,\gamma)}$, but still preserves $=,\in$.
If $X \in \mathcal{V}_{\beta+1}$, then $Q \in \sigma(j_{E(\kappa,\beta)\circ\mathcal{V}_{\beta+1}}(X))$.
Apply σ to Cohen functions. Changing value, say of $j_{E(\kappa,\beta)\circ\mathcal{V}_{\beta+1}}(f_\kappa)$ on $i_{E(\kappa,\beta)}{}''[id]_{\mathcal{V}_{\beta+1}}$ to κ will translates to changing the value of $j_{W^\gamma(\kappa,\beta)}(f_\kappa)$ on Q to κ. Similar for the rest of generators of $E(\kappa,\beta)$.[3]

Let $W^{\gamma*}(\kappa,\beta)$ be the least such extension (in $\prec$).

Let now $\gamma<\delta<\omega_1$. Then $W^{\gamma*}(\kappa,\beta) \subseteq W^{\delta*}(\kappa,\beta)$, since

$$k_{E(\kappa,\gamma),E(\kappa,\delta)}(W^{\gamma*}(\kappa,\beta)) = W^{\delta*}(\kappa,\beta).$$

Note that the critical point of $k_{E(\kappa,\gamma),E(\kappa,\delta)}$ is $(\kappa^{+3})^{N_{E(\kappa,\gamma)}} > i_{E(\kappa,\beta+1)}(\kappa^{++})$.

Set

$$W^*(\kappa,\beta) := k_{E(\kappa,\gamma)}(W^{\gamma*}(\kappa,\beta)).$$

Then $W^*(\kappa,\beta)$ is a fine κ-complete ultrafilter over $\mathcal{P}_\kappa(i_{E(\kappa,\beta+1)}(\kappa^{++}))$ in V. In addition it extends every $W^{\delta*}(\kappa,\beta)$.

[3]We have $\kappa^{++}-$many generators. For a generator τ we use the Cohen function f_τ.

Let $\beta+1<\gamma<\omega_1$. Denote by

$$j_{W^{\gamma*}(\kappa,\beta)}: N_{E(\kappa,\gamma)} \to M^{\gamma}_{W^{\gamma*}(\kappa,\beta)} \simeq \mathrm{Ult}(V, W^{\gamma*}(\kappa,\beta))$$

corresponding to $W^{\gamma*}(\kappa,\beta)$ elementary embedding and ultrapower. Similar, let

$$j_{W^*(\kappa,\beta)}: V \to M_{W^*(\kappa,\beta)} \simeq \mathrm{Ult}(V, W^*(\kappa,\beta))$$

corresponding to $W^*(\kappa,\beta)$ elementary embedding and ultrapower.
For every $\beta'<\beta$, $E(\kappa,\beta') \in M^{\gamma}_{W^{\gamma*}(\kappa,\beta)}$ and $E(\kappa,\beta') \in M_{W^*(\kappa,\beta)}$, since $E(\kappa,\beta') \triangleleft E(\kappa,\beta)$ and $M^{\gamma}_{W^{\gamma*}(\kappa,\beta)}, M_{W^*(\kappa,\beta)}$ start with the ultrapower by $E(\kappa,\beta)$.
By definability, then

$$W^{\gamma*}(\kappa,\beta') \in M^{\gamma}_{W^{\gamma*}(\kappa,\beta)} \text{ and } W^*(\kappa,\beta') \in M_{W^*(\kappa,\beta)}.$$

Also, for every $\beta' \leq \beta$ and for every finite a with the measure $E(\kappa,\beta')(a)$ over $\kappa^{|a|}$ defined, we have

$$E(\kappa,\beta')(a) \leq_{RK} W^{\gamma*}(\kappa,\beta) \text{ and } E(\kappa,\beta')(a) \leq_{RK} W^*(\kappa,\beta).$$

Again, this holds since the ultrapower starts with those by $E(\kappa,\beta)$.

The above allows to reflect the sequences

$$\langle E(\kappa,\beta) \mid \beta<\omega_1\rangle, \langle W^{\gamma*}(\kappa,\beta) \mid \beta+1<\gamma<\omega_1\rangle \text{ and } \langle W^*(\kappa,\beta) \mid \beta<\omega_1\rangle$$

down below κ and to define

$$\langle E(\alpha,\beta) \mid \beta<\omega_1\rangle, \langle W^{\gamma*}(\alpha,\beta) \mid \beta+1<\gamma<\omega_1\rangle \text{ and } \langle W^*(\alpha,\beta) \mid \beta<\omega_1\rangle,$$

