Applications of pcf for mild large cardinals to elementary embeddings.

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Abstract

The following pcf results are proved:
1. Assume that $\kappa > \aleph_0$ is a weakly compact cardinal. Let $\mu > 2^{\kappa}$ be a singular cardinal of cofinality $\kappa$. Then for every regular $\lambda < \text{pp}_{\Gamma(\kappa)}(\mu)$ there is an increasing sequence $(\lambda_i \mid i < \kappa)$ of regular cardinals converging to $\mu$ such that $\lambda = \text{tcf}(\prod_{i<\kappa} \lambda_i, <, J_{bd\kappa})$.

2. Let $\kappa$ be a strong limit cardinal and $\theta$ a cardinal above $\mu$. Suppose that at least one of them has an uncountable cofinality. Then there is $\sigma_\ast < \mu$ such that for every $\chi < \theta$ the following holds:

$$\theta > \sup\{\sup \text{pcf}_{\sigma_\ast, \text{complete}}(a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi) \text{ and } |a| < \mu\}.$$ 

As an application we show that:

if $\kappa$ is a measurable cardinal and $j : V \rightarrow M$ is the elementary embedding by a $\kappa$-complete non-trivial ultrafilter over $\kappa$, then for every $\tau$ the following holds:

1. if $j(\tau)$ is a cardinal then $j(\tau) = \tau$;
2. $|j(\tau)| = |j(j(\tau))|$;
3. for any $\kappa$-complete ultrafilter $W$ on $\kappa$, $|j(\tau)| = |j_W(\tau)|$.

The first two items provide affirmative answers to questions from [2] and the third to a question of D. Fremlin.

1 Introduction

We address here the following question:

Suppose $\kappa$ is a measurable cardinal, $U$ a $\kappa$-complete non-trivial ultrafilter over $\kappa$ and $j : V \rightarrow M$ the corresponding elementary embedding. Can one characterize cardinals moved by $j$?

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There are trivial answers. For example:

τ is moved by j iff \( \text{cof}(\tau) = \kappa \) or there is some \( \delta < \tau \) with \( j(\delta) \geq \tau \).

Also, assuming GCH, it is not hard to find a characterization in terms not mentioning \( j \).

However, it turns out that an answer is possible in terms not mentioning \( j \) already in ZFC (Theorem 3.12):

Let \( \tau \) be a cardinal. Then either

1. \( \tau < \kappa \) and then \( j(\tau) = \tau \),
   or

2. \( \kappa \leq \tau \leq 2^\kappa \) and then \( j(\tau) > \tau \), \( 2^\kappa < j(\tau) < (2^\kappa)^+ \),
   or

3. \( \tau \geq (2^\kappa)^+ \) and then \( j(\tau) > \tau \) iff there is a singular cardinal \( \mu \leq \tau \) of cofinality \( \kappa \) above \( 2^\kappa \) such that \( \text{pp}(\text{Pr}(\kappa)) \geq \tau \), and if \( \tau^* \) denotes the least such \( \mu \), then \( \tau \leq \text{pp}(\text{Pr}(\kappa))(\tau^*) < j(\tau) < \text{pp}(\text{Pr}(\kappa))(\tau^*)^+ \).

Straightforward conclusions of this result provide affirmative answers to questions mentioned in the abstract.

A crucial tool here is PCF–theory and specially Revisited GCH Theorem [5] Sh460.

A new result involving weakly compact cardinal is obtained (Theorem 2.1):

Assume that \( \kappa > \aleph_0 \) is a weakly compact cardinal. Let \( \mu > 2^\kappa \) be a singular cardinal of cofinality \( \kappa \). Then for every regular \( \lambda < \text{pp}(\text{Pr}(\kappa)) \) there is an increasing sequence \( \langle \lambda_i \mid i < \kappa \rangle \) of regular cardinals converging to \( \mu \) such that \( \lambda = \text{tcf}(\prod_{i<\kappa} \lambda_i, \prec_{J_{\text{bd}}(\kappa)}) \).

Also a bit sharper version of [5] Sh460, 2.1 for uncountable cofinality is proved (Theorem 2.5):

Let \( \mu \) be a strong limit cardinal and \( \theta \) a cardinal above \( \mu \). Suppose that at least one of them has an uncountable cofinality. Then there is \( \sigma_* < \mu \) such that for every \( \chi < \theta \) the following holds:

\[ \theta > \sup \{ \sup \text{pcf}_{\sigma_*-\text{complete}}(a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi) \text{ and } |a| < \mu \}. \]

The first author proved a version of 3.12 assuming certain weak form of the Shelah Weak Hypothesis (SWH)\(^1\) and using [3] Sh371. Then the second author was able to show that the actual assumption used holds in ZFC. All PCF results of the paper are due solely to him.

\(^1\)Consistency of negations of SWH is widely open except very few instances.
Let us recall the definitions of few basic notions of PCF theory that will be used here.

Let $a$ be a set of regular cardinals above $|a|$.

$$\text{pcf}(a) = \{\text{tcf}((\prod a, <_J)) \mid J \text{ is an ideal on } a$$

and $(\prod a, <_J)$ has true cofinality $\}.$

Let $\rho$ a cardinal.

$$\text{pcf}_{\rho-\text{complete}}(a) = \{\text{tcf}((\prod a, <_J)) \mid J \text{ is a } \rho - \text{complete ideal on } a$$

and $(\prod a, <_J)$ has true cofinality $\}.$

Let $\eta$ be a cardinal.

$$J_{<_\eta}[a] = \{b \subseteq a \mid \text{for every ultrafilter } D \text{ on } b, \text{cf}(\prod b, <_D) < \lambda\}.$$

Let $\lambda$ be a singular cardinal.

$$\text{pp}_{\Gamma(\kappa)}^{+}(\lambda) = \text{pp}_{\Gamma(\kappa, \kappa)}^{+}(\lambda) = \sup\{\text{tcf}((\prod a, <_J)) \mid a \text{ is a set of } \kappa \text{ regular cardinals unbounded in } \lambda,$$

$J$ is a $\kappa - \text{complete ideal on } a$ which includes $J_{\text{bd}}^a$ and $(\prod a, <_J)$ has true cofinality $\}.$

$$\text{pp}_{\Gamma(\kappa)}^{+}(\lambda)$$

denotes the first regular without such representation. $^2$

\section{PCF results.}

\textbf{Theorem 2.1} Assume that $\kappa > \aleph_0$ is a weakly compact cardinal. Let $\mu > 2^{\kappa}$ be a singular cardinal of cofinality $\kappa$. Then for every regular $\lambda < \text{pp}_{\Gamma(\kappa)}^{+}(\mu)$ there is an increasing sequence $\langle \lambda_i \mid i < \kappa \rangle$ of regular cardinals converging to $\mu$ such that $\lambda = \text{tcf}(\prod_{i<\kappa} \lambda_i, <_{\text{bd}}^\mu).$

\textbf{Remark 2.2} It is possible to remove the assumption $\mu > 2^{\kappa}$. Just [4](Sh430) § 6, 6.7A should be used to find the pcf-generators in the proof below. See also 6.3 of Abraham -Magidor handbook article [1].

$^2$Note that $\text{pp}_{\Gamma(\kappa)}^{+}(\lambda) \leq (\text{pp}_{\Gamma(\kappa)}(\lambda))^{+}$ and it is open if $\text{pp}_{\Gamma(\kappa)}^{+}(\lambda) < (\text{pp}_{\Gamma(\kappa)}(\lambda))^{+}$ can ever occur (see [3],Sh355, p.41.)
Proof. By No Hole Theorem (2.3, p.57 [3]), there are a $\kappa$–complete ideal $I_1$ on $\kappa$ and a sequence of regular cardinals $\vec{\lambda}^1 = \langle \lambda^1_i \mid i < \kappa \rangle$ with $\mu = \lim_{I_1} \vec{\lambda}^1$ such that $\lambda = \text{tcf}(\prod_{i<\kappa} \lambda^1_i, <_{I_1})$.

