Applications of pcf for mild large cardinals to elementary embeddings.

Moti Gitik* and Saharon Shelah†

Abstract

The following pcf results are proved:

1. Assume that $\kappa > \aleph_0$ is a weakly compact cardinal. Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Then for every regular $\lambda < \mathrm{pp}^+_{\Gamma(\kappa)}(\mu)$ there is an increasing sequence $\langle \lambda_i \mid i < \kappa \rangle$ of regular cardinals converging to μ such that $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{bd}_\kappa})$.

2. Let μ be a strong limit cardinal and θ a cardinal above μ. Suppose that at least one of them has an uncountable cofinality. Then there is $\sigma_* < \mu$ such that for every $\chi < \theta$ the following holds:

$$\theta > \sup\{\sup \mathrm{pcf}_{\sigma_*-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi) \text{ and } |\mathfrak{a}| < \mu\}.$$

As an application we show that:
if κ is a measurable cardinal and $j : V \to M$ is the elementary embedding by a κ–complete ultrafilter over a measurable cardinal κ, then for every τ the following holds:

1. *if $j(\tau)$ is a cardinal then $j(\tau) = \tau$;*
2. *$|j(\tau)| = |j(j(\tau))|$;*
3. *for any κ–complete ultrafilter W on κ, $|j(\tau)| = |j_W(\tau)|$.*

The first two items provide affirmative answers to questions from [2] and the third to a question of D. Fremlin.

1 Introduction

We address here the following question:

Suppose κ is a measurable cardinal, U a κ–complete non-trivial ultrafilter over κ and $j : V \to M$ the corresponding elementary embedding. Can one characterize cardinals moved by j?

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There are trivial answers. For example:
τ is moved by j iff $\mathrm{cof}(\tau) = \kappa$ or there is some $\delta < \tau$ with $j(\delta) \geq \tau$.
Also, assuming GCH, it is not hard to find a characterization in terms not mentioning j.

However, it turns out that an answer is possible in terms not mentioning j already in ZFC (Theorem 3.12):

Let τ be a cardinal. Then either

1. $\tau < \kappa$ *and then* $j(\tau) = \tau$,
   or

2. $\kappa \leq \tau \leq 2^\kappa$ *and then* $j(\tau) > \tau$, $2^\kappa < j(\tau) < (2^\kappa)^+$,
   or

3. $\tau \geq (2^\kappa)^+$ *and then* $j(\tau) > \tau$ *iff there is a singular cardinal* $\mu \leq \tau$ *of cofinality* κ *above* 2^κ *such that* $\mathrm{pp}_{\Gamma(\kappa)}(\mu) \geq \tau$, *and if* τ^* *denotes the least such* μ, *then*
   $\tau \leq \mathrm{pp}_{\Gamma(\kappa)}(\tau^*) < j(\tau) < \mathrm{pp}_{\Gamma(\kappa)}(\tau^*)^+$.

Straightforward conclusions of this result provide affirmative answers to questions mentioned in the abstract.

A crucial tool here is PCF–theory and specially Revisited GCH Theorem [5] Sh460.
A new result involving weakly compact cardinal is obtained (Theorem 2.1):

Assume that $\kappa > \aleph_0$ *is a weakly compact cardinal. Let* $\mu > 2^\kappa$ *be a singular cardinal of cofinality* κ. *Then for every regular* $\lambda < \mathrm{pp}^+_{\Gamma(\kappa)}(\mu)$ *there is an increasing sequence* $\langle \lambda_i \mid i < \kappa \rangle$ *of regular cardinals converging to* μ *such that* $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{bd}_\kappa})$.

Also a bit sharper version of [5] Sh460, 2.1 for uncountable cofinality is proved (Theorem 2.5):

Let μ *be a strong limit cardinal and* θ *a cardinal above* μ. *Suppose that at least one of them has an uncountable cofinality. Then there is* $\sigma_* < \mu$ *such that for every* $\chi < \theta$ *the following holds:*

$$\theta > \sup\{\sup \mathrm{pcf}_{\sigma_*-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi) \textit{ and } |\mathfrak{a}| < \mu\}.$$

The first author proved a version of 3.12 assuming certain weak form of the Shelah Weak Hypothesis (SWH)[1] and using [3] Sh371. Then the second author was able to show that the actual assumption used holds in ZFC. All PCF results of the paper are due solely to him.

[1]Consistency of negations of SWH is widely open except very few instances.

Let us recall the definitions of few basic notions of PCF theory that will be used here. Let $\mathfrak{a}$ be a set of regular cardinals above $|\mathfrak{a}|$.

$$\begin{aligned}\mathrm{pcf}(\mathfrak{a}) = \{\mathrm{tcf}((\prod \mathfrak{a}, <_J)) \mid J \text{ is an ideal on } \mathfrak{a} \\ \text{and } (\prod \mathfrak{a}, <_J) \text{ has true cofinality } \}.\end{aligned}$$

Let ρ a cardinal.

$$\begin{aligned}\mathrm{pcf}_{\rho-\mathrm{complete}}(\mathfrak{a}) = \{\mathrm{tcf}((\prod \mathfrak{a}, <_J)) \mid J \text{ is a } \rho-\text{complete ideal on } \mathfrak{a} \\ \text{and } (\prod \mathfrak{a}, <_J) \text{ has true cofinality } \}.\end{aligned}$$

Let η be a cardinal.

$$J_{<\eta}[\mathfrak{a}] = \{\mathfrak{b} \subseteq \mathfrak{a} \mid \text{ for every ultrafilter } D \text{ on } \mathfrak{b}, \mathrm{cf}(\prod \mathfrak{b}, <_D) < \lambda\}.$$

Let λ be a singular cardinal.

$$\begin{aligned}\mathrm{pp}_{\Gamma(\kappa)}(\lambda) = \mathrm{pp}_{\Gamma(\kappa^+,\kappa)}(\lambda) = \sup\{tcf((\prod \mathfrak{a}, <_J)) \mid \mathfrak{a} \text{ is a set of } \kappa \text{ regular cardinals unbounded in } \lambda, \\ J \text{ is a } \kappa-\text{complete ideal on } \mathfrak{a} \text{ which includes } J^{\mathrm{bd}}_{\mathfrak{a}} \text{ and } (\prod \mathfrak{a}, <_J) \text{ has true cofinality } \}.\end{aligned}$$

$\mathrm{pp}^+_{\Gamma(\kappa)}(\lambda)$ denotes the first regular without such representation. [2]

2 PCF results.

Theorem 2.1 *Assume that $\kappa > \aleph_0$ is a weakly compact cardinal. Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Then for every regular $\lambda < \mathrm{pp}^+_{\Gamma(\kappa)}(\mu)$ there is an increasing sequence $\langle \lambda_i \mid i < \kappa \rangle$ of regular cardinals converging to μ such that $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{bd}_\kappa})$.*

Remark 2.2 It is possible to remove the assumption $\mu > 2^\kappa$. Just [4](Sh430) § 6, 6.7A should be used to find the pcf-generators in the proof below. See also 6.3 of Abraham -Magidor handbook article [1].

[2] Note that $\mathrm{pp}^+_{\Gamma(\kappa)}(\lambda) \leq (\mathrm{pp}_{\Gamma(\kappa)}(\lambda))^+$ and it is open if $\mathrm{pp}^+_{\Gamma(\kappa)}(\lambda) < (\mathrm{pp}_{\Gamma(\kappa)}(\lambda))^+$ can ever occur (see [3],Sh355, p.41.)

Proof. By No Hole Theorem (2.3, p.57 [3]), there are a κ–complete ideal I_1 on κ and a sequence of regular cardinals $\vec{\lambda}^1 = \langle \lambda_i^1 \mid i < \kappa \rangle$ with $\mu = \lim_{I_1} \vec{\lambda}^1$ such that
$\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i^1, <_{I_1})$.

Denote the set $\{\lambda_i^1 \mid i < \kappa\}$ by $\mathfrak{a}^1$. Let $\mathfrak{a}^2 = \mathrm{pcf}(\mathfrak{a}^1)$.
Without loss of generality assume that $\lambda = \max \mathrm{pcf}(\mathfrak{a}^1)$. Note that by [3] the following holds:

1. $\mathfrak{a}^1 \subseteq \mathfrak{a}^2 \subseteq Reg \setminus \kappa^{++}$,
2. $\mathrm{pcf}(\mathfrak{a}^2) = \mathfrak{a}^2$,
3. $|\mathrm{pcf}(\mathfrak{a}^2)| \leq 2^\kappa$.

