Simpler Short Extenders Forcing - arbitrary gap (January version).

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Abstract

Our aim is to present a version of short extenders forcing which splits between cardinals above and below the critical cardinal and allows to blow up the power arbitrary higher.

1 The Main Preparation Forcing

The definition of the forcing below follows the ideas of [1], [2], [3], [4]. We extend the method of [4] to gaps up to κ^{++}. The forcing adds a certain kind of a simplified morass with linear limits with a gap up to κ^{++}. The object added by a parallel forcing in [4] adds Velleman's [5] simplified morass with linear limits of the gap 1. We do not know if there exists an analog of Velleman morass for higher gaps. A generic set for the forcing below probably may serve as such morass.

Fix two cardinals κ and θ such that $\kappa < \theta$ and θ is regular.

Definition 1.1 The set $\mathcal{P}'$ consists of all sequences of triples.

$$\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$$

such that

1. s is a closed set of cardinals from the interval $[\kappa^+, \theta]$ satisfying the following:

(a) $|s \cap \delta| < \delta$ for each inaccessible $\delta \in [\kappa^+, \theta]$

(b) $\kappa^+, \theta \in s$

(c) if $\rho^+ \in s$ and $\rho \geq \kappa^+$, then $\rho \in s$

(d) if $\rho \in s$ is singular, then s is unbounded in ρ and $\rho^+ \in s$.

2. For every $\tau \in s$, $A^{0\tau}$ is an elementary submodel of $\langle H(\theta^+), \in, <, \kappa \rangle$

   (a) $|A^{0\tau}| = \tau$ and $A^{0\tau} \supseteq \tau$

   (b) ${}^{cf\tau>}A^{0\tau} \subseteq A^{0\tau}$

3. If $\tau, \tau' \in s$ and $\tau < \tau'$ then $A^{0\tau} \subseteq A^{0\tau'}$

4. If τ is a limit point of s, then $A^{0\tau} = \cup\{A^{0\rho} \mid \rho \in s \cap \tau\}$.

5. For every $\tau \in s$, $A^{1\tau}$ is a set of at most τ many elementary submodels of $A^{0\tau}$ such that

   (a) $A^{0\tau} \in A^{1\tau}$ and each element of $A^{1\tau} \backslash \{A^{0\tau}\}$ belongs to $A^{0\tau}$

   (b) if $B \in A^{1\tau}$, then $\tau \subseteq B$

   (c) if $B \in A^{1\tau}$ then $\tau \in B$

   (d) if $A, B \in A^{1\tau}$ and $B \subset A$, then $B \in A$

   In particular, the above condition (d) imply that $\langle A^{1\tau}, \subseteq \rangle$ is well founded.

   Let $A \in A^{1\tau}$. We define $otp_\tau(A)$ to be $\sup\{otp(C) \mid C \subseteq \mathcal{P}(A) \cap A^{1\tau}$ and C is a chain under the inclusion relation$\}$.

   Further, we shall list more properties of $A^{1\tau}$. Let us now turn to C^τ.

6. For every $\tau \in s$, $C^\tau : A^{1\tau} \to \mathcal{P}(A^{1\tau})$ is a function such that

   (a) (Closure and maximality condition) for each $A \in A^{1\tau}$, $C^\tau(A)$ is a closed chain (under inclusion) of elements of $\mathcal{P}(A) \cap A^{1\tau}$ of the length $otp_\tau(A)$ and there is no chain in $\mathcal{P}(A) \cap A^{1\tau}$ that properly includes $C^\tau(A)$.

   In particular, this means that there are chains of the maximal length (i.e. $otp_\tau(A)$ which was defined as a supremum is really maximum) and $C^\tau(A)$ is one of them.

   (b) (Coherency condition) if $B \in C^\tau(A)$ then $C^\tau(B)$ is the initial segment of $C^\tau(A)$ which starts with B.

(c) (Unboundedness condition) If $otp_\tau(A) - 1$ is a limit ordinal (note that A itself is always the last member of $C^\tau(A)$, hence $otp_\tau(A)$ is always a successor ordinal) then $C^\tau(A)\backslash\{A\}$ is unbounded in A, i.e. $\cup(C^\tau(A)\backslash\{A\}) = A$.

We call A in such a case **a limit model** and otherwise **a successor** one. Note that if $B \in A^{1\tau}$, $B \subsetneqq A$ then $B \in A$ and hence B is included in a member of $C^\tau(A)\backslash\{A\}$.

(d) if $A \in A^{1\tau}$ is a successor model, then ${}^{cf(\tau)>}A \subseteq A$

7. If $A, B \in A^{1\tau}$ then $otp(A) = otp(B)$ iff $otp_\tau(A) = otp_\tau(B)$.

Let us introduce one basic notion - Δ - system type.

Let $F_0, F_1, F \in A^{1\mu}$ for some $\mu \in s$. We say then that F_0, F_1, F are of a **Δ-system type** iff

(a) F is a successor model

(b) F_0, F_1 are its immediate predecessors (under the inclusion relation) and $otp_\mu F_0 = otp_\mu F_1$ (in particular, the conclusion of (27) above holds and in particular F_0, F_1 are isomorphic over $F_0 \cap F_1$)

(c) $F \in C^\mu(A^{0\mu})$

(d) one of F_0, F_1 is in $C^\mu(F)$

(e) there are $G_0, G_0^*, G_1, G_1^*, G^* \in C^\tau(A^{0\tau})$,

for $\tau = \min(s\backslash\mu + 1)$ such that

(i) $G_0 \in F_0$

(ii) $G_1 \in F_1$

(iii) $F_0 \cap F_1 = F_0 \cap G_0 = F_1 \cap G_1$

(iv) $F_0 \in G_0^*, F_1 \in G_1^*, F \in G^*$ and G_0^*, G_1^*, G^* are the least under the inclusion elements of $A^{1\tau}$ including F_0, F_1, F respectively

(v) $G_0 \in F_0 \in G_0^* \in G_1 \in F_1 \in G_1^* \in F \in G^*$.

Note that $\tau = \mu^+$ unless it is an inaccessible.

We will say that F_0, F_1, F are of a **Δ-system type according to a chain** X if the conditions (a) - (e) above are satisfied, only in (e) we have $C^\tau(A^{0\tau})$ replaced by X.

Let us call a triple $F_0, F_1, F \in A^{1\mu}$ **a suitable for switching** iff

(a) F_0, F_1, F are of a Δ-system type

(b) for each $\tau \in s \cap \mu$, $F \in A^{0\tau}$ and if $A \in C^\tau(A^{0\tau})$ is the first with $F \in A$, then its immediate predecessor A^- in $C^\tau(A^{0\tau})$ is in F. Moreover, if there are $A_0, A_1 \in A^{1\tau}$ such that the triple A_0, A_1, A is a is of a Δ- system type, then $\sup(A_0) < \sup(A_1)$, implies $A_0 \in F \in A_1$.

    Note that in the last case, i.e. if there are $A_0, A_1 \in A^{1\tau}$ such that the triple A_0, A_1, A is of a Δ- system type and $\sup(A_0) < \sup(A_1)$, then it will be impossible to have $A_1 \in F \in A$ by 1.1(11). Also, by 1.1(29), we will must to have $F_0, F_1 \in A_1$ as well.

Let us say that F_0, F_1, F are **suitable for switching according to a chain** X if the above conditions are satisfied, with C^μ replaced by X.

Let us state some preliminary definitions.

Definition 1.2 Suppose now that a triple F_0, F_1, F is a suitable for switching, $F \in C^\mu(A^{0\mu}), F_0 \in C^\mu(F)$. Define

$$\langle C^\nu(A^{0\nu})_F | \nu \in s\rangle,$$

the switch of

$$\langle C^\nu(A^{0\nu}) | \nu \in s\rangle,$$

by F as follows:

$$C^\nu(A^{0\nu})_F = C^\nu(A^{0\nu}),$$

for each $\nu \in s \backslash \mu + 1$,

$$C^\mu(A^{0\mu})_F = (C^\mu(A^{0\mu}) \backslash C^\mu(F_0))^\frown C^\mu(F_1),$$

$$C^\nu(A^{0\nu})_F = (C^\nu(A^{0\nu}) \backslash C^\nu(A))^\frown C^\nu(\pi_{F_0 F_1}[A]),$$

for each $\nu \in s \cap \mu$, where $A \in C^\nu(A^{0\nu})$ is the maximal element of $C^\nu(A^{0\nu})$ contained in F_0.

Note that for $\nu \in s \cap \mu$, $C^\nu(A^{0\nu})_F$ is still continuous. It is also increasing due to the choice of F_0, F_1, F as a suitable for switching pair and further condition 1.1(29).

Definition 1.3 Let us call

$$\bigcup\{C^\nu(A^{0\nu}) | \nu \in s\}$$

the **central line**.

Suppose now that a triple B_0, B_1, B is a suitable for switching, $B \in C^\mu(A^{0\mu}), B_0 \in C^\mu(B)$. Define the **line 1 generated by** B to be

$$\bigcup\{C^\nu(A^{0\nu})_B | \nu \in s\}.$$

Continue, let a triple B_0^1, B_1^1, B^1 be a suitable for switching according to the line 1, i.e. according to increasing parts of $C^{\mu^1}(A^{0\mu^1})_B$, for some $\mu^1 \in s$, $B^1 \in C_B^{\mu^1}(A^{0\mu^1}), B_0^1 \in C^{\mu^1}(B^1)$. Define the **line 2 generated by** B, B^1.

It will be

$$\bigcup\{C^\nu(A^{0\nu})_{BB^1} | \nu \in s\},$$

where

$$C^\nu(A^{0\nu})_{BB^1} = C^\nu(A^{0\nu})_B,$$

for each $\nu \in s \backslash \mu^1 + 1$,

$$C^{\mu^1}(A^{0\mu^1})_{BB_1} = (C^{\mu^1}(A^{0\mu^1})_B \backslash C^{\mu^1}(B_0^1))^\frown C^{\mu^1}(B_1^1),$$

$$C^\nu(A^{0\nu})_{BB^1} = (C^\nu(A^{0\nu})_B \backslash C^\nu(A))^\frown C^\nu(\pi_{B_0^1 B_1^1}[A]),$$

for each $\nu \in s \cap \mu^1$, where $A \in C^\nu(A^{0\nu})_B$ is the maximal element of $C^\nu(A^{0\nu})_B$ contained in B_0^1.

Continue by induction and define **line** n for each $n < \omega$.

Definition 1.4 (General distance)

Let $A \in A^{1\nu}$ for some $\nu \in s$. Define $gd(A)$ **the general distance from the central line** to be 0 if $A \in C^\nu(A^{0\nu})$. If $A \notin C^\nu(A^{0\nu})$ then let $gd(A)$ be the least $n < \omega$ such that there exist $B^1, B^2, ..., B^n$ with $C^\nu(A^{0\nu})_{B^1,...,B^n}$ defined and with $A \in C^\nu(A^{0\nu})_{B^1,...,B^n}$.

Note that by further generation condition 1.1(17) and 1.1(31), $gd(A)$ will always be defined.

Let us formulate a similar to the Δ - system type (but a bit weaker) notion. The only difference will be that we replace in the clause (e) of the definition of a Δ - system type $C^\tau(A^{0\tau})$ by the k -line version for some $k < \omega$.

Let $F_0, F_1, F \in A^{1\mu}$ for some $\mu \in s$. We say then that F_0, F_1, F are of a **weak Δ-system type** iff F_0, F_1, F are of a Δ-system type

or the following holds:

(a) F is a successor model

(b) F_0, F_1 are its immediate predecessors and $otp_\mu F_0 = otp_\mu F_1$ (in particular, the conclusion of (28) above holds)

(c) F_0, F_1 are isomorphic over $F_0 \cap F_1$

(d) $gd(F)$ is defined and equal to some $k, 0 < k < \omega$.

(e) there is a sequence of models $B^1, ..., B^k$ witnessing $gd(F) = k$ such that

  (i) $F \in C^\mu(A^{0\mu})_{B^1,...,B^k}$

  (ii) one of F_0, F_1 in $C^\mu(F)$

  (iii) there are $G_0, G_0^*, G_1, G_1^*, G^*$ all in $C^\tau(A^{0\tau})_{B_1,...,B_k}$, for $\tau = \min(s \backslash \mu + 1)$ such that

    (α) $G_0 \in F_0$

    (β) $G_1 \in F_1$

    (γ) $F_0 \cap F_1 = F_0 \cap G_0 = F_1 \cap G_1$

    (δ) $F_0 \in G_0^*, F_1 \in G_1^*, F \in G^*$ and G_0^*, G_1^*, G^* are the least under the inclusion elements of $A^{1\tau}$ including F_0, F_1, F respectively

    (ϵ) $G_0 \in F_0 \in G_0^* \in G_1 \in F_1 \in G_1^* \in F \in G^*$.

  Note that $\tau = \mu^+$ unless it is an inaccessible.

Further we shall require that a small adjustment turns a weak Δ-system type into a Δ-system type.

Let us call F for which there are F_0, F_1 with F_0, F_1, F of a Δ-system type or of a weak Δ-system type— a **splitting point**.

The next condition guarantees the uniqueness for triples as above.

8. (Immediate predecessors condition)

   Let F be in $A^{1\mu}$ for some $\mu \in s$. Suppose that there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a weak Δ-system type with F being the largest model, then F_0, F_1 are unique.

   Let us state now a condition that deals with extensions of a Δ-system type models.

9. (Bigger models condition)

   Let F be in $C^\mu(A^{0\mu})$ for some $\mu \in s$. Suppose that there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a Δ-system type with F being the largest model. Let $\tau = \min(s \backslash \mu + 1)$.

If F' is one of F_0, F_1, F and G' is the smallest element of $C^\tau(A^{0\tau})$ including F' then the following hold

(a) if G' is not the first element of $C^\tau(G')$, then the immediate predecessor $\hat{G}'$ of G' in $C^\tau(G')$ belongs to F' as well as $C^\tau(\hat{G}')$. In particular, $\tau \in F'$

(b) if $H \in C^\tau(A^{0\rho})$ and $H \supseteq F'$, for some $\rho \in s\backslash\mu + 1$, then $H \supseteq G'$.

(c) if $H \in C^\tau(A^{0\rho})$, $H \supseteq F'$, for some $\rho \in s\backslash\mu + 1$ and H is the first like this in $C^\tau(A^{0\rho})$, then then the immediate predecessor of H in $C^\tau(A^{0\rho})$ (if exists) is in F'

The following condition says that once we have models of a Δ-system type then it is impossible to have models of smaller cardinalities in between.

10. (No small models condition) Let F_0, F_1, F be as in (8) and $F_0 \in C^\mu(F)$. If for some $\xi \in s \cap \mu$ we have $A \in C^\xi(A^{0\xi})$ with $A \subseteq F$, then $A \in F_0$.

    Further it will be shown that the above is true for $A^{1\mu}$ replacing C^μ and $A^{1\xi}$ replacing C^ξ.

11. (No splittings between a model and its immediate predecessor of maximal supremum) Let $F_0, F_1, F \in A^{1\mu}$ be of a weak Δ-system type. Suppose that $\sup(F_0) < \sup(F_1)$. Then there is no splitting points between F_1 and F, i.e. there is no $\rho \in s$ and a splitting point $B \in A^{1\rho}$ with $F_1 \in B \in F$. But there may (and actually will be many) splitting points B with $F^- \in B \in F_1$.

    Let $F \in A^{1\mu}$ be a successor model. We denote by F^- its immediate predecessor in $C^\mu(F)$. Let us define now the set $Pred(F)$.

    Suppose first that there are no $F_0, F_1 \in A^{1\mu} \cap F$ such that F_0, F_1, F are of a weak Δ-system type. Assume that $gd(F)$ is defined (actually, the generation condition (17) will guarantee that this is always the case). Let $gd(F) = k$. Fix the smallest (or simplest) sequence of models $B^1, ..., B^k$ witnessing this. Then $F, F^- \in C^\mu(A^{0\mu})_{B^1,...,B^k}$.

    Set then

$$Pred_0(F) = \{F^-\},$$

$$Pred_{n+1}(F) = \bigcup_{i<\omega} Pred_{n+1,i}(F),$$

where

$$Pred_{n+1,0}(F) = Pred_n(F)$$

and

$$Pred_{n+1,i+1}(F) = Pred_{n+1,i}(F) \cup \{\pi_{B_0B_1}[G] | G \in Pred_{n,i}(F), B_0, B_1, B \in F \cap A^{1\rho}$$

$$\text{are of a weak } \Delta - \text{system type for some } \rho \in s \backslash \mu + 1 \text{ and } G \subset B_0 (G \in (A^{1\mu})^{B_0}),$$

$$\text{the general distance of } B \text{ relatively to } F \text{ is at most } i \text{ with a witnessing sequence inside } F$$

$$(\text{ i.e. relatively to } C^\mu(A^{0\mu})_{B^1,...,B^k} \text{ , or in other words } k - i \leq gd(B) \leq k + i)\},$$

for each $n < \omega$.

Suppose now that there are $F_0, F_1 \in A^{1\mu} \cap F$ such that F_0, F_1, F are of a weak Δ-system type. Assume that $\sup(F_1) > \sup(F_0)$, otherwise just switch between them.

Assume that $gd(F_1)$ is defined (actually, the generation condition (17) will guarantee that this is always the case). Let $gd(F_1) = k$. Fix the smallest (or simplest) sequence of models $B^1, ..., B^k$ witnessing this.