for $\alpha<\kappa$ in a set $\mathcal{A}$ of measure one for the normal measure U over κ.
The point is that U is the normal measure over κ of every strongly compact measure $W^*(\kappa,\beta)$.
Denote the projection by to U by nor_β. There are only ω_1 many strongly compact measures $W^*(\kappa,\beta)$, so we can assume that there is a single function nor that combines all nor_β's.
For every $\delta<\omega_1$ there is a set A_δ of $W^*(\kappa,\delta)-$measure one such that for every $P \in A_\delta$ the sequences

$$\langle E(\kappa,\beta) \mid \beta<\delta\rangle, \langle W^{\gamma*}(\kappa,\beta) \mid \beta+1<\gamma<\delta\rangle \text{ and } \langle W^*(\kappa,\beta) \mid \beta<\delta\rangle$$

will reflect down an ordinal $\alpha = nor(P)$. Let

$$B := \bigcap_{\delta<\omega_1} nor\text{''} A_\delta \text{ and } A'_\delta := A_\delta \cap nor^{-1}\text{''} B.$$

By shrinking A'_δ's more, if necessary, we can assume that for any $\tau < \delta < \omega_1$ and any $\alpha \in B$, the restriction to τ of the sequences projected from A'_δ is exactly the the sequences projected from A'_τ. Let $\mathcal{A}$ be such B.

Let $\beta + 1 < \gamma < \omega_1$. Consider $k_{E(\kappa,\beta+1),E(\kappa,\gamma)} : N_{E(\kappa,\beta+1)} \to N_{E(\kappa,\gamma)}$. By elementarity,

$$k_{E(\kappa,\beta+1),E(\kappa,\gamma)}(i_{E(\kappa,\beta+1)}(\kappa^{++})) = i_{E(\kappa,\gamma)}(\kappa^{++}).$$

In addition,

$$k_{E(\kappa,\beta+1),E(\kappa,\gamma)}{}''(i_{E(\kappa,\beta+1)}(\kappa^{++}))$$

is unbounded in $i_{E(\kappa,\gamma)}(\kappa^{++})$, since

$$i_{E(\kappa,\beta+1)}(\kappa^{++}) = \sup\{i_{E(\kappa,\beta+1)}(f)(\kappa) \mid f : \kappa \to \kappa^{++}\}$$

and

$$i_{E(\kappa,\gamma)}(\kappa^{++}) = \sup\{i_{E(\kappa,\gamma)}(f)(\kappa) \mid f : \kappa \to \kappa^{++}\}.$$

We will use $k_{E(\kappa,\beta+1),E(\kappa,\gamma)}$ to move from $\mathcal{P}_\kappa(i_{E(\kappa,\beta+1)}(\kappa^{++}))$ to $\mathcal{P}_\kappa(i_{E(\kappa,\xi)}(\kappa^{++}))$, once $\gamma = \xi + 1$.

A crucial thing is that once we have $\beta + 1 < \gamma, \gamma + 1 < \delta < \omega_1$, then $k_{E(\kappa,\beta+1),E(\kappa,\gamma+1)}$ is in $M_{W^*(\kappa,\delta)} \simeq \mathrm{Ult}(V, W^*(\kappa,\delta))$, since it starts with $E(\kappa,\delta+1)$ and $k_{E(\kappa,\beta+1),E(\kappa,\gamma+1)}$ is in $N_{E(\kappa,\delta+1)}$, the ultrapower by $E(\kappa,\delta+1)$.

2 Forcing.

We define here a strongly compact version of the Magidor supercompact forcing based on sequences of filters and ultrafilters

$$\langle W^{\gamma *}(\kappa,\beta) \mid \beta + 1 < \gamma < \omega_1\rangle, \langle W^*(\kappa,\beta) \mid \beta < \omega_1\rangle,$$

$$\langle W^{\gamma *}(\alpha,\beta) \mid \beta + 1 < \gamma < \omega_1\rangle \text{ and } \langle W^*(\alpha,\beta) \mid \beta < \omega_1\rangle,$$

for $\alpha < \kappa$ in $\mathcal{A}$.

A major compensation on luck of normality here is that each $W^*(\alpha,\beta)$ starts with $E(\alpha,\beta)$, which is a coherent sequence of $(\alpha,\alpha^{++})-$extenders.
Further,once we decide to preserve κ^{++}, then the extenders $E(\kappa,\beta)$'s κ will be replaced by subextenders of lengthes below κ^{++} and $\langle W^{\gamma *}(\kappa,\beta) \mid \beta + 1 < \gamma < \omega_1\rangle, \langle W^*(\kappa,\beta) \mid \beta < \omega_1\rangle$ will be redefined accordingly.