Denote the set $\{\lambda^1_i \mid i < \kappa \}$ by $\vec{a}^1$. Let $a^2 = \text{pcf}(\vec{a}^1)$. Without loss of generality assume that $\lambda = \text{max} \text{pcf}(\vec{a}^1)$. Note that by [3] the following holds:

1. $a^1 \subseteq a^2 \subseteq \text{Reg} \setminus \kappa^+$,
2. $\text{pcf}(a^2) = a^2$,
3. $|\text{pcf}(a^2)| \leq 2^\kappa$.

By [3][[Sh345a, 3.6, 3.8(3)] there is a smooth and closed generating sequence for $a^1$ (here we use $2^\kappa < \mu$) which means a sequence $\langle b_\theta \mid \theta \in a^2 \rangle$ such that

1. $\theta \in b_\theta \subseteq a^2$,
2. $\theta \not\in \text{pcf}(a^2 \setminus b_\theta)$,
3. $b_\theta = \text{pcf}(b_\theta)$,
4. $\theta_1 \in b_{\theta_2}$ implies $b_{\theta_1} \subseteq b_{\theta_2}$,
5. $\theta = \text{max} \text{pcf}(b_\theta)$.

Then by [3][[Sh345a,3.2(5)]:

(*)$_1$: if $c \subseteq a^2$, then for some finite $\delta \subseteq \text{pcf}(c)$ we have $c \subseteq \text{pcf}(c) \subseteq \bigcup \{b_\theta \mid \theta \in \delta\}$.

The next claim is a consequence of [5][Sh460, 2.1]:

**Claim 1** There is $\sigma_* < \kappa$ such that for every $\vec{a} \subset \text{Reg} \cap (\kappa^+, \mu)$ of cardinality less than $\kappa$ there is a sequence $\langle a_\alpha \mid \alpha < \sigma_* \rangle$ such that

1. $\vec{a} = \bigcup_{\alpha<\sigma_*} a_\alpha$,
2. $\text{max} \text{pcf}(a_\alpha) < \mu$, for every $\alpha < \sigma_*$.

Proof. The cardinal $\kappa$ is a strong limit, so we can apply [5][Sh460, 2.1] to $\kappa$ and $\mu$. Hence there is $\sigma_* < \kappa$ such that for every $\vec{a} \subset \text{Reg} \cap (\kappa^+, \mu)$ of cardinality less than $\kappa$ we have $\text{pcf}_{\sigma_*+}$–complete$(\vec{a}) \subseteq \mu$. This means that the $\sigma_*+–$complete ideal generated by $J_{<\mu}(\vec{a})$ is everything, i.e. $\mathcal{P}(\vec{a})$. See 8.5 of [1] for the detailed argument. So there are $a_\alpha$’s in $J_{<\mu}(\vec{a})$, for
\( \alpha < \sigma_s \) such that \( a = \bigcup_{\alpha < \sigma_s} a_\alpha \). But then also \( \max \operatorname{pcf}(a_\alpha) < \mu \), for every \( \alpha < \sigma_s \).

\( \Box \) of the claim.

Let \( \sigma_s < \kappa \) be given by the claim. Let \( i < \kappa \). Apply the claim to the set \( a_i^1 := \{ \lambda_j^1 \mid j < i \} \). So there is a sequence \( \langle a_{\alpha} \mid \alpha < \sigma_s \rangle \) such that

1. \( a_i^1 = \bigcup_{\alpha < \sigma_s} a_{\alpha} \),
2. \( \max \operatorname{pcf}(a_{\alpha}) < \mu \), for every \( \alpha < \sigma_s \).

Now, by \((\ast)_1\), for every \( \alpha < \sigma_s \),

\[ \operatorname{pcf}(a_{\alpha}) \subseteq \bigcup \{ b_\theta \mid \theta \in \mathcal{d}_{\alpha} \}, \]

for some finite \( \mathcal{d}_{\alpha} \subseteq \operatorname{pcf}(a_{\alpha}) \).

Set \( \mathcal{d}_i = \bigcup_{\alpha < \sigma_s} \mathcal{d}_{\alpha} \). Then \( \mathcal{d}_i \) is a subset of \( \mu \) of cardinality \( \leq \sigma_s \). In addition we have \( \mathcal{d}_i \subseteq \operatorname{pcf}(a_i^1) \) and \( a_i^1 \subseteq \bigcup \{ b_\theta \mid \theta \in \mathcal{d}_i \} \).

Let \( \langle \theta_{i, \epsilon} \mid \epsilon < \sigma_s \rangle \) be a listing of \( \mathcal{d}_i \).

**Claim 2** There are a function \( g \) and \( \vec{u} = \langle u_\epsilon \mid \epsilon < \sigma_s \rangle \) such that

1. \( g : \kappa \to \kappa \) is increasing,
2. \( \xi < g(\xi) \), for every \( \xi < \kappa \),
3. \( \kappa = \bigcup_{\epsilon < \sigma_s} u_\epsilon \),
4. for any \( \epsilon < \sigma_s \) and \( \xi < \eta < \kappa \) the following holds:
   \[ \lambda_\xi^1 \in b_{\theta_{g(\eta), \epsilon}} \text{ iff } \xi \in u_\epsilon. \]

**Proof.** Here is the place to use the weak compactness of \( \kappa \).

We will define a \( \kappa \)--tree \( T \) and then will use its \( \kappa \)--branch.

Fix \( \eta < \kappa \). Let \( P \subseteq \sigma_s \times \eta \). Define a set

\[ A_P := \{ \alpha \in (\eta, \kappa) \mid \forall \xi < \eta \forall \epsilon < \sigma_s (\langle \epsilon, \xi \rangle \in P \iff \lambda_\xi^1 \in b_{\theta_{g(\eta), \epsilon}}) \}. \]

Note that always there is \( P \subseteq \sigma_s \times \eta \) with \( |A_P| = \kappa \). Just \( |\mathcal{P}(\sigma_s \times \eta)| < \kappa \), so the function

\[ \alpha \mapsto \langle \langle \epsilon, \xi \rangle \mid \epsilon < \sigma_s, \xi < \eta \text{ and } \lambda_\xi^1 \in b_{\theta_{g(\eta), \epsilon}} \rangle \]
is constant on a set of cardinality \( \kappa \).

Also for such \( P \) we will have \( \text{rng}(P) = \eta \), i.e. for every \( \xi < \eta \) there is \( \epsilon < \sigma_\epsilon \) (which may be not unique) such that \((\epsilon, \xi) \in P \). Thus pick \( \alpha \in A_P \). Then \( \alpha > \eta > \xi \) and \( \alpha_\epsilon = \bigcup \{ b_\theta \mid \theta \in d_\alpha \} \).

Clearly \( \lambda_\epsilon^1 \) appears in \( a_\alpha^1 = \{ \lambda_\epsilon^1 \mid \nu < \alpha \} \). Hence there is \( \epsilon < \sigma_\epsilon \) such that \( \lambda_\epsilon^1 \in b_{\theta_\alpha,\epsilon} \), and so \((\epsilon, \xi) \in P \).

Let

\[
T := \{ P \mid \exists \eta < \kappa (P \subseteq \sigma_* \times \eta \text{ and } |A_P| = \kappa) \}.
\]

If \( P \subseteq \sigma_* \times \eta, P' \subseteq \sigma_* \times \eta' \) are both in \( T \) then set \( P <_T P' \) iff

- \( \eta < \eta' \),
- \( P' \cap (\sigma_* \times \eta) = P \).

Then \( \langle T, <_T \rangle \) is a \( \kappa \)-tree. Let \( X \subseteq \sigma_* \times \kappa \) be a \( \kappa \)-branch. Define now an increasing function \( g : \kappa \to \kappa \). Set \( g(\eta) = \min(AX\cap(\sigma_* \times \eta)) \setminus \sup(\{ g(\eta') \mid \eta' < \eta \}) \).

Let now \( \epsilon < \sigma_* \). Define \( u_\epsilon \) as follows:

\[
\xi \in u_\epsilon \text{ iff for some } \eta > \xi \text{ and some (every)} \alpha \in A_{X\cap(\sigma_* \times \eta)}, \lambda_\xi^1 \in b_{\theta_\alpha,\epsilon}.
\]

Then for any \( \epsilon < \sigma_* \) and \( \xi < \eta < \kappa \) the following holds:

\[
\lambda_\xi^1 \in b_{\theta(g(\eta),\epsilon)} \text{ iff } \xi \in u_\epsilon.
\]

Finally \( |X| = \kappa \) implies that for every \( \xi < \kappa \) there is \( \epsilon < \sigma_* \) with \( \xi \in u_\epsilon \). Thus let \( \xi < \kappa \).