Set

$$Pred_0(F) = \{F_0, F_1\}$$

$$Pred_{n+1}(F) = \bigcup_{i<\omega} Pred_{n+1,i}(F),$$

where

$$Pred_{n+1,0}(F) = Pred_n(F)$$

and

$$Pred_{n+1,i+1}(F) = Pred_{n+1,i}(F) \cup \{\pi_{B_0B_1}[G] | G \in Pred_{n,i}(F), B_0, B_1, B \in F_1 \cap A^{1\rho}$$

$$\text{are of a weak } \Delta - \text{system type for some } \rho \in s \backslash \mu + 1 \text{ and } G \subset B_0 (G \in (A^{1\mu})^{B_0}),$$

$$\text{the general distance of } B \text{ relatively to } F_1 \text{ is at most } i \text{ with a witnessing sequence inside } F_1$$

$$(\text{ i.e. relatively to } C^\mu(A^{0\mu})_{B^1,...,B^k} \text{ , or in other words } k - i \leq gd(B) \leq k + i)\},$$

for each $n < \omega$.

We required in (11) that in this case there is no splittings between F_1 and F, i.e. there is no splitting point B with $F_1 \in B \in F$. But there may (and actually will be many) splitting points B with $F_0 \in B \in F_1$. Also we require in (15) that F^- is in $Pred_n(F)$ for some $n < \omega$.

Let us define in both cases

$$Pred(F) = \bigcup_{n<\omega} Pred_n(F).$$

The next condition requires a kind of a weak homogeneity.

12. (The weak homogeneity) Let $B \in C^\rho(A^{0\rho})$ be a splitting point as witnessed by B_0, B_1, for some $\rho \in s$ and let $\mu \in s \cap \rho$. Suppose that for some successor model $F \in C^\mu(A^{0\mu})$ the triple B_0, B_1, B is as in the definition of $Pred(F)$. Then for each $\eta \in s \cap \mu + 1$ we have $X \in A^{1\eta} \cap \mathcal{P}(B_0)$ iff $\pi_{B_0 B_1}[X] \in A^{1\eta} \cap \mathcal{P}(B_1)$.

   Intuitively, this means that everything of cardinality at most μ is copied by the isomorphism $\pi_{B_0 B_1}$ from B_0- side to B_1 - side and vise verse. This condition is crucial for preserving GCH.

   The next condition describes the structure of bigger models inside a splitting.

13. (Bigger models over splitting points) Let F_1 be as in (11) with $F \in C^\mu(A^{0\mu})$ and $\rho \in s \backslash \mu + 1$. Suppose that B is the least element of $C^\rho(A^{0\rho})$ including F. Then B is a successor point, moreover B^- is a successor point as well, $Pred(B) = \{B^-\}, Pred(B^-) = \{(B^-)^-\}$ and $F \in B, F_1 \in B^-$. In addition, if $\rho \in F_1$ then

$$(B^-)^- \in F_1 \in B^- \in F \in B.$$

   If $\rho \in F \backslash F_1$ then

$$F_1 \in B^- \in F \in B.$$

14. (No splittings at limits) If $\rho \in s$ is a limit point of s, then no model in $A^{1\rho}$ can be a splitting model.

15. Let $F \in C^\mu(A^{0\mu})$ be a successor model and F^- be its immediate predecessor in $C^\mu(A^{0\mu})$, for some $\mu \in s$. Then $F^- \in Pred(F)$.

   Note that this condition is relevant only when F splits, otherwise $F^- \in Pred_0(F)$ by the definition.

16. (No small models condition 2)

   Let F be a successor point in $C^\mu(A^{0\mu})$ and $A \in A^{1\xi} \cap F$, for some $\xi \in s \cap \mu$. Then there is $G \in Pred(F)$ with $A \in G$.

Let us define the sets $A_k^{1\mu}(A)$, for $A \in A^{1\mu}$ and $1 \le k < \omega$.

$$
\begin{aligned}
&A_k^{1\mu}(A) = \bigcup_{n<\omega} C_{kn}^{\mu}(A) \ , \quad \text{where} \\
&C_{k0}^{\mu}(A) = C^{\mu}(A) \\
&C_{k2n}^{\mu}(A) = \{E \mid \exists F \in C_{k2n-1}^{\mu}(A) \quad E \in C^{\mu}(F)\} \\
&C_{k2n+1}^{\mu}(A) = \{E \in A^{1\mu} \mid \exists F \in C_{k2n}^{\mu}(A) \quad E \in Pred(F) \backslash C^{\mu}(F) \\
&\ \text{and the generalized distance of } E \text{ from } C^{\mu}(F) \text{ is at most } k\}.
\end{aligned}
$$

We define $A_0^{1\mu}(A)$ similar only with

$$C_{02n+1}^{\mu}(A) = \{E \in A^{1\mu} \mid \exists F \in C_{k2n}^{\mu}(A) \quad E \in Pred(F) \backslash C^{\mu}(F)$$

and $E, F,$ the immediate predecessor F^- of F in $C^{\mu}(F)$ are of a Δ- system type$\}$.

In particular, $A_0^{1\mu}(A)$ is defined using only models of cardinality μ.

Set $A^{1\mu}(A) = \bigcup_{k<\omega} A_k^{1\mu}(A)$. It is possible to define $A^{1\mu}(A)$ also as follows:

$$
\begin{aligned}
&A^{1\mu}(A) = \bigcup_{n<\omega} C_n^{\mu}(A) \ , \quad \text{where} \\
&C_0^{\mu}(A) = C^{\mu}(A) \\
&C_{2n}^{\mu}(A) = \{E \mid \exists F \in C_{2n-1}^{\mu}(A) \quad E \in C^{\mu}(F)\} \\
&C_{2n+1}^{\mu}(A) = \{E \in A^{1\mu} \mid \exists F \in C_{2n}^{\mu}(A) \quad E \in Pred(F) \backslash C^{\mu}(F)\}.
\end{aligned}
$$

The next condition describes the way in which elements of $A^{1\mu}$ are generated.

17. (Generation condition)

    Let $\mu \in s$. Then $A^{1\mu} = A^{1\mu}(A^{0\mu})$.

    Set also $A_k^{1\mu} = A_k^{1\mu}(A^{0\mu})$ and $C_k^{\mu} = C_k^{\mu}(A^{0\mu})$ for each $k < \omega$.

    This condition implies that we can reconstruct everything just from the top models (i.e. $A^{0\xi}$'s), $C^{\xi}(A^{0\xi})$'s and the splitting points over $C^{\xi}(A^{0\xi})$'s.

    The next condition provides a weak form of elementarity.

18. If for some $\tau, \xi \in s$ we have $A \in C^{\tau}(A^{1\tau})$ and $B \in C^{\xi}(A^{1\xi}) \cap A$, then $C^{\xi}(B) \in A$, $A^{1\xi}(B) \in A$ as well. Also for each $E \in C^{\tau}(A)$, if there is an element of $C^{\xi}(B)$ including E, then the first such element is in A.

19. Let A be a set in $C^\tau(A^{0\tau})$ and $F \in C^\mu(A^{0\mu})$ be a member of $C^\mu(A^{0\mu})$ including A, for some $\tau, \mu \in s, \tau < \mu$. Then for each $\xi \in s$, $\tau < \xi \leq \mu$ implies that there is $G \in C^\xi(A^{0\xi})$ such that
$$A \subseteq G \subseteq F.$$

20. Let $\rho < \tau$ be in s and $A \in C^\rho(A^{0\rho})$ be a successor model. Suppose $B \in C^\tau(A^{0\tau})$ is the least with $A \subset B$. Then B is a successor model. Suppose that B is not the least element of $C^\tau(A^{0\tau})$. Let B^- be the immediate predecessor of B in $C^\tau(A^{0\tau})$. If $\tau \in A$ then $B^- \in A$. Moreover, if A is the least in $C^\rho(A^{0\rho})$ with B^- inside then $C^\rho(A)\backslash\{A\} \subseteq B^-$.

21. Let $\rho < \tau$ be in s and $A \in C^\rho(A^{0\rho})$ be a limit model. Suppose $B \in C^\tau(A^{0\tau})$ is the least with $A \subset B$. Then B is a limit model. In addition, if $\tau \in A$, then $A \cap (C^\tau(B)\backslash\{B\})$ is cofinal in B.

    Intuitively the last two conditions mean that the sequences $C^\tau(A^{0\tau})$ and $C^\rho(A^{0\rho})$ mix together nicely. Note that $C^\rho(A^{0\rho})$ is closed. Hence always, if $F \cap C^\rho(A^{0\rho})$ is not empty, then there is a maximal $A \in C^\rho(A^{0\rho})$ which is a subset of F.

22. (Least model including a successor one must be a successor model) Let $\rho < \tau$ be in s, $A \in C^\rho(A^{0\rho})$ be a successor model and $B \in C^\tau(A^{0\tau})$ be the least with $A \subset B$. Then B must be a successor model and $A \in B$.

23. (Local maximal models) Let $\rho < \tau$ be in s, $A \in C^\rho(A^{0\rho})$ be a successor model, $\tau \in A$ and $B \in C^\tau(A^{0\tau})$ be the least with $A \subset B$. Suppose that B is not the least element of $C^\tau(A^{0\tau})$. Let B^- be the immediate predecessor of B in $C^\tau(A^{0\tau})$. Then for every $X \in A \cap A^{1\tau}$ we have $X \in A^{1\tau}(B^-)$.

    This means that B^- is a local (relatively to A) version of $A^{0\tau}$.

    The next three conditions provide a kind of linearity over the central line.

24. Let $\rho < \mu < \tau$ be in s and $A \in C^\rho(A^{0\rho})$. Suppose that F, G are the least elements of $C^\mu(A^{0\mu})$ and $C^\tau(A^{0\tau})$ respectively including A. Then G includes F and it is the least such element of $C^\tau(A^{0\tau})$.

25. Let $\rho < \mu < \tau$ be in s, $F \in C^\tau(A^{0\tau})$, $F_1 \in C^\mu(A^{0\mu})$ be the maximal element of $C^\mu(A^{0\mu})$ contained in F (if exists) and F_2 be the maximal element of $C^\rho(A^{0\rho})$ contained in F_1, if exists. Then, if F_1, F_2 exist, then F_2 is the maximal element of $C^\rho(A^{0\rho})$ contained in F.

26. (Continuity at limit points) Suppose that ρ is a limit point of s. Let $\langle F_{\rho\alpha} \mid \alpha < \delta \rangle$ be an increasing enumeration of $C^\rho(A^{0\rho})$. For each $\alpha < \delta$ and $\xi \in s \cap \rho$ let $F_{\xi\alpha}$ be the largest element of $C^\xi(A^{0\xi})$ included in $F_{\rho\alpha}$, if it exists.

    Then for each $\alpha < \delta$ the following hold

    (a) $F_{\xi\alpha}$ exists for all but boundedly many $\xi \in \rho \cap s$

    (b) the sequence

    $$\langle F_{\xi\alpha} \mid \xi \in \rho \cap s, F_{\xi\alpha} \text{ exists} \rangle$$

    is increasing continuous with limit $F_{\rho\alpha}$

27. (Isomorphism condition) If A, B, C are of a Δ-system type for some $A \in C^\tau(C), C \in C^\tau(A^{0\tau})$ then the structures

    $$\langle A, <, \in, \subseteq, \kappa, \tau, C^\tau(A), A^{1\tau}(A), f_A \ , \ \langle A^{1\rho} \cap A \mid \rho \in s\backslash\tau \rangle, \langle C^\rho \restriction A^{1\rho} \cap A \mid \rho \in s\backslash\tau \rangle \rangle$$

    and

    $$\langle B, <, \in, \subseteq \kappa, \tau, C^\tau(B), A^{1\tau}(B), f_B, \ \langle A^{1\rho} \cap B | \rho \in s\backslash\tau \rangle, \langle C^\rho \restriction A^{1\rho} \cap B | \rho \in s\backslash\tau \rangle \rangle$$

    are isomorphic over $A \cap B$, where $f_A : \tau \leftrightarrow A$, $f_B : \tau \leftrightarrow B$ are some fixed in advance enumerations (for example, least such is the well-ordering $<$).

    Let π_{AB} denotes the unique isomorphism. Note that, in particular, $A \cap \tau^+ = B \cap \tau^+$, since both are ordinals and, so π_{AB} is the isomorphism between them.

    Let us state a similar condition. The main difference will be that $\langle C^\rho \restriction A^{1\rho} \cap A \mid \rho \in s\backslash\tau \rangle$ will not be mentioned. The reason is that the switching, which will be defined later, may change C^ρ's which are in one of the models without effecting an other model at all (unless $\tau^+ = \theta$, for example in Gap 4 case). For τ with $\tau^+ = \theta$ (27) suffice.

28. (General Isomorphism Condition) If $A, B \in A^{1\tau}$ and $otp_\tau(A) = otp_\tau(B)$ (equivalently, by (7) $otp(A) = otp(B)$) then the structures

    $$\langle A, <, \in, \subseteq, \kappa, \tau, C^\tau(A), A^{1\tau}(A), f_A \ , \ \langle A^{1\rho} \cap A \mid \rho \in s\backslash\tau \rangle, C^\tau \restriction A^{1\tau} \cap A \rangle$$

    and

    $$\langle B, <, \in, \subseteq \kappa, \tau, C^\tau(B), A^{1\tau}(B), f_B, \ \langle A^{1\rho} \cap B | \rho \in s\backslash\tau \rangle, C^\tau \restriction A^{1\tau} \cap B | \rangle$$

are isomorphic where $f_A : \tau \leftrightarrow A$, $f_B : \tau \leftrightarrow B$ are some fixed in advance enumerations (for example, least such is the well-ordering $<$).

The next condition is weak version of elementarity.

29. (Weak elementarity condition)

    Let $\tau, \mu \in s$, $A \in A^{1\tau}$ and $B \in A^{1\mu}$. If $B \in A$, then $A^{1\mu}(B)$ and $C^{\mu}(B)$ are in A. In addition, if $x \in A$ and for some $C \in C^{\mu}(B)$ we have $x \in C$, then the first member of $C^{\mu}(B)$ with this property is in A. Also require that if $B \in A$, then the function f_B as in (27) is in A. If $B' \in A$ and $otp(B) = otp(B')$ then the isomorphism $\pi_{BB'}$ is in A as well.

    Let define now one more basic notion and then use it to state the requirement on a weak Δ-system type.

    Let B_0, B_1, B with $B \in C^{\rho}(A^{0\rho}), B_0 \in C^{\rho}(B)$ be a suitable for switching triple, for some $\rho \in s$. We define the **switch by** B or $sw(B)$ of the functions $C^{\tau}, \tau \in s \cap \rho + 1$ as follows: $C^{\rho}_B(B) = C^{\rho}(B_1) \cup \{B\}$ and for each $E \in C^{\rho}(A^{0\rho}) \backslash C^{\rho}(B)$ let $C^{\rho}_B(E) = (C^{\rho}(E) \backslash C^{\rho}(B)) \cup C^{\rho}_B(B)$.
    Let now $\tau \in s \cap \rho$. Pick the first element A of $C^{0\tau}(A^{0\tau})$ with $B \in A$. Its immediate predecessor A^- in $C^{\tau}(A^{0\tau})$ is in B, by our assumption. Then $A^- \subset B_0$. Leave $C^{\tau}(A^-)$ unchanged as well all its initial segments. Set $C^{\tau}_B(A^{0\tau}(q)) = (C^{\tau}(A^{0\tau}) \backslash C^{\tau}(A^-)) \cup \pi_{B_0B_1}[C^{\tau}(A^-)]$. In order to obtain the full function C^{τ}_B we just move the defined already portions via isomorphisms of the models in $A^{1\tau}$.
    Remember that $B \in A$, hence $\pi_{B_0B_1}[A^-]$ remains inside $Pred(A)$.
    Note that the above definition extends the definition 1.2, where we dealt only with $C^{\tau}(A^{0\tau})$.

    We define now $sw(B^0, ..., B^n)$ by induction to be the result of the application of B^n to $sw(B^0, ..., B^{n-1})$.

    Note that the application of a same B twice leaves the functions C^{τ} unchanged, i.e $C^{\tau}_{BB} = C^{\tau}$.

    Let us require the following:

30. Suppose that $F_0, F_1, F \in A^{1\mu}$ are of a weak Δ -system type with $\sup(F_1) > \sup(F_0)$, for some $\mu \in s$. Then there are $B^0, ..., B^n$ with each B^i either in F_1 or $F \in B^i$, such that $sw(B^0, ..., B^n)$ turns $F_0, F_1, F \in A^{1\mu}$ into a Δ -system type triple with all the relevant conditions above satisfied according to the new C^{τ}'s, i.e. $C^{\tau}_{B^0,...,B^n}$'s.

31. Let $A \in A^{1\mu}$, for some $\mu \in s$. Then there are $B^0, ..., B^n$ such that $sw(B^0, ..., B^n)$ moves A to the central line, i.e. $A \in C^{\mu}_{B^0,...,B^n}(A^{0\mu})$.

$\square$ of the definition.

Lemma 1.5 *For each $\mu \in s$ and $A \in A^{1\mu}$ there is $C \in C^{\mu}(A^{0\mu})$ with $otp(A) = otp(C)$.*

Proof. We prove the statement by induction. Let n be the least with $A \in C^{\mu}_{n}$. If $n = 0$ then take $C = A$.

If $n > 0$ is even, then there is $B \in C^{\mu}_{n-1}$ with $A \in C^{\mu}(B)$. By induction, then there is $D \in C^{\mu}(A^{0\mu})$ of the order type equal to $otp(B)$. Now use 1.1(7) for B and D. Let $A' = \pi_{BD}[A]$. Then $A' \in C^{\mu}(D)$ which is an initial segment of $C^{\mu}(A^{0\mu})$. So we are done.

If n is odd then there is $F \in C^{\mu}_{n-1}$ with $A \in Pred(F) \backslash C^{\mu}(F)$. Now, if F is not a splitting a point, then $otp(F^{-}) = otp(A)$, where F^{-} is the immediate predecessor of F in $C^{\mu}(F)$. Now we apply the induction to F and use 1.1(28).

If F is a splitting point, then let $F_0 \in C^{\mu}(F), F_1$ be witnessing this. Again, $otp(A) = otp(F_0)$ and we can apply the induction to F and use 1.1(28).