For each $\alpha \in \mathcal{A} \cup \{\kappa\}$ let us fix disjoint sets

$$\langle A(\alpha, \beta) \mid \beta < \omega_1 \rangle$$

such that $A(\alpha, \beta) \in W^{\beta+2*}(\alpha, \beta)$. Recall that

$$W^{\beta+2*}(\alpha, \beta) \subseteq W^{\gamma*}(\alpha, \beta) \subseteq W^*(\alpha, \beta),$$

for every $\gamma, \beta + 2 \leq \gamma < \omega_1$. Further, let us always shrink to subsets of $A(\alpha, \beta)$ once dealing with sets of $W^{\gamma*}(\alpha, \beta)-$measure one.
For $P \in \bigcup_{\beta<\omega_1} A(\alpha, \beta)$, denote by $o(P)$ the unique β with $P \in A(\alpha, \beta)$. Denote by $nor(P)$ the projection of P to the normal measure over κ, i.e. the image of P under the projection map of $W^*(\alpha, o(P))$ to $E(\alpha, \beta)(\alpha)$. Note that typically $nor(P) < P \cap \alpha$.

Definition 2.1 Let $\alpha \in \mathcal{A} \cup \{\kappa\}$, $\eta = \omega_1$, if $\alpha = \kappa$ and $\eta < \omega_1$, if $\alpha < \kappa$. We call a subtree of $[\mathcal{P}_\alpha(\theta)]^{<\omega}$ (where θ is large enough) *a nice* $(\alpha, \eta)-$*tree* iff

1. $\mathrm{Lev}_0(T) \in \bigcap_{\beta<\eta} W^*(\alpha, \beta)$,
2. $P \in T$ implies $o(P) < \eta$,
3. for every $P \in T$, $\mathrm{Suc}_T(P) \in \bigcap_{o(P)\leq\beta<\eta} W^*(\alpha, \beta)$.
   Denote $\mathrm{Suc}_T(P) \cap A(\alpha, \beta)$ by $\mathrm{Suc}^\beta_T(P)$.
4. For every $P \in T$ which comes from a level > 0, and every $\beta, o(P) \leq \beta < \eta$, we require $\mathrm{Suc}^\beta_T(P) \subseteq \mathrm{Suc}^\beta_T(P^-)$, where P^- is the immediate predecessor of P in T.

Define now $(\alpha, \eta)-$*good sets* by induction on $\alpha \in \mathcal{A} \cup \{\kappa\}$ and $\eta \leq \omega_1$.

Definition 2.2
1. If $\eta = 1$, then an $(\alpha, \eta)-$good set is just the same as a nice $(\alpha, \eta)-$tree, which in this case has splitting only in $W^*(\alpha, 0)$.
2. if $\eta \geq 2$, then an $(\alpha, \eta)-$good set X is a pair $\langle T, F\rangle$, where
   (a) T is a nice $(\alpha, \eta)-$tree,
   (b) F is a function with domain $\{P \in T \mid o(P) > 0\}$ such that for every $P \in \mathrm{dom}(F)$, $F(P)$ is an $(nor(P), o(P))-$good set.

Define now a direct extension order. We deal first with trees.

Definition 2.3 Let $\alpha \in \mathcal{A} \cup \{\kappa\}$, $\eta = \omega_1$, if $\alpha = \kappa$ and $\eta < \omega_1$, if $\alpha < \kappa$. Let T_1, T_2 be nice $(\alpha, \eta)-$trees. Set $T_1 \leq^* T_2$ iff T_2 is obtained from T_1 by shrinking its levels.

Now we use induction in order to define a direct extension order on $(\alpha, \eta)-$good sets.

Definition 2.4 Let $X_1 = \langle T_1, F_1 \rangle, X_2 = \langle T_2, F_2 \rangle$ be $(\alpha, \eta)-$good sets. Set $X_1 \leq^* X_2$ iff

1. $T_1 \leq^* T_2$,
2. for every $P \in \mathrm{dom}(F_2)$, $F_1(P) \leq^* F_2(P)$.

Let $X = \langle T, F \rangle$ be an $(\alpha, \eta)-$good set and $P \in \mathrm{Lev}_0(T)$. Define a one step extension $X^\frown P$ of X by P.

Definition 2.5 Define $X^\frown P$ to be a pair $\langle T^\frown P, F^\frown P \rangle$, where

1. $T^\frown P = \{Q \in T \mid Q >_T P\}$,
2. $F^\frown P = (F \restriction T^\frown P) \cup \{(P, F(P))\}$.

Intuitively - the Magidor sequence will start now with P, everything in the tree T above P will remain (we will be allowed to shrink things there). In addition, we would like to keep the information below P, i.e. $F(P)$.

Let now $X^\frown P$ be a one step extension of an $(\alpha, \eta)-$good set. Define a one step extension of $X^\frown P$ as follows:

Definition 2.6 There are two possibilities:

1. $Q \in \mathrm{Suc}_T(P)$ and we define $X^\frown P^\frown Q$ to be a pair $\langle T^\frown P^\frown Q, F^\frown P^\frown Q \rangle$, where
   (a) $T^\frown P^\frown Q = \{R \in T \mid R >_T Q\}$,
   (b) $F^\frown P^\frown Q = F \restriction T^\frown P^\frown Q$.

   Or

2. $Q \in \mathrm{Lev}_0(T^P)$ (where $F(P) = \langle T^P, F^P \rangle$, i.e. T^P denotes the tree part of $F(P)$ and F^P its function part)
   and we define $X^\frown P^\frown Q$ to be a pair $\langle T^\frown P^\frown Q, F^\frown P^\frown Q \rangle$, where
   (a) $T^\frown P^\frown Q = T^\frown P$,

(b) $F^\frown P^\frown Q = (F \restriction T^\frown P \setminus \{\langle P, F(P)\rangle\}) \cup \{\langle P, F(P)^\frown Q\rangle\} \cup \{\langle Q, F^P(Q)\rangle\}$.

The intuition behind the first item is clear. In the second one, we move from α to $nor(P)$ and add Q there. $F^P(Q)$ is a $(nor(P), o(Q))-$good set. Its first coordinate is a tree. We prefer not to add it to T explicitly in order to keep T fully over α and not to mix with elements over $nor(Q)$. However, it will be allowed to use elements of the tree of $F^P(Q)$ in further extensions.

If the second possibility occurs, then instead of writing $X^\frown P^\frown Q$ let us write $X^\frown Q^\frown P$, and this way preserve the sequence increasing.
If the first possibility occurs, then let us replace P with its modified version P^Q which we describe below. Note that if one prefer to dealing with ordinals instead of members of $\mathcal{P}_\alpha(\theta)$ and to develop a non-normal version of Magidor forcing, then there is no need in P^Q.
Set

$$P^Q = (P \cap nor(Q)) \cup \{C_\eta(Q) \mid \eta \in P \setminus nor(Q)\},$$

where C_η is the Cohen function which links $[id]$ with η. This way P is turned into a typical member of a set of measure one over $\mathcal{P}_{nor(Q)}(Q \cap \alpha)$.

Continue by induction. Suppose that $X^\frown P_1^\frown ...^\frown P_n$ is defined. Define $n+1-$extension.

Definition 2.7 1. $Q \in \mathrm{Suc}_T(P_n)$ and we define $X^\frown P_1^\frown ...^\frown P_n^\frown Q$ to be a pair $\langle T^\frown P_1^\frown ...^\frown P_n^\frown Q, F^\frown P_1^\frown ...^\frown P_n^\frown Q\rangle$, where

(a) $T^\frown P_1^\frown ...^\frown P_n^\frown Q = \{R \in T \mid R >_T Q\}$,

(b) $F^\frown P_1^\frown ...^\frown P_n^\frown Q = F \restriction T^\frown P_1^\frown ...^\frown P_n^\frown Q$.

Or

2. $Q \in \mathrm{Lev}_0(T^{P_i})$, for some $i, 1 \leq i \leq n$ (where $F(P_i) = \langle T^{P_i}, F^{P_i}\rangle$, i.e. T^{P_i} denotes the tree part of $F(P_i)$ and F^{P_i} its function part) and we define $X^\frown P_1^\frown ...^\frown P_n^\frown Q$ to be a pair $\langle T^\frown P_1^\frown ...^\frown P_n^\frown Q, F^\frown P_1^\frown ...^\frown P_n^\frown Q\rangle$, where

(a) $T^\frown P_1^\frown ...^\frown P_n^\frown Q = T^\frown P_1^\frown ...^\frown P_n$,

(b) $F^\frown P_1^\frown ...^\frown P_n^\frown Q = (F \restriction T^\frown P_1^\frown ...^\frown P_n \setminus \{\langle P_i, F(P_i)\rangle\}) \cup \{\langle P_i, F(P_i)^\frown Q\rangle\} \cup \{\langle Q, F^{P_i}(Q)\rangle\}$.

Again, if the second possibility occurs, then instead of writing $X^\frown P_1^\frown ...^\frown P_n^\frown Q$ let us write $X^\frown P_1^\frown ...^\frown P_{i-1}^\frown Q^\frown P_i^\frown ...^\frown P_n$ and this way preserve the sequence increasing.

If the first possibility occurs, then let us replace $P_j, j \leq i$ with their modified versions P_j^Q as it was done above.

Define a direct order extension $\leq^*$ on the set of $n-$extensions exactly as in Definition 2.4

Define now our forcing notion.

Definition 2.8 Let $\mathcal{P}$ consists of all $n-$extensions of all $(\kappa, \omega_1)-$good sets, for every $n < \omega$.

Definition 2.9 Let $X^\frown P_1{}^\frown ...{}^\frown P_n, Y^\frown Q_1{}^\frown ...{}^\frown Q_m \in \mathcal{P}$. Set

$$X^\frown P_1{}^\frown ...{}^\frown P_n \geq^* Y^\frown Q_1{}^\frown ...{}^\frown Q_m$$

iff

1. $n = m$,
2. $X^\frown P_1{}^\frown ...{}^\frown P_n \geq^* Y^\frown Q_1{}^\frown ...{}^\frown Q_n$, as $n-$extensions.

Define now the forcing order on $\mathcal{P}$.

Definition 2.10 Let $X^\frown P_1{}^\frown ...{}^\frown P_n, Y^\frown Q_1{}^\frown ...{}^\frown Q_m \in \mathcal{P}$. Set

$$X^\frown P_1{}^\frown ...{}^\frown P_n \geq Y^\frown Q_1{}^\frown ...{}^\frown Q_m$$

iff

1. $n \geq m$,
2. $P_i = Q_i$, for every $i, 1 \leq i \leq m$,
3. $Y^\frown P_1{}^\frown ...{}^\frown P_m{}^\frown P_{m+1}{}^\frown ...{}^\frown P_n$ is an $(n-m)-$extension of $Y^\frown P_1{}^\frown ...{}^\frown P_m$,
4. $Y^\frown P_1{}^\frown ...{}^\frown P_m{}^\frown P_{m+1}{}^\frown ...{}^\frown P_n \leq^* X^\frown P_1{}^\frown ...{}^\frown P_m{}^\frown P_{m+1}{}^\frown ...{}^\frown P_n$, as $n-$extensions.

Notation 2.11 Let us return to common notation and instead of writing $X^\frown P_1{}^\frown ...{}^\frown P_n$ write $\langle P_1, ..., P_n, X\rangle$.

Lemma 2.12 $\langle \mathcal{P}, \leq, \leq^* \rangle$ *satisfies the Prikry condition.*

Proof. Let σ be a statement of the forcing language and $p \in \mathcal{P}$. Suppose for simplicity that the trunk of p is empty, i.e. p is of the form $\langle <>, X\rangle$.

Let us call a condition $\langle P_1, ..., P_n, Z\rangle$ a good condition iff all its 1–extensions which come from the same measure conclude the same about σ, i.e.

- all of them force σ,
  or
- all of them force $\neg\sigma$,
  or
- all of them do not decide σ.

Claim 1 Let $\langle P_1, ..., P_n, Y\rangle \in \mathcal{P}$. Then there is $\langle P_1, ..., P_n, Z\rangle \geq^* \langle P_1, ..., P_n, Y\rangle$ which is a good condition.

Proof. Just shrink all relevant measure one sets.
$\square$ of the claim.

Claim 2 Let $\langle <>, Y\rangle \in \mathcal{P}$. Then there is $\langle <>, Z\rangle \geq^* \langle <>, Y\rangle$ such that every $\langle P_1, ..., P_n, Z'\rangle \geq \langle <>, Z\rangle$ is a good condition.

Proof. First apply Claim 1 to $\langle <>, Y\rangle$ and find a direct extension $\langle <>, Z_0\rangle$ which is good. Then apply Claim 1 to each 1–element extension of $\langle <>, Z_0\rangle$ and find its direct extension $\langle <>, Z_1\rangle$ such that any one element extension of $\langle <>, Z_1\rangle$ is a good condition.
Continue by induction and for every $n < \omega$ find $\langle <>, Z_n\rangle$ such that any $n-$element extension of $\langle <>, Z_n\rangle$ is a good condition.
Finally set $Z = \bigcap_{n<\omega} Z_n$.
$\square$ of the claim.

Let us turn now to two element extensions. In contrast to one element extensions, we will have here a new principal situation to consider.

We call a condition $\langle P_1, ..., P_n, Z\rangle$ a 2–good condition iff all its 2–extensions which come from the same measures conclude the same about σ, i.e.

- all of them force σ,
  or
- all of them force $\neg\sigma$,
  or
- all of them do not decide σ.

Let $\langle <>, Z\rangle$ be a condition as in Claim 2, i.e. such that every $\langle P_1, ..., P_n, Z'\rangle \geq \langle <>, Z\rangle$ is a good condition. Denote by T_Z the tree part of Z and by F_Z its function part, i.e. $Z = \langle T_Z, F_Z\rangle$. Suppose that $\langle P, Z\rangle$ is a one element extension of $\langle <>, Z\rangle$ and we extend it further by adding some Q from a higher measure than those of P. In such extension P should be replaced by P^Q. So this two element extension will be $\langle P^Q, Q, Z\rangle$.
Now this can be done an other way around. Thus we can first extend by adding Q, i.e. to $\langle Q, Z\rangle$ and only then pick an element P^Q from $F_Z(Q)$, assuming that it is there. Both ways result in the same condition $\langle P^Q, Q, Z\rangle$. So we need to argue either decides the same way.

Claim 3 Let $\langle <>, Z\rangle$ be as above and $\beta < \gamma < \omega_1$. Then there is $\langle <>, Z^*\rangle \geq^* \langle <>, Z\rangle$ such that any two element extension of $\langle <>, Z^*\rangle$ which comes from measures β and γ provides the same conclusion about σ without any dependence on the way it was created.

Proof. First we shrink the $\gamma-$th measure one set of $\mathrm{Lev}_0(T_Z)$ such that for any Q_1, Q_2 the decisions by $\beta-$th measure one set of $\mathrm{Lev}_0(F_Z(Q_1))$ and those of of $\mathrm{Lev}_0(F_Z(Q_2))$ are the same. Denote the result by Z'. Next we shrink Z' to Z'' such that for $\beta-$th measure one set of $\mathrm{Lev}_0(T_{Z''})$ we will have the decisions by $\gamma-$th measure one set of $\mathrm{Suc}_{T_{Z''}}(P_1)$ and those of of $\mathrm{Suc}_{T_{Z''}}(P_2)$ are the same, for any $P_1, P_2 \in \mathrm{Lev}_{0,\beta}(T_{Z''})$.
We claim now that $Z^* := Z''$ is as desired. Suppose otherwise.
Then there are $\langle P_1, Q_1, Z^*\rangle, \langle P_2, Q_2, Z^*\rangle$ 2–element extensions of $\langle <>, Z^*\rangle$ from measures β, γ which disagree about σ, i.e. one, say $\langle P_1, Q_1, Z^*\rangle$ decides σ and $\langle P_2, Q_2, Z^*\rangle$ does not decide it or decide σ in the opposite fashion. Let us assume that $\langle P_1, Q_1, Z^*\rangle \Vdash \sigma$ and $\langle P_2, Q_2, Z^*\rangle$ does not decide σ.
This type of situation can occur only when this two conditions were obtained in the two different ways. Split into two cases.

Case 1. $\langle P_1, Q_1, Z^*\rangle$ *was obtained by first picking an element of* β *and only then of* γ. Then $\langle P_2, Q_2, Z^*\rangle$, necessarily, was obtained by first picking an element of γ and only then of β. By goodness and the choice of Z^*, then any two element extension which was obtained by first picking an element of β and only then of γ will force σ and any two element extension which was obtained by first picking an element of γ and only then of β will not decide σ.

Denote $\mathrm{Lev}_{0\gamma}(T_{Z^*})$ by A. For every $Q \in A$, denote $\mathrm{Lev}_{0\beta}(T_{F_{Z^*}(Q)})$ by B_Q. Then the function $Q \mapsto B_Q$ represents a set $B \in W^{*\gamma}(\kappa, \beta)$. But recall that $W^{*\gamma}(\kappa, \beta) \subseteq W^*(\kappa, \beta)$. Hence $B \in W^*(\kappa, \beta)$. In particular, $B \cap \mathrm{Lev}_{0\beta}(T_{Z^*}) \neq \emptyset$. Pick some $P \in B \cap \mathrm{Lev}_{0\beta}(T_{Z^*})$. Then the function $Q \mapsto P^Q$ represents P in $\mathrm{Ult}(V, W^*(\kappa, \gamma))$. So, the set $E := \{Q \mid P^Q \in B_Q\}$ is in $W^*(\kappa, \gamma)$. Pick now some $Q \in A \cap \mathrm{Suc}_{T_{Z^*,\gamma}}(P) \cap E$. Then $\langle P^Q, Q, Z^*\rangle \Vdash \sigma$, as two step extension of $\langle <>, Z^* \rangle$ obtained by first picking an element of β and only then of γ. On the other hand $P^Q \in B_Q$, and so $\langle P^Q, Q, Z^* \rangle$ can be viewed as a step extension of $\langle <>, Z^* \rangle$ obtained by first picking an element of γ and only then of β. But this contradicts our assumption that extensions which are obtained this way do not decide σ.

Case 2. $\langle P_1, Q_1, Z^* \rangle$ *was obtained by first picking an element of* γ *and only then of* β. Similar to the previous case.

$\square$ of the claim.

Next we apply Claim 3 to all possible $\beta < \gamma$. As a result a condition
$\langle <>, Z_2 \rangle \geq^* \langle <>, Z \rangle$ will be obtained such any two element extensions of it, which come from same measures agree about σ.

We proceed further by straightforward induction from $n-$extensions to $n+1-$extensions. Let us only deal with the following type of commutativity.

Consider 3−extensions. Let $\beta < \gamma < \delta < \omega_1$. Suppose that $Z^\frown P^\frown Q^\frown R$ is a 3−element extension of Z with P being from $\beta-$th measure, Q being from $\gamma-$th measure and R being from $\delta-$th measure. Now, if P was picked first, than Q and finally R, then the result will be $\langle (P^Q)^R, Q^R, R, Z \rangle$. Note first that $(P^Q)^R = P^Q$, since $P^Q \subseteq Q \cap \kappa < nor(R)$, and so it is not effected by switching from Q to Q^R.

Suppose now that P was added first, R after it and only then Q^R. So we have now $\langle (P^R)^{Q^R}, Q^R, R, Z \rangle$.

Let argue that for most Q's, $\quad (P^R)^{Q^R} = P^{Q^R}$.

Consider the function $R \mapsto Q^R$ which represents Q in the ultrapower by the $\delta-$th measure. P is represented by $R \mapsto P^R$. Let us look at the function $R \mapsto (P^R)^{Q^R}$. It represents P^Q. But note that $P^Q \subset Q \cap \kappa < nor(R)$ and $(P^R)^{Q^R} \subset nor(R)$. So P^Q does not move. Hence $(P^R)^{Q^R} = P^Q$.

$\square$

Let $\langle P_\beta \mid \beta < \omega_1 \rangle$ be a generic sequence. Denote $nor(P_\beta)$ by κ_β, for every $\beta < \omega_1$.

The next lemma is obvious.

Lemma 2.13 *The sequence* $\langle \kappa_\beta \mid \beta < \omega_1 \rangle$ *is an increasing continuous unbounded in* κ *sequence.*

Let us deal now with successors and double successors of κ'_βs.

Lemma 2.14 *For every limit* $\beta < \omega_1$, *both* $(\kappa_\beta^+)^V$ *and* $(\kappa_\beta^{++})^V$ *change their cofinality to* ω, *and both* κ^+ *and* κ^{++} *change their cofinality to* ω_1.

Proof. Let $\beta < \omega_1$ be a limit ordinal or $\beta = \omega_1$. In the last case κ will be just κ_{ω_1}. We use $k_{E(\kappa_\beta,\gamma),E(\kappa_\beta,\delta)}$ in order to move P_γ to P_δ, for $\gamma < \delta < \beta$. Note that, if $\gamma < \delta < \eta < \beta$, then $k_{\kappa_\beta,\gamma,\delta}$ belongs basically to to the ultrapower with $\eta-$th measure. The direct limit of the system

$$\langle\langle P_\gamma \mid \gamma < \beta\rangle, \langle k_{E(\kappa_\beta,\gamma),E(\kappa_\beta,\delta)} \mid \gamma < \delta < \beta\rangle\rangle$$

will produce the desired cofinal sequence. Denote it by $\langle P^\beta_\gamma \mid \gamma < \beta\rangle$.
The point is that the measures that are used start with $(\kappa_\beta, \kappa^{++}_\beta)-$extenders. So we have a nice representation of all the ordinals below κ^{++}_β. Actually, the ordinals below κ^+_β are represented by the canonical functions, but in order to get to κ^{++}_β the extenders are used.
Note that $P_\gamma \cap \kappa_\beta$ does not move. It is the most important over κ it self. Thus, we will need $P^{\omega_1}_\alpha \cap (\kappa^+)^V$, which cardinality is at least $|P_\alpha| >> nor(P_\alpha)^{++}$ (in V), in order to cover the set $\{\sup(P^{\omega_1}_\gamma \cap (\kappa^+)^V) \mid \gamma < \alpha\}$, for a limit $\alpha < \omega_1$. We refer to [2] where situations with coverings of small cardinalities were studied.

Deal with the principal case $\beta = \omega_1$. The case $\beta < \omega_1$ is similar.
Let us proceed as follows. Consider P_0, P_1 and P_2. We have $P_0 \cap nor(P_1)$ is an ordinal below $nor(P_1)$. The rest of P_0 is spread inside the interval $[nor(P_1), (nor(P_1))^{+3})$. Note that $(nor(P_1))^{+3} < P_1 \cap nor(P_2)$.
We are interested in $(P_0 \setminus nor(P_1)) \cap (nor(P_1))^{++}$.
Recall that $P_0 \in \mathcal{P}_{nor(P_1)}((i_{E(nor(P_1),o(P_0))}(nor(P_1))^{++}))$,
which corresponds over κ to $\mathcal{P}_\kappa(i_{E(\kappa,o(nor(P_0))}(\kappa^{++}))$. The embedding $k_{E(\kappa,o(P_0)),E(\kappa,o(P_1))}$ moves the ordinal $i_{E(\kappa,o(P_0))}(\kappa)$ to $i_{E(\kappa,o(P_1))}(\kappa)$. The critical point of
$k_{E(\kappa,o(P_0)),E(\kappa,o(P_1))}$ is $(\kappa^{+3})^{N_{E(\kappa,o(P_0))}}$. So, κ^{++} does not move.

Let us denote $i_{E(\kappa,o(P_\gamma))}(\kappa)$ by $\eta_\gamma, \gamma < \omega_1$. Then, $\eta_\gamma + \kappa^{++}$ will move to $\eta_\delta + \kappa^{++}$, whenever $\gamma \leq \delta < \omega_1$. Each of P_γ's will contribute its part in the interval $[\eta_\gamma, \eta_\gamma + \kappa^{++})$ and this way κ^{++} will be eventually covered.
By a simple density argument, for every $\tau < \kappa^{++}$ there will be $n < \omega, \gamma_1 < ... < \gamma_n < \omega_1$ and $Q \in \mathcal{P}_\kappa(i_{E(\kappa,o(Q))}(\kappa^{++}))$ such that

- $\langle P_{\gamma_1}, ..., P_{\gamma_n}, Q, X\rangle \in G(\mathcal{P})$,
- $i_{E(\kappa,o(Q))}(\kappa^{++}) + \tau \in Q$.

Suppose now that $\langle P_{\gamma_1}, ..., P_{\gamma_n}, Q, X\rangle \leq \langle P_{\gamma_1}, ..., P_{\gamma_n}, Q^R, R, X\rangle \in G(\mathcal{P})$. Then in R, $i_{E(\kappa,o(Q))}(\kappa^{++}) + \tau$ corresponds to $i_{E(\kappa,o(Q))}(\kappa^{++}) + \tau$. This means, in particular, that different τ's will create different sequences (in the direct limit).
Now each sequence is generated by an element of one of P_γ's, for $\gamma < \omega_1$. Hence,

$\bigcup_{\gamma<\omega_1} P_\gamma$ will actually cover a set of size κ^{++}.
$\square$

Our next tusk will be to change slightly the above setting in order to preserve κ^{++} while still collapsing $\kappa_\alpha^+, \kappa_\alpha^{++}$ etc., for α's below ω_1.
It will be achieved by replacing the extenders $E(\kappa,\beta), \beta < \omega_1$, by their subextenders of lengthes below κ^{++}.

Let $\mathfrak{A}$ be an elementary submodel of some H_θ, with θ big enough, of cardinality κ^+, closed under $\kappa-$sequences and with everything relevant inside. We cut all the extenders to $\mathfrak{A}$. Namely each $E(\kappa,\beta), \beta < \omega_1$ is replaced by $\tilde{E}(\kappa,\beta) = E(\kappa,\beta) \restriction \mathfrak{A} := E(\kappa,\beta) \restriction \kappa^{++} \cap \mathfrak{A}$. Consider $i_{\tilde{E}(\kappa,\beta)} : V \to N_{\tilde{E}(\kappa,\beta)} \simeq \mathrm{Ult}(V, \tilde{E}(\kappa,\beta))$. Let $\tilde{\eta}_{\kappa\beta} = i_{\tilde{E}(\kappa,\beta)}(\kappa^{++} \cap \mathfrak{A})$.
Then we define filters and ultrafilters as before but instead of $\mathcal{P}_\kappa(\eta_{\kappa\beta})$ they will be on $\mathcal{P}_\kappa(\tilde{\eta}_{\kappa\beta})$, where $\eta_{\kappa\beta} = i_{E(\kappa,\beta)}(\kappa^{++})$.
The definability of this filters and ultrafilters allows to apply elementary embedding

$$k_{\tilde{E}(\kappa,\beta),E(\kappa,\beta)} : N_{\tilde{E}(\kappa,\beta)} \to N_{E(\kappa,\beta)}$$

in order to move the things to $N_{\tilde{E}(\kappa,\beta)}$.

Define the forcing $\mathcal{P}$ as before only implementing the change made over κ. κ^{++} will not be collapsed now since the present $\mathcal{P}$ satisfies $\kappa^{++}-$c.c. The point is that $\tilde{\eta}_{\kappa\beta} < \kappa^{++}$, for every $\beta < \omega_1$.

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