Pick some \( \eta, \xi < \eta < \kappa \). Consider \( X \cap (\sigma_* \times \eta) \). Then, as was observed above, there are \( \alpha \in A_{X\cap(\sigma_* \times \eta)} \) and \( \epsilon < \sigma_* \) such that \( \lambda_\xi^1 \in b_{\theta_\alpha,\epsilon} \). Hence \( \xi \in u_\epsilon \).

\( \Box \) of the claim.

Claim 3 Suppose that \( u_\epsilon \in I^+_1 \), for some \( \epsilon < \sigma_* \). Then \( |u_\epsilon| = \kappa \) and the quasi order \( \prod_{i \in u_\epsilon} (\theta_{g(i),\epsilon}, <_{j_{u_\epsilon}^{\eta_\epsilon}}) \) has true cofinality \( \lambda \).

Proof. \( \kappa \)-completeness of \( I_1 \) implies that \( |u_\epsilon| = \kappa \), since clearly \( \{ \xi \} \in I_1 \), for every \( \xi < \kappa \).

Suppose now that the quasi order \( \prod_{i \in u_\epsilon} (\theta_{g(i),\epsilon}, <_{j_{u_\epsilon}^{\eta_\epsilon}}) \) does not have a true cofinality or it has true cofinality \( \neq \lambda \). Recall that \( \lambda = \max \pcf(a_1) \). So by \( [3](\text{Sh345a}) \) there is an unbounded subset \( v \) of \( u \) such that \( \prod_{i \in v} (\theta_{g(i),\epsilon}, <_{j_{u_\epsilon}^{\eta_\epsilon}}) \) has a true cofinality \( \lambda_\ast < \lambda \). We can take \( \lambda_\ast \) to be just the least \( \delta \) such that an unbounded subset of \( u_\epsilon \) appears in \( J_{\leq \delta}[u_\epsilon] \).

Without loss of generality we can assume that \( \lambda_\ast = \max \pcf(\{ \theta_{g(i),\epsilon} \mid i \in v \}) \). We have
\( \lambda_* \in \pcf(\{\theta_{g(i)}, \epsilon \mid i \in v\}) \subseteq \pcf(a_1) = a_2. \) Set \( v_1 := \{ i \in v \mid \theta_{g(i)}, \epsilon \in b_{\lambda_*} \}. \) Then \( v_1 \) is unbounded in \( v. \) By smoothness of the generators, \( i \in v_1 \) implies \( b_{\theta_{g(i)}, \epsilon} \subseteq b_{\lambda_*}. \) Then

\[
i \in v_1 \text{ and } \xi \in u_* \cap i \text{ imply } \lambda^i_\xi \in b_{\lambda_*}.
\]

But \( v_1 \) is unbounded in \( \kappa, \) hence for every \( \xi \in u_* \) there is \( i \in v_1, i > \xi. \) So, \( \{ \lambda^i_\xi \mid \xi \in u_* \} \subseteq b_{\lambda_*}. \) By the closure of the generators, \( \pcf(b_{\lambda_*}) = b_{\lambda_*}. \) Hence \( \pcf(\{ \lambda^i_\xi \mid \xi \in u_* \}) \subseteq b_{\lambda_*}. \) This impossible since \( u_* \in I_1^+ \) and so \( \lambda \in \pcf(\{ \lambda^i_\xi \mid \xi \in u_* \}), \) but \( \lambda_* < \lambda. \) Contradiction.

\( \square \) of the claim.

**Claim 4** There is \( \epsilon < \sigma_* \) such that \( u_* \in I_1^+ \) and \( \mu = \lim_{\theta_{g(i)}, \epsilon} (\theta_{g(\xi)}, i < \kappa). \)

**Proof.** Suppose otherwise. Set \( s := \{ \epsilon < \sigma_* \mid u_* \in I_1^+ \}. \) Then for every \( \epsilon \in s \) there is \( v_* \) an unbounded subset of \( \kappa \) such that \( \theta^*_\epsilon := \sup \{ \theta_{g(i)}, \epsilon \mid i \in v_* \} \) is below \( \mu. \) Set

\[
\theta_* := \sup \{ \theta^*_\epsilon \mid \epsilon \in s \}. \text{ Then } \theta_* < \mu, \text{ since } \cof(\mu) = \kappa > \sigma_*.
\]

Set \( w_1 := \bigcup \{ u_* \mid \epsilon \in \sigma_* \setminus s \}. \) Then \( w_1 \in I_1 \) as a union of less than \( \kappa \) of its members. Also the set \( w_2 := \{ i < \kappa \mid \lambda^i_i \leq \theta_* \} \) belongs to \( I_1 \) because \( \mu = \lim_{\theta_{g(i)}, \epsilon} \{ \lambda^i_\xi \mid i < \kappa \}. \) Hence \( w := w_1 \cup w_2 \in I_1. \)

Let \( \xi \in \kappa \setminus w. \) Then

\[
\lambda^1_\xi \in \{ \lambda^1_\rho \mid \rho < \xi + 1 \} \subseteq \bigcup \{ b_{\theta_{g(\xi + 1), \xi}} \mid \epsilon < \sigma_* \}.
\]

Hence for some \( \epsilon < \sigma_* \), \( \lambda^1_\xi \in b_{\theta_{g(\xi + 1), \xi}}. \) Then \( \xi \in u_* \). Now, \( \xi \notin w \) and so \( \xi \notin w_1. \) Hence \( \epsilon \in s. \) Pick some \( \tau \in v_* \), \( \tau > \xi. \) Then \( \lambda^1_\xi \in b_{\theta_{g(\tau), \xi}}, \) since \( \xi \in u_* \). Then

\[
\lambda^1_\xi \leq \max \{ b_{\theta_{g(\tau), \xi}} \mid \theta_{g(\tau), \epsilon} \leq \theta^*_\epsilon \leq \theta_*. \}
\]

But then \( \xi \in w_2. \) Contradiction.

\( \square \) of the claim.

\[
\square
\]

**Proposition 2.3** Let \( a \) be a set of regular cardinals with \( \min(a) > 2^{|a|}. \) Let \( \sigma < \theta \leq |a|. \) Suppose that \( \lambda \in \pcf_{\sigma, \text{complete}}(a), \mu < \lambda \) and \( \pcf_{\theta, \text{complete}}(a) \subseteq \mu. \) Then there is \( c \subseteq \pcf_{\theta, \text{complete}}(a) \) such that \( |c| < \theta, \ c \subseteq \mu \) and \( \lambda \in \pcf_{\sigma, \text{complete}}(c). \)

**Remark 2.4** It is possible to replace the assumption \( \min(a) > 2^{|a|} \) by \( \min(a) > |a| \) using [4](Sh430) § 6, 6.7A in order to find the pcf-generators used in the proof.
Proof. Let \( \langle b_\xi \mid \xi \in \text{pcf}(a) \rangle \) be a set of generators as in Theorem 2.1. We have \( \lambda \in \text{pcf}_{\sigma\text{-complete}}(a) \subseteq \text{pcf}(a) \), hence \( b_\lambda \) is defined and \( \text{max} \text{pcf}(b_\lambda) = \lambda \in \text{pcf}_{\sigma\text{-complete}}(a) \subseteq \text{pcf}(a) \).

By [4], 6.7F(1), there is \( c \subseteq \text{pcf}_{\theta\text{-complete}}(a \cap b_\lambda) \subseteq \mu \) of cardinality \( < \theta \) such that \( b_\lambda \cap a \subseteq \bigcup \{b_\xi \mid \xi \in c\} \). Then, by smoothness, \( \xi \in c \Rightarrow b_\xi \subseteq b_\lambda \). Hence \( \text{pcf}(c) \subseteq \text{pcf}(b_\lambda) = b_\lambda \).

Hence \( \text{max} \text{pcf}(c) \leq \lambda \).

Now, if \( \lambda \in \text{pcf}_{\sigma\text{-complete}}(c) \), then we are done. Suppose otherwise. Then there are \( j(*) < \sigma \) and \( \theta_j \in \lambda \cap \text{pcf}_{\sigma\text{-complete}}(c) \), for every \( j < j(*) \), such that \( c \subseteq \bigcup \{b_{\theta_j} \mid j < j(*)\} \).

So if \( \eta \in b_\lambda \cap a \), then for some \( \chi \in c \) we have \( \eta \in b_\chi \), as \( b_\lambda \cap a \subseteq \bigcup \{b_\xi \mid \xi \in c\} \). Hence for some \( j < j(*) \), \( \chi \in b_{\theta_j} \), and so \( b_\chi \subseteq b_{\theta_j} \) and \( \eta \in b_{\theta_j} \).

Then \( b_\lambda \cap a \subseteq \bigcup_{j < j(*)} b_{\theta_j} \). Recall that \( j(*) < \sigma \) and \( \theta_j < \lambda \), for every \( j < j(*) \).

Note that \( \lambda \in \text{pcf}_{\sigma\text{-complete}}(a) \) implies that \( \lambda \in \text{pcf}_{\sigma\text{-complete}}(b_\lambda \cap a) \), see for example 4.14 of [1]. So there is a \( \sigma\text{-complete ideal} \ J \) on \( b_\lambda \cap a \) such that \( \lambda = \text{tcf}(\prod (b_\lambda \cap a), <j) \). Then for some \( j < j(*) \), \( b_{\theta_j} \in J^+ \) which is impossible since \( \text{max} \text{pcf}(b_{\theta_j}) = \theta_j < \lambda \). Contradiction.

The next result follows from 2.1 of [5] Sh460.

**Theorem 2.5** Let \( \mu \) be a strong limit cardinal and \( \theta \) a cardinal above \( \mu \). Suppose that at least one of them has an uncountable cofinality. Then there is \( \sigma_\ast < \mu \) such that for every \( \chi < \theta \) the following holds:

\[
\theta > \sup \{ \sup \text{pcf}_{\sigma_\ast\text{-complete}}(a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi) \text{ and } |a| < \mu \}.
\]

**Proof.** Assume first that \( \text{cof}(\mu) \neq \text{cof}(\theta) \). Suppose on contrary that

\[
\forall \mu^* < \mu \exists \chi < \theta (\theta \leq \sup \{ \sup \text{pcf}_{\mu^*_\text{-complete}}(a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi) \text{ and } |a| < \mu \}).
\]

If \( \text{cof}(\theta) < \text{cof}(\mu) \), then there will be \( \chi < \theta \) such that for every \( \mu^* < \mu \)

\[
\theta \leq \sup \{ \sup \text{pcf}_{\mu^*_\text{-complete}}(a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi) \text{ and } |a| < \mu \}.
\]

But this is impossible by 2.1 of [5] applied to \( \mu \) and \( \chi \).

If \( \text{cof}(\theta) > \text{cof}(\mu) \), then still there will be \( \chi < \theta \) such that for every \( \mu^* < \mu \)

\[
\theta \leq \sup \{ \sup \text{pcf}_{\mu^*_\text{-complete}}(a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi) \text{ and } |a| < \mu \}.
\]

Just for every \( \mu^* < \mu \) pick some \( \chi_{\mu^*} \) such that

\[
\theta \leq \sup \{ \sup \text{pcf}_{\mu^*_\text{-complete}}(a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi_{\mu^*}) \text{ and } |a| < \mu \}.
\]
and set $\chi = \bigcup_{\mu^* < \mu} \chi_{\mu^*}$.

So let us assume that $\text{cof}(\theta) = \text{cof}(\mu)$. Denote this common cofinality by $\kappa$. By the assumption of the theorem $\kappa > \aleph_0$.

Let $\langle \mu_i \mid i < \kappa \rangle$ be an increasing continuous sequence with limit $\mu$ such that each $\mu_i$ is a strong limit cardinal. Let $\theta > \mu$ be singular cardinal of cofinality $\kappa$. Fix an increasing continuous sequence $\langle \theta_i \mid i < \kappa \rangle$ with limit $\theta$ such that $\theta_0 > \mu$.

Suppose that there are no $\sigma_i^* < \mu$ which satisfies the conclusion of the theorem. In particular, for every $i < \kappa$, $\mu_i$ cannot serve as $\sigma_i^*$. Hence there is $\chi_i < \theta$ such that

$$\theta = \sup \{ \sup \text{pcf}_{\mu_i} (a) \mid a \subseteq \text{Reg} \cap (\mu^+, \chi_i) \text{ and } |a| < \mu \}.$$ 

So, for each $j < \kappa$, there is $a_{i,j} \subseteq \text{Reg} \cap (\mu^+, \chi_i)$ of cardinality less than $\mu$ such that $\text{pcf}_{\mu_i} (a_{i,j}) \not\subseteq \theta_j$.

Set $\theta_\kappa := \theta$. For every $i \leq \kappa$, we apply Theorem 2.1 of [5] to $\mu_i$ and $\theta_i$. There is $\sigma_i^* < \mu$ such that

$$\text{if } a \subseteq \text{Reg} \cap (\mu^+, \theta_i) \text{ and } |a| < \mu \text{ then } \text{pcf}_{\sigma_i^*} (a) \subseteq \theta_i.$$

Define now by induction a sequence $\langle i(n) \mid n < \omega \rangle$ such that

1. $i(n) < i(n+1) < \kappa$,
2. $\sigma_\kappa^* < \mu_{i(0)}$,
3. $\sigma_{i(n)}^* < \mu_{i(n+1)}$,
4. $\chi_{i(n)} < \theta_{i(n+1)}$.

Let $i(\omega) = \bigcup_{n<\omega} i(n)$. Then $i(\omega) < \kappa$, since $\kappa$ is a regular above $\aleph_0$. So $\theta_{i(\omega)} < \theta$. Now, for every $j < \kappa$ and $n < \omega$ the following holds:

$$a_{i(n), j} \subseteq \text{Reg} \cap (\mu^+, \chi_i(n)) \subseteq \text{Reg} \cap (\mu^+, \theta_{i(n+1)}) \subseteq \text{Reg} \cap (\mu^+, \theta_{i(\omega)}) \text{ and } \text{pcf}_{\sigma_{i(n+1)}} (a_{i(n), j}) \subseteq \theta_{i(n+1)} < \theta_{i(\omega)}.$$ 

Let $n < \omega$ and $j \in (i(\omega), \kappa)$. Then by the choice of $a_{i(n), j}$ the following holds:

$$a_{i(n), j} \subseteq \text{Reg} \cap (\mu^+, \chi_i(n)) \subseteq \text{Reg} \cap (\mu^+, \theta_{i(n+1)}) \text{ and } \text{pcf}_{\mu_{i(n)}} (a_{i(n), j}) \not\subseteq \theta_j.$$

By the choice of $\sigma_{i(n+1)}^*$, we have

$$\text{pcf}_{\sigma_{i(n+1)}} (a_{i(n), j}) \subseteq \theta_{i(n+1)}.$$
Proof sequence of regular cardinals $\langle \eta, \mu < \eta < j \rangle$.

Let $n_\star < \omega$ with $\mu_{(n_\star)} > \sigma_\star$. Then $b_{i(n_\star), j} \subseteq \text{Reg} \cap (\mu_{(i)}^+, \theta_{(i)})$ and $|b_{i(n_\star), j}| < \mu_{(i)}$, but $\text{pcf}_{\mu_{(n_\star)}}(b_{i(n_\star), j}) \subsetneq \theta_j > \theta_{(i)}$. Which is impossible. Contradiction.

\[ \square \]

3 Applications.

Let $\kappa$ be a measurable cardinal, $U$ be a $\kappa$–complete non-principle ultrafilter over $\kappa$ and let $j_U : V \to M \simeq ^*V/U$ be the corresponding elementary embedding. Denote $j_U$ further simply by $j$.

Lemma 3.1 Let $\mu > 2^\kappa$ be a singular cardinal of cofinality $\kappa$. Then $j(\mu) \geq \text{pp}_{\Gamma(\kappa)}(\mu)$.

Proof. Let $\lambda < \text{pp}_\Gamma(\mu)$ be a regular cardinal. Then, by Theorem 2.1, there is an increasing sequence of regular cardinals $\langle \lambda_i \mid i < \kappa \rangle$ converging to $\mu$ such that $\lambda = \text{tcf}(\prod_{i < \kappa} \lambda_i, <_{j_U})$. The ultrafilter $U$ clearly extends the dual to $j_U^{bd}$. Hence $[\langle \lambda_i \mid i < \kappa \rangle]_U$ represents an ordinal below $j(\mu)$ of cofinality $\lambda$. Hence $j(\mu) > \lambda$ and we are done.

\[ \square \]

Let us denote for a singular cardinal $\mu$ of cofinality $\kappa$ by $\mu^*$ the least singular $\xi \leq \mu$ of cofinality $\kappa$ above $2^\kappa$ such that $\text{pp}_{\Gamma(\kappa)}(\xi) \geq \mu$.

Then, by [3](Sh 355, 2.3(3), p.57), $\text{pp}_{\Gamma(\kappa)}(\mu) \leq^+ \text{pp}_{\Gamma(\kappa)}(\mu^*)$.

Lemma 3.2 Let $\mu > 2^\kappa$ be a singular cardinal of cofinality $\kappa$. Then $j(\mu) \geq \text{pp}_{\Gamma(\kappa)}(\mu^*)$.

Proof. By 3.1, $j(\mu^*) \geq \text{pp}_{\Gamma(\kappa)}(\mu^*)$. But $\mu^* \leq \mu$, hence $j(\mu^*) \leq j(\mu)$.

\[ \square \]

Lemma 3.3 Let $\mu > 2^\kappa$ be a singular cardinal of cofinality $\kappa$. Let $\eta, \mu < \eta < j(\mu)$ be a regular cardinal. Then $\eta \leq \text{pp}_{\Gamma(\kappa)}(\mu^*)$.
Proof.
Let $\eta, \mu < \eta < j(\mu)$ be a regular cardinal. Let $f_\eta : \kappa \to \mu$ be a function which represents $\eta$ in $M$, i.e. $[f_\eta]_U = \eta$. We can assume that $\text{rng}(f_\eta) \subseteq \text{Reg} \cap ((2^\kappa)^+, \mu)$, since $|j(2^\kappa)| = 2^\kappa$ and so $j(2^\kappa) < \mu < \eta$. Set $\tau := \text{U–limit of } \text{rng}(f_\eta)$. Then $\tau > 2^\kappa$.

Note that $\text{cof}(\tau) = \kappa$. Otherwise, $f_\eta$ is just a constant function mod $U$. Let $\delta$ be the constant value. Then $\delta < j(\delta) = \eta$. By elementarity $\delta$ must be a regular cardinal. But then $j''\delta$ is unbounded in $\eta$, which means that $\eta$ is a singular cardinal. Contradiction.

Denote $f(\alpha)$ by $\tau_\alpha$, for every $\alpha < \kappa$. Then each $\tau_\alpha$ is a regular cardinal in the interval $((2^\kappa)^+, \tau)$ and $\tau = \lim_U \langle \tau_\alpha \mid \alpha < \kappa \rangle$. We have $\eta = \text{tcf}(\prod_{\alpha < \kappa} \tau_\alpha, <_U)$.

We will show in the next lemma (3.4) that this does not affect $\text{pp}(\kappa)(\tau)$.

Namely, $\eta = \text{tcf}(\prod_{\alpha < \kappa} \tau_\alpha, <_\kappa)$ implies $\text{pp}(\kappa)(\tau) \geq \eta > \mu$. Then, by the choice of $\mu^*$, we have $\mu^* \leq \tau$ By [3](Sh 355, 2.3(3), p.57), $\text{pp}(\kappa)(\mu^*) \geq \text{pp}(\kappa)(\tau)$.

□

**Lemma 3.4**
Let $\kappa$ be a regular cardinal and $\tau$ be a singular cardinal of cofinality $\kappa$. Then

$$\text{pp}(\kappa)(\tau) = \sup \{ \text{tcf}(\prod_{\alpha < \kappa} \tau_\alpha, <_I) \mid \langle \tau_\alpha \mid \alpha < \kappa \rangle \text{ is a sequence of regular cardinals with }$$

$$\lim_I \langle \tau_\alpha \mid \alpha < \kappa \rangle = \tau, I \text{ is a } \kappa \text{ complete ideal over } \kappa \text{ which extends } J^\text{bd}_\kappa \}.$$  

*Proof.* Clearly,

$$\text{pp}(\kappa)(\tau) \leq \text{sup} \{ \text{tcf}(\prod_{\alpha < \kappa} \tau_\alpha, <_I) \mid \langle \tau_\alpha \mid \alpha < \kappa \rangle \text{ is a sequence of regular cardinals with }$$

$$\lim_I \langle \tau_\alpha \mid \alpha < \kappa \rangle = \tau, I \text{ is a } \kappa \text{ complete ideal over } \kappa \text{ which extends } J^\text{bd}_\kappa \}.$$  

Just if $\eta = \text{tcf}(\prod a, <_J)$, where $a$ is a set of $\kappa$ regular cardinals unbounded in $\tau$, $J$ is a $\kappa$–complete ideal on $a$ which includes $J^\text{bd}_a$. Then we can view $a$ as a $\kappa$–sequence.

---

3It is possible to force a situation where such $\tau < \mu$. Start with a $\eta^{++}$–strong $\tau, \kappa < \tau < \mu$. Use the extender based Magidor to blow up the power of $\tau$ to $\eta^{+}$ simultaneously changing the cofinality of $\tau$ to $\kappa$. The forcing satisfies $\kappa^{++}$–c.c., so it will not effect pp structure of cardinals different from $\tau$.

4Actually, the original definition of pp ([3]II,Definition 1.1, p.41) involves sequences rather than sets.

5A version of this lemma was suggested by Menachem Magidor.
Let us deal with the opposite direction. Suppose that \( \eta = \text{tcf}(\prod_{\alpha < \kappa} \tau_{\alpha} < I) \), where \( \langle \tau_{\alpha} \mid \alpha < \kappa \rangle \) is a sequence of regular cardinals with \( \lim_{I} \langle \tau_{\alpha} \mid \alpha < \kappa \rangle = \tau \), \( I \) is a \( \kappa \) complete ideal over \( \kappa \) which extends \( J_{\kappa}^{\text{bd}} \). Without loss of generality we can assume that \( \kappa < \tau_{\alpha} < \tau \), for every \( \alpha < \kappa \). Set \( a = \{ \tau_{\alpha} \mid \alpha < \kappa \} \). Define a projection \( \pi : \kappa \to a \) by setting \( \pi(\alpha) = \tau_{\alpha} \). Let

\[ J := \{ X \subseteq a \mid \pi^{-1} X \in I \}. \]

Then \( J \) will be a \( \kappa \)–complete ideal on \( a \) which extends \( J_{a}^{\text{bd}} \).

Let us argue that \( \eta = \text{tcf}(\prod a, < J) \). Fix a scale \( \langle f_{i} \mid i < \eta \rangle \) which witnesses \( \eta = \text{tcf}(\prod_{\alpha < \kappa} \tau_{\alpha} < I) \). Define for a function \( f \in \prod_{\alpha < \kappa} \tau_{\alpha} \) a function \( \bar{f} \in \prod_{\alpha < \kappa} \tau_{\alpha} \) as follows:

\[ \bar{f}(\alpha) = \sup\{ f(\beta) \mid \tau_{\beta} = \tau_{\alpha} \}. \]

Note that for every \( \alpha < \kappa \), \( \bar{f}(\alpha) < \tau_{\alpha} \), since \( \tau_{\alpha} \) is a regular cardinal above \( \kappa \).

Consider the sequence \( \langle \bar{f}_{i} \mid i < \kappa \rangle \). It need not be a scale, since the sequence need not be \( I \)–increasing. But this is easy to fix. Just note that for every \( i < \eta \) there will be \( i', i \leq i' < \eta \), such that

\[ f_{i} \leq \bar{f}_{i} \leq \bar{f}_{i'}. \]

Just given \( i < \eta \), find some \( i', i \leq i' < \eta \), such that \( \bar{f}_{i} \leq \bar{f}_{i'} \). Then \( \bar{f}_{i} \leq \bar{f}_{i'} \leq \bar{f}_{i''} \). Now by induction it is easy to shrink the sequence \( \langle \bar{f}_{i} \mid i < \kappa \rangle \) and to obtain an \( I \)–increasing subsequence \( \langle g_{\xi} \mid \xi < \eta \rangle \) which is a scale in \( (\prod_{\alpha < \kappa} \tau_{\alpha}, < I) \).

For every \( \xi < \eta \) define \( h_{\xi} \in \prod a \) as follows:

\[ h_{\xi}(\rho) = g_{\xi}(\alpha), \text{ if } \rho = \tau_{\alpha}, \text{ for some (every) } \alpha < \kappa. \]

It is well defined since \( g_{\xi}(\alpha) = g_{\xi}(\beta) \) once \( \tau_{\alpha} = \tau_{\beta} \).

Let us argue that \( \langle h_{\xi} \mid \xi < \eta \rangle \) is a scale in \( (\prod a, < J) \).

Clearly, \( \xi < \xi' \) implies \( h_{\xi} <_{J} h_{\xi'} \), since \( g_{\xi} <_{I} g_{\xi'} \).

Let \( h \in \prod a \). Consider \( g \in \prod_{\alpha < \kappa} \tau_{\alpha} \) defined by setting \( g(\alpha) = h(\tau_{\alpha}) \). There is \( \xi < \eta \) such that \( g <_{I} g_{\xi} \). Then \( h <_{J} h_{\xi} \), since

\[ \pi^{-1} \{ \rho \in a \mid h(\rho) < h_{\xi}(\rho) \} \supseteq \{ \alpha < \kappa \mid g(\alpha) < g_{\xi}(\alpha) \}. \]

\( \square \)

**Theorem 3.5** Let \( \mu > 2^{\kappa} \) be a singular cardinal of cofinality \( \kappa \). Then \( \text{pp}_{\Gamma(\kappa)}(\mu^{+}) \leq j(\mu) < \text{pp}_{\Gamma(\kappa)}(\mu^{+})^{+} \).
Proof. Note that \( j(\mu) \) is always singular. Just \( \mu \) is a singular cardinal, hence \( j(\mu) \) is a singular in \( M \) and so in \( V \). Now the conclusion follows by 3.2,3.3.

\[ \square \]

We can deduce now an affirmative answer to a question of D. Fremlin for cardinals of cofinality \( \kappa \).

**Corollary 3.6** Let \( W \) be a non-principal \( \kappa \)-complete ultrafilter on \( \kappa \) and \( j_W : V \to M_W \) the corresponding elementary embedding. Then for every \( \mu \) of cofinality \( \kappa \), \( |j(\mu)| = |j_W(\mu)| \).

**Proof.** Let \( \mu \) be a cardinal of cofinality \( \kappa \). If \( \mu < 2^\kappa \), then \( 2^\kappa < j_W(\mu) < j_W(2^\kappa) < (2^\kappa)^+ \), for any non-principal \( \kappa \)-complete ultrafilter \( W \) on \( \kappa \).

If \( \mu > 2^\kappa \), then, by 3.5, \( pp_{\Gamma(\kappa)}(\mu^+) \leq j(\mu) < pp_{\Gamma(\kappa)}(\mu^+) \). But recall that \( j \) was the elementary embedding of an arbitrary non-principal \( \kappa \)-complete ultrafilter \( U \) on \( \kappa \) and the bounds do not depend on it. Hence if \( W \) is an other non-principal \( \kappa \)-complete ultrafilter on \( \kappa \), then \( pp_{\Gamma(\kappa)}(\mu^+) \leq j_W(\mu) < pp_{\Gamma(\kappa)}(\mu^+) \).

\[ \square \]

**Corollary 3.7** For every \( \mu \) of cofinality \( \kappa \), \( |j(\mu)| = |j(j(\mu))| \).

**Proof.** It follows from 3.6. Just take \( W = U^2 \) and note that \( j(j(\mu)) = j_{U^2}(\mu) \).

\[ \square \]

Our next tusk will be to show that the fist inequality is really a strict inequality.

**Lemma 3.8** Let \( \mu > 2^\kappa \) be a singular cardinal of cofinality \( \kappa \). Then \( pp_{\Gamma(\kappa)}(\mu) \leq (pp_{\Gamma(\kappa)}(\mu))^M \).\(^7\)

**Proof.** Let \( \eta, \mu < \eta < pp_{\Gamma(\kappa)}^+(\mu) \) be a regular cardinal.

By Theorem 2.1, there is an increasing converging to \( \mu \) sequence \( \langle \eta_i \mid i < \kappa \rangle \) of regular cardinals such that
\[
\eta = tcf(\prod_{i<\kappa} \eta_i, <_{j^{bd}}).
\]

Note that both \( \langle \eta_i \mid i < \kappa \rangle \) and \( J^{bd}_\kappa \) are in \( M \). Also \( ^eM \subseteq M \), hence each function of the witnessing scale is in \( M \), however the scale itself may be not in \( M \). Still we can work inside \( M \) and define a scale recursively using functions from the \( V \)-scale.

---

\(^6\)Readers interested only in a full answer to Fremlin’s question can jump after the corollary directly to 3.12. The non-strict inequality in its conclusion suffices.

\(^7\)\( (pp_{\Gamma(\kappa)}(\mu))^M \) stands for \( pp_{\Gamma(\kappa)}(\mu) \) as computed in \( M \). Note that it is possible to have \( (pp_{\Gamma(\kappa)}(\mu))^M > pp_{\Gamma(\kappa)}(\mu) \), just as \( (2^\kappa)^M > 2^\kappa \).
Thus let \( \langle f_\tau \mid \tau < \eta \rangle \) be a scale mod \( J^\text{bd}_\kappa \) which witnesses \( \eta = \text{tcf}(\prod_{i < \kappa} \eta_i, <_{j^\text{bd}_i}) \). Work in \( M \) and define recursively an increasing mod \( J^\text{bd}_\kappa \) sequence of functions \( \langle g_\xi \mid \xi < \eta' \rangle \) in \( \prod_{i < \kappa} \eta_i \) as far as possible.

We claim first that \( \text{cof}(\eta') = \eta \), as computed in \( V \). Thus if \( \eta < \text{cof}(\eta') \), then there will be \( \tau^* < \eta \) such that \( f_{\tau^*} \geq_{J^\text{bd}_\kappa} g_\xi \), for every \( \xi < \eta' \), since for every \( \xi < \eta' \) there is \( \tau < \eta \) such that \( f_\tau \geq_{J^\text{bd}_\kappa} g_\xi \). But having \( f_{\tau^*} \geq_{J^\text{bd}_\kappa} g_\xi \), for all \( \xi < \eta' \), we can continue and define \( g_{\eta'} \) to be \( f_{\tau^*} \).

If \( \eta > \text{cof}(\eta') \), then again there will be \( \tau^* < \eta \) such that \( f_{\tau^*} \geq_{J^\text{bd}_\kappa} g_\xi \), for every \( \xi < \eta' \), and again we can continue and define \( g_{\eta'} \) to be \( f_{\tau^*} \).

So \( \text{cof}(\eta') = \eta \). Let \( \langle \eta'_\tau \mid \tau < \eta \rangle \) be a cofinal in \( \eta' \) sequence (in \( V \)). Now, for every \( \tau < \eta \) there is \( \tau', \tau \leq \tau' < \eta \) such that \( f_{\tau'} \geq_{J^\text{bd}_\kappa} g_{\tau'} \), since the sequence \( \langle g_\xi \mid \xi < \eta' \rangle \) is maximal.

Hence there is \( A_\tau \subseteq \kappa, |A_\tau| = \kappa \) such that \( f_\tau \upharpoonright A_\tau <_{J^\text{bd}_\kappa} g_{\eta'_\tau}, \upharpoonright A_\tau \). But \( \eta > \mu > 2^\kappa \), hence there is \( A^* \subseteq \kappa \) such that for \( \eta \) many \( \tau \)'s we have \( A^* = A_\tau \). Then for every \( \tau < \eta \) there is \( \tau'' \), \( \tau \leq \tau'' < \eta \) such that \( f_\tau \upharpoonright A^* <_{J^\text{bd}_\kappa} g_{\eta''_\tau}, \upharpoonright A^* \).

It follows that the sequence \( \langle g_\xi \mid A^* \upharpoonright \xi < \eta' \rangle \) is a scale in \( \text{tcf}(\prod_{i < A^*} \eta_i, <_{j^\text{bd}_i}) \). Hence, in \( M \), \( \eta' < \text{pp}_{\Gamma(\kappa)}(\mu) \). But \( \text{cof}(\eta') = \eta \), hence, in \( M \), \( \eta \leq \eta' < \text{pp}_{\Gamma(\kappa)}(\mu) \).

\( \square \)

**Lemma 3.9** Let \( \mu > 2^\kappa \) be a singular cardinal of cofinality \( \kappa \) such that \( \mu^* = \mu \).

Then \( j(\xi) < \mu \) for every \( \xi < \mu \).

**Proof.** Suppose otherwise. Then there is \( \xi < \mu \) such that \( j(\xi) \geq \mu \). Necessarily \( \xi > 2^\kappa \).

Let \( \eta \) be a regular cardinal \( \xi \leq \eta < \mu \). Pick a function \( f_\eta : \kappa \rightarrow \xi \) which represents \( \eta \) in \( M \). Without loss of generality we can assume that \( \text{min}(\text{rng}(f_\eta)) > 2^\kappa \). Let \( \delta_\eta \leq \xi \) be the \( U \)-limit of \( \text{rng}(f_\eta) \). Then \( \text{cof}(\delta_\eta) = \kappa \) and \( j(\delta_\eta) > \eta \). Also \( \eta \leq \text{pp}_{\Gamma(\kappa)}(\delta_\eta) \), by the definition of \( \text{pp}_{\Gamma(\kappa)}(\delta_\eta) \). By Lemma 3.2, we have \( j(\delta_\eta) \geq \text{pp}_{\Gamma(\kappa)}((\delta_\eta)^*) \), and by [3] (Sh 355, 2.3(3), p.57), \( \text{pp}_{\Gamma(\kappa)}(\delta_\eta) \leq \text{pp}_{\Gamma(\kappa)}((\delta_\eta)^*) \). Set

\[
\delta := \text{min}\{\delta_\eta \mid \xi \leq \eta < \mu \text{ and } \eta \text{ is a regular cardinal}\}.
\]

Then \( \text{pp}_{\Gamma(\kappa)}(\delta) \geq \text{pp}_{\Gamma(\kappa)}(\delta_\eta) \), for every regular \( \eta, \xi \leq \eta < \mu \). But \( \text{pp}_{\Gamma(\kappa)}(\delta_\eta) \geq \eta \). Hence \( \text{pp}_{\Gamma(\kappa)}(\delta) \geq \mu \) which is impossible since \( \mu^* = \mu \). Contradiction.

\( \square \)

**Lemma 3.10** Let \( \mu > 2^\kappa \) be a singular cardinal of cofinality \( \kappa \).

Then \( \text{pp}_{\Gamma(\kappa)}(\mu^*) < j(\mu) \).
Proof. By 3.2 we have $j(\mu) \geq \text{pp}_{\Gamma(\kappa)}(\mu^*)$. Suppose that $j(\mu) = \text{pp}_{\Gamma(\kappa)}(\mu^*)$. Then $\mu = \mu^*$, since by 3.2 we have $j(\mu^*) \geq \text{pp}_{\Gamma(\kappa)}(\mu^*)$. By Theorem 2.5, there is $\sigma_* < \kappa$ such that

$$\forall \chi < \mu > \sup\{\text{pcf}_{\sigma_* - \text{complete}}(a) \;|\; a \subseteq \text{Reg} \cap (\kappa^+, \chi) \land |a| < \kappa\}.$$  

Then, by elementarity,

$$M \models \forall \chi < j(\mu) > \sup\{\text{pcf}_{j(\sigma_*) - \text{complete}}(a) \;|\; a \subseteq \text{Reg} \cap (j(\kappa^+), \chi) \land |a| < j(\kappa)\}.$$  

Clearly, $j(\sigma_*) = \sigma_*$. Take $\chi = \mu$. Let $\eta$ be a regular cardinal (i.e. of $V$) such that

$$(*) \quad M \models j(\mu) > \eta > \sup\{\text{pcf}_{\sigma_* - \text{complete}}(a) \;|\; a \subseteq \text{Reg} \cap (j(\kappa^+), \mu) \land |a| < j(\kappa)\}.$$  

Note that there are such $\eta$’s since $j(\mu)$ is a singular cardinal of cofinality $\text{cof}(j(\kappa))$. By Lemma 3.3, then $\eta \leq \text{pp}_{\Gamma(\kappa)}(\mu)$. Now, by Lemma 3.8, $\text{pp}_{\Gamma(\kappa)}(\mu) \leq (\text{pp}_{\Gamma(\kappa)}(\mu))^M$. Hence $M \models \eta \leq \text{pp}_{\Gamma(\kappa)}(\mu)$. But then there is $a \in M$ such that

$$M \models a \subseteq \text{Reg} \cap (j(\kappa^+), \mu) \land |a| = \kappa \land \eta \leq \max \text{pcf}_{\kappa - \text{complete}}(a).$$  

Which clearly contradicts $(*)$.

$\square$

So we proved the following:

**Theorem 3.11** Let $\mu > 2^\kappa$ be a singular cardinal of cofinality $\kappa$. Then $\text{pp}_{\Gamma(\kappa)}(\mu^*) < j(\mu) < \text{pp}_{\Gamma(\kappa)}(\mu^*)^+$.  

Deal now with cardinals of arbitrary cofinality.

**Theorem 3.12** Let $\tau$ be a cardinal. Then either

1. $\tau < \kappa$ and then $j(\tau) = \tau$,

   or

2. $\kappa \leq \tau \leq 2^\kappa$ and then $j(\tau) > \tau$, $2^\kappa < j(\tau) < (2^\kappa)^+$;

   or

3. $\tau \geq (2^\kappa)^+$ and then $j(\tau) > \tau$ iff there is a singular cardinal $\mu \leq \tau$ of cofinality $\kappa$ above $2^\kappa$ such that $\text{pp}_{\Gamma(\kappa)}(\mu) \geq \tau$, and if $\tau^*$ denotes the least such $\mu$, then $\tau \leq \text{pp}_{\Gamma(\kappa)}(\tau^*) < j(\tau) < \text{pp}_{\Gamma(\kappa)}(\tau^*)^+$.  

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Proof. Suppose otherwise. Let $\tau$ be the least cardinal witnessing this. Clearly then $\tau > (2^\kappa)^+$. If $\text{cof}(\tau) = \kappa$, then we apply 3.11 to derive the contradiction. Suppose that $\text{cof}(\tau) \neq \kappa$.

Claim 5 There is a singular cardinal $\xi$ of cofinality $\kappa$ such that $j(\xi) > \tau$.

Proof. Thus let $f_\tau : \kappa \to \tau$ be a function which represents $\tau$ in $M$. Without loss of generality we can assume that

$$\nu \in \text{rng}(f_\tau) \Rightarrow (\nu > 2^\kappa \text{ and } \nu \text{ is a cardinal}).$$

Then either $f_\tau$ is a constant function mod $U$ or $\xi := \text{limit } \text{rng}(f_\tau)$ has cofinality $\kappa$.

Suppose first that $f_\tau$ is a constant function mod $U$ with value $\xi$. If $\xi = \tau$, then $j(\xi) = \tau$. Suppose that $\xi < \tau$. Then $j(\xi) = \tau > \xi$ and also $\xi$ is a cardinal above $2^\kappa$. By minimality of $\tau$ then $\xi^*$ exists and

$$\text{pp}_{\Gamma(\kappa)}(\xi^*) < \tau = j(\xi) < \text{pp}_{\Gamma(\kappa)}(\xi^*)^+.$$ 

But this is impossible since $\tau$ is a cardinal. Contradiction. So $\text{cof}(\xi) = \kappa$ and $j(\xi) > \tau$.

Now affirmative answers to a question of D. Fremlin and to questions 4, 5 of [2] follow easily.

Corollary 3.13 Let $W$ be a non-principal $\kappa$–complete ultrafilter on $\kappa$ and $j_W : V \to M_W$ the corresponding elementary embedding. Then for every $\tau$, $|j(\tau)| = |j_W(\tau)|$.

Proof. Let $W$ be a non-principal $\kappa$–complete ultrafilter on $\kappa$ and $j_W : V \to M_W$ the corresponding elementary embedding. Let $\tau$ be an ordinal. Without loss of generality we

\footnote{Non strict inequality $\text{pp}_{\Gamma(\kappa)}(\tau^*) \leq j(\tau) < \text{pp}_{\Gamma(\kappa)}(\tau^*)^+$ suffices for a question of D. Fremlin and 4 of [2].}
can assume that $\tau$ is a cardinal, otherwise just replace it by $|\tau|$. Now by 3.12, $j(\tau) > \tau$ iff $j_W(\tau) > \tau$ and if $j(\tau) > \tau$ then either $j(\tau), j_W(\tau) \in (2^\kappa, (2^\kappa)^+)$, or $j(\tau), j_W(\tau) \in (\text{pp}_{\Gamma(\kappa)}(\tau^*), \text{pp}_{\Gamma(\kappa)}(\tau^*)^+)$. 

\[ \square \]

**Corollary 3.14** For every $\tau$, $|j(\tau)| = |j(j(\tau))|$. 

*Proof.* Apply 3.13 with $W = U^2$. 

\[ \square \]

It is straightforward to extend this to arbitrary iterated ultrapowers of $U$:

**Corollary 3.15** Let $\tau$ be a cardinal with $j(\tau) > \tau$. Let $\alpha \leq 2^\kappa$, if $\tau \leq 2^\kappa$, and $\alpha \leq \text{pp}_{\Gamma(\kappa)}(\tau^*)$, if $\tau > 2^\kappa$. Then $|j(\tau)| = |j_\alpha(\tau)|$, where $j_\alpha : V \rightarrow M_\alpha$ denotes the $\alpha$-th iterated ultrapower of $U$. 

**Corollary 3.16** For every $\tau$, if $j(\tau) \neq \tau$, then $j(\tau)$ is not a cardinal. 

*Proof.* Follows immediately from 3.12. 

\[ \square \]

The following question looks natural:

*Let $\alpha$ be any ordinal. Suppose $j(\alpha) > \alpha$. Let $W$ be a non-principal $\kappa$-complete ultrafilter on $\kappa$ and $j_W : V \rightarrow M_W$ the corresponding elementary embedding. Does then $j_W(\alpha) > \alpha$?*

Next statement answers it negatively assuming that $o(\kappa)$—the Mitchell order of $\kappa$ is at least 2.

**Proposition 3.17** Let $W$ be a non-principal $\kappa$-complete ultrafilter on $\kappa$ and $j_W : V \rightarrow M_W$ the corresponding elementary embedding. Suppose that $U \triangleleft W$, i.e. $U \in M_W$. Then $j_W(\alpha) > \alpha = j(\alpha)$, for some $\alpha < (2^\kappa)^+$. 

*Proof.* Let $\alpha = j_\omega(\kappa)$, i.e. the $\omega$-th iterate of $\kappa$ by $U$. Then $j(\alpha) = \alpha$, since $j_\omega(\kappa) = \cup_{n<\omega} j_n(\kappa)$. Let us argue that $j_W(\alpha) > \alpha$. Thus we have $U$ in $M_W$. So $j_\omega(\kappa)$ as computed in $M_W$ is the real $j_\omega(\kappa)$. In addition

\[ M_W \models |j_\omega(\kappa)| = 2^\kappa < (2^\kappa)^+ < j_W(\kappa), \]

and so $\kappa < \alpha = j_\omega(\kappa) < j_W(\kappa)$. Hence

\[ j_W(\alpha) = j_W(j_\omega(\kappa)) > j_W(\kappa) > \alpha. \]
Let us note that the previous proposition is sharp.

**Proposition 3.18** Suppose that there is no inner model with a measurable of the Mitchell order $\geq 2$. Let $W$ be a non-principal $\kappa$-complete ultrafilter on $\kappa$ and $j_W : V \to M_W$ the corresponding elementary embedding. Then $j(\alpha) > \alpha$ iff $j_W(\alpha) > \alpha$, for every ordinal $\alpha$.

**Proof.** Assume that $U$ is normal or just replace it by such. Let $W$ be a non-principal $\kappa$–complete ultrafilter on $\kappa$ and $j_W : V \to M_W$ the corresponding elementary embedding. The assumption that there is no inner model with a measurable of the Mitchell order $\geq 2$ guarantees that there exists the core model. Denote denote it by $K$. Let $U^* = U \cap K$. Then it is a normal ultrafilter over $\kappa$ in $K$. Denote by $j^*$ its elementary embedding. Then $j^*_W | K = j^*_n$, for some $n < \omega$, since $\omega M_W \subset M_W$ there are no measurable cardinals in $K$ of the Mitchell order 2.

Hence we need to argue that

$$j^*(\alpha) > \alpha \iff j^*_n(\alpha) > \alpha,$$

for every ordinal $\alpha$ and every $n < \omega$. But this is trivial, since $j^*(\alpha) > \alpha$ implies $j^*_n(\alpha) = j^*(j^*(\alpha)) > j^*(\alpha) > \alpha$ and in general $j^*_{k+1}(\alpha) = j^*(j^*_k(\alpha)) > j^*_k(\alpha) > \alpha$, for every $k, 0 < k < \omega$. On the other hand, if $j^*(\alpha) = \alpha$, then $j^*_\xi(\alpha) = \alpha$, for every $\xi$.

\[ \square \]

### 4 Concluding remarks and open problems.

**Question 1.** Is weak compactness really needed for Theorem 2.1? Or explicitly:

Let $\kappa$ a regular cardinal. Let $\mu > 2^\kappa$ be a singular cardinal of cofinality $\kappa$. Suppose that $\lambda < \text{pp}_{\Gamma(\kappa)}(\mu)$. Is there an increasing sequence $\langle \lambda_i | i < \kappa \rangle$ of regular cardinals converging to $\mu$ such that $\lambda = \text{tcf}(\prod_{i<\kappa} \lambda_i, <_{\text{jbd}})$?

See [3] pp.443-444, 5.7 about the related results.

**Question 2.** Does Theorem 2.5 remain true assuming $\text{cof}(\mu) = \text{cof}(\theta) = \omega$?

Suppose now that we have an $\omega_1$-saturated $\kappa$-complete ideal on $\kappa$ instead of a $\kappa$-complete ultrafilter. The following generic analogs of questions 4,5 of [2] and of a question of Fremlin are natural:

**Question 3.** Let $W$ be an $\omega_1$-saturated filter on $\kappa$. Does each the following hold:

1. $\| W^+ \forall \tau (\widetilde{j}_W(\tau) > \tau \implies \tau \text{ is not a cardinal}).$
2. \( \vartriangleleft_{w+} \forall \tau (|j_{\vartriangleleft w}(\tau)| = |j_{\vartriangleleft w}(j_{\vartriangleleft w}(\tau))|) \).

3. Let \( W_1 \) be an other \( \omega_1 \)-saturated filter on \( \kappa \). Suppose that for some \( \tau \) we have \( \delta, \delta_1 \) such that

- \( \vartriangleleft_{w+} j_{\vartriangleleft w}(\tau) = \bar{\delta} \),
- \( \vartriangleleft_{w_1^+} j_{\vartriangleleft w_1}(\tau) = \bar{\delta}_1 \).

Then \( |\delta| = |\delta_1| \).

Note that in such situation \( 2^{\aleph_0} \geq \kappa \) and so 2.1 does not apply. Assuming variations of SWH and basing on \([3]\), Sh371, it is possible to answer positively this questions for \( \tau > 2^\kappa \).

Recall a question of similar flavor from \([2]\) (Problem 6):

**Question 4.** Let \( W \) be an \( \omega_1 \)-saturated filter on \( \kappa \). Can the following happen:

\( \vartriangleleft_{w+} j_{\vartriangleleft w}(\kappa) \) is a cardinal? Or even \( \vartriangleleft_{w+} j_{\vartriangleleft w}(\kappa) = \kappa^{++} \)?

**References**