□

This lemma together with 1.1(28) allow to transfer the conditions of 1.1 stated for elements of $C^{\mu}(A^{0\mu})$ to those of $A^{1\mu}$. Thus for example the following general version of 1.1(23) holds:

Lemma 1.6 *Let $\rho < \tau$ be in s, $A \in A^{1\rho}$ be a successor model. Suppose that $A \cap A^{1\tau} \neq \emptyset$ Then there is $E \in A \cap A^{1\tau}$ such that for every $X \in A \cap A^{1\tau}$ we have $X \in A^{1\tau}(E)$.*

Proof. Using 1.5 find $A' \in C^{\rho}(A^{0\rho})$ of the same order type as those of A. By 1.1(23), there is the maximal element E' of $A' \cap A^{1\tau}$. Then we can use 1.1(28) to move it to A, i.e. set $E = \pi_{A'A}[E']$.

□

Notation 1.7 Denote further the maximal model of $A \cap A^{1\tau}$ by $(A^{0\tau})^{A}$.

Lemma 1.8 *Let A be as in the lemma 1.6. Suppose that $A^{1\rho}(A) \cap (A^{0\tau})^{A} \neq \emptyset$, then for each $\tau' \in s \cap [\rho, \tau] \cap A$ the maximal model $(A^{0\tau'})^{A}$ exists.*

Proof. It follows by 1.5 and 1.1(19).

□

Lemma 1.9 *Let $\rho < \tau$ be in s, $A \in A^{1\rho}$ be a successor model. Suppose that*

$$A \cap \bigcup \{A^{1\tau} | \tau \in s \backslash \rho\} \neq \emptyset.$$

Then there are $n < \omega$, $\tau_n > ... > \tau_0$ in $A \cap (s \backslash \rho)$ and the maximal models $(A^{0\tau_n})^{A} \in ... \in (A^{0\tau_0})^{A}$ such that for each $\tau \in s \backslash \rho$ we have $(A^{0\tau})^{A} \subseteq (A^{0\tau_k})^{A}$ (if defined), for some $k \leq n$.

Proof. By **??** and 1.1(28), it is enough to deal with $A \in C^{\rho}(A^{0\rho})$. Now it follows by 1.1(4, 25).
□

The next lemma follows easily from the definition of $Pred$.

Lemma 1.10 *Let $\rho \in s, A \in A^{1\rho}$ be a successor model. Then $A^{1\rho}(A) = \bigcup\{A^{1\rho}(X)|X \in Pred(A)\}$.*

Lemma 1.11 *Let $\rho < \tau$ be in s, $A \in A^{1\rho}$ be a successor model. Then there are $n < \omega$, $\tau_n > ... > \tau_0$ in $A \cap (s\backslash\rho)$ and the maximal models $(A^{0\tau_n})^A \in ... \in (A^{0\tau_0})^A$ such that for each $B \in A^{1\rho}(A)\backslash\{A\}$ we have $B \in (A^{0\tau_k})^A$ for some $k \leq n$.*

Proof. Let B be in $A^{1\rho}(A)$. Then by 1.10 there is $X \in Pred(A)$ with $B \in A^{1\rho}(X)$. Let $n < \omega$, $\tau_n > ... > \tau_0$ in $A \cap (s\backslash\rho)$ and the maximal models $(A^{0\tau_n})^A \in ... \in (A^{0\tau_0})^A$ be as in 1.9. Now, by the definition of $Pred$ and 1.9, $B \in (A^{0\tau_k})^A$ for some $k \leq n$.
□

The following is a consequence of 1.1(7), (8) and the previous lemma.

Lemma 1.12 *Let F be in $A^{1\mu}$ for some $\mu \in s$. Suppose that there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a Δ-system type with F being the largest model, then F_0, F_1 are unique.*

The following lemmas follow easily from the definition of $A^{1\mu}(A)$.

Lemma 1.13 *Let $\mu \in s, F \in A^{1\mu}$ be a successor model with unique immediate predecessor F^- in $C^{1\mu}(F)$. Then $A^{1\mu}_0(F) = A^{1\mu}_0(F^-) \cup \{F\}$*

and

$A^{1\mu}(F) = \bigcup\{A^{1\mu}(X)|X \in Pred(F)$.

Lemma 1.14 *Let $\mu \in s, F \in A^{1\mu}$ be a successor model which is a splitting point in $A^{1\mu}(F)$ i.e. there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a Δ-system type with F being the largest model. Then $A^{1\mu}(F) = A^{1\mu}(F_0) \cup A^{1\mu}(F_1) \cup \{F\}$.*

Lemma 1.15 *Let $\mu \in s, F \in A^{1\mu}$ be a limit model. Then $A^{1\mu}(F) = \bigcup\{A^{1\mu}(D)|D \in C^{\mu}(F)\}$.*

Lemma 1.16 *(Identity on the common part) Suppose $\mu \in s, A, B \in A^{1\mu}_0$ and $otp(A) = otp(B)$. Then π_{AB} is the identity on $A \cap B$.*

Proof. Suppose that $A \neq B$. Consider the walks from $A^{0\mu}$ to A and to B. Let G be the last common model of the walks. Then it must be a splitting point. Let G_0, G_1 be its immediate predecessors witnessing this with $G_0 \in C^\mu(G)$. So, G_0, G_1, G are of a weak Δ-system type. In particular $\pi_{G_0G_1}$ is the identity on $G_0 \cap G_1$. Suppose that $A \in A^{1\mu}(G_0)$ and $B \in A^{1\mu}(G_1)$. Set $B_0 = \pi_{G_1G_0}[B]$. If $x \in A \cap B$, then $x \in G_0 \cap G_1$ and so in B_0. B_0 is simpler then B so we can apply induction to A, B_0. Hence, π_{AB_0} is the identity on $A \cap B_0$. In particular, $\pi_{AB_0}(x) = x$. But $x \in G_0 \cap G_1$. So $\pi_{G_0G_1}(x) = x$. Then $\pi_{B_0B}(x) = x$, since $\pi_{G_0G_1}$ extends $\pi B_0 B$. Now

$$\pi_{AB}(x) = \pi_{B_0B}(\pi_{AB_0}(x)) = \pi_{B_0B}(x) = x.$$

□

Remark 1.17 (1) Note that in the gap 4 case we have $A_0^{1\mu} = A^{1\mu}$, for $\mu = \kappa^{++}$. Hence, any two elements of $A^{1\mu}$ of the same order type are isomorphic over their common intersection. This breaks down for $\mu = \kappa^+$ even in the gap 4 case.
(2) The argument of the lemma can be used in more general situations. Once having a splitting point G we can replace B by $\pi_{G_1G_0}[B]$. The crucial is that $\pi_{G_1G_0}$ is the identity on $G_0 \cap G_1$ and this is true always for splitting points.

Definition 1.18 (The general walk between models) Let $\nu \in s$. Define a function gwk on elements A of $A^{1\nu}$. We will call $gwk(A)$ a **general walk from** $A^{0\nu}$ **to** A . The definition is by induction on the general distance of A from the central line, i.e. on $gd(A)$ simultaneously on each $\nu \in s$ and $A \in A^{1\nu}$.

(a) if $gd(A) = 0$ then set $gwk(A) = \langle A \rangle$

(b) if $gd(A) = n > 0$, then let pick simplest in the general walk sense models $B^1, ..., B^n$ such that $A \in C^\nu(A^{0\nu})_{B^0,...,B^n}$ (they exist by 1.1(31)). Consider the triple B_0^n, B_1^n, B^n (forming a Δ - system type as in 1.3). Set $A_0 = \pi_{B_1^nB_0^n}[A]$. Note that $A \subseteq B_1^n$, since otherwise there will be now need in B^n. Set

$$gwk(A) = gwk(A_0)^\frown gwk(B^n)^\frown B_0^{n\frown} B_1^{n\frown} A.$$

Let us make now one technical definition which relates to intersections of models.

Definition 1.19 Let $\xi, \zeta \in s, A \in A^{1\xi}$ and $B \in A^{1\zeta}$. We say that A **satisfies the intersection property with respect to** B or shortly $ip(A, B)$ iff either

(1) $\xi > \zeta$

or

(2) $\xi \leq \zeta$ and $A \subseteq B$

or

(3) $\xi = \zeta$ and $B \subseteq A$

or

(4) $\xi \leq \zeta, A \not\subseteq B, B \not\subseteq A$ and then there are $A' \in A \cup \{A\}$ and $D_1 \in (A^{1\rho_1})^A, ..., D_n \in (A^{1\rho_n})^A$, for some $\rho_1, ..., \rho_n \in s \backslash \xi + 1$ such that

(a) $A' = A$ unless $\xi = \zeta$ and $otp_\xi(A) > otp_\xi(B)$.
If this is the case (i.e. $otp_\xi(A) > otp_\xi(B)$), then $otp(A') = otp(B)$ and ($A' \in (A^{1\xi})^A$ or A' is an image of an element of $(A^{1\xi})^A$ under isomorphisms $\pi_{G_0G_1}$ for models $G_0, G_1 \in A$).

(b) $A \cap B = A \cap A' \cap D_1 \cap D_2 \cap ... \cap D_n$.

(c) $A' \in A^{1\xi}$
or
$A' = \pi_{IJ}[A'' \cap H_1 ... \cap H_k]$, for some $A'' \in A^{1\xi}(A), H_1 \in (A^{1\eta_1})^A, ..., H_k \in (A^{1\eta_k})^A, I, J \in (A^{1\eta})^A$, for some $\eta, \eta_1, ..., \eta_k \in s \backslash \xi + 1$.

Let $ipb(A, B)$ denotes that both $ip(A, B)$ and $ip(B, A)$ hold.

Lemma 1.20 *(General Intersection Lemma) Let $\xi, \zeta \in s, A \in A^{1\xi}$ and $B \in A^{1\zeta}$. Then $ipb(A, B)$.*

Proof. At least one of A and B is not on the central line. Without loss of generality we can assume that one of A, B is on the central line. Otherwise make finitely many switches that lead to this situation. We put the model of the least cardinality between A and B on the central line. Let A be such a model. We like to show $ip(A, B)$.

Consider the walk from $A^{0\zeta}$ to B. Let Z be the last model in $C^\zeta(A^{0\zeta})$ of this walk. Then Z must be a successor model. Let Z^- be the immediate predecessor of Z in $C^\zeta(A^{0\zeta})$ and $Z_1 \in Pred(Z)$ be the next point in the walk leading to B. If Z^-, Z_1 are isomorphic over $Z^- \cap Z_1$ then we would like to use $\pi_{Z_1 Z^-}$ to move B to a simpler (according to the generalized distance gd) model B_0 and from $ip(A, B_0)$ deduce $ip(A, B)$. Also in general case we would like to replace B by a simpler model. Proceed as follows. If Z^-, Z_1, Z are of a weak Δ - system type, then denote Z_1 by G_B and let F_A be the smallest model in $C^\zeta(A^{0\zeta})$ including A. If Z^-, Z_1, Z are not of a weak Δ - system type, then let $G_B \in A^{1\rho}$ be the last model used to generate Z_1 in $Pred(Z)$ with some $\rho \in s \backslash \zeta$. Let F_A be the smallest model in $C^\zeta(A^{0\rho})$ including A.

Compare now G_B and F_A.

Case 1. $F_A \notin A^{1\rho}(G_B)$ and $G_B \notin A^{1\rho}(F_A)$.

Consider the last common point of the walks to G_B and to F_A from $A^{0\rho}$. Let E denotes this point. Then it must be a successor point.

Subcase 1.1. E does not have immediate predecessors of a weak Δ - system type or it does but at least one of F_A, G_B is not in $A^{1\rho}$ of them.

Suppose that F_A is such. Then there are $\eta \in s \backslash \rho + 1$ and a model $H_A \in A^{1\eta}$ with immediate predecessors H_{A0}, H_{A1} of a weak Δ - system type such that F_A is on the H_{A1} - side. Pick the smallest model K_B in the moved (according the way of moving to (or generating) G_B) $C^\eta(A^{0\eta})$ with G_B inside $(A^{1\rho})^{K_B}$.

Now again we compare H_A and K_B according to the walks from $A^{0\eta}$. Note that the models under the consideration are simpler than F_A, G_B since they are more close to the central (beginning) line, i.e. gd decreases. So we can reduce the situation (either induction or finitely many applications of the process used above) to the negation of the present subcase.

Subcase 1.2. E has immediate predecessors E_0, E_1 of a weak Δ - system type with $F_A \in A^{1\rho}(E_0)$ and $G_B \in A^{1\rho}(E_1)$.

By the definition of a Δ - system type, there will be $D_{01} \in E_0 \cap A^{1\zeta}, D_{10} \in E_1 \cap A^{1\zeta}$ such that

$$E_0 \cap E_1 = E_0 \cap D_{01} = E_1 \cap D_{10}$$

and E_0, E_1 are isomorphic over $E_0 \cap E_1$. Let E_0 be the one in $C^\rho(E)$.

Now we move G_B and B to E_0 side. Set $G_B^0 = \pi_{E_1 E_0}[G_B]$ and $B^0 = \pi_{E_1 E_0}[B]$. Then

$$A \cap B = A \cap F_A \cap B \cap G_B = A \cap F_A \cap E_0 \cap E_1 \cap G_B \cap B =$$

$$A \cap F_A \cap E_0 \cap D_{01} \cap G_B^0 \cap B^0 = A \cap B^0 \cap D_{01}.$$

Induction can be applied to A, B^0, D_{01}, since at least B^0 and D_{01} are simpler than B again according to the distance from the basic central line, i.e. gd.

Case 2. $F_A \in A^{1\rho}(G_B)$ or $G_B \in A^{1\rho}(F_A)$.

Let $G_B \in A^{1\rho}(F_A)$.

Subcase 2.1. $G_B \notin A$.

Denote by G_B^0 the model used at the last step together with G_B to move (construct) B. Then there is $G \in A^{1\rho}$ such that G_B^0, G_B, G are of a weak Δ - system type. Here we have $G_B^0 \in C^\rho(G)$ and $G \in A^{1\rho}(F_A)$.

Subsubcase 2.1.1 $\rho \in A$.

By minimality of F_A, then also $G \in A^{1\rho}(F')$, for some $F' \in A \cap C^\rho(F)$. We use here 1.1(20) or (21). If F_A is a successor model, then F_A^- exists, it is in A and is equal to $(A^{0\rho})^A$. Consider the walk from F' to G. We assume that no models of bigger than ρ cardinalities are involved here (otherwise we are back in the situation considered in Case 1) and so the walk is entirely in $A_0^{1\rho}$. Let F be the last point of this walk in A and E the very next point of this walk. Then F must be a limit point. Let

$$\tilde{F} = \bigcup \{X | X \in A \cap C^\rho(F) \backslash \{F\}\}.$$

E must be a splitting point with two immediate predecessors E_0, E_1 of a Δ - system type, $E_0 \in C^\rho(E), G \in A^{1\rho}(E_1)$. We would like to move to E_0 side simplifying the situation. By the definition of a Δ - system type, there will be $D_{01} \in E_0 \cap A^{1\zeta}, D_{10} \in E_1 \cap A^{1\zeta}$ such that

$$E_0 \cap E_1 = E_0 \cap D_{01} = E_1 \cap D_{10}$$

and E_0, E_1 are isomorphic over $E_0 \cap E_1$. Set $G_B^0 = \pi_{E_1 E_0}[G_B]$ and $B^0 = \pi_{E_1 E_0}[B]$. Then

$$A \cap B = A \cap F_A \cap B \cap G_B = A \cap F_A \cap E \cap E_1 \cap B = A \cap F' \cap E \cap E_1 \cap B =$$

$$A \cap \tilde{F} \cap E_1 \cap B = A \cap E_0 \cap E_1 \cap B = A \cap F' \cap E_0 \cap E_1 \cap G_B \cap B =$$

$$A \cap F' \cap D_{01} \cap G_B^0 \cap B^0 = A \cap F' \cap B^0 \cap D_{01}.$$

Induction can be applied to A, B^0, D_{01}, since at least B^0 and D_{01} are simpler than B again according to the distance from the basic central line.

Subsubcase 2.1.2. $\rho \notin A$.

Let $\delta = \min(A \cap (s \backslash \rho)$. Let F_A^δ be the least element of $C^\delta(A^{0\delta})$ including A. Then by 1.1(24), $F_A \subseteq F_A^\delta$. The walk from $A^{0\rho}$ to G goes via F_A. Assume again that no models of cardinalities above ρ are involved in this walk. Let E be the last model of the walk inside $C^\rho(A^{0\rho})$. Now, $E \subseteq F_A$, since the walk passes F_A. Moreover, $E \in F_A$, since F_A is the least member of $C^\rho(A^{0\rho})$ including A and A is on the central line as well. Let H be the least element of $C^\delta(A^{0\delta})$ including E. Then $H \subset F_A^\delta$. By minimality of F_A^δ, then also $H \in C^\delta(F')$, for some $F' \in A \cap C^\delta(F_A^\delta)$. We use here 1.1(20) or (21). If F_A^δ is a successor model, then $(F_A^\delta)^-$ exists, it is in A and is equal to $(A^{0\delta})^A$. Pick the smallest $F \in A \cap C^\delta(F')$ with

$H \subseteq F$. Note that in the present case F need not be a limit point. Thus it may be equal to H and since δ is a limit point of s, H will be an increasing continuous union of models smaller cardinalities in s. We set

$$\tilde{F} = \bigcup\{X | X \in A \cap C^{\nu}(A^{0\nu}) \cap F, \nu \in s \cap \delta\}.$$

E must be a splitting point with two immediate predecessors E_0, E_1 of a Δ - system type, $E_0 \in C^{\rho}(E), G \in A^{1\rho}(E_1)$. We would like to move to E_0 side simplifying the situation. By the definition of a Δ - system type, there will be $D_{01} \in E_0 \cap A^{1\zeta}, D_{10} \in E_1 \cap A^{1\zeta}$ such that

$$E_0 \cap E_1 = E_0 \cap D_{01} = E_1 \cap D_{10}$$

and E_0, E_1 are isomorphic over $E_0 \cap E_1$. Set $G_B^0 = \pi_{E_1 E_0}[G_B]$ and $B^0 = \pi_{E_1 E_0}[B]$. Then, using 1.1(26), we obtain

$$A \cap B = A \cap F_A \cap B \cap G_B = A \cap F_A \cap E \cap E_1 \cap B = A \cap F \cap E \cap E_1 \cap B =$$

$$A \cap \tilde{F} \cap E_1 \cap B = A \cap E_0 \cap E_1 \cap B = A \cap F \cap E_0 \cap E_1 \cap G_B \cap B =$$

$$A \cap F' \cap D_{01} \cap G_B^0 \cap B^0 = A \cap F' \cap B^0 \cap D_{01}.$$

Induction can be applied to A, B^0, D_{01}, since at least B^0 and D_{01} are simpler than B again according to the distance from the basic central line.

Subcase 2.2. $G_B \in A$.

Let G_B^0, G be as in the previous case. Then they also are in A. Now we deal with G_B^0 and G_B exactly as in the appropriate case of the third intersection lemma (or see below). This allows to replace G_B (and so B) by a simpler (closer to the central line) model G_B^0 (and B by $B_0 = \pi_{G_B G_B^0}[B]$). Let us reproduce the argument of the third intersection lemma. Denote for simplicity G_B by G_1 and G_B^0 by G_0. Let $B_0 = \pi_{G_1 G_0}[B]$.

Recall that $G_0 = f_{G_0}[\rho]$ and $G_1 = f_{G_1}[\rho]$, where f_{G_0} and f_{G_1} are the fixed functions from ρ one to one onto G_0 and G_1 respectively. Also, they are respected by isomorphism $\pi_{G_0 G_1}$ of the structures and are in A by the elementarity condition 1.1(29). Set $T_0 = f_{G_0}^{-1}[B_0]$ and $T_1 = f_{G_1}^{-1}[B]$. Then $\pi_{G_0^i G_1^i}[T_0] = T_1$, but $T_0, T_1 \subseteq \rho$ and $\pi_{G_0 G_1} \restriction \rho = id$, since $\rho \subseteq G_0^i \cap G_1^i$. Hence $T_0 = T_1$. Note that $A \cap B = f_{G_1}[A \cap T_1]$, since $\alpha \in A \cap B$ iff $f_{G_1}^{-1}(\alpha) \in A$ and $f_{G_1}^{-1}(\alpha) \in T_1$ iff $f_{G_1}^{-1}(\alpha) \in A \cap T_1$, also $A \cap G_1 = f_{G_1}[A \cap \rho]$. Similar, $A \cap B_0 = f_{G_1}[A \cap T_0]$.

Now

$$A \cap B = f_{G_1}[A \cap T_1] = \pi_{G_0 G_1}(f_{G_0}[A \cap T_0]) =$$

$$\pi_{G_0 G_1}[A \cap B_0],$$

since $\alpha \in f_{G_1}[A \cap T_1]$ iff $\alpha \in f_{G_1}[T_1]$ and $\alpha \in f_{G_1}[A \cap \rho]$ iff $\pi_{G_1 G_0}(\alpha) \in f_{G_0}[T_0]$ and $\pi_{G_1 G_0}(\alpha) \in f_{G_0}[A \cap \rho]$. iff $\pi_{G_1 G_0}(\alpha) \in f_{G_0}[T_0 \cap A] = A \cap B_0$.

Note only that $\pi_{G_1 G_0}(\alpha) \in A$ iff $\alpha \in A \cap G_1$, since $\pi_{G_1 G_0} \in A$. It is crucial that $\pi_{G_1 G_0} \restriction \rho = id$ and that G_0, $G_1 \in A$ implies $f_{G_1}[A \cap \rho] = A \cap G_1$, $f_{G_0^i}[A \cap \rho] = A \cap G_0$.
□

The proof of the next lemma is similar to those of 1.20.

Lemma 1.21 *Let A, B be sets in $A^{1\tau}$ for some $\tau \in s$ and $B \subset A$. Then $B \in A^{1\tau}(A)$.*

Proof. Suppose otherwise. Without loss of generality we can assume that one of the models A, B is on the central line. Let then E be the last common model of the walks from $A^{0\tau}$ to A and to B (or just the last model of the walk to B in $C^\tau(A^{0\tau})$, if A is in the central line, i.e. $A \in C^\tau(A^{0\tau})$) . Then E must be a successor model. Suppose that E is a splitting point. The non splitting case is treated similar. Let E_0, E_1 be the immediate predecessors of E such that the triple E_0, E_1, E is of a weak Δ - system type. If $A \in A^{1\tau}(E_0)$ and $B \in A^{1\tau}(E_1)$ (or $A \in A^{1\tau}(E_1)$ and $B \in A^{1\tau}(E_0)$), then $B \subseteq E_0 \cap E_1$ and so $\pi_{E_0 E_1}$ does not move B, since the triple E_0, E_1, E is of a weak Δ - system type. It is impossible to have now $E_0 \in C^\tau(E)$, since then the common walk can be continued further to E_0. Let us replace A by $A' = \pi_{E_0 E_1}(A)$. Then $A' \supset B$. Applying induction, we will have $B \in A^{1\tau}(A')$. Now, moving back, B (which does not move) will be in $A^{1\tau}(A)$.

Suppose now that at least one of A, B is not in $A^{1\tau}(E_i)$ for $i \in 2$. Let $\sup(E_1) > \sup(E_0)$. Then there is $X \in Pred(E) \backslash Pred_0(E)$ with A or B inside $A^{1\tau}(X)$. Consider models $H_0, H_1, H \in E_1 \cap A^{1\rho}$ of a weak Δ - system type generating X as in the definition of $Pred$. If $H \in A$, then $\pi_{H_1 H_0}[B] \subset A$. Induction applies then to A and $\pi_{H_1 H_0}[B]$. Hence, $\pi_{H_1 H_0}[B] \in A^{1\tau}(A)$. Then also $B \in A^{1\tau}(A)$.
Note that it is impossible to have in the present situation the following:

$$A = E_1, B = \pi_{H_0 H_1}[E_0].$$

Since then $E_0 \subset A = E_1$. Which implies that $E_0 = E_1$.
Suppose now that $H \notin A$. Assume that $B \in A^{1\tau}(X)$. The case $A \in A^{1\tau}(X)$ is similar. Let F_A be the smallest model in the moved (according the way of moving to A from the central line) $C^\rho(A^{0\rho})$ with A inside $(A^{1\tau})^F_A$. Compare F_A with H.

Case 1. $F_A \notin A^{1\rho}(H)$ and $H \notin A^{1\rho}(F_A)$.

Consider the last common point K of the walks to F_A and to H. Proceeding as in 1.20, we can assume that K has immediate predecessors K_0, K_1 of a weak Δ - system type with

$F_A \in A^{1\rho}(K_0)$ and $H \in A^{1\rho}(K_1)$. By the definition of a Δ - system type, there will be $D_{01} \in K_0 \cap A^{1\zeta}, D_{10} \in K_1 \cap A^{1\zeta}$ such that

$$K_0 \cap K_1 = K_0 \cap D_{01} = K_1 \cap D_{10}$$

and K_0, K_1 are isomorphic over $K_0 \cap K_1$. Let K_0 be the one in $C^\rho(K)$.

Now we move H and B to K_0 side. Set $H^0 = \pi_{K_1 K_0}[H]$ and $B^0 = \pi_{K_1 K_0}[B]$. But $B \subseteq K_0 \cap K_1$, since $B \subseteq A \subseteq F_A \subseteq K_0$. Hence $B_0 = B$. This contradicts the choice of H as the simplest possible, since we found a simpler replacement H^0.

Case 2. $F_A \in A^{1\rho}(H)$ or $H \in A^{1\rho}(F_A)$.

Let $H \in A^{1\rho}(F_A)$. Assume that $\rho \in A$. The case $\rho \notin A$ is similar and repeats Subsubcase 2.1.2 of 1.20. By minimality of F_A, then also $H \in A^{1\rho}(F')$, for some $F' \in A \cap C^\rho(F)$. We use here 1.1(20) or (21). If F_A is a successor model, then F_A^- exists, it is in A and is equal to $(A^{0\rho})^A$. Consider the walk from F' to H. We assume that no models of bigger than ρ cardinalities are involved here (otherwise we are back in the situation considered in Case 1) and so the walk is entirely in $A_0^{1\rho}$. Let F be the last point of this walk in A and Y the very next point of this walk. Then F must be a limit point. Let

$$\tilde{F} = \bigcup \{X | X \in A \cap C^\rho(F) \backslash \{F\}\}.$$

Y must be a splitting point with two immediate predecessors Y_0, Y_1 of a Δ - system type, $Y_0 \in C^\rho(Y), G \in A^{1\rho}(Y_1)$. We would like to move to Y_0 side simplifying the situation. By the definition of a Δ - system type, there will be $D_{01} \in Y_0 \cap A^{1\zeta}, D_{10} \in Y_1 \cap A^{1\zeta}$ such that

$$Y_0 \cap Y_1 = Y_0 \cap D_{01} = Y_1 \cap D_{10}$$

and Y_0, Y_1 are isomorphic over $Y_0 \cap Y_1$. Then

$$A \cap B = A \cap F_A \cap B \cap H = A \cap F_A \cap Y \cap Y_1 \cap B = A \cap F' \cap Y \cap Y_1 \cap B =$$

$$A \cap \tilde{F} \cap Y_1 \cap B = A \cap Y_0 \cap Y_1 \cap B = A \cap \tilde{F} \cap Y_0 \cap Y_1 \cap H \cap B =$$

$$A \cap \tilde{F} \cap D_{10} \cap B.$$

Now, since $B \subseteq A$ we must have $B \subseteq D_{10}$. So, $B \subseteq Y_0$ and we can move everything to the Y_0 - side simplifying the situation.

□

The following two lemmas extend similar statements for $A_0^{1\tau}$. Their prove follows the lines of 1.20.

Lemma 1.22 *Let A be a set in $A^{1\tau}$ for some $\tau \in s$. Then the following holds: for each $\rho \in s \backslash \tau + 1$ there is $F \in A^{1\rho}$ such that*

(1) $A \subseteq F$

(2) $gd(F) \leq gd(A)$

(3) if $G \in A^{1\xi}$, for some $\xi \in s \backslash \rho$ and $G \supseteq A$, then $A \subseteq F \subseteq G$.

Proof. Suppose that $A \in C^{\tau}(A^{0\tau})$, otherwise just move it to the central line by doing finitely many switches. Pick F to be the least model in $C^{\rho}(A^{0\rho})$ including A. We claim that F is as desired. Thus let $G \in A^{1\xi}$, for some $\xi \in s \backslash \rho$ and $G \supseteq A$. Assume that $F \not\supseteq G$. By 1.20, we have $ip(F, G)$ and by the definition 1.19 of $ip(F, G)$ there will be $D \in F \cap A^{1\xi}$ with $F \cap D \supseteq F \cap G \supseteq A$, for some $\xi \in s \backslash \rho + 1$. Let $E \in C^{\xi}(A^{0\xi})$ be the least model including A. Then $E \supset F$, by 1.1(24),as $\xi > \rho$ and both models E and F are on the central line. Hence $D \subset E$. But $D \supseteq A$ and E was the least model of $C^{\xi}(A^{0\xi})$ including A this is impossible by 1.1(20, 21, 6(a)).
□

Lemma 1.23 *Let A be a set in $A^{1\tau}$ for some $\tau \in s$. Then the following holds: if $H \in A^{1\xi}$, for some $\xi \in s \backslash \tau + 1$, and $H \supseteq A$, then for each $\rho \in s, \tau < \rho < \xi$ there is $F \in A^{1\rho}$ with $A \subseteq F \subseteq H$.*

Proof. Pick $F \in A^{1\rho}$ and $E \in A^{1\xi}$ satisfying the conclusion of 1.22 with ρ and with ξ respectively. Then, by 1.22(3) (for F), we obtain

$$A \subset F \subset G.$$

But 1.22(3) for E implies $H \supseteq G$. So,

$$A \subseteq F \subseteq H$$

and we are done.
□

We turn now to the definition, of the order on $\mathcal{P}'$.

Let us give a preliminary definition.

Definition 1.24 Let $p = \langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle \in \mathcal{P}'$ and $B \in C^{\rho}(A^{0\rho})$ for some $\rho \in s$. Define **the switching of** p **by** B , or shortly- $swt(p, B)$ to be $q = \langle\langle A^{0\tau}(q), A^{1\tau}(q), C^{\tau}(q) \mid \tau \in s(q)\rangle$ so that $q = p$ unless the following condition is satisfied:

(*) B is a successor point having two immediate predecessors $B_0 \in C^{\rho}(B)$ and B_1 such that the triple B_0, B_1, B is suitable for switching (see 1.2) i.e.

for each $\tau \in s \cap \rho$, $B \in A^{0\tau}$ and if $A \in C^{\tau}(A^{0\tau})$ is the first with $B \in A$, then its immediate predecessor A^- in $C^{\tau}(A^{0\tau})$ is in B. Moreover, if A is a splitting point as witnessed by A_0, A_1 and $\sup(A_0) < \sup(A_1)$, then $A_0 \in B \in A_1$.

Note that in the last case, i.e. if A is a splitting point as witnessed by A_0, A_1 and $\sup(A_0) < \sup(A_1)$, then it is impossible to have $A_1 \in B \in A$ by 1.1(11). Also, by 1.1(29), we must have $B_0, B_1 \in A_1$ as well. It is not hard to construct B's that fail to satisfy the second part of (b). What is needed is a chain of models of the length $> \tau$ which splits more than τ many times and two successive models $A^-, A = A^{0\tau}$ with $A^- \in C^{\tau}(A)$, and the chain inside both A^- and A. Now any splitting point of this chain $B \in A$ which is above $\sup(A^- \cap \rho)$ will do the job.

If (*) holds then q will be obtained from p by switching B_0 and B_1. Thus $s(q) = s$, $A^{0\tau}(q) = A^{0\tau}, A^{1\tau}(q) = A^{1\tau}$ for each $\tau \in s$, $C^{1\tau}(q) = C^{1\tau}$ for every $\tau \in s\backslash\rho+1$. Only $C^{\tau}(q)$'s for $\tau \in s \cap \rho + 1$ may be different.

Let $C^{\rho}(q)(B) = C^{\rho}(B_1) \cup \{B\}$ and for each $E \in C^{\rho}(A^{0\rho})\backslash C^{\rho}(B)$ let $C^{\rho}(q)(E) = (C^{\rho}(E)\backslash C^{\rho}(B)) \cup C^{\rho}(q)(B)$.

Let now $\tau \in s \cap \rho$. Pick the first element A of $C^{0\tau}(A^{0\tau})$ with $B \in A$. Its immediate predecessor A^- in $C^{\tau}(A^{0\tau})$ is in B, by (b). Then $A^- \subset B_0$. Leave $C^{\tau}(A^-)$ unchanged as well all its initial segments. Set $C^{\tau}(q)(A^{0\tau}(q)) = (C^{\tau}(A^{0\tau})\backslash C^{\tau}(A^-)) \cup \pi_{B_0 B_1}[C^{\tau}(A^-)]$. In order to obtain the full function $C^{\tau}(q)$ we just move the defined already portions via isomorphisms of the models in $A^{1\tau}$. Remember that $B \in A$, hence $\pi_{B_0 B_1}[A^-]$ remains inside $Pred(A)$.

It is not hard to see that such defined q is in $\mathcal{P}'$.

Note that in particular, $C^{\tau}(q)(A^-) = C^{\tau}(A^-)$. Also, if A is a splitting point as witnessed by A_0, A_1 and $\sup(A_0) < \sup(A_1)$, then, as it was pointed above, we have $A_0 \in B \in A_1$, by 1.1(11) and so, by 1.1(29), $B_0, B_1 \in A_1$ as well. Now, suppose that $A_0 \in C^{\tau}(A)$. Then $A_0 = A^-$ and, so $C^{\tau}(A_0)$ does not change. Then also $C^{\tau}(A_1)$ does not change, since the models A_0, A_1 are isomorphic. Note that in this situation $\langle A_0, C^\rho(q) \restriction A^{1\rho} \cap A_0 \rangle = \langle A_0, C^\rho \restriction A^{1\rho} \cap A_0 \rangle$ is not isomorphic to $\langle A_1, C^\rho(q) \restriction A^{1\rho} \cap A_1 \rangle$, since B_0 and B_1 switched and both are in A_1.

$\square$ of Definition 1.24.

Remark 1.25 (1) It is problematic to deal here only with models for which being of the same order type implies isomorphism over a common part. The switches that preserve this condition are not suffice. Thus Strategic Closure and Chain Condition Lemmas below break down. Let us illustrate this in the gap 4 case. Suppose that we have $p \in \mathcal{P}'$ of the following form: $\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p) = \{A^{0\kappa^+}(p), A\}, C^{\kappa^+}(p) = \{A^{0\kappa^+}(p), A\}, A^{0\kappa^{++}}(p), A^{1\kappa^{++}}(p) = \{A^{0\kappa^{++}}(p), G, G_0, G_1\}, C^{\kappa^{++}}(p) = \{A^{0\kappa^{++}}(p), G, G_0\}, ...\rangle$, with G_0, G_1, G of a Δ-system type and $G_0, G_1, G \in A^{0\kappa^+}(p), A \in G_0$. Then $swt(p, G) \in \mathcal{P}'$. Let $A' = \pi_{G_0G_1}[A]$. But suppose that we like (in order to show κ^{+++}-c.c. of $\mathcal{P}'_{\leq\kappa^+}$) to combine p with a similar condition q but with $A^{0\kappa^+}(q) \subset G_0$ and $A^{0\kappa^+}(q) \not\subset G_1$. Let r be such combination. Now if we need to preform the switch of G in order to show the strategic closure (for example, if we need to replace A by A') , then there is a problem. Thus $swt(r, G) \notin \mathcal{P}'$, since $\pi_{G_0G_1}[A^{0\kappa^+}(q)]$ will have the same order type as those of $A^{0\kappa^+}(p)$ but will not be isomorphic to it by the isomorphism which is the identity on the common part.

(2) Note that Chain Conditions Lemmas require switchings with models satisfying the condition $(*)$ of 1.24.

Note that $swt(swt(p, B), B) = p$, where swt of $swt(p, B)$ is defined as above in 1.24.

We define also $swt(p, B_0, \ldots, B_n)$. Just use an induction on the length of the finite sequence of models $B_0, \ldots, B_n$. Thus, if $r = swt(p, B_0, \ldots, B_m)$ is defined then set

$$swt(p, B_0, \ldots, B_m, B_{m+1}) = swt(r, B_{m+1}) \ .$$

Definition 1.26 Let $p, r \in \mathcal{P}'$. Then $p \geq r$ iff there are $B_0, \ldots, B_n$ such that $q = swt(p, B_0, \ldots, B_n)$ is defined and the following holds:

(1) $s(q) \supseteq s(r)$

(2) for every $\tau \in s(r)$

  (a) $A^{1\tau}(q) \supseteq A^{1\tau}(r)$

  (b) $C^\tau(q) \restriction A^{1\tau}(r) = C^\tau(r)$

(c) $A^{0\tau}(r) \in C^{\tau}(q)(A^{0\tau}(q))$

(e) for each $A \in A^{1\tau}(r)$ we have $A^{1\tau}(r)(A) = A^{1\tau}(q)(A)$.

This means that no changes can be made inside models that were already chosen.

Remark 1.27 (1) Note that if $t = swt(p, B_0, \ldots, B_n)$, then $t \geq p$ and

$$p = swt(swt(p, B_0, \ldots, B_n), B_n, B_{n-1}, \ldots, B_0) = swt(t, B_n, \ldots, B_0) \geq t.$$

Hence the switching produces equivalent conditions.

(2) We need to allow $swt(p, B)$ for the Δ-system argument. Since in this argument two conditions are combined into one and so C^0 should pick one of them only.

(3) The use of finite sequences $B_0, \ldots, B_n$ is needed in order to insure transitivity of the order $\leq$ on $\mathcal{P}'$.

Let us start with a lemma that provides a simple way to extend conditions.

Lemma 1.28 *(Extension Lemma)*

Let $p = \langle\langle A^{0\nu}, A^{1\nu}, C^{\nu}\rangle \mid \nu \in s\rangle \in \mathcal{P}'$. *Suppose that* $\langle B(\nu) \mid \nu \in s\rangle$ *is an increasing continuous sequence such that*

(a) $|B(\nu)| = \nu$

(b) $B(\nu) \supseteq \nu$

(c) ${}^{cf\nu>}B(\nu) \subseteq B(\nu)$

(d) $B(\nu) \prec H(\theta)$

(e) $p \in B(\kappa^+)$

Then the extension $p^\frown\langle B(\nu) \mid \nu \in s\rangle$*, defined in the obvious fashion, is in* $\mathcal{P}'$ *and is stronger than* p*, where for* $\nu \in s$ *we just replace* $A^{0\nu}$ *by* $B(\nu)$*, add* $B(\nu)$ *to* $A^{1\nu}$ *and extend* C^{ν} *by adding* $B(\nu)$*.*

Proof. All the conditions of 1.1 hold easily here. Also 1.26 is trivially satisfied.
□

The next lemma is needed (or is nontrivial) only if there are more than κ^+ cardinals between κ and θ or even if there are inaccessible cardinals between κ and θ. If the number of the cardinals between κ and θ is less than κ^{++}, then then the support of conditions can be fixed. Thus we can use always s to be the set of all regular cardinals of the interval $[\kappa^+, \theta]$ and require that each model of a condition includes s.

Lemma 1.29 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$ *be in* $\mathcal{P}'$ *and* $\rho \in [\kappa^+, \theta]$ *be a regular cardinal. Then there is* $q = \langle\langle B^{0\tau}, B^{1\tau}, D^{\tau}\rangle \mid \tau \in t\rangle$ *extending* p *and with* $\rho \in t$*.*

Proof. Clearly, we can assume that $\rho \notin s$. Let $\rho^* = \min(s\backslash\rho + 1)$. Recall that θ is always in support of any condition. So, $\rho^* \leq \theta$. By 1.1(1), ρ^* should be an inaccessible. Let $\rho' = \max(s \cap \rho)$. If ρ is itself an inaccessible or if $\rho = (\rho')^+$, then set $t = s \cup \{\rho\}$. Otherwise we are forced to add together with ρ some additional cardinals. If there are no inaccessibles in the interval $(\rho', \rho]$, then set $t = s \cup \{\xi \in (\rho', \rho] \mid \xi$ is a cardinal $\}$. If there are inaccessibles inside the interval (ρ', ρ), but ρ is not an inaccessible, then let $\rho'' = \sup\{\xi < \rho \mid \xi$ is an inaccessible $\}$. Now, if ρ'' itself is an inaccessible (i.e. if there is maximal inaccessible below ρ) then set $t = s \cup \{\xi \in [\rho'', \rho] \mid \xi$. If ρ'' is singular then pick a cofinal closed sequence

$\langle \rho_i \mid i < cf\rho'' \rangle$ in ρ'' such that for each $i, \rho_i \in (\rho', \rho'')$ and ρ_{i+1} is an inaccessible. Set then $t = s \cup \{\rho_i | cf\rho_i \geq \kappa^+\} \cup \{\xi \in [\rho'', \rho] | \xi \text{ is a cardinal }\}$.

Turn now to the definition of q. We concentrate on the central line. The full condition will be obtained by mapping it using isomorphisms over splitting points. So the issue will be to satisfy 1.1(19). Thus for each $A \in C^\tau(A^{0\tau})$, with $\tau \in s \cap \rho^*$, $F \in C^{\rho^*}(A^{0\rho^*})$ and $\xi \in t \backslash s$ we need to add a model G such that $A \subseteq G \subseteq F$ with $|G| = \xi$. It is enough to deal only with $A \in C^{\rho'}(A^{0\rho'}), F \in C^{\rho^*}(A^{0\rho^*})$ such that F is the least element of $C^{\rho^*}(A^{0\rho^*})$ including A and A on the other hand is the maximal element of $C^{\rho'}(A^{0\rho'})$ included in F. Denote by S the set of all such pairs $\langle A, F \rangle$. Clearly the cardinality of S is at most ρ'.
By induction let us pick for each $\langle A, F \rangle$ the smallest possible increasing continuous chain $\langle B_\mu | \mu \in t \backslash s \rangle$ of elementary submodels of $\langle F, p \cap F \rangle$ such that

(0) $A \in B_{(\rho')^+}$

(1) $|B_\mu| = \mu$ and $B_\mu \supseteq \mu$

(2) ${}^{cf\mu>}B_\mu \subseteq B_\mu$

(3) if μ is nonlimit then $\langle B_{\mu'} \mid \mu' < \mu \rangle \in B_\mu$

(4) $B_{(\rho')^+}$ includes models added (if any) for each pair $\langle A', F' \rangle \in S$ with $A' \in A$, as well as $A', F' \rangle$.

Let $q = \langle \langle B^{0\tau}, B^{1\tau}, D^\tau \rangle \mid \tau \in t \rangle$ be the set obtained from p by adding the sequences defined above to the central line and then mapping the result by isomorphisms over splitting points.

□

Now we turn to splittings of $\mathcal{P}'$.

Definition 1.30 Let $\tau \in (\kappa, \theta]$ be a cardinal. Set

$$\mathcal{P}'_{\geq\tau} = \{\langle\langle A^{0\rho}, A^{1\rho}, C^\rho \rangle \mid \rho \in s \backslash \tau \rangle \mid \exists \langle\langle A^{0\nu}, A^{1\nu}, C^\nu \rangle \mid \nu \in s \cap \tau \rangle \, \langle\langle A^{0\mu}, A^{1\mu}, C^\mu \rangle \mid \mu \in s \rangle \in \mathcal{P}\} .$$

Let $G(\mathcal{P}'_{\geq\tau})$ be generic. Define

$$\begin{aligned}\mathcal{P}'_{<\tau} =& \{\langle\langle A^{0\nu}, A^{1\nu}, C^\nu \rangle \mid \nu \in s \cap \tau \rangle \mid \exists \langle\langle A^{0\rho}, A^{1\rho}, C^\rho \rangle \mid \rho \in s \backslash \tau \rangle \in G(\mathcal{P}'_{\geq\tau}) \\ & \langle\langle A^{0\mu}, A^{1\mu}, C^\mu \rangle\rangle \mid \mu \in s \rangle \in \mathcal{P}'\} .\end{aligned}$$

Note that it is not immediate here that $\mathcal{P}'$ splits into $\mathcal{P}'_{\geq\tau} * \underset{\sim}{\mathcal{P}}'_{<\tau}$.

Let τ be a regular cardinal. If $p \in \mathcal{P}'$, then $p\backslash\tau$- the part of p above τ, is defined as follows:

$$p\backslash\tau = \langle\langle A^{0\xi}(p), A^{1\xi}(p), C^{\xi}(p)\rangle \mid \xi \in s(p)\backslash\tau\rangle$$

Similarly, define $p \restriction \tau$ to be the part of p consisting of its elements below τ, i.e.

$$p \restriction \tau = \langle\langle A^{0\xi}(p), A^{1\xi}(p), C^{\xi}(p)\rangle \mid \xi \in s(p) \cap \tau\rangle$$

Note that $\mathcal{P}'$ is not $\mathcal{P}'_{<\tau} \times \mathcal{P}_{\geq\tau}$ where $\mathcal{P}_{<\tau} = \{p \restriction \tau \mid p \in \mathcal{P}'\}$. The complication here is due to the way of interconnections between models. So, instead of product let us deal with the iteration. Thus in $V^{\mathcal{P}'}_{\geq\tau}$ we define $\mathcal{P}'_{<\tau}$ to be the set of all $p \restriction \tau$ for $p \in \mathcal{P}'$ such that $p\backslash\tau$ is in the generic set $G(\mathcal{P}'_{\geq\tau}) \subseteq \mathcal{P}'_{\geq\tau}$. The next lemma shows that the map $p \mapsto p\backslash\tau$ is a projection map and so $\mathcal{P}'_{\geq\tau}$ is a nice suborder of $\mathcal{P}'$.

For $p \in \mathcal{P}'$ and $q \in \mathcal{P}'_{\geq\tau}$ let $q^\frown p$ denotes the set obtained by combining p and q in the obvious fashion. Note that such a set need not be in general a condition in $\mathcal{P}'$, but in reasonable cases it will.

Lemma 1.31 *(The Splitting Lemma) Let $p \in \mathcal{P}'$, τ be a regular cardinal in $(\kappa, \theta] \cap s(p)$ and $q \in \mathcal{P}'_{\geq\tau}$. If $q \geq_{\mathcal{P}'_{\geq\tau}} p\backslash\tau$, then $q^\frown p \in \mathcal{P}'$ and extends p.*

Proof. Let $p = \langle\langle A^{0\xi}, A^{1\xi}\ ,\ C^{\xi}\rangle \mid \xi \in s\rangle$. Note that $q^\frown p$ need not be a condition since 1.1 may break badly. Thus for example, switching inside $\mathcal{P}'_{\geq\tau}$ may move models in a way that when adding back $A^{0\xi}$'s (for $\xi < \tau$) C^{ξ}'s cannot be moved. In order to deal with such situations, we first replace q by an equivalent condition (switching it into such condition) satisfying 1.26 (1,2) with $p\backslash\tau$ and only then add the full p. Once $A^{0\xi}(p) \in C^{\xi}(q)(A^{0\xi}(q))$ and $C^{\xi}(q)$ extends $C^{\xi}(p)$ for $\xi \in s\backslash\tau$ the problem above disappears.

The rest easily follows from 1.1.

□

Let us show now a strategic closure of the forcing.

Lemma 1.32 *(Strategic Closure Lemma) Let $\rho \in (\kappa, \theta]$ be a regular cardinal. Then $\langle\mathcal{P}'_{\geq\rho}, \leq\rangle$ is ρ^+ – strategically closed.*

Proof. We define a winning strategy for the player playing at even stages. Thus suppose $\langle p_j \mid j < i\rangle$ is a play according to this strategy up to an even stage i. Define p_i.

Let for each $j < i$

$$p_j = \langle\langle A_j^{0\tau}, A_j^{1\tau}, C_j^\tau\rangle \mid \tau \in s_j\rangle \ .$$

Case 1 i is a successor ordinal.
Pick a sequence $\langle B(\tau) \mid \tau \in s_{i-1}\rangle$ satisfying the conditions (a) – (d) of 1.28 with p replaced by p_{i-1}. Let p_i be the extension of p_{i-1} by $\langle B(\tau) \mid \tau \in s_{i-1}\rangle$.

Case 2 i is a limit ordinal.
Replacing each p_j $(j < i)$ by a switched condition if necessary, we can assume p_j's satisfy the conditions of (1),(2) of 1.26, i.e. one extends another in the natural sense. Define first $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ as follows: set $s = \bigcup_{j<i} s_j$, $A^{0\tau} = \bigcup_{j<i,\tau\in s_j} A_j^{0\tau}$, $A^{1\tau} = \bigcup_{j<i,\tau\in s_j} A_j^{1\tau} \cup \{A^{0\tau}\}$ and $C^\tau = \bigcup_{j<i,\tau\in s_j} C_j^\tau \cup \{\langle A^{0\tau}, \cup\{C_j^\tau(A_j^{0\tau}) \mid j \text{ is even and } \tau \in s_j\rangle\}$, for $\tau \in s$.

Such defined p is not necessarily a condition. Thus, for example, 1.1(2(b)) may fail. We fix this by defining p_i from p as follows. Set $B(\rho) = A^{0\rho}$ and for each $\tau \in (\rho, \theta] \cap s$ we chose $B(\tau)$ to be a model such that

(i) $A^{0\tau} \in B(\tau)$

(ii) $|B(\tau)| = \tau$, $B(\tau) \supseteq \tau$

(iii) ${}^{cf\tau>}B(\tau) \subseteq B(\tau)$

(iv) if $\tau < \tau'$ then $B(\tau) \subseteq B(\tau')$

(v) if τ is a limit point of s then $B(\tau) = \cup\{B(\tau') \mid \tau' \in s \cap \tau\}$

(vi) $\langle p_j \mid j < i\rangle, p,\ B(\rho) \in B(\tau)$ for every $\tau \in (\rho, \theta] \cap s$.

Let p_i be obtained from p by adding the sequence $\langle B(\tau) \mid \tau \in [\rho, \theta) \cap s\rangle$. We define

$$C^\tau(p_i)(B(\tau)) = C^\tau \cup \{\langle B(\tau)\ ,\ C^\tau(A^{0\tau})^\frown B(\tau)\rangle\} \ .$$

Such defined p_i is a condition. The proof as those of 1.28 follows easily. Note that here we have $\{p_j \mid j < i\} \subseteq A^{0\tau}$ for each $\tau \in s$.
□

Let us turn now to the chain conditions.

Lemma 1.33 *(Chain Condition Lemma) Let τ be a regular cardinal in $[\kappa^+, \theta]$. Then, in $V^{\mathcal{P}'_{\geq\tau}}$ the forcing $\mathcal{P}_{<\tau}$ satisfies τ^+-chain condition.*

Proof. Suppose otherwise. Let us assume that

$$\phi \Vdash_{\mathcal{P}'_{\geq\tau}} (\underset{\sim}{p}_\alpha = \langle\langle \underset{\sim}{A}^{0\xi}_\alpha, \underset{\sim}{A}^{1\xi}_\alpha, \underset{\sim}{C}^\xi_\alpha\rangle | \xi \in \underset{\sim}{s}_\alpha\rangle | \alpha < \tau^+\rangle \text{ is an antichain in } \underset{\sim}{\mathcal{P}}'_{<\tau}) \ .$$

Define by induction, using the strategy of 1.4 for $\mathcal{P}'_{\geq\tau}$, an increasing sequence of conditions $\langle q_\alpha | \alpha < \tau^+\rangle$, $q_\alpha = \langle\langle A^{0\xi}_\alpha, A^{1\xi}_\alpha, C^\xi_\alpha\rangle | \xi \in t_\alpha\rangle$ and a sequence $\langle p_\alpha \mid \alpha < \tau^+\rangle, p_\alpha = \langle\langle A^{0\xi}_\alpha, A^{1\xi}_\alpha, C^\xi_\alpha\rangle | \xi \in s_\alpha\rangle$ so that for every $\alpha < \tau^+$

$$q_\alpha \Vdash_{\mathcal{P}'_{\geq\tau}} \langle\langle \underset{\sim}{A}^{0\xi}_\alpha, \underset{\sim}{A}^{1\xi}_\alpha, \underset{\sim}{C}^\xi_\alpha\rangle \mid \xi \in \underset{\sim}{s}_\alpha\rangle = \check{p}_\alpha \ .$$

For a limit $\alpha < \tau^+$ let

$$\overline{q}_\alpha = \langle\langle \overline{A}_\alpha^{0\xi}, \overline{A}_\alpha^{1\xi}, \overline{C}_\alpha^{\xi}\rangle \mid \xi \in \overline{t}_\alpha\rangle$$

be the condition produced by the strategy and q_α be its extension deciding p_α. We form a Δ-system now stabilizing as many parts of the conditions as possible. Note that $s_\alpha \subseteq \tau$ and $|s_\alpha| < \tau$ since τ is regular, for each $\alpha < \tau^+$. Hence we can assume that all s_α's are the same and equal to some s. Let $\alpha < \beta < \tau^+$, $cf\alpha = cf\beta = \tau$ be in the system. We like to show then the compatibility of $q_\alpha^\frown p_\alpha$ and $q_\beta^\frown p_\beta$ or since $q_\beta \geq q_\alpha$ the compatibility of $q_\beta^\frown p_\alpha$ and $q_\beta^\frown p_\beta$.

Let $\hat{\tau} = \max(\tau \cap s)$, which exists and is regular since τ is regular by the definition of a support. First pick $B^{\hat{\tau}}(0) \prec A^{0\tau}_{\beta+1}$ of cardinality $\hat{\tau}$ with $q_\beta, p_\alpha, p_\beta \in B^{\hat{\tau}}(0)$ and ${}^{\hat{\tau}>}B^{\hat{\tau}}(0) \subseteq B^{\hat{\tau}}(0)$. Then we define by induction on $\xi \in s$ sets B^ξ such that

(1) $|B^\xi| = \xi$, ${}^{cf\xi>}B^\xi \subseteq B^\xi$

(2) $B^{\hat{\tau}}(0) \in B^\xi$

(3) $B^\xi \prec A^{0\tau}_{\beta+1}$

(4) $\langle B^{\xi'} \mid \xi' \in s \cap \xi\rangle \in B^\xi$.

Define now a common extension

$$p = \langle\langle B^{0\xi}, B^{1\xi}, D^\xi\rangle \mid \xi \in s \cup t_\beta\rangle$$

as follows. For each $\xi \in s$ let

$$B^{0\xi} = B^\xi, B^{1\xi} = A^{1\xi}_\alpha \cup A^{1\xi}_\beta \cup \{B^\xi\} \ ,$$

if $\xi \neq \hat{\tau}$ and

$$B^{1\hat{\tau}} = A^{1\hat{\tau}}_\alpha \cup A^{1\hat{\tau}}_\beta \cup \{B^{\hat{\tau}}(0), B^{\hat{\tau}}\},$$

$$D^\xi = C^\xi_\alpha \cup C^\xi_\beta \cup \{\langle B^\xi, \langle C^\xi_\beta(A^{0\xi}_\beta)^\frown B^\xi\rangle\}$$

(if $\xi = \hat{\tau}$, then we add also $B^{\hat{\tau}}(0)$).
For every $\xi \in t_\beta$ let

$$B^{0\xi} = A^{0\xi}_{\beta+1}, B^{1\xi} = A^{1\xi}_\beta \cup \{A^{0\xi}_{\beta+1}\} \text{ and } D^\xi = C^\xi_\beta \cup \{\langle A^{0\xi}_{\beta+1}\}, \langle C^\xi_\beta(A^{0\xi}_\beta)^\frown A^{0\xi}_{\beta+1}\rangle\}.$$

We need to check that such defined p is in $\mathcal{P}'$.

Note that $B^{\hat{\tau}}(0)$ will be the immediate successor of $A^{0\hat{\tau}}_\alpha, A^{0\hat{\tau}}_\beta$ and the triple $A^{0\hat{\tau}}_\beta, A^{0\hat{\tau}}_\alpha, B^{\hat{\tau}}(0)$ will be of a Δ-system type over $C^\tau(A^{0\tau}_{\beta+1})$. Also, $B^{\hat{\tau}}(0) \in B^{0\xi}$ for each $\xi \in s \cup t_\beta$. Hence the requirements of 1.1 related to splittings of models are satisfied here, as well as the requirement (b) on switching of 1.24. The rest of the conditions hold trivially in the present context.
□

The next lemma shows GCH in $V^{\mathcal{P}'}$. The forcing $\mathcal{P}'$ was designed specially to make this true.

Lemma 1.34 *(GCH Lemma) Let τ be a regular cardinal in $[\kappa^+, \theta]$. Then in $V^{\mathcal{P}'}$ we have $2^\tau = \tau^+$.*

Proof. Let $N \prec H((2^\lambda)^+)$ for λ large enough such that $\mathcal{P}' \in N$, $|N| = \tau^+$ and ${}^\tau N \subseteq N$. Using τ^{++}-strategic closure of $\mathcal{P}'_{\geq\tau^+}$ we find $p^N_{\geq\tau^+} \in \mathcal{P}'_{\geq\tau^+}$ which is N-generic for $\mathcal{P}'_{\geq\tau^+}$. Let $G(\mathcal{P}'_{\geq\tau^+})$ be a generic subset of $\mathcal{P}'_{\geq\tau^+}$ with $p_{\geq\tau^+} \in G(\mathcal{P}'_{\geq\tau^+})$. Then, $N[p_{\geq\tau^+}] \prec V_\lambda[G(\mathcal{P}'_{\geq\tau^+})]$. By Lemma 1.8, $\mathcal{P}'_{<\tau^+}$ satisfies τ^{++}-c.c in $V[G(\mathcal{P}'_{\geq\tau^+})]$. In particular, $\mathcal{P}_{=\tau}$ satisfies τ^{++}-c.c. Let $G(\mathcal{P}'_{=\tau})$ be a generic subset of $\mathcal{P}_{=\tau}$ over $V[G(\mathcal{P}'_{\geq\tau^+})]$. Denote $N[p_{\geq\tau^+}]$ by N_1. Then $N_1[N_1 \cap G(\mathcal{P}'_{=\tau})] \prec V[G(\mathcal{P}'_{\geq\tau^+})][G(\mathcal{P}'_{=\tau})]$, since each antichain for $\mathcal{P}'_{=\tau}$ has cardinality at most τ^+. Hence, if it belongs to N_1 then it is also contained in N_1. Denote $N_1[N_1 \cap G(\mathcal{P}'_{=\tau})]$ by N_2. We now consider $\mathcal{P}'_{<\tau} \cap N_2$. Clearly this is a forcing of cardinality τ^+. We claim that it is equivalent to $\mathcal{P}'_{<\tau}$. Thus, by Lemma 1.8, $\mathcal{P}'_{<\tau}$ satisfies τ^+-c.c., so $\mathcal{P}'_{<\tau} \cap N_2$ is a nice suborder of $\mathcal{P}'_{<\tau}$. Let $G \subseteq \mathcal{P}'_{<\tau}$ be generic over $V[G(\mathcal{P}'_{\geq\tau^+})][G(\mathcal{P}'_{=\tau})]$ and $H = G \cap N_2$. Then H is $\mathcal{P}'_{<\tau} \cap N_2$ generic over $V[G(\mathcal{P}'_{\geq\tau^+})][G(\mathcal{P}'_{=\tau})]$. Thus, if $A \subseteq \mathcal{P}'_{<\tau} \cap N_2$ is a maximal antichain, then A is antichain also in $\mathcal{P}'_{<\tau}$, since N_2 is an elementary submodel. Hence $|A| \leq \tau$. But then $A \in N_2$, and so $N_2 \vDash$ (A is a maximal antichain in $\mathcal{P}'_{<\tau}$). By elementary, A is a maximal antichain in $\mathcal{P}'_{<\tau}$. So there is $p \in G \cap A$. Finally, $A \subseteq N_2$ implies that $p \in N_2$ and hence $p \in H$.

We claim that each subset of τ is already in $N_2[G]$. It is enough since $|N_2[G]| = |N| = \tau^+$. Let $\underset{\sim}{a}$ be a name of a function from τ to 2. Work in V. Define by induction (using the strategic closure of the forcings and τ^+-c.c. of $\mathcal{P}'_{<\tau}$) sequences of ordinals

$$\langle \delta_\beta | \beta < \tau \rangle, \langle \gamma(\alpha, \beta) | \beta < \tau, \alpha < \delta_\beta \rangle$$

and sequences of conditions

$$\langle p_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau), \langle p(\beta) | \beta < \tau \rangle$$

such that

(1) for each $\beta < \tau$, $\delta_\beta < \tau^+$

(2) for each $\beta < \tau$, $\langle p_\beta(\alpha)_{\geq \tau} | \alpha < \delta_\beta \rangle$ is increasing sequence of elements of $\mathcal{P}'_{\geq \tau}$ and $p(\beta)$ is its upper bound obtained as in the Strategic Closure Lemma

(3) $p_0(0)_{\geq \tau^+} \geq p^N_{\geq \tau^+}$

(4) the sequence $\langle p(\beta) | \beta < \tau \rangle$ is increasing

(5) for each $\beta < \tau$ and $\alpha < \delta_\beta$, $p_\beta(\alpha)$ forces "$\underset{\sim}{a}(\beta) = \gamma(\alpha, \beta)$"

(6) if some $p \in \mathcal{P}'$ is stronger than $p(\beta)_{\geq \tau}$ where top models of cardinalities below τ are viewed as empty or trivial, then there is $\alpha < \delta$ such that the conditions $p, p_\beta(\alpha)$ are compatible. (I.e. $\{p_\beta(\alpha)_{<\tau} | \alpha < \delta_\beta\}$ is a pre-dense set as forced by $p(\beta)_{\geq \tau}$).

Set $p(\tau)$ to be the upper bound of $\langle p(\beta) | \beta < \tau \rangle$ as in the Strategic Closure Lemma. Let L denotes the top model of cardinality τ of $p(\tau)_{\geq \tau}$, i.e. $A^{1\tau}(p(\tau)_{\geq \tau})$. Pick $K \in N$ realizing the same type as those of L in $H(\lambda)[G_{\geq \tau^+}]$. Let

$$\langle q(\beta) | \beta < \tau \rangle, \langle q_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau)$$

be the sequences corresponding to

$$\langle p_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau), \langle p(\beta) | \beta < \tau \rangle.$$

Define a name $\underset{\sim}{b}$ of a subset of τ to be

$$\{ < q_\beta(\alpha), \gamma(\alpha, \beta) > | \alpha < \delta_\beta, \beta < \tau \}.$$

Clearly, $\underset{\sim}{b}$ is in N. Combine now K, L into one condition making them a splitting point. Let M be a model of cardinality τ such that $K, L \in M$ as well as the sequences

$$\langle p_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau), \langle p(\beta) | \beta < \tau \rangle$$

and

$$\{< q_\beta(\alpha), \gamma(\alpha,\beta) > | \alpha < \delta_\beta, \beta < \tau\}.$$

Let $\langle A(\xi) | \xi \in s\rangle$ be an increasing continuous sequence of models with $|A^\xi| = \xi$ and $K, L, M, \langle p_\beta(\alpha) | \alpha < \delta_\beta\rangle(\beta < \tau), \langle p(\beta) | \beta < \tau\rangle$ and $\{< q_\beta(\alpha), \gamma(\alpha,\beta) > | \alpha < \delta_\beta, \beta < \tau\} \in A(\kappa^+)$. Put this sequence to be the top sequence of such combined condition which we denote by r.

Claim 1.34.1 $r \Vdash \underset{\sim}{a} = \underset{\sim}{b}$.

Proof. Let G be a generic subset of $\mathcal{P}'$ with $r \in G$. Then also $p(\tau)_{\geq\tau}, q(\tau)_{\geq\tau} \in G$. Now, for each $\beta < \tau$ there is $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$ (just otherwise there will be a condition t in G forcing that for some β there is no $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$. Extend it to t' deciding the value $\underset{\sim}{a}(\beta)$. By (6) there is α such that $t', p_\beta(\alpha)$ are compatible). Let $r' \in G$ be a common extension of r and $p_\beta(\alpha)$. Now M will be a splitting point witnessed by L, K in r' and the isomorphism π_{LK} moves $p_\beta(\alpha)$ to $q_\beta(\alpha)$. Hence $q_\beta(\alpha) \leq r'$. But then $q_\beta(\alpha) \in G$.
□ of the claim.
□

The rest mainly repeats those of [4]. The property $icb(A, B)$ is crucial for the main forcing.

2 Types of Models

The basic approach here is as in [1] but instead of dealing with types of ordinals we shall consider elementary submodels of $H(\chi^{+k})$ for some χ big enough, $k \leq \omega$ and types of such models.

Fix $n < \omega$. Set $\delta_n = \kappa_n^{+\kappa_n^{+n+2}+1}$. Denote by δ_n^- the immediate predecessor of δ_n, i.e. $\kappa_n^{+\kappa_n^{+n+2}}$. Fix using GCH an enumeration $\langle a_\alpha \mid \alpha < \kappa_n\rangle$ of $[\kappa_n]^{<\kappa_n}$ so that for every successor cardinal $\delta < \kappa_n$ the initial segment $\langle a_\alpha \mid \alpha < \delta\rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_n(\langle a_\alpha \mid \alpha < \kappa_n\rangle) = \langle a_\alpha \mid \alpha < j_n(\kappa_n)\rangle$ where j_n is the canonical embedding of the (κ_n, δ_n^+)-extender E_n. Then $\langle a_\alpha \mid \alpha < \delta_n^+\rangle$ will enumerate $[\delta_n^+]^{\leq\delta_n}$ and we fix this enumeration. For each $k \leq \omega$ consider a structure

$$\mathfrak{A}_{n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, E_n, \kappa_n, \delta_n, \delta_n^+, \chi, \langle a_\alpha \mid \alpha < \delta_n^+\rangle, 0, 1, \ldots, \alpha, \ldots \mid \alpha < \kappa_n^{+k}\rangle$$

in the appropriate language $\mathcal{L}_{n,k}$ with a large enough regular cardinal χ.

Remark 2.1 It is possible to use κ_n^{++} here (as well as in [1]) instead of κ_n^{+k}. The point is that there are only κ_n^{++} many ultrafilters over κ_n and we would like that equivalent conditions use the same ultrafilter. The only parameter that that need to vary is k in $H(\chi^{+k})$.

Let $\mathcal{L}'_{n,k}$ be the expansion of $\mathcal{L}_{n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less or equal than δ_n let $\mathfrak{A}_{n,k,a}$ be the expansion of $\mathfrak{A}_{n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less or equal than δ_n. Denote by $tp_{n,k}(b)$ the $\mathcal{L}_{n,k}$-type realized by b in $\mathfrak{A}_{n,k}$. Further we identify it with the ordinal coding it and refer to it as the k-type of b. Let $tp_{n,k}(a, b)$ be a the $\mathcal{L}'_{n,k}$-type realized by b in $\mathfrak{A}_{n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [1].

Lemma 2.2 *(a)* $|\{tp_{n,k}(b) \mid b \in H(\chi^{+k})\}| = \kappa_n^{+k+1}$

(b) $|\{tp_{n,\kappa}(a, b) \mid a, b \in H(\chi^{+k})\}| = \kappa_n^{+k+1}$

Proof. (a) The cardinality of the language $\mathcal{L}_{n,k}$ is κ_n^{+k} so the number of formulas is κ_n^{+k}. Now the number of types is $2^{\kappa_n^{+k}} = \kappa_n^{+k+1}$.

(b) The same argument.

□

Remark 2.3 In particular the lemma above implies that the number of types of cardinals

$$\{\kappa_n^{+\alpha+1} | \alpha < \kappa^{+n+2}\},$$

is also small - κ^{+k+1}. The purpose of choosing $\delta_n = \kappa_n^{+\kappa_n^{+n+2}+1}$ was to insure this.

This lemma implies immediately the following:

Lemma 2.4 *Let $A \prec \mathfrak{A}_{n,k+1}$ and $|A| \geq \kappa_n^{+k+1}$. Then the following holds:*

(a) for every $a, b \in H(\chi^{+k})$ there $c, d \in A \cap H(\chi^{+k})$ with $tp_{n,k}(a, b) = tp_{n,k}(c, d)$

(b) for every $a \in A$ and $b \in H(\chi^{+k})$ there is $d \in A \cap H(\chi^{+k})$ so that $tp_{n,k}(a \cap H(\chi^{+k}),$ $b)$=$tp_{n,k}(a \cap H(\chi^{+k}),$ $d)$.

Proof. (a) Note that $tp_{n,k}(a, b) \in A$, by 2.1 and since $|A| \geq \kappa_n^{+k+1}$, so $A \supseteq \kappa_n^{+k+1}$. Now, $H(\chi^{+k+1}) \models (\exists x, y \in H(\chi^{+k}) \forall \varphi(v, u) \in tp_{n,k}(a, b)\ (H(\chi^{+k}) \models \varphi(x, y)))$. But $A \prec H(\chi^{+k+1})$. So

$$A \models (\exists x, y \in H(\chi^{+k}) \forall \varphi(x, y) \in tp_{n,k}(a, b)(H(\chi^{+k}) \models \varphi(x, y)))\ .$$

Pick $c, d \in A$ satisfying this formula. Then $c, d \in H(\chi^{+k})$ and $tp_{n,k}(c, d) = tp_{n,k}(a, b)$.

(b) Similar. □

Lemma 2.5 *Suppose that $A \prec \mathfrak{A}_{n,k+1}, |A| \geq \kappa_n^{+k+1}$. Let τ be a cardinal in the interval $[\kappa_n, \delta_n]$ those $k+1$-type is realized unboundedly often below δ_n^-. Then there are $\tau' < \tau$ and $A' \prec A \cap H(\chi^{+k})$ such that $\tau', A' \in A$ and $\langle \tau', A' \rangle$ and $\langle \tau, A \cap H(\chi^{+k}) \rangle$ realize the same $tp_{n,k}$. Moreover, if $|A| \in A$, then we can find such A' of cardinality $|A|$.*

Proof. Then τ is a limit of cardinals realizing the same k- type as τ does or τ is a successor cardinal and then the maximal limit cardinal below τ is a limit of cardinals realizing the same k- type as τ does.

We can pick now a cardinal $\tau' < \tau$ in A realizing $tp_{n,k}(\tau)$. It exists by elementarity of A and since A contains all k-types. Now, there will be $A' \in A$, $A' \prec A \cap H(\chi^{+k})$ such that $\langle \tau', A' \rangle$ and $\langle \tau, A \cap H(\chi^{+k}) \rangle$ realize the same $tp_{n,k}$. Since

$$H(\chi^{+k+1}) \models \exists X \prec (H(\chi^{+k}) \quad (tp_{n,k}(\langle \tau', X \rangle) = tp_{n,k}(\langle \tau, A \cap H(\chi^{+k}) \rangle))).$$

Just $A \cap H(\chi^{+k})$ witnesses this. By elementarity, then there is such a set in A.
□.

The next lemma will be crucial further for the chain condition arguments.

Lemma 2.6 *Suppose that $A \prec \mathfrak{A}_{n,k+1}, |A| \geq \kappa_n^{+k+1}$, $B \prec \mathfrak{A}_{n,k}$, and $C \in \mathcal{P}(B) \cap A \cap H(\chi^{+k})$. Then there is D so that*

(a) $D \in A$

(b) $C \subseteq D$

(c) $D \prec A \cap H(\chi^{+k}) \prec H(\chi^{+k})$.

(d) $tp_{n,k}(C, B) = tp_{n,k}(C, D)$.

Proof. As in 2.2., the following formula is true in $H(\chi^{+k+1})$:

$$\exists x \subseteq H(\chi^{+k})((x \prec H(\chi^{+k})) \wedge (x \supseteq C) \wedge (\forall \varphi(y, z) \in tp_{n,k}(C, B) H(\chi^{+k}) \models \varphi(C, x))).$$

Then the same holds in A. Let D witness this. Hence $D \in A$, $D \supseteq C$, $D \prec A \cap H(\chi^{+k}) \prec H(\chi^{+k})$ and $tp_{n,k}(C, B) = tp_{n,k}(C, D)$.
□

Further we shall add models $B \cap H(\chi^{+k})$ with $B \prec H(\chi^{+k+1})$ or models realizing the same $tp_{n,k}(a,-)$ as those of elementary submodel of $H(\chi^{+k+1})$ intersected with $H(\chi^{+k})$ for any a inside. We will require that for every $k < \omega$, each condition p has an equivalent condition q with every model in it being an elementary submodel of $H(\chi^k)$.

The next definition is similar to those of [1], but deals with cardinals rather than ordinals. The first two cases are added here for notational simplicity.

Definition 2.7 Let $k \leq n$ and $\nu = \kappa_n^{+\beta+1}$ for some $\beta \leq \kappa_n^{+n+2}$. The cardinal ν is called k-**good** iff $\nu = \kappa_n^{+n+1}$ (i.e. $\beta = n+1$) or $\nu = \delta_n$ (i.e.$\beta = \kappa_n^{+n+2}$) or the following holds

(1) β is a limit ordinal of cofinality at least κ_n^{++}

(2) for every $\gamma < \beta \quad tp_{n,k}(\gamma, \beta)$ is realized unboundedly many times in κ_n^{+n+2} or equivalently $tp_{n,k}(\kappa_n^{+\gamma+1}, \nu)$ is realized unboundedly many times in δ_n^-.

ν is called **good** iff for some $k \leq n \quad \nu$ is k-good.

The following lemma was proved in [1] in context of ordinals, but is true easily for cardinals as well.

Lemma 2.8 *Suppose that a cardinal $\nu = \kappa_n^{+\beta+1}$ is k-good for some $k, 0 < k \leq n$ and $\beta, n+1 < \beta < \kappa_n^{+n+2}$. Then there are arbitrary large $k-1$- good cardinals below $\kappa_n^{+\beta}$.*

3 The Main Forcing

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.

Fix $n < \omega$. Following [2, Sec.3]. We define first Q_{n0}.

For each $n < \omega$ let $\delta_n = \kappa_n^{+\kappa_n^{+n+2}+1}$.

Definition 3.1 Let Q_{n0} be the set of the triples $\langle a, A, f\rangle$ so that:

1. f is partial function from θ^+ to κ_n of cardinality at most κ
2. a is a partial function of cardinality less than κ_n so that
   (a) There is $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle | \tau \in s\rangle \in G(\mathcal{P}')$ which we call it further **a background condition of** A, such that for each $\tau \in s \quad A^{0\tau}$ is a successor model having unique immediate predecessor $(A^{0\tau})^-$ (i.e. $Pred(A^{0\tau}) = \{(A^{0\tau})^-\}$) and $\langle A^{0\tau}\rangle^- | \tau \in s\rangle \in A^{0\kappa^+}$. The same holds for $\langle\langle (A^{0\tau})^-, A^{1\tau}\backslash\{A^{0\tau}\}, C^\tau \restriction A^{0\tau}\rangle | \tau \in s\rangle$, i.e. for each $\tau \in$

s $(A^{0\tau})^-$ is a successor model having unique immediate predecessor $((A^{0\tau})^-)^-$ (i.e. $Pred((A^{0\tau})^-) = \{((A^{0\tau'})^-)^-\}$) and $\langle ((A^{0\tau})^-)^- | \tau \in s\rangle \in (A^{0\kappa^+})^-$.

dom(a) consists of models appearing in $A^{1\kappa^+}$ and in $(A^{1\tau})^-$, $\tau \in s$.

Note that conditions as above are dense in $\mathcal{P}'$. Let us refer to them further as **conditions of the right form**.

(b) for each $X \in \mathrm{dom}(a)$ there is $k \leq \omega$ so that $a(X) \subseteq H(\chi^{+k})$.

Also the following holds

(i) $|X| = \kappa^+$ implies $|a(X)| = \kappa_n^{+n+1}$

(ii) $|X| = \theta$ implies $|a(X)| = \delta_n$ and $a(X) \cap \delta_n^+ \in ORD$

(iii) $A^{0\kappa^+}, (A^{0\kappa^+})^-, (A^{0\theta})^- \in \mathrm{dom}(a)$.

This way we arranged that κ_n^{+n+1} will correspond to κ^+ and δ_n to θ.

Further let us refer to A^{0,κ^+} as **the maximal model of the domain of** a and to $\langle (A^{0\tau})^- | (A^{0\tau})^- \in \mathrm{dom}(a)\rangle$ as **the maximal sequence of the domain of** a. Denote the first as $\max(\mathrm{dom}(a))$ and the second as $\vec{\max}(\mathrm{dom}(a))$ (or just $\max(a), \vec{\max}(a)$).

Further passing from Q_{0n} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

(c) (Models come from $A^{0\kappa^+}$) If $X \in \mathrm{dom}(a)$ and $X \neq A^{0\kappa^+}$ then $X \in A^{0\kappa^+}$.

The condition puts restriction on models in dom(a) and allows to control them via the maximal model of cardinality κ^+.

(d) (All the cardinalities are inside $A^{0\kappa^+}$) If $(A^{0\tau})^- \in \mathrm{dom}(a)$, then $\tau \in A^{0\kappa^+}$.

(e) (No holes) If $X \in A^{1\tau} \cap \mathrm{dom}(a)$, for some $\tau \in s$, then $(A^{0\tau})^- \in \mathrm{dom}(a)$ as well.

This means that in order to add $X \in A^{1\tau}$ to dom(a) we need first to insure that the maximal model of cardinality as those of X is inside.

(f) If $A, B \in \mathrm{dom}(a)$, $A \in B$ (or $A \subseteq B$) and k is the minimal so that $a(A) \subseteq H(\chi^{+k})$ or $a(B) \subseteq H(\chi^{+k})$, then $a(A) \cap H(\chi^{+k}) \in a(B) \cap H(\chi^{+k})$ (or $a(A) \cap H(\chi^{+k}) \subseteq a(B) \cap H(\chi^{+k})$).

The intuitive meaning is that a is supposed to preserve membership and inclusion. But we cannot literally require this since $a(A)$ and $a(B)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(g) Let $A, B \in \mathrm{dom}(a)$. Then

  (i) $|A| = |B|$ implies $|a(A)| = |a(B)|$

  (ii) $|A| < |B|$ implies $|a(A)| < |a(B)|$

(h) The set
$$\{\nu \in s | (A^{0\nu})^- \in \mathrm{dom}(a)\}$$
is closed.

(i) The image by a of $A^{0\kappa^+}$, i.e. $a(A^{0\kappa^+})$, intersected with δ_n^+ is above all the rest of $\mathrm{rng}(a)$ restricted to δ_n^+ in the ordering of the extender E_n (via some reasonable coding by ordinals).

Recall that the extender E_n acts on δ_n^+ and our main interest is in Prikry sequences it will produce. So, parts of $\mathrm{rng}(a)$ restricted to δ_n^+ will play the central role.

(j) If $A \in \mathrm{dom}(a)$ then $C^{|A|}(A) \cap \mathrm{dom}(a)$ is a closed chain. Let $\langle A_i | i < j\rangle$ be its increasing continuous enumeration. For each $l < j$ consider the final segment $\langle A_i | l \leq i < j\rangle$ and its image $\langle a(A_i) | l \leq i < j\rangle$. Find the minimal k so that
$$a(A_i) \subseteq H(\chi^{+k}) \text{ for each } i, l \leq i < j.$$
Then the sequence
$$\langle a(A_i) \cap H(\chi^{+k}) | l \leq i < j\rangle$$
is increasing and continuous.

Note that k here may depend on l, i.e. on the final segment.

(k) (The walk is in the domain) If $A \in \mathrm{dom}(a) \cap A^{1\nu}$, for some $\nu \in s$, then the general walk from $(A^{0\nu})^-$ to A is in $\mathrm{dom}(a)$.

(l) If $A \in \mathrm{dom}(a) \cap A^{1\nu}$, for some $\nu \in s$ is a limit model and $\mathrm{cof}(otp_\nu(A) - 1) < \kappa_n$ (i.e. the cofinality of the sequence $C^\nu(A) \backslash \{A\}$ under the inclusion relation is less than κ_n) then a closed cofinal subsequence of $C^{\kappa^+}(A) \backslash \{A\}$ is in $\mathrm{dom}(a)$. The images of its members under a form a closed cofinal in $a(A)$ sequence.

(m) If $\langle X_i | i < j\rangle$ is an increasing (under the inclusion) sequence of elements of $\mathrm{dom}(a)$ with $X_i \in C^{\tau_i}(A^{0\tau_i})$, $i < j$, then $\bigcup_{i<j} X_i \in \mathrm{dom}(a)$ as well.

Note that $\bigcup_{i<j} X_i \in C^{\cup_{i<j}\tau_i}(A^{0\cup_{i<j}\tau_i})$. So, in particular, by 2e also $A^{0\cup_{i<j}\tau_i} \in \mathrm{dom}(a)$.

(n) (The minimal models condition) Suppose that $X \in \mathrm{dom}(a) \cap C^{\xi}(A^{0\xi})$, for some $\xi \in s \backslash \kappa^+ + 1$. Let $\tau \in s$ and $X^* \in C^{\tau}(A^{0\tau})$ be such that $\tau < \xi$, $X \in X^*$ and for each $\rho, \tau \leq \rho < \xi$, $Z \in C^{\rho}(A^{0\rho})$ we have $X \in Z$ implies $X^* \in Z$ or $X^* = Z$. Then $X^* \in \mathrm{dom}(a)$ as well as $(X^*)^-$-its immediate predecessor in $C^{\tau}(A^{0\tau})$.

In addition, we require the following:

if $(X^*)^- \notin X$, then for each $H \in a((X^*)^-)$ there is $H' \in a((X^*)^-)$ with $H \in H'$ and $a(X) \subseteq H'$. Moreover, if $|a(X)| \in a((X^*)^-)$, then $|H'| = |a(X)|$. If $|a(X)| \notin a((X^*)^-)$, then $|H'| = \min(a((X^*)^-) \cap ORD \backslash |a(X)|)$.

Note that $X \in A^{0\kappa^+} \in \mathrm{dom}(a)$, by 2e. So X^* always exists.

The second part of the condition insures that there will be enough models in $a((X^*)^-)$ to allow extensions which will include $a(X)$.

(o) (Minimal cover condition) Let $A \in A^{0\xi} \cap \mathrm{dom}(a), X \in A^{0\tau} \cap \mathrm{dom}(a)$ for some $\xi < \tau$ in s. Suppose that $A \not\subseteq X$. Then

- $\tau \in A$ implies that the smallest model of $A \cap C^{\tau}(A^{0\tau})$ including X is in $\mathrm{dom}(a)$
- $\tau \notin A$ implies that the smallest model of $A \cap C^{\rho}(A^{0\rho})$ including X is in $\mathrm{dom}(a)$, for $\rho = \min(A \cap s \backslash \tau)$.

(p) (The first models condition) Suppose that $A \in dom(a) \cap C^{\tau}(A^{0\tau}), B \in \mathrm{dom}(a) \cap C^{\rho}(A^{0\rho})$, $\sup(A) > \sup(B)$ and $B \notin A$, for some $\tau < \rho, \tau, \rho \in s$. Let $\eta = \min((A \cap s) \backslash \rho)$. Then the first model $E \in A \cap C^{\eta}(A^{0\eta})$ which includes B is in $\mathrm{dom}(a)$.

(q) (Models witnessing Δ-system type are in the domain) If $F_0, F_1, F \in A^{1\mu} \cap \mathrm{dom}(a)$ is a triple of a Δ - system type, for some $\mu \in s$, then the corresponding models $G_0, G_0^*, G_1, G_1^*, G^*$, as in the definition of a Δ - system type (see 1.1(**??**)), are in $\mathrm{dom}(a)$ as well and

$$a(F_0) \cap a(F_1) = a(F_0) \cap a(G_0) = a(F_1) \cap a(G_1).$$

(r) If $F_0, F_1, F \in A^{1\mu}$ is a triple of a Δ - system type, for some $\mu \in s$ and $F, F_0 \in \mathrm{dom}(a)$ (or $F, F_1 \in \mathrm{dom}(a)$), then $F_1 \in \mathrm{dom}(a)$ (or $F_0 \in \mathrm{dom}(a)$).

(s) (The isomorphism condition) Let $F_0, F_1, F \in A^{1\mu} \cap \mathrm{dom}(a)$ be a triple of a Δ - system type, for some $\mu \in s$. Then

$$\langle a(F_0) \cap H(\chi^{+k}), \in \rangle \simeq \langle a(F_1) \cap H(\chi^{+k}), \in \rangle$$

where k is the minimal so that $a(F_0) \subseteq H(\chi^{+k})$ or $a(F_1) \subseteq H(\chi^{+k})$.

Note that it is possible to have for example $a(F_0) \prec H(\chi^{+6})$ and $a(F_1) \prec H(\chi^{+18})$. Then we take $k = 6$.

Let π be the isomorphism between

$$\langle a(F_0) \cap H(\chi^{+k}), \in \rangle, \langle a(F_1) \cap H(\chi^{+k}), \in \rangle$$

and $\pi_{F_0 F_1}$ be the isomorphism between F_0 and F_1. Require that for each $Z \in F_0 \cap \mathrm{dom}(a)$ we have $\pi_{F_0 F_1}(Z) \in F_1 \cap \mathrm{dom}(a)$ and

$$\pi(a(Z) \cap H(\chi^{+k})) = a(\pi_{F_0 F_1}(Z)) \cap H(\chi^{+k}).$$

3. $\{\alpha < \kappa^{+3} \mid \alpha \in \mathrm{dom}(a)\} \cap \mathrm{dom}(f) = \emptyset$.

4. $A \in E_{n,a(\max(a))}$.

5. for every ordinals α, β, γ which are elements of $\mathrm{rng}(a)$ or actually the ordinals coding models in $\mathrm{rng}(a)$ we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \quad \text{implies}$$
$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho))$$

for every $\rho \in \pi``_{\max \mathrm{rng}(a),\alpha}(A)$.

We define now Q_{n1} and $\langle Q_n, \leq_n, \leq^*_n \rangle$ as in [2, Sec.2].

Definition 3.2 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) for every $n < \omega$ $p_n \in Q_n$

(2) there is $\ell(p) < \omega$ such that

(i) for every $n < \ell(p)$ $\quad p_n \in Q_{n1}$

(ii) for every $n \geq \ell(p)$ $\quad p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$

(iii) for every $n, m \geq \ell(p)$ $\quad \max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$ and the maximal models of cardinality κ^+ are the same.

This implies in particular that the maximal sequence of a_m is a subsequence of those of of a_n.

(3) for every $n \geq m \geq \ell(p)$ $\text{dom}(a_m) \subseteq \text{dom}(a_n)$

(4) for every n, $\ell(p) \leq n < \omega$, and $X \in \text{dom}(a_n)$ the following holds:
for each $k < \omega$ the set

$$\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k})) \text{ or } |a_m(X)| \text{ is not } k - good\}$$

is finite.

Lemma 3.3 *Let $p = \langle p_k \mid k < \omega \rangle \in \mathcal{P}$, $p_k = \langle a_k, A_k, f_k \rangle$ for $k \geq \ell(p)$ and X be a model appearing in an element of $G(\mathcal{P}')$. Suppose that*

(a) $X \notin \bigcup_{\ell(p) \leq k < \omega} \text{dom}(a_k) \cup \text{dom}(f_k)$.

(b) X is a successor model or if it is a limit one with $cof(otp_{\kappa^+}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega \rangle$, $q_k = \langle b_k, B_k, g_k \rangle$ for $k \geq \ell(q)$, of p so that starting with some $n \geq \ell(q)$ we have $X \in \text{dom}(b_k)$ for each $k \geq n$.

Remark We would like to avoid at this stage adding limit models of small cofinality since by 3.1(2l) this will require additional adding of sequences of models.

Proof.

We split the proof into few cases.

Case 3.1. $(A^{0\kappa^+})^- \subseteq X \subseteq (A^{0\theta})^-$ for a background condition $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle | \tau \in s \rangle \in G(\mathcal{P}')$ for p, with $A^{0\kappa^+}$ being the maximal model of p.

If no models of cardinalities above $|X|$ appear in $\text{dom}(a_n)$ except those of cardinality θ, then let us pick an $n - good$ cardinality $\nu_n < \delta_n$ above all the cardinalities of the models in $\text{rng}(a_n)$ (remember that there are less than κ_n). Now chose an elementary submodel of $H(\chi^{+\omega})$ of cardinality ν_n including $\text{rng}(a_n)$ as an element. Map X to such a model.

Suppose now that there are models of cardinalities above $|X|$ in $\text{dom}(a_n)$. Then this true for each $m > n$ also. Pick the least possible cardinal $\rho, |X| < \rho < \theta$ such that a model of cardinality ρ appears in $\text{dom}(a_m)$ for some $m \geq \ell(p)$. Find now $m^* \geq \ell(p)$ with a model of cardinality ρ in $\text{dom}(a_m)$ and so that for each $l \geq m^*$ the cardinality of the image under a_l of a model of cardinality ρ is at least 8-good. For each $l \geq m^*$ we can use 2.8 in order to find a k_l-good cardinal ν_l which is below the cardinality of the image under a_l of a model of cardinality ρ and above all the cardinalities of the models in $\text{rng}(a_l)$, where $\langle k_i | m^* \leq i < \omega \rangle$ is a nondecreasing sequence of the natural numbers converging to infinity and with $k_{m^*} > 5$.

What remains is to chose an elementary submodel of $H(\chi^{+\omega})$ of cardinality ν_l including $\mathrm{rng}(a_l)$ as an element and to map X to such a model.

Case 3.2. $X = A^{0\xi}$, $\xi \in s$ for some condition $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle | \tau \in s\rangle \in G(\mathcal{P}')$ of the right form and such that the maximal model of p of cardinality $|X|$ exists and belongs to $C^{|X|}(X)$.

Then, by the definition of the order 1.26, for each $n \geq \ell(p)$, each element Y the maximal sequence of a_n must be in $C^{|Y|}(A^{0|Y|})$. Note that the maximal sequence of a_n belongs to $A^{0\kappa^+}$, since since the maximal element of cardinality κ^+ is in $A^{0\kappa^+}$ then by weak elementarity (1.1(29) the sequence consisting of the smallest elements of $C^{\eta}((A^{0\eta}))^{A^{0\kappa^+}})$ $(\eta \in s \cap A^{0\kappa^+})$ including the maximal element of cardinality κ^+ of a_n belongs to $A^{0\kappa}$. But the maximal sequence of a_n is its subsequence, since by 3.1(2a) the maximal sequence has a right form.

Now we extend $\mathrm{dom}(a_n)$ by adding to it $\langle A^{0\tau} |$ there is a model of cardinality τ in $\mathrm{dom}(a_n)\rangle$ as a new maximal sequence. Map it to an increasing continuous sequence of elementary submodels of $H(\chi^{+\omega})$ of corresponding cardinalities with $\mathrm{rng}(a_n)$ inside all of them.

Case 3.3. Not Case 3.1 or 3.2.

Applying Cases 3.1 and 3.2 if necessary we can assume that $X \in A^{1\xi} \cap A^{0\kappa^+}$, $\xi \in s$, for some condition $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle | \tau \in s\rangle \in G(\mathcal{P}')$ of the right form and such that the maximal sequence of a_n is a subsequence of $\langle A^{0\tau} | \tau \in s\rangle$ including $A^{0\xi}$, for each n big enough. Assume for simplicity that this holds for each n.

We deal with the present case by induction on the general distance $gd(X)$, i.e. on the distance from the central line of $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle | \tau \in s\rangle$.

Let us start with models of cardinality κ^+ from $C^{\kappa^+}(A^{0\kappa^+})$. There is no problem to add such models unless they are splitting points. Continue by induction on cardinality to non-splitting models of cardinalities above κ^+. We need now to be able to satisfy 3.1(2n). Thus in order to add a model X we need first to add models X^* and $(X^*)^-$. In order to do this $(X^*)^*, ((X^*)^*)^-$ etc. need be added. Note that the process is finite since we go down in cardinalities of models that are supposed to be added. So the induction applies. Note only how to deal with the following situation: suppose that $X^*, (X^*)^- \in C^{\rho}(A^{0\rho})$ are already in $\mathrm{dom}(a_n)$ and we like to add $X \in C^{\xi}(A^{0\xi})$ satisfying $(X^*)^- \notin X$. In order to add such a model X, let us first add $(A^{0\xi})^{(X^*)^-}$, if exists. Then $X \in C^{\xi}((A^{0\xi})^{(X^*)^-})$ and by 1.1(29) we have $C^{\xi}((A^{0\xi})^{(X^*)^-}) \in (X^*)^-$ and hence in X^* as well. Pick the smallest $Y \in C^{\xi}((A^{0\xi})^{(X^*)^-}) \cap (X^*)^-$. Now, $X \notin (X^*)^-$ implies that Y is a limit model and if the length of $C^{\xi}(Y)$ is α, then $\sup((X^*)^- \cap (\alpha - 1) < \sup(X^*) \cap (\alpha - 1) < \alpha - 1$. Remember that $Y \in C^{\xi}(Y)$ and so this length is always a successor ordinal. By induction we can assume that $Y \in \mathrm{dom}(a_n)$, just Y is simpler than X in a sense that Y is a member of a smaller model

$((X^*)^-)$. Consider $a_n(Y)$, $a_n((X^*)^-)$ and $a_n(X^*)$. Now inside $a_n(X^*)$ we pick an elementary submodel M of $a_n(Y)$ which includes a_n"$Y \cap C^\xi(Y)$ has cardinality as those of $a_n(Y)$ and is in $a_n(Y)$. Note that the very last requirement may case the necessity of reducing of $H(\chi^+k)$ to $H(\chi^{k'})$, where $k' = k-1$, if $k < \omega$ or $k' < \omega$, but large enough, if $k = \omega$.

Let us turn now to splitting points. Thus suppose that F_0, F_1, F are of a Δ - system type for some $F_0, F \in C^\xi(A^{0\xi})$, $A^{0\xi} \in \mathrm{dom}(a_n)$ and $F \in A^{0\kappa^+}$. Assume that F is already in $\mathrm{dom}(a_n)$. We need to add F_0, F_1 and witnessing models $G_0, G_0^*, G_1, G_1^* \in C^\rho(A^{0\rho})$ as in the definition of a Δ-system type. Also 3.1(2q) should be satisfied.
Then add the one of F_0, F_1 which is in $C^\xi(F)$. Suppose that F_0 is such a model. Then we add G_0, G_0^*. Consider two possibilities $\sup(F_0) < \sup(F_1)$ and $\sup(F_1) < \sup(F_0)$.

Possibility A. $\sup(F_0) < \sup(F_1)$.

Consider then $a_n(F_0) \cap a_n(G_0)$. More precisely, we need first take the same $H(\chi^{+k})$ for all the relevant models, but assume for simplicity that all of them have the same k already.

Claim A1. There is an elementary submodel M of $a_n(F)$, $M \in a_n(F)$ such that

(a) $|M| = |a_n(F_0)|$

(b) $\sup(M \cap \delta_n) > \sup(a_n(F_0) \cap \delta_n)$

(c) M realize the same $k-1$ type over $a_n(F_0) \cap a_n(G_0) \cap H(\chi^{+k-1})$ as $a_n(F_0) \cap H(\chi^{+k-1})$ does

(d) $otp(M \cap \delta_n) = otp(a_n(F_0) \cap \delta_n)$

(e) let N be the model corresponding in M to $a_n(G_0)$ via the isomorphism between M and $a_n(F_0)$. Then $N \supseteq a_n(F_0)$.

Proof. Note first that for each $D \in a_n(G^0)$ there is $E \in a_n(G_0), E \prec a_n(G_0) \cap H(\chi^{+k-1})$ such that $E, a_n(F_0)$ realize the same $k-1$ -type, are of the same order type once intersected with $H(\chi^{+k-1})$ and if $H \in E$ corresponds to $a_n(G_0)$ via the isomorphism then $H \supseteq D$. Thus if the above does not hold, then pick $D \in a_n(G_0)$ witnessing the failure. We will have, using the elementarity of $a_n(G_0)$, that

$$H(\chi^{+k}) \vDash (\forall x((x \prec H(\chi^{+k-1}) \& tp_{n,k-1}(x) = tp_{n,k-1}(a_n(F_0) \cap H(\chi^{+k-1})) \& x \cap \delta_n \approx a_n(F_0) \cap \delta_n) \rightarrow$$

(the model corresponding to $a_n(G_0)$ in x does not contain D))).

But this is impossible. Just take x to be $a_n(F_0)$. By the choice of D it is in $a_n(G_0)$.

Now recall that $a_n(G_0), a_n(F) \prec a_n(G_1^*)$ and $a_n(G_0), a_n(F_0), a_n(G_0^*) \in a_(F) \in a_n(G_1^*)$. By elementarity we can find now $M \prec a_n(F), M \in a_n(F)$ as desired. Just inside $a_n(F)$ we apply the observation above to $D = a_n(G_0^*)$.
$\square$ of the claim.

Extend now a_n by mapping F_1 to M and G_1 to the element of M corresponding to $a_n(G_0)$. By the claim we will have

$$a_n(G_0^*) \subset a_n(G_1),$$

as desired. This completes the Possibility A.

Possibility B. $\sup(F_0) > \sup(F_1)$.

It is similar to Possibility A and a bit easier. Thus we pick M inside $a_n(G_0) \cap a_n(F)$ to be its elementary submodel (again more precisely reducing k by 1) so that $(a), (c)$ and (d) of Claim A1 are satisfied. Then also $\sup(M \cap \delta_n) < \sup(a_n(G_0) < \sup(a_n(F_0) \cap \delta_n)$ and the model corresponding to $a_n(G_0)$ in M will be subset of $a_n(G_0)$. Let N denotes this model. Pick also an elementary extension of N_1 of the same cardinality inside $a_n(G_0) \cap a_n(F)$ which is an elementary submodel of $a_n(G_0)$ and with $M \in N_1$.
Now we extend a_n by mapping F_1 to M, G_1 to N and G_1^* to N_1.

This completes the second possibility.

Suppose now that A is a splitting point and the set of the immediate predecessors of A-$Pred(A)$ contains A_0, A_1 with A_0, A_1, A of a weak Δ-system type(i.e. $Pred_0(A) = \{A_0, A_1\}$) but the immediate predecessor $A^-$of A in $C^{|A|}(A)$ differs from both A_0, A_1. Suppose that $\sup(A_0) < \sup(A_1)$. Let $|A| = \mu$.

Assume first that $A^- \in Pred_1(A)$. Then there are $B_0, B_1, B \in A_1 \cap A^{1\rho}$ of a Δ - system type for some $\rho \in s \backslash \mu + 1$, such that

- $B_0 \in C^\rho(B) \in C^\rho(A^{0\rho})$
- $A_0 \subseteq B_0$
- $A^- = \pi_{B_0 B_1}[A_0]$.

We may assume that $A^{0\mu}$ and $A^{0\rho}$ are already in $\mathrm{dom}(a_n)$. Now add B_0, B_1, B to $\mathrm{dom}(a_n)$ so that the images are in $a_n(A)$. Then let us add A^- and map it to an elementary submodel of $a_n(B_1)$ of the appropriate cardinality. Add A_0 and map it to the isomorphic image of $a_n(A^-)$ under the isomorphism $\pi_{a_n(B_1), a_n(B_0)}$ between $a_n(B_1), a_n(B_0)$.

Now we need to add A_1 and the models G_0, G_0^*, G_1, G_1^* witnessing that A_0, A_1, A are of a Δ- system type. First add G_0, G_0^*, then add A_1, G_1, G_1^* as it was done in **Possibility A** considered above.

The general case - $A^- \in Pred_n(A)$, $n > 1$ is handled similar. Lemmas **??** and **??** used to simplify the situation.

□

The ordering $\leq^*$ on $\mathcal{P}$ and $\leq_n$ on Q_{n0} is not closed in the present situation. Thus it is possible to find an increasing sequence of $\aleph_0$ conditions $\langle\langle a_{ni}, A_{ni}, f_{ni}\rangle \mid i < \omega\rangle$ in Q_{n0} with no upperbound. The reason is that the union of maximal models of these conditions, i.e. $\bigcup_{i<\omega} \max(\operatorname{dom} a_{ni})$ need not be in A_{11} for any A_{11} in $G(\mathcal{P}')$. The next lemma shows that still $\leq_n$ and so also $\leq^*$ share a kind of strategic closure. The proof is similar to those of [**?**, 3.5].

Lemma 3.4 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_n \rangle$ does not add new sequences of ordinals of the length $< \kappa_n$, i.e. it is (κ_n, ∞) – distributive.*

Now as in [**?**] we obtain the following:

Lemma 3.5 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 3.6 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Finally we define $\rightarrow$ on $\mathcal{P}$ similar to those of [1] or [2].

The arguments of [2, 3.19], [**?**, 3.8] can be used to derive the following.

Lemma 3.7 *$\langle \mathcal{P}, \rightarrow \rangle$ satisfies κ^{++}-c.c.*

Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++}\rangle$ be a name of an antichain of the length κ^{++}. Using **??** we find an increasing sequence $\langle\langle A_\alpha^{0\tau}, A_\alpha^{1\tau}, C_\alpha^{\tau}\rangle \mid \tau \in s_\alpha, \alpha < \kappa^{++}\rangle$ of elements of $\mathcal{P}'$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ so that for every $\alpha < \kappa^{++}$ the following holds:

(a) $\langle\langle A_{\alpha+1}^{0\tau}, A_{\alpha+1}^{1\tau}, C_{\alpha+1}^{\tau}\rangle \mid \tau \in s_{\alpha+1}\rangle \Vdash \underset{\sim}{p}_\alpha = \check{p}_\alpha$

(b) $s_\alpha = \bigcup_{\beta<\alpha} s_\beta$

(c) $\bigcup\{A_\beta^{0\tau} | \beta < \alpha, \tau \in s_\beta\} = A_\alpha^{0\tau}$, for each $\tau \in s_\alpha$

(d) ${}^{\tau>}A_{\alpha+1}^{0\tau} \subseteq A_{\alpha+1}^{0\tau}$, for each $\tau \in s_{\alpha+1}$

(e) $A_{\alpha+1}^{0\tau}$ is a successor model, for each $\tau \in s_{\alpha+1}$

(f) $\langle\langle\cup A^{1\tau}_\beta \mid \tau \in s_\beta\rangle \mid \beta < \alpha\rangle \in (A^{0\kappa^+}_{\alpha+1})^-$

(g) for every $\alpha \le \beta < \kappa^{++}, \tau \in s_\beta$ we have

$$A^{0\tau}_\alpha \in C^\beta(A^{0\tau}_\beta)$$

(h) $A^{0\tau}_{\alpha+2}$ is not an immediate successor model of $A^{0\tau}_{\alpha+1}$, for every $\alpha < \kappa^{++}, \tau \in s_{\alpha+1}$.

(i) $p_\alpha = \langle p_{\alpha n} | n < \omega\rangle$

(j) for every $n \ge \ell(p_\alpha)$ the maximal model of $\mathrm{dom}(a_{\alpha n})$ is $A^{0\kappa^+}_{\alpha+1}$ and the maximal sequence of $\mathrm{dom}(a_{\alpha n})$ is a subsequence of $\langle (A^{0\tau}_{\alpha+1})^- | \tau \in s_{\alpha+1}\rangle$, where $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle$

Let $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle$ for every $\alpha < \kappa^{++}$ and $n \ge \ell(p_\alpha)$. Extending by 3.3 if necessary, let us assume that $A^{0\kappa^+}_\alpha \in \mathrm{dom}(a_{\alpha n})$, for every $n \ge \ell(p_\alpha)$. Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^+$ the following holds:

(1) $\ell = \ell(p_\alpha) = \ell(p_\beta)$

(2) for every $n < \ell \quad p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1} i.e. $p_{\alpha n} \cup p_{\beta n}$ is a function.

(3) for every n, $\ell \le n < \omega \quad \langle \mathrm{dom}(a_{\alpha n}), \mathrm{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$

(4) for every n, $\omega > n \ge \ell \quad \mathrm{rng}(a_{\alpha n}) = \mathrm{rng}(a_{\beta n})$.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let α be in S. For each $n, \ell \le n < \omega$, there will be $\beta(\alpha, n) < \alpha$ such that

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_\alpha \subseteq A^{0\kappa^+}_{\beta(\alpha,n)}.$$

Just recall that $|a_{\alpha n}| < \kappa_n$. Shrink S to a stationary subset S^* so that for some $\alpha^* < \min S^*$ of cofinality κ^+ we will have $\beta(\alpha, n) < \alpha^*$, whenever $\alpha \in S^*, \ell \le n < \omega$. Now, the cardinality of $A^{0\kappa^+}_{\alpha^*}$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\alpha, \beta \in S^*, \ell \le n < \omega$

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_\alpha = \mathrm{dom}(a_{\beta n}) \cap A^{0\kappa^+}_\beta.$$

Let us add $A^{0\kappa^+}_{\alpha^*}$ to each $p_\alpha, \alpha \in S^*$. By 3.3, it is possible to do this without adding other additional models except the images of $A^{0\kappa^+}_{\alpha^*}$ under isomorphisms. Thus, $A^{0\kappa^+}_{\alpha^*} \in C^{\kappa^+}(A^{0\kappa^+}_\alpha)$

and $A^{0\kappa^+}_\alpha \in \mathrm{dom}(a_{\alpha n}) \cap C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1})$. So, 3.1(2n) was already satisfied after adding $A^{0\kappa^+}_\alpha$. The rest of 3.1 does not require adding additional models in the present situation.

Denote the result for simplicity by p_α as well. Note that (again by 3.3 and the argument above) any $A^{0\kappa^+}_\gamma$ for $\gamma \in S^* \cap (\alpha^*, \alpha)$ or, actually any other successor or limit model $X \in C^{\kappa^+}(A^{0\kappa}_\alpha)$ with $cof(otp_{\kappa^+}(X)) = \kappa^+$, which is between $A^{0\kappa^+}_{\alpha^*}$ and $A^{0\kappa^+}_\alpha$ can be added without adding other additional models or ordinals except the images of it under isomorphisms.

Let now $\beta < \alpha$ be ordinals in S^*. We claim that p_β and p_α are compatible in $\langle \mathcal{P}, \rightarrow \rangle$. First extend p_α by adding $A^{0\kappa^+}_{\beta+2}$. As it was remarked above, this will not add other additional models or ordinals except the images of $A^{0\kappa^+}_{\beta+2}$ under isomorphisms to p_α.

Let p be the resulting extