On regularity of accumulation points.

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Abstract

Regularity of accumulation points is shown.

1 Introduction

W. Mitchell in his celebrating Covering Lemma for sequences of measures (see for example [5]) introduced a notion of an accumulation point. Their existence was shown later to be consistent in [1], and from the optimal assumptions much later in [2]. The purpose of this note is to show regularity of accumulation points under some mild cardinal arithmetic assumptions.

This results (and certainly the techniques) are likely to be known to Mitchell, but we were unable to find any written account.

We refer to Mitchell's excellent handbook article [5] for basic definitions and to an introduction to the core model $\mathcal{K}(\mathcal{F})$ for sequences of measures.

The following notion was defined in [3]:

Definition 1.1 A *generating sequence* for a measure $\mathcal{F}(\kappa, \lambda)$ is a cofinal subset C of κ having a function g such that $g(\nu) \leq o(\nu)$ for all sufficiently large $\nu \in C$ and such that any set $x \in \mathcal{K}(\mathcal{F})$ is in $\mathcal{F}(\kappa, \lambda)$ if and only if for every sufficiently large $\nu \in C$ we have

$$\nu \in x, \text{ if } g(\nu) = o(\nu),$$

$$x \cap \nu \in \mathcal{F}(\nu, g(\nu)), \text{ if } g(\nu) < o(\nu).$$

Let $\beta \leq o(\kappa)$ be the least without a generating sequence. We deal with the case when $\text{cof}(\beta) > \kappa$ and κ changes its cofinality. By Mitchell [3], then $\text{cof}(\kappa) = \omega$, as witnessed by a cofinal in κ sequence of accumulation points.

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2 The argument

Suppose that κ is a strong limit cardinal of cofinality ω which was regular in $\mathcal{K}(\mathcal{F})$. Let $\langle \delta_n \mid n < \omega \rangle$ be an increasing sequence of accumulation points cofinal in κ. Assume that all of them are singular.

Lemma 2.1 $\sup(\{\mathrm{cof}(\delta_n) \mid n < \omega\}) = \kappa$.

Proof. Suppose otherwise. Then $\sup(\{\mathrm{cof}(\delta_n) \mid n < \omega\}) = \delta < \kappa$. Take a covering model N closed under $\delta-$sequences of its elements. Then a final segment of δ_n's will not be accumulation points in N. Contradiction.
$\square$

Now we can assume that the sequence $\{\mathrm{cof}(\delta_n) \mid n < \omega\}$ is increasing. Also assume that $\mathrm{cof}(\delta_0) > \omega$.

Pick a covering model $N \prec H_{\chi^+}$, for some regular χ large enough with $\{\delta_n \mid n < \omega\} \in N$ and ${}^\omega N \subseteq N$. Assume for simplicity that $|N| < \mathrm{cof}(\delta_0)$.
By elementarity, for every $n < \omega$, there is a covering model N_n for δ_n inside N. Then, also, the club C^{N_n} of indiscernibles and the Skolem function h^{N_n} of N_n will be in N. Assume also that $\min(C_{N_n}) > \delta_{n-1}$.
Note that $\sup(C^{N_n} \cap N) < \delta_n$ and $\sup(C^{N_n} \cap N)$ is a limit point of C^{N_n} of uncountable cofinality.
Denote $C^{N_n} \cap N$ by C_n.

Lemma 2.2 *There is a club $E_n \subseteq C_n$ which consists of indiscernibles of N for κ*

Proof. This follows from the fact that a limit of indiscernibles for κ which is below κ is itself an indiscernible for κ. Note that δ_n is an accumulation point, so for every $\xi < \delta_n, \xi \in N$ there is an indiscernible c for κ in N such that $\xi < c < \delta_n$.
$square$

Let prove a stronger statement, however the previous lemma will suffice for our purposes.

Lemma 2.3 *Assume only that δ_n is an indiscernible in N for κ. Then there is $\delta'_n < \sup(C^{N_n} \cap N)$ such that every $c \in C^{N_n} \cap N$ is an indiscernible in N for κ.*

Proof. First note that it is impossible to have a single $\alpha < \kappa$ inside N such that $\alpha^N(c) \leq \alpha$, for an unbounded set of c's in C_n, since then α must be at least δ_n, but $\kappa = \min(h^{N''}\delta_n \setminus \delta_n)$.

Suppose now that there is an increasing sequence $\{c_i \mid i < \omega\} \subseteq C_n$ such that $\bigcup_{i<\omega} c_i = \bigcup_{i<\omega} \alpha^N(c_i)$ and each $\alpha^N(c_i)$ is less than the supremum. Denote $c_\omega = \bigcup_{i<\omega} c_i$. By elementarity, there is $g \in N_n \cap \mathcal{K}(\mathcal{F})$ such that for every $i < \omega$, $\min(g''c_i \setminus c_i) = \alpha^N(c_i) < c_\omega$. We have $c_\omega \in C^{N_n}$ and using coherency inside N_n, for all but finitely many $i < \omega$, the set $\alpha^N(c_i) + 1$ should be in a measure over c_ω, since $c_i \in \alpha^N(c_i) + 1$. This is clearly impossible. $\square$

Fix the sequence $\langle \delta'_n \mid n < \omega \rangle$ given by the lemma.

Consider $\beta_\omega = \sup_{n<\omega} \beta^N(\delta_n)$. Recall that $\beta \leq o(\kappa)$ was the least without a generating sequence and $\mathrm{cof}(\beta) > \kappa$. So, $\beta_\omega < \beta$.

(*)**Assume that the sequence $\langle \mathcal{F}(\kappa, \zeta) \mid \zeta < \beta \rangle$ has no weak repeat point.**

Then there is a set $Z \subseteq \kappa$ such that

1. $Z \in \mathcal{F}(\kappa, \zeta)$, for every $\zeta \leq \beta_\omega$,
2. $Z \notin \mathcal{F}(\kappa, \zeta)$, for every $\zeta, \beta_\omega < \zeta < \beta$.

Set

$$Y = \{\nu \in Z \mid \forall \xi < o(\nu)(Z \cap \nu \in \mathcal{F}(\nu, \xi))\}.$$

Then $Y \in \mathcal{F}(\kappa, \zeta)$, for every $\zeta \leq \beta_\omega$, since otherwise there will be some $\zeta \leq \beta_\omega$ such that $\kappa \setminus Y \in \mathcal{F}(\kappa, \zeta)$. Then, in the ultrapower $\mathrm{Ult}(\mathcal{K}(\mathcal{F}), \mathcal{F}(\kappa, \zeta))$, $o(\kappa) = \zeta$ and $Z \notin \mathcal{F}(\kappa, \xi)$, for some $\xi < \zeta$. This is impossible by the choice of Z, since $\zeta \leq \beta_\omega$.

Then, starting with some $n^* < \omega$, $\delta_n \in Y$ and then $C^{N_n} \subseteq Z \cap \delta_n$. Just take n^* to be such that $\min(C^{N_{n^*}})$ is above the supports of Y, Z.

Now fix any $n \leq n^*$.

Construct by induction two increasing sequences $\langle c_i \mid i < \omega \rangle$ and $\langle a_i \mid i < \omega \rangle$ such that, for every $i < \omega$, the following hold:

1. $c_i \in C_n \setminus \delta'_n + 1$,
2. $c_i < a_i < c_{i+1}$,
3. a_i is an indiscernible for κ in N with $\beta^N(a_i) > \beta_\omega$.

It is easy to do this since δ_n is an accumulation point.

Consider $c = \bigcup_{i<\omega} c_i = \bigcup_{i<\omega} a_i$. Then $c \in C_n$, and so $c \in Z$. On the other hand, $\beta^N(c) \geq \sup_{i<\omega} \alpha^N(a_i) > \beta_\omega$, and so $c \in \kappa \setminus Z$. Which is impossible.

Let us now work without assuming (*).

Define c as above. Then, by its definition, $\beta^N(c) \geq \beta_\omega$ and $\beta^{N_n}(c) < o(\delta_n)$. Moreover,

$\beta^{N_n}(c) \in h^{N_n}{}''c$ and $h^{N_n} \in N \cap \mathcal{K}(\mathcal{F})$. Hence, $h^{N_n} = h^N(d_1, ..., d_\ell, \delta_n)$, for some indiscernibles $d_1, ..., d_\ell$ of N below δ_n. We can assume that $d_1, ..., d_\ell$ are already below c. Now in the argument of the Covering Lemma for N, at the stage δ of the iteration, we will have both $\beta^{N_n}(c)$ and the pre-image of $\beta^N(c)$. By continuing the iteration to the final model, $\beta^{N_n}(c)$ will be mapped to some $\beta' < \beta^N(\delta_n) < \beta^N(c)$. Then, $\mathcal{F}(\kappa, \beta') \neq \mathcal{F}(\kappa, \beta^N(c))$. There will be $A \in \mathcal{F}(\kappa, \beta') \setminus \mathcal{F}(\kappa, \beta^N(c))$ which has a pre-image below c, since both ultrafilters have pre-images at stage before c. Hence, $c \in A \cap \delta_n$, by using h^{N_n} and $c \notin A$, by using h^N. This is impossible. Contradiction.

References

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