By [3]([Sh345a, 3.6, 3.8(3)) there is a smooth and closed generating sequence for $\mathfrak{a}^1$ (here we use $2^\kappa < \mu$) which means a sequence $\langle \mathfrak{b}_\theta \mid \theta \in \mathfrak{a}^2 \rangle$ such that

1. $\theta \in \mathfrak{b}_\theta \subseteq \mathfrak{a}^2$,
2. $\theta \notin \mathrm{pcf}(\mathfrak{a}^2 \setminus \mathfrak{b}_\theta)$,
3. $\mathfrak{b}_\theta = \mathrm{pcf}(\mathfrak{b}_\theta)$,
4. $\theta_1 \in \mathfrak{b}_{\theta_2}$ implies $\mathfrak{b}_{\theta_1} \subseteq \mathfrak{b}_{\theta_2}$,
5. $\theta = \max \mathrm{pcf}(\mathfrak{b}_\theta)$.

Then by [3][Sh345a,3.2(5)]:
$(*)_1$: if $\mathfrak{c} \subseteq \mathfrak{a}^2$, then for some finite $\mathfrak{d} \subseteq \mathrm{pcf}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{c}) \subseteq \bigcup\{\mathfrak{b}_\theta \mid \theta \in \mathfrak{d}\}$.
The next claim is a consequence of [5](Sh460, 2.1):

Claim 1 There is $\sigma_* < \kappa$ such that for every $\mathfrak{a} \subset Reg \cap (\kappa^+, \mu)$ of cardinality less than κ there is a sequence $\langle \mathfrak{a}_\alpha \mid \alpha < \sigma_* \rangle$ such that

1. $\mathfrak{a} = \bigcup_{\alpha<\sigma_*} \mathfrak{a}_\alpha$,
2. $\max \mathrm{pcf}(\mathfrak{a}_\alpha) < \mu$, for every $\alpha < \sigma_*$.

Proof. The cardinal κ is a strong limit, so we can apply [5](Sh460, 2.1) to κ and μ. Hence there is $\sigma_* < \kappa$ such that for every $\mathfrak{a} \subset Reg \cap (\kappa^+, \mu)$ of cardinality less than κ we have $\mathrm{pcf}_{\sigma_*^+-\mathrm{complete}}(a) \subseteq \mu$. This means that the σ_*^+–complete ideal generated by $J_{<\mu}(\mathfrak{a})$ is everything, i.e. $\mathcal{P}(\mathfrak{a})$. See 8.5 of [1] for the detailed argument. So there are $\mathfrak{a}_\alpha$'s in $J_{<\mu}(\mathfrak{a})$, for

$\alpha < \sigma_*$ such that $\mathfrak{a} = \bigcup_{\alpha<\sigma_*} \mathfrak{a}_\alpha$. But then also $\max \mathrm{pcf}(\mathfrak{a}_\alpha) < \mu$, for every $\alpha < \sigma_*$.
$\square$ of the claim.

Let $\sigma_* < \kappa$ be given by the claim. Let $i < \kappa$. Apply the claim to the set $\mathfrak{a}^1_i := \{\lambda^1_j \mid j < i\}$. So there is a sequence $\langle \mathfrak{a}_{i\alpha} \mid \alpha < \sigma_* \rangle$ such that

1. $\mathfrak{a}^1_i = \bigcup_{\alpha<\sigma_*} \mathfrak{a}_{i\alpha}$,

2. $\max \mathrm{pcf}(\mathfrak{a}_{i\alpha}) < \mu$, for every $\alpha < \sigma_*$.

Now, by $(*)_1$, for every $\alpha < \sigma_*$,

$$\mathrm{pcf}(\mathfrak{a}_{i\alpha}) \subseteq \bigcup \{\mathfrak{b}_\theta \mid \theta \in \mathfrak{d}_{i\alpha}\},$$

for some finite $\mathfrak{d}_{i\alpha} \subseteq \mathrm{pcf}(\mathfrak{a}_{i\alpha})$.
Set $\mathfrak{d}_i = \bigcup_{\alpha<\sigma_*} \mathfrak{d}_{i\alpha}$. Then $\mathfrak{d}_i$ is a subset of μ of cardinality $\leq \sigma_*$. In addition we have $\mathfrak{d}_i \subseteq \mathrm{pcf}(\mathfrak{a}^1_i)$ and $\mathfrak{a}^1_i \subseteq \bigcup \{\mathfrak{b}_\theta \mid \theta \in \mathfrak{d}_i\}$.
Let $\langle \theta_{i,\epsilon} \mid \epsilon < \sigma_* \rangle$ be a listing of $\mathfrak{d}_i$.

Claim 2 There are a function g and $\vec{u} = \langle u_\epsilon \mid \epsilon < \sigma_* \rangle$ such that

1. $g : \kappa \to \kappa$ is increasing,

2. $\xi \leq g(\xi)$, for every $\xi < \kappa$,

3. $\kappa = \bigcup_{\epsilon<\sigma_*} u_\epsilon$,

4. for any $\epsilon < \sigma_*$ and $\xi < \eta < \kappa$ the following holds:

$$\lambda^1_\xi \in \mathfrak{b}_{\theta_{g(\eta),\epsilon}} \text{ iff } \xi \in u_\epsilon.$$

Proof. Here is the place to use the weak compactness of κ.

We will define a κ–tree T and then will use its κ–branch.

Fix $\eta < \kappa$. Let $P \subseteq \sigma_* \times \eta$. Define a set

$$A_P := \{\alpha \in (\eta, \kappa) \mid \forall \xi < \eta \forall \epsilon < \sigma_* (\langle \epsilon, \xi \rangle \in P \Leftrightarrow \lambda^1_\xi \in \mathfrak{b}_{\theta_{\alpha,\epsilon}})\}.$$

Note that always there is $P \subseteq \sigma_* \times \eta$ with $|A_P| = \kappa$. Just $|\mathcal{P}(\sigma_* \times \eta)| < \kappa$, so the function

$$\alpha \longmapsto \langle \langle \epsilon, \xi \rangle \mid \epsilon < \sigma_*, \xi < \eta \text{ and } \lambda^1_\xi \in \mathfrak{b}_{\theta_{\alpha,\epsilon}} \rangle$$

is constant on a set of cardinality κ.
Also for such P we will have $\mathrm{rng}(P)=\eta$, i.e. for every $\xi<\eta$ there is $\epsilon<\sigma_*$ (which may be not unique) such that $(\epsilon,\xi)\in P$. Thus pick $\alpha\in A_P$. Then $\alpha>\eta>\xi$ and $\mathfrak{a}^1_\alpha\subseteq\bigcup\{\mathfrak{b}_\theta\mid\theta\in\mathfrak{d}_\alpha\}$. Clearly λ^1_ξ appears in $\mathfrak{a}^1_\alpha=\{\lambda^1_\nu\mid\nu<\alpha\}$. Hence there is $\epsilon<\sigma_*$ such that $\lambda^1_\xi\in b_{\theta_{\alpha,\epsilon}}$, and so $(\epsilon,\xi)\in P$.

Let

$$T:=\{P\mid\exists\eta<\kappa(P\subseteq\sigma_*\times\eta\text{ and }|A_P|=\kappa)\}.$$

If $P\subseteq\sigma_*\times\eta, P'\subseteq\sigma_*\times\eta'$ are both in T then set $P<_T P'$ iff

- $\eta<\eta'$,
- $P'\cap(\sigma_*\times\eta)=P$.

Then $\langle T,<_T\rangle$ is a κ–tree. Let $X\subseteq\sigma_*\times\kappa$ be a κ–branch. Define now an increasing function $g:\kappa\to\kappa$. Set $g(\eta)=\min(A_{X\cap(\sigma_*\times\eta)}\setminus\sup\{g(\eta')\mid\eta'<\eta\})$.
Let now $\epsilon<\sigma_*$. Define u_ϵ as follows:

$$\xi\in u_\epsilon\text{ iff for some }\eta>\xi\text{ and some (every)}\alpha\in A_{X\cap(\sigma_*\times\eta)},\lambda^1_\xi\in\mathfrak{b}_{\theta_{\alpha,\epsilon}}.$$

Then for any $\epsilon<\sigma_*$ and $\xi<\eta<\kappa$ the following holds:

$$\lambda^1_\xi\in\mathfrak{b}_{\theta_{g(\eta),\epsilon}}\text{ iff }\xi\in u_\epsilon.$$

Finally $|X|=\kappa$ implies that for every $\xi<\kappa$ there is $\epsilon<\sigma_*$ with $\xi\in u_\epsilon$. Thus let $\xi<\kappa$. Pick some $\eta,\xi<\eta<\kappa$. Consider $X\cap(\sigma_*\times\eta)$. Then, as was observed above, there are $\alpha\in A_{X\cap(\sigma_*\times\eta)}$ and $\epsilon<\sigma_*$ such that $\lambda^1_\xi\in b_{\theta_{\alpha,\epsilon}}$. Hence $\xi\in u_\epsilon$.
$\square$ of the claim.

Claim 3 Suppose that $u_\epsilon\in I_1^+$, for some $\epsilon<\sigma_*$. Then $|u_\epsilon|=\kappa$ and the quasi order $\prod_{i\in u_\epsilon}(\theta_{g(i),\epsilon},<_{J^{\mathrm{bd}}_{u_\epsilon}})$ has true cofinality λ.

Proof. κ–completeness of I_1 implies that $|u_\epsilon|=\kappa$, since clearly $\{\xi\}\in I_1$, for every $\xi<\kappa$. Suppose now that the quasi order $\prod_{i\in u_\epsilon}(\theta_{g(i),\epsilon},<_{J^{\mathrm{bd}}_{u_\epsilon}})$ does not have a true cofinality or it has true cofinality $\neq\lambda$. Recall that $\lambda=\max\mathrm{pcf}(\mathfrak{a}_1)$. So by [3](Sh345a) there is an unbounded subset v of u such that $\prod_{i\in v}(\theta_{g(i),\epsilon},<_{J^{\mathrm{bd}}_v})$ has a true cofinality $\lambda_*<\lambda$. We can take λ_* to be just the least δ such that an unbounded subset of u_ϵ appears in $J_{\le\delta}[u_\epsilon]$. Without loss of generality we can assume that $\lambda_*=\max\mathrm{pcf}(\{\theta_{g(i),\epsilon}\mid i\in v\})$. We have

$\lambda_* \in \mathrm{pcf}(\{\theta_{g(i),\epsilon} \mid i \in v\}) \subseteq \mathrm{pcf}(\mathfrak{a}_1) = \mathfrak{a}_2$. Set $v_1 := \{i \in v \mid \theta_{g(i),\epsilon} \in \mathfrak{b}_{\lambda_*}\}$. Then v_1 is unbounded in v. By smoothness of the generators, $i \in v_1$ implies $\mathfrak{b}_{\theta_{g(i),\epsilon}} \subseteq \mathfrak{b}_{\lambda_*}$. Then

$$i \in v_1 \text{ and } \xi \in u_\epsilon \cap i \text{ imply } \lambda^1_\xi \in \mathfrak{b}_{\lambda_*}.$$

But v_1 is unbounded in κ, hence for every $\xi \in u_\epsilon$ there is $i \in v_1, i > \xi$. So, $\{\lambda^1_\xi \mid \xi \in u_\epsilon\} \subseteq \mathfrak{b}_{\lambda_*}$. By the closure of the generators, $\mathrm{pcf}(\mathfrak{b}_{\lambda_*}) = \mathfrak{b}_{\lambda_*}$. Hence $\mathrm{pcf}(\{\lambda^1_\xi \mid \xi \in u_\epsilon\}) \subseteq \mathfrak{b}_{\lambda_*}$. This impossible since $u_\epsilon \in I_1^+$ and so $\lambda \in \mathrm{pcf}(\{\lambda^1_\xi \mid \xi \in u_\epsilon\})$, but $\lambda_* < \lambda$. Contradiction.
$\square$ of the claim.

Claim 4 There is $\epsilon < \sigma_*$ such that $u_\epsilon \in I_1^+$ and $\mu = \lim_{J^{\mathrm{bd}}_\kappa + (\kappa \setminus u_\epsilon)} \langle \theta_{g(i),\epsilon} \mid i < \kappa \rangle$.

Proof. Suppose otherwise. Set $s := \{\epsilon < \sigma_* \mid u_\epsilon \in I_1^+\}$. Then for every $\epsilon \in s$ there is v_ϵ an unbounded subset of κ such that $\theta^*_\epsilon := \sup\{\theta_{g(i),\epsilon} \mid i \in v_\epsilon\}$ is below μ. Set
$\theta_* := \sup\{\theta^*_\epsilon \mid \epsilon \in s\}$. Then $\theta_* < \mu$, since $\mathrm{cof}(\mu) = \kappa > \sigma_*$.
Set $w_1 := \bigcup\{u_\epsilon \mid \epsilon \in \sigma_* \setminus s\}$. Then $w_1 \in I_1$ as a union of less than κ of its members. Also the set $w_2 := \{i < \kappa \mid \lambda^1_i \leq \theta_*\}$ belongs to I_1 because $\mu = \lim_{I_1}\{\lambda^1_i \mid i < \kappa\}$. Hence $w := w_1 \cup w_2 \in I_1$.
Let $\xi \in \kappa \setminus w$. Then

$$\lambda^1_\xi \in \{\lambda^1_\rho \mid \rho < \xi + 1\} \subseteq \bigcup\{\mathfrak{b}_{\theta_{g(\xi+1),\epsilon}} \mid \epsilon < \sigma_*\}.$$

Hence for some $\epsilon < \sigma_*$, $\lambda^1_\xi \in \mathfrak{b}_{\theta_{g(\xi+1),\epsilon}}$. Then $\xi \in u_\epsilon$. Now, $\xi \notin w$ and so $\xi \notin w_1$. Hence $\epsilon \in s$. Pick some $\tau \in v_\epsilon, \tau > \xi$. Then $\lambda^1_\xi \in \mathfrak{b}_{\theta_{g(\tau),\epsilon}}$, since $\xi \in u_\epsilon$. Then

$$\lambda^1_\xi \leq \max(\mathfrak{b}_{\theta_{g(\tau),\epsilon}}) = \theta_{g(\tau),\epsilon} \leq \theta^*_\epsilon \leq \theta_*.$$

But then $\xi \in w_2$. Contradiction.
$\square$ of the claim.
$\square$

Proposition 2.3 *Let $\mathfrak{a}$ be a set of regular cardinals with $\min(\mathfrak{a}) > 2^{|\mathfrak{a}|}$. Let $\sigma < \theta \leq |\mathfrak{a}|$. Suppose that $\lambda \in \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{a})$, $\mu < \lambda$ and $\mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a}) \subseteq \mu$. Then there is $\mathfrak{c} \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$ such that $|\mathfrak{c}| < \theta$, $\mathfrak{c} \subseteq \mu$ and $\lambda \in \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{c})$.*

Remark 2.4 It is possible to replace the assumption $\min(\mathfrak{a}) > 2^{|\mathfrak{a}|}$ by $\min(\mathfrak{a}) > |\mathfrak{a}|$ using [4](Sh430) § 6, 6.7A in order to find the pcf-generators used in the proof.

Proof. Let $\langle \mathfrak{b}_\xi \mid \xi \in \mathrm{pcf}(\mathfrak{a})\rangle$ be a set of generators as in Theorem 2.1. We have $\lambda \in \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{a}) \subseteq \mathrm{pcf}(\mathfrak{a})$, hence $\mathfrak{b}_\lambda$ is defined and $\max\mathrm{pcf}(\mathfrak{b}_\lambda) = \lambda \in \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{a}) \subseteq \mathrm{pcf}(\mathfrak{a})$.
By [4], 6.7F(1), there is $\mathfrak{c} \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a}\cap\mathfrak{b}_\lambda) \subseteq \mu$ of cardinality $< \theta$ such that $\mathfrak{b}_\lambda \cap \mathfrak{a} \subseteq \bigcup\{\mathfrak{b}_\xi \mid \xi \in \mathfrak{c}\}$. Then, by smoothness, $\xi \in \mathfrak{c} \Rightarrow \mathfrak{b}_\xi \subseteq \mathfrak{b}_\lambda$. Also $\mathrm{pcf}(\mathfrak{c}) \subseteq \mathrm{pcf}(\mathfrak{b}_\lambda) = \mathfrak{b}_\lambda$. Hence $\max\mathrm{pcf}(\mathfrak{c}) \leq \lambda$.

Now, if $\lambda \in \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{c})$, then we are done. Suppose otherwise. Then there are $j(*) < \sigma$ and $\theta_j \in \lambda \cap \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{c})$, for every $j < j(*)$, such that $\mathfrak{c} \subseteq \bigcup\{\mathfrak{b}_{\theta_j} \mid j < j(*)\}$. So if $\eta \in \mathfrak{b}_\lambda \cap \mathfrak{a}$, then for some $\chi \in \mathfrak{c}$ we have $\eta \in \mathfrak{b}_\chi$, as $\mathfrak{b}_\lambda \cap \mathfrak{a} \subseteq \bigcup\{\mathfrak{b}_\xi \mid \xi \in \mathfrak{c}\}$. Hence for some $j < j(*)$, $\chi \in \mathfrak{b}_{\theta_j}$, and so $\mathfrak{b}_\chi \subseteq \mathfrak{b}_{\theta_j}$ and $\eta \in \mathfrak{b}_{\theta_j}$.
Then $\mathfrak{b}_\lambda \cap \mathfrak{a} \subseteq \bigcup_{j<j(*)} \mathfrak{b}_{\theta_j}$. Recall that $j(*) < \sigma$ and $\theta_j < \lambda$, for every $j < j(*)$.
Note that $\lambda \in \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{a})$ implies that $\lambda \in \mathrm{pcf}_{\sigma-\mathrm{complete}}(\mathfrak{b}_\lambda \cap \mathfrak{a})$, see for example 4.14 of [1]. So there is a σ–complete ideal J on $\mathfrak{b}_\lambda \cap \mathfrak{a}$ such that $\lambda = \mathrm{tcf}(\prod(\mathfrak{b}_\lambda \cap \mathfrak{a}), <_J)$. Then for some $j < j(*)$, $\mathfrak{b}_{\theta_j} \in J^+$ which is impossible since $\max\mathrm{pcf}(\mathfrak{b}_{\theta_j}) = \theta_j < \lambda$. Contradiction.
□

The next result follows from 2.1 of [5] Sh460.

Theorem 2.5 *Let μ be a strong limit cardinal and θ a cardinal above μ. Suppose that at least one of them has an uncountable cofinality. Then there is $\sigma_* < \mu$ such that for every $\chi < \theta$ the following holds:*

$$\theta > \sup\{\sup\mathrm{pcf}_{\sigma_*-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi) \textit{ and } |\mathfrak{a}| < \mu\}.$$

Proof. Assume first that $\mathrm{cof}(\mu) \neq \mathrm{cof}(\theta)$. Suppose on contrary that

$$\forall \mu^* < \mu \exists \chi < \theta (\theta \leq \sup\{\sup\mathrm{pcf}_{\mu^*-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi) \text{ and } |\mathfrak{a}| < \mu\}).$$

If $\mathrm{cof}(\theta) < \mathrm{cof}(\mu)$, then there will be $\chi < \theta$ such that for every $\mu^* < \mu$

$$\theta \leq \sup\{\sup\mathrm{pcf}_{\mu^*-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi) \text{ and } |\mathfrak{a}| < \mu\}.$$

But this is impossible by 2.1 of [5] applied to μ and χ.
If $\mathrm{cof}(\theta) > \mathrm{cof}(\mu)$, then still there will be $\chi < \theta$ such that for every $\mu^* < \mu$

$$\theta \leq \sup\{\sup\mathrm{pcf}_{\mu^*-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi) \text{ and } |\mathfrak{a}| < \mu\}.$$

Just for every $\mu^* < \mu$ pick some χ_{μ^*} such that

$$\theta \leq \sup\{\sup\mathrm{pcf}_{\mu^*-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi_{\mu^*}) \text{ and } |\mathfrak{a}| < \mu\},$$

and set $\chi = \bigcup_{\mu^* < \mu} \chi_{\mu^*}$.

So let us assume that $\mathrm{cof}(\theta) = \mathrm{cof}(\mu)$. Denote this common cofinality by κ. By the assumption of the theorem $\kappa > \aleph_0$.

Let $\langle \mu_i \mid i < \kappa \rangle$ be an increasing continuous sequence with limit μ such that each μ_i is a strong limit cardinal. Let $\theta > \mu$ be singular cardinal of cofinality κ. Fix an increasing continuous sequence $\langle \theta_i \mid i < \kappa \rangle$ with limit θ such that $\theta_0 > \mu$.

Suppose that there are no $\sigma_* < \mu$ which satisfies the conclusion of the theorem. In particular, for every $i < \kappa$, μ_i cannot serve as σ_*. Hence there is $\chi_i < \theta$ such that

$$\theta = \sup\{\sup \mathrm{pcf}_{\mu_i-\mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \chi_i) \text{ and } |\mathfrak{a}| < \mu\}.$$

So, for each $j < \kappa$, there is $\mathfrak{a}_{i,j} \subseteq \mathrm{Reg} \cap (\mu^+, \chi_i)$ of cardinality less than μ such that $\mathrm{pcf}_{\mu_i-\mathrm{complete}}(\mathfrak{a}_{i,j}) \not\subseteq \theta_j$.

Set $\theta_\kappa := \theta$. For every $i \leq \kappa$, we apply Theorem 2.1 of [5] to μ and θ_i. There is $\sigma_i^* < \mu$ such that

$$\text{if } \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu^+, \theta_i) \text{ and } |\mathfrak{a}| < \mu \text{ then } \mathrm{pcf}_{\sigma_i^*-\mathrm{complete}}(\mathfrak{a}) \subseteq \theta_i.$$

Define now by induction a sequence $\langle i(n) \mid n < \omega \rangle$ such that

1. $i(n) < i(n+1) < \kappa$,
2. $\sigma_\kappa^* < \mu_{i(0)}$,
3. $\sigma_{i(n)}^* < \mu_{i(n+1)}$,
4. $\chi_{i(n)} < \theta_{i(n+1)}$.

Let $i(\omega) = \bigcup_{n<\omega} i(n)$. Then $i(\omega) < \kappa$, since κ is a regular above $\aleph_0$. So $\theta_{i(\omega)} < \theta$. Now, for every $j < \kappa$ and $n < \omega$ the following holds:

$$\mathfrak{a}_{i(n),j} \subseteq \mathrm{Reg} \cap (\mu^+, \chi_{i(n)}) \subseteq \mathrm{Reg} \cap (\mu^+, \theta_{i(n+1)}) \subseteq \mathrm{Reg} \cap (\mu^+, \theta_{i(\omega)}) \text{ and}$$

$$\mathrm{pcf}_{\sigma^*_{i(n+1)}-\mathrm{complete}}(\mathfrak{a}_{i(n),j}) \subseteq \theta_{i(n+1)} < \theta_{i(\omega)}.$$

Let $n < \omega$ and $j \in (i(\omega), \kappa)$. Then by the choice of $\mathfrak{a}_{i(n),j}$ the following holds:

$$\mathfrak{a}_{i(n),j} \subseteq \mathrm{Reg} \cap (\mu^+, \chi_{i(n)}) \subseteq \mathrm{Reg} \cap (\mu^+, \theta_{i(n+1)}) \text{and } \mathrm{pcf}_{\mu_{i(n)}-\mathrm{complete}}(\mathfrak{a}_{i(n),j}) \not\subseteq \theta_j.$$

By the choice of $\sigma^*_{i(n+1)}$, we have

$$\mathrm{pcf}_{\sigma^*_{i(n+1)}-\mathrm{complete}}(\mathfrak{a}_{i(n),j}) \subseteq \theta_{i(n+1)}.$$

By 2.3 there is $\mathfrak{b}_{i(n),j} \subseteq \mathrm{pcf}_{\sigma^*_{i(n+1)}-\mathrm{complete}}(\mathfrak{a}_{i(n),j})$ such that
$|\mathfrak{b}_{i(n),j}| < \sigma^*_{i(n+1)} < \mu_{i(n+2)} < \mu_{i(\omega)}$ and $\mathrm{pcf}_{\mu_{i(n)}-\mathrm{complete}}(\mathfrak{b}_{i(n),j}) \not\subseteq \theta_j$.
Obviously, $\mathfrak{b}_{i(n),j} \subseteq \mathrm{Reg} \cap (\mu^+, \theta_{i(n+1)})$, since
$\mathrm{pcf}_{\sigma^*_{i(n+1)}-\mathrm{complete}}(\mathfrak{a}_{i(n),j}) \subseteq \theta_{i(n+1)}$.

Apply Theorem 2.1 of [5] to $\mu_{i(\omega)}$ (recall that it is a strong limit) and $\theta_{i(\omega)}$. So, there is $\sigma_* < \mu_{i(\omega)}$ such that

$$\text{if } \mathfrak{b} \subseteq \mathrm{Reg} \cap (\mu^+_{i(\omega)}, \theta_{i(\omega)}) \text{ and } |\mathfrak{b}| < \mu_{i(\omega)} \text{ then } \mathrm{pcf}_{\sigma_*-\mathrm{complete}}(\mathfrak{b}) \subseteq \theta_{i(\omega)}.$$

Now take $n_* < \omega$ with $\mu_{i(n_*)} > \sigma_*$. Then $\mathfrak{b}_{i(n_*),j} \subseteq \mathrm{Reg} \cap (\mu^+_{i(\omega)}, \theta_{i(\omega)})$ and $|\mathfrak{b}_{i(n_*),j}| < \mu_{i(\omega)}$, but $\mathrm{pcf}_{\mu_{i(n_*)}-\mathrm{complete}}(\mathfrak{b}_{i(n_*),j}) \not\subseteq \theta_j > \theta_{i(\omega)}$. Which is impossible. Contradiction.
□

3 Applications.

Let κ be a measurable cardinal, U be a κ–complete non-principle ultrafilter over κ and let $j_U : V \to M \simeq {}^\kappa V/U$ be the corresponding elementary embedding. Denote j_U further simply by j.

Lemma 3.1 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Then $j(\mu) \geq \mathrm{pp}_{\Gamma(\kappa)}(\mu)$.*

Proof. Let $\lambda < \mathrm{pp}^+_{\Gamma(\kappa)}(\mu)$ be a regular cardinal. Then, by Theorem 2.1, there is an increasing sequence of regular cardinals $\langle \lambda_i \mid i < \kappa \rangle$ converging to μ such that $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$. The ultrafilter U clearly extends the dual to J^{bd}_κ. Hence $[\langle \lambda_i \mid i < \kappa \rangle]_U$ represents an ordinal below $j(\mu)$ of cofinality λ. Hence $j(\mu) > \lambda$ and we are done.
□

Let us denote for a singular cardinal μ of cofinality κ by μ^* the least singular $\xi \leq \mu$ of cofinality κ above 2^κ such that $\mathrm{pp}_{\Gamma(\kappa)}(\xi) \geq \mu$.
Then, by [3](Sh 355, 2.3(3), p.57), $\mathrm{pp}_{\Gamma(\kappa)}(\mu) \leq^+ \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)$.

Lemma 3.2 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Then $j(\mu) \geq \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)$.*

Proof. By 3.1, $j(\mu^*) \geq \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)$. But $\mu^* \leq \mu$, hence $j(\mu^*) \leq j(\mu)$.
□

Lemma 3.3 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Let $\eta, \mu < \eta < j(\mu)$ be a regular cardinal. Then $\eta \leq \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)$.*

Proof.
Let $\eta, \mu < \eta < j(\mu)$ be a regular cardinal. Let $f_\eta : \kappa \to \mu$ be a function which represents η in M, i.e. $[f_\eta]_U = \eta$. We can assume that $\mathrm{rng}(f_\eta) \subseteq \mathrm{Reg} \cap ((2^\kappa)^+, \mu)$, since $|j(2^\kappa)| = 2^\kappa$ and so $j(2^\kappa) < \mu < \eta$. Set $\tau := U$–limit of $\mathrm{rng}(f_\eta)$.[3] Then $\tau > 2^\kappa$.
Note that $\mathrm{cof}(\tau) = \kappa$. Otherwise, f_η is just a constant function mod U. Let δ be the constant value. Then $\delta < j(\delta) = \eta$. By elementarity δ must be a regular cardinal. But then $j''\delta$ is unbounded in η, which means that η is a singular cardinal. Contradiction.
Denote $f(\alpha)$ by τ_α, for every $\alpha < \kappa$. Then each τ_α is a regular cardinal in the interval $((2^\kappa)^+, \tau)$ and $\tau = \lim_U \langle \tau_\alpha \mid \alpha < \kappa \rangle$. We have $\eta = \mathrm{tcf}(\prod_{\alpha<\kappa} \tau_\alpha, <_U)$.
Note that once U is not normal we cannot claim that the function $\alpha \mapsto \tau_\alpha$ is one to one. So there is a slight tension between the true cofinalities of the sequence $\langle \tau_\alpha \mid \alpha < \kappa \rangle$ and of the set $\{\tau_\alpha \mid \alpha < \kappa\}$.
We will show in the next lemma (3.4) that this does not effect $\mathrm{pp}_{\Gamma(\kappa)}(\tau)$.
Namely, $\eta = \mathrm{tcf}(\prod_{\alpha<\kappa} \tau_\alpha, <_U)$ implies $\mathrm{pp}_{\Gamma(\kappa)}(\tau) \geq \eta > \mu$.[4]
Then, by the choice of μ^*, we have $\mu^* \leq \tau$ By [3](Sh 355, 2.3(3), p.57), $\mathrm{pp}_{\Gamma(\kappa)}(\mu^*) \geq \mathrm{pp}_{\Gamma(\kappa)}(\tau)$.
□

Lemma 3.4 [5] *Let κ be a regular cardinal and τ be a singular cardinal of cofinality κ. Then*

$$\mathrm{pp}_{\Gamma(\kappa)}(\tau) = \sup\{\mathrm{tcf}(\prod_{\alpha<\kappa} \tau_\alpha, <_I) \mid \langle \tau_\alpha \mid \alpha < \kappa \rangle \textit{ is a sequence of regular cardinals with}$$

$$\lim_I \langle \tau_\alpha \mid \alpha < \kappa \rangle = \tau, I \textit{ is a } \kappa \textit{ complete ideal over } \kappa \textit{ which extends } J^{\mathrm{bd}}_\kappa\}.$$

Proof. Clearly,

$$\mathrm{pp}_{\Gamma(\kappa)}(\tau) \leq \sup\{\mathrm{tcf}(\prod_{\alpha<\kappa} \tau_\alpha, <_I) \mid \langle \tau_\alpha \mid \alpha < \kappa \rangle \text{ is a sequence of regular cardinals with}$$

$$\lim_I \langle \tau_\alpha \mid \alpha < \kappa \rangle = \tau, I \text{ is a } \kappa \text{ complete ideal over } \kappa \text{ which extends } J^{\mathrm{bd}}_\kappa\}.$$

Just if $\eta = tcf((\prod \mathfrak{a}, <_J))$, where $\mathfrak{a}$ is a set of κ regular cardinals unbounded in τ, J is a $\kappa-$ complete ideal on $\mathfrak{a}$ which includes $J^{\mathrm{bd}}_{\mathfrak{a}}$. Then we can view $\mathfrak{a}$ as a κ–sequence.

[3] It is possible to force a situation where such $\tau < \mu$. Start with a η^{++}–strong $\tau, \kappa < \tau < \mu$. Use the extender based Magidor to blow up the power of τ to η^+ simultaneously changing the cofinality of τ to κ. The forcing satisfies κ^{++}-c.c., so it will not effect pp structure of cardinals different from τ.
[4] Actually, the original definition of pp ([3]II,Definition 1.1, p.41) involves sequences rather than sets.
[5] A version of this lemma was suggested by Menachem Magidor.

Let us deal with the opposite direction. Suppose that $\eta = \mathrm{tcf}(\prod_{\alpha<\kappa} \tau_\alpha, <_I)$, where $\langle \tau_\alpha \mid \alpha < \kappa \rangle$ is a sequence of regular cardinals with $\lim_I \langle \tau_\alpha \mid \alpha < \kappa \rangle = \tau$, I is a κ complete ideal over κ which extends J^{bd}_κ. Without loss of generality we can assume that $\kappa < \tau_\alpha < \tau$, for every $\alpha < \kappa$. Set $\mathfrak{a} = \{\tau_\alpha \mid \alpha < \kappa\}$. Define a projection $\pi : \kappa \to \mathfrak{a}$ by setting $\pi(\alpha) = \tau_\alpha$. Let

$$J := \{X \subseteq \mathfrak{a} \mid \pi^{-1}{}''X \in I\}.$$

Then J will be a κ–complete ideal on $\mathfrak{a}$ which extends $J^{\mathrm{bd}}_{\mathfrak{a}}$.

Let us argue that $\eta = \mathrm{tcf}(\prod \mathfrak{a}, <_J)$.

Fix a scale $\langle f_i \mid i < \eta \rangle$ which witnesses $\eta = \mathrm{tcf}(\prod_{\alpha<\kappa} \tau_\alpha, <_I)$. Define for a function $f \in \prod_{\alpha<\kappa} \tau_\alpha$ a function $\bar{f} \in \prod_{\alpha<\kappa} \tau_\alpha$ as follows:

$$\bar{f}(\alpha) = \sup\{f(\beta) \mid \tau_\beta = \tau_\alpha\}.$$

Note that for every $\alpha < \kappa$, $\quad \bar{f}(\alpha) < \tau_\alpha$, since τ_α is a regular cardinal above κ.
Consider the sequence $\langle \bar{f}_i \mid i < \kappa \rangle$. It need not be a scale, since the sequence need not be I–increasing. But this is easy to fix. Just note that for every $i < \eta$ there will be $i', i \leq i' < \eta$, such that

$$f_i \leq \bar{f}_i \leq_I \bar{f}_{i'}.$$

Just given $i < \eta$, find some $i', i \leq i' < \eta$, such that $\bar{f}_i \leq_I f_{i'}$. Then $\bar{f}_i \leq_I f_{i'} \leq \bar{f}_{i'}$. Now by induction it is easy to shrink the sequence $\langle \bar{f}_i \mid i < \kappa \rangle$ and to obtain an I–increasing subsequence $\langle g_\xi \mid \xi < \eta \rangle$ which is a scale in $(\prod_{\alpha<\kappa} \tau_\alpha, <_I)$.
For every $\xi < \eta$ define $h_\xi \in \prod \mathfrak{a}$ as follows:

$$h_\xi(\rho) = g_\xi(\alpha), \text{ if } \rho = \tau_\alpha, \text{ for some (every) } \alpha < \kappa.$$

It is well defined since $g_\xi(\alpha) = g_\xi(\beta)$ once $\tau_\alpha = \tau_\beta$.
Let us argue that $\langle h_\xi \mid \xi < \eta \rangle$ is a scale in $(\prod \mathfrak{a}, <_J)$.
Clearly, $\xi < \xi'$ implies $h_\xi <_J h_{\xi'}$, since $g_\xi <_I g_{\xi'}$.
Let $h \in \prod \mathfrak{a}$. Consider $g \in \prod_{\alpha<\kappa} \tau_\alpha$ defined by setting $g(\alpha) = h(\tau_\alpha)$. There is $\xi < \eta$ such that $g <_I g_\xi$. Then $h <_J h_\xi$, since

$$\pi^{-1}{}''\{\rho \in \mathfrak{a} \mid h(\rho) < h_\xi(\rho)\} \supseteq \{\alpha < \kappa \mid g(\alpha) < g_\xi(\alpha)\}.$$

□

Theorem 3.5 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ.*
Then $\mathrm{pp}_{\Gamma(\kappa)}(\mu^*) \leq j(\mu) < \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)^+$.

Proof. Note that $j(\mu)$ is always singular. Just μ is a singular cardinal, hence $j(\mu)$ is a singular in M and so in V. Now the conclusion follows by 3.2,3.3.
□

We can deduce now an affirmative answer to a question of D. Fremlin for cardinals of cofinality κ:[6]

Corollary 3.6 *Let W be a non-principal κ–complete ultrafilter on κ and $j_W : V \to M_W$ the corresponding elementary embedding. Then for every μ of cofinality κ, $|j(\mu)| = |j_W(\mu)|$.*

Proof. Let μ be a cardinal of cofinality κ. If $\mu < 2^\kappa$, then $2^\kappa < j_W(\mu) < j_W(2^\kappa) < (2^\kappa)^+$, for any non-principal κ–complete ultrafilter W on κ.
If $\mu > 2^\kappa$, then, by 3.5, $\mathrm{pp}_{\Gamma(\kappa)}(\mu^*) \leq j(\mu) < \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)^+$. But recall that j was the elementary embedding of an arbitrary non-principal κ–complete ultrafilter U on κ and the bounds do not depend on it. Hence if W is an other non-principal κ–complete ultrafilter on κ, then $\mathrm{pp}_{\Gamma(\kappa)}(\mu^*) \leq j_W(\mu) < \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)^+$.
□

Corollary 3.7 *For every μ of cofinality κ, $|j(\mu)| = |j(j(\mu))|$.*

Proof. It follows from 3.6. Just take $W = U^2$ and note that $j(j(\mu)) = j_{U^2}(\mu)$.
□

Our next tusk will be to show that the fist inequality is really a strict inequality.

Lemma 3.8 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Then $pp_{\Gamma(\kappa)}(\mu) \leq (pp_{\Gamma(\kappa)}(\mu))^M$.*[7]

Proof. Let $\eta, \mu < \eta < pp^+_{\Gamma(\kappa)}(\mu)$ be a regular cardinal.
By Theorem 2.1, there is an increasing converging to μ sequence $\langle \eta_i \mid i < \kappa \rangle$ of regular cardinals such that

$$\eta = \mathrm{tcf}(\prod_{i<\kappa} \eta_i, <_{J^{\mathrm{bd}}_\kappa}).$$

Note that both $\langle \eta_i \mid i < \kappa \rangle$ and J^{bd}_κ are in M. Also ${}^\kappa M \subseteq M$, hence each function of the witnessing scale is in M, however the scale itself may be not in M. Still we can work inside M and define a scale recursively using functions from the V-scale.

[6] Readers interested only in a full answer to Fremlin's question can jump after the corollary directly to 3.12. The non-strict inequality in its conclusion suffices.

[7] $(pp_{\Gamma(\kappa)}(\mu))^M$ stands for $pp_{\Gamma(\kappa)}(\mu)$ as computed in M. Note that it is possible to have $(pp_{\Gamma(\kappa)}(\mu))^M > pp_{\Gamma(\kappa)}(\mu)$, just as $(2^\kappa)^M > 2^\kappa$.

Thus let $\langle f_\tau \mid \tau < \eta\rangle$ be a scale mod $J^{\rm bd}_\kappa$ which witnesses $\eta = {\rm tcf}(\prod_{i<\kappa}\eta_i, <_{J^{\rm bd}_\kappa})$. Work in M and define recursively an increasing mod $J^{\rm bd}_\kappa$ sequence of functions $\langle g_\xi \mid \xi < \eta'\rangle$ in $\prod_{i<\kappa}\eta_i$ as far as possible.

We claim first that ${\rm cof}(\eta') = \eta$, as computed in V. Thus if $\eta < {\rm cof}(\eta')$, then there will be $\tau^* < \eta$ such that $f_{\tau^*} \geq_{J^{\rm bd}_\kappa} g_\xi$, for every $\xi < \eta'$, since for every $\xi < \eta'$ there is $\tau < \eta$ such that $f_\tau \geq_{J^{\rm bd}_\kappa} g_\xi$. But having $f_{\tau^*} \geq_{J^{\rm bd}_\kappa} g_\xi$, for all $\xi < \eta'$, we can continue and define $g_{\eta'}$ to be f_{τ^*}. If $\eta > {\rm cof}(\eta')$, then again there will be $\tau^* < \eta$ such that $f_{\tau^*} \geq_{J^{\rm bd}_\kappa} g_\xi$, for every $\xi < \eta'$, and again we can continue and define $g_{\eta'}$ to be f_{τ^*}.

So ${\rm cof}(\eta') = \eta$. Let $\langle \eta'_\tau \mid \tau < \eta\rangle$ be a cofinal in η' sequence (in V). Now, for every $\tau < \eta$ there is $\tau', \tau \leq \tau' < \eta$ such that $f_\tau \not\geq_{J^{\rm bd}_\kappa} g_{\tau'}$, since the sequence $\langle g_\xi \mid \xi < \eta'\rangle$ is maximal. Hence there is $A_\tau \subseteq \kappa, |A_\tau| = \kappa$ such that $f_\tau \restriction A_\tau <_{J^{\rm bd}_\kappa} g_{\eta'_{\tau'}} \restriction A_\tau$. But $\eta > \mu > 2^\kappa$, hence there is $A^* \subseteq \kappa$ such that for η many τ's we have $A^* = A_\tau$. Then for every $\tau < \eta$ there is $\tau'', \tau \leq \tau'' < \eta$ such that $f_\tau \restriction A^* <_{J^{\rm bd}_\kappa} g_{\eta'_{\tau''}} \restriction A^*$.

It follows that the sequence $\langle g_\xi \restriction A^* \mid \xi < \eta'\rangle$ is a scale in ${\rm tcf}(\prod_{i\in A^*}\eta_i, <_{J^{\rm bd}_{A^*}})$. Hence, in M, $\eta' < pp^+_{\Gamma(\kappa)}(\mu)$. But ${\rm cof}(\eta') = \eta$, hence, in M, $\eta \leq \eta' < pp^+_{\Gamma(\kappa)}(\mu)$.

□

Lemma 3.9 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ such that $\mu^* = \mu$. Then $j(\xi) < \mu$ for every $\xi < \mu$.*

Proof. Suppose otherwise. Then there is $\xi < \mu$ such that $j(\xi) \geq \mu$. Necessarily $\xi > 2^\kappa$. Let η be a regular cardinal $\xi \leq \eta < \mu$. Pick a function $f_\eta : \kappa \to \xi$ which represents η in M. Without loss of generality we can assume that $\min({\rm rng}(f_\eta)) > 2^\kappa$. Let $\delta_\eta \leq \xi$ be the U–limit of ${\rm rng}(f_\eta)$. Then ${\rm cof}(\delta_\eta) = \kappa$ and $j(\delta_\eta) > \eta$. Also $\eta \leq {\rm pp}_{\Gamma(\kappa)}(\delta_\eta)$, by the definition of ${\rm pp}_{\Gamma(\kappa)}(\delta_\eta)$. By Lemma 3.2, we have $j(\delta_\eta) \geq pp_{\Gamma(\kappa)}((\delta_\eta)^*)$, and by [3](Sh 355, 2.3(3), p.57), ${\rm pp}_{\Gamma(\kappa)}(\delta_\eta) \leq {\rm pp}_{\Gamma(\kappa)}((\delta_\eta)^*)$. Set

$$\delta := \min\{\delta_\eta \mid \xi \leq \eta < \mu \text{ and } \eta \text{ is a regular cardinal }\}.$$

Then ${\rm pp}_{\Gamma(\kappa)}(\delta) \geq {\rm pp}_{\Gamma(\kappa)}(\delta_\eta)$, for every regular $\eta, \xi \leq \eta < \mu$. But ${\rm pp}_{\Gamma(\kappa)}(\delta_\eta) \geq \eta$. Hence ${\rm pp}_{\Gamma(\kappa)}(\delta) \geq \mu$ which is impossible since $\mu^* = \mu$. Contradiction.

□

Lemma 3.10 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ. Then ${\rm pp}_{\Gamma(\kappa)}(\mu^*) < j(\mu)$.*

Proof. By 3.2 we have $j(\mu) \geq \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)$.
Suppose that $j(\mu) = \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)$. Then $\mu = \mu^*$, since by 3.2 we have $j(\mu^*) \geq \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)$. By Theorem 2.5, there is $\sigma_* < \kappa$ such that

$$\forall \chi < \mu (\mu > \sup\{\sup \mathrm{pcf}_{\sigma_* - \mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (\kappa^+, \chi) \wedge |\mathfrak{a}| < \kappa\}).$$

Then, by elementarity,

$$M \models \forall \chi < j(\mu)(j(\mu) > \sup\{\sup \mathrm{pcf}_{j(\sigma_*) - \mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (j(\kappa^+), \chi) \wedge |\mathfrak{a}| < j(\kappa)\}).$$

Clearly, $j(\sigma_*) = \sigma_*$. Take $\chi = \mu$. Let η be a regular cardinal (i.e. of V) such that

$$(*) \quad M \models j(\mu) > \eta > \sup\{\sup \mathrm{pcf}_{\sigma_* - \mathrm{complete}}(\mathfrak{a}) \mid \mathfrak{a} \subseteq \mathrm{Reg} \cap (j(\kappa^+), \mu) \wedge |\mathfrak{a}| < j(\kappa)\}.$$

Note that there are such η's since $j(\mu)$ is a singular cardinal of cofinality $\mathrm{cof}(j(\kappa))$. By Lemma 3.3, then $\eta \leq \mathrm{pp}_{\Gamma(\kappa)}(\mu)$. Now, by Lemma 3.8, $\mathrm{pp}_{\Gamma(\kappa)}(\mu) \leq (\mathrm{pp}_{\Gamma(\kappa)}(\mu))^M$. Hence $M \models \eta \leq \mathrm{pp}_{\Gamma(\kappa)}(\mu)$. But then there is $\mathfrak{a} \in M$ such that

$$M \models \mathfrak{a} \subseteq \mathrm{Reg} \cap (j(\kappa^+), \mu) \wedge |\mathfrak{a}| = \kappa \wedge \eta \leq \max \mathrm{pcf}_{\kappa - \mathrm{complete}}(\mathfrak{a}).$$

Which clearly contradicts $(*)$.
$\square$

So we proved the following:

Theorem 3.11 *Let $\mu > 2^\kappa$ be a singular cardinal of cofinality κ.*
Then $\mathrm{pp}_{\Gamma(\kappa)}(\mu^*) < j(\mu) < \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)^+$.

Deal now with cardinals of arbitrary cofinality.

Theorem 3.12 *Let τ be a cardinal. Then either*

1. *$\tau < \kappa$ and then $j(\tau) = \tau$,*
   or
2. *$\kappa \leq \tau \leq 2^\kappa$ and then $j(\tau) > \tau$, $2^\kappa < j(\tau) < (2^\kappa)^+$,*
   or
3. *$\tau \geq (2^\kappa)^+$ and then $j(\tau) > \tau$ iff there is a singular cardinal $\mu \leq \tau$ of cofinality κ above 2^κ such that $\mathrm{pp}_{\Gamma(\kappa)}(\mu) \geq \tau$, and if τ^* denotes the least such μ, then*
   $\tau \leq \mathrm{pp}_{\Gamma(\kappa)}(\tau^*) < j(\tau) < \mathrm{pp}_{\Gamma(\kappa)}(\tau^*)^+$.

Proof. Suppose otherwise. Let τ be the least cardinal witnessing this. Clearly then $\tau > (2^\kappa)^+$. If $\mathrm{cof}(\tau) = \kappa$, then we apply 3.11 to derive the contradiction. Suppose that $\mathrm{cof}(\tau) \neq \kappa$.

Claim 5 There is a singular cardinal ξ of cofinality κ such that $j(\xi) > \tau$.

Proof. Thus let $f_\tau : \kappa \to \tau$ be a function which represents τ in M. Without loss of generality we can assume that

$$\nu \in \mathrm{rng}(f_\tau) \Rightarrow (\nu > 2^\kappa \text{ and } \nu \text{ is a cardinal }).$$

Then either f_τ is a constant function mod U or $\xi := U$–limit $\mathrm{rng}(f_\tau)$ has cofinality κ.
Suppose first that f_τ is a constant function mod U with value ξ. If $\xi = \tau$, then $j(\tau) = \tau$. Suppose that $\xi < \tau$. Then $j(\xi) = \tau > \xi$ and also ξ is a cardinal above 2^κ. By minimality of τ then ξ^* exists and

$$\mathrm{pp}_{\Gamma(\kappa)}(\xi^*) < \tau = j(\xi) < \mathrm{pp}_{\Gamma(\kappa)}(\xi^*)^+.$$

But this is impossible since τ is a cardinal. Contradiction. So $\mathrm{cof}(\xi) = \kappa$ and $j(\xi) > \tau$.
□ of the claim.

Let $\mu \leq \tau$ be the least singular cardinal above 2^κ of cofinality κ such that $j(\mu) > \tau$. We claim that $\mu = \mu^*$. Note that by 3.11, we have $\mathrm{pp}_{\Gamma(\kappa)}(\mu^*) < j(\mu^*) \leq j(\mu) < \mathrm{pp}_{\Gamma(\kappa)}(\mu^*)^+$.
τ is a cardinal below $j(\mu)$, hence $\tau \leq \mathrm{pp}_{\Gamma(\kappa)}(\mu^*) < j(\mu^*)$. The minimality of μ implies then that $\mu = \mu^*$. Note that also $\tau^* = \mu$. Thus $\mathrm{pp}_{\Gamma(\kappa)}(\tau^*) \geq \tau \geq \mu = \mu^*$, and so $\tau^* \geq \mu$. Also $\tau \leq \mathrm{pp}_{\Gamma(\kappa)}(\mu)$ implies $\tau^* \leq \mu$.
Apply finally 3.7. It follows that $|j(j(\mu))| = |j(\mu)|$, but $j(\mu) > \tau$, hence $j(j(\mu)) > j(\tau) > j(\mu)$. So

$$\mathrm{pp}_{\Gamma(\kappa)}(\mu) < j(\mu) < j(\tau) < \mathrm{pp}_{\Gamma(\kappa)}(\mu)^+,$$

and we are done.
□

Now affirmative answers to a question of D. Fremlin and to questions 4,5 of [2] follow easily.[8]

Corollary 3.13 *Let W be a non-principal κ–complete ultrafilter on κ and $j_W : V \to M_W$ the corresponding elementary embedding. Then for every τ, $|j(\tau)| = |j_W(\tau)|$.*

Proof. Let W be a non-principal κ–complete ultrafilter on κ and $j_W : V \to M_W$ the corresponding elementary embedding. Let τ be an ordinal. Without loss of generality we

[8]Non strict inequality $\mathrm{pp}_{\Gamma(\kappa)}(\tau^*) \leq j(\tau) < \mathrm{pp}_{\Gamma(\kappa)}(\tau^*)^+$ suffices for a question of D. Fremlin and 4 of [2].

can assume that τ is a cardinal, otherwise just replace it by $|\tau|$. Now by 3.12, $j(\tau) > \tau$ iff $j_W(\tau) > \tau$ and if $j(\tau) > \tau$ then either $j(\tau), j_W(\tau) \in (2^\kappa, (2^\kappa)^+)$,
or $j(\tau), j_W(\tau) \in (\mathrm{pp}_{\Gamma(\kappa)}(\tau^*), \mathrm{pp}_{\Gamma(\kappa)}(\tau^*)^+)$.
□

Corollary 3.14 *For every τ, $|j(\tau)| = |j(j(\tau))|$.*

Proof. Apply 3.13 with $W = U^2$.
□

It is straightforward to extend this to arbitrary iterated ultrapowers of U:

Corollary 3.15 *Let τ be a cardinal with $j(\tau) > \tau$. Let $\alpha \leq 2^\kappa$, if $\tau \leq 2^\kappa$, and $\alpha \leq \mathrm{pp}_{\Gamma(\kappa)}(\tau^*)$, if $\tau > 2^\kappa$. Then $|j(\tau)| = |j_\alpha(\tau))|$, where $j_\alpha : V \to M_\alpha$ denotes the α-th iterated ultrapower of U.*

Corollary 3.16 *For every τ, if $j(\tau) \neq \tau$, then $j(\tau)$ is not a cardinal.*

Proof. Follows immediately from 3.12.
□

The following question looks natural:

Let α be any ordinal. Suppose $j(\alpha) > \alpha$. Let W be a non-principal κ–complete ultrafilter on κ and $j_W : V \to M_W$ the corresponding elementary embedding. Does then $j_W(\alpha) > \alpha$?

Next statement answers it negatively assuming that $o(\kappa)$– the Mitchell order of κ is at least 2.

Proposition 3.17 *Let W be a non-principal κ–complete ultrafilter on κ and $j_W : V \to M_W$ the corresponding elementary embedding. Suppose that $U \triangleleft W$, i.e. $U \in M_W$. Then $j_W(\alpha) > \alpha = j(\alpha)$, for some $\alpha < (2^\kappa)^+$.*

Proof. Let $\alpha = j_\omega(\kappa)$, i.e. the ω-th iterate of κ by U. Then $j(\alpha) = \alpha$, since $j_\omega(\kappa) = \cup_{n<\omega} j_n(\kappa)$. Let us argue that $j_W(\alpha) > \alpha$. Thus we have U in M_W. So $j_\omega(\kappa)$ as computed in M_W is the real $j_\omega(\kappa)$. In addition

$$M_W \models |j_\omega(\kappa)| = 2^\kappa < (2^\kappa)^+ < j_W(\kappa),$$

and so $\kappa < \alpha = j_\omega(\kappa) < j_W(\kappa)$. Hence

$$j_W(\alpha) = j_W(j_\omega(\kappa)) > j_W(\kappa) > \alpha.$$

□

Let us note that the previous proposition is sharp.

Proposition 3.18 *Suppose that there is no inner model with a measurable of the Mitchell order* ≥ 2*. Let* W *be a non-principal* κ*–complete ultrafilter on* κ *and* $j_W : V \to M_W$ *the corresponding elementary embedding. Then* $j(\alpha) > \alpha$ *iff* $j_W(\alpha) > \alpha$*, for every ordinal* α*.*

Proof. Assume that U is normal or just replace it by such. Let W be a non-principal κ–complete ultrafilter on κ and $j_W : V \to M_W$ the corresponding elementary embedding.
The assumption that there is no inner model with a measurable of the Mitchell order ≥ 2 guarantees that there exists the core model. Denote denote it by $\mathcal{K}$. Let $U^* = U \cap \mathcal{K}$. Then it is a normal ultrafilter over κ in $\mathcal{K}$. Denote by j^* its elementary embedding. Then $j_W \restriction \mathcal{K} = j^*_n$, for some $n < \omega$, since ${}^\omega M_W \subset M_W$ there are no measurable cardinals in $\mathcal{K}$ of the Mitchell order 2.
Hence we need to argue that

$$j^*(\alpha) > \alpha \Leftrightarrow j^*_n(\alpha) > \alpha,$$

for every ordinal α and every $n < \omega$. But this is trivial, since $j^*(\alpha) > \alpha$ implies $j^*_2(\alpha) = j^*(j^*(\alpha)) > j^*(\alpha) > \alpha$ and in general $j^*_{k+1}(\alpha) = j^*(j^*_k(\alpha)) > j^*_k(\alpha) > \alpha$, for every $k, 0 < k < \omega$. On the other hand, if $j^*(\alpha) = \alpha$, then $j^*_\xi(\alpha) = \alpha$, for every ξ.
□

4 Concluding remarks and open problems.

Question 1. *Is weak compactness really needed for Theorem 2.1? Or explicitly:*
Let κ *a regular cardinal. Let* $\mu > 2^\kappa$ *be a singular cardinal of cofinality* κ*. Suppose that* $\lambda < \mathrm{pp}^+_{\Gamma(\kappa)}(\mu)$*. Is there an increasing sequence* $\langle \lambda_i \mid i < \kappa \rangle$ *of regular cardinals converging to* μ *such that* $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{bd}_\kappa})$*?*

See [3] pp.443-444, 5.7 about the related results.

Question 2. *Does Theorem 2.5 remain true assuming* $\mathrm{cof}(\mu) = \mathrm{cof}(\theta) = \omega$*?*

Suppose now that we have an ω_1-saturated κ–complete ideal on κ instead of a κ–complete ultrafilter. The following generic analogs of questions 4,5 of [2] and of a question of Fremlin are natural:

Question 3. *Let* W *be an* ω_1*–saturated filter on* κ*. Does each the following hold:*

1. $\Vdash_{W^+} \forall \tau (\underset{\sim}{j}_W(\tau) > \tau \longrightarrow \tau$ *is not a cardinal*$)$.

2. $\Vdash_{W^+} \forall\tau(|\underset{\sim}{j}_W(\tau)| = |\underset{\sim}{j}_W(\underset{\sim}{j}_W(\tau))|)$.

3. *Let W_1 be an other ω_1–saturated filter on κ. Suppose that for some τ we have δ, δ_1 such that*

   - $\Vdash_{W^+} \underset{\sim}{j}_W(\tau) = \check{\delta}$,
   - $\Vdash_{W_1^+} \underset{\sim}{j}_{W_1}(\tau) = \check{\delta}_1$.

   Then $|\delta| = |\delta_1|$.

Note that in such situation $2^{\aleph_0} \geq \kappa$ and so 2.1 does not apply. Assuming variations of SWH and basing on [3], Sh371, it is possible to answer positively this questions for $\tau > 2^\kappa$.

Recall a question of similar flavor from [2] (Problem 6):

Question 4. *Let W be an ω_1–saturated filter on κ. Can the following happen:*
$\Vdash_{W^+} \underset{\sim}{j}_W(\kappa)$ *is a cardinal? Or even* $\Vdash_{W^+} \underset{\sim}{j}_W(\kappa) = \kappa^{++}$?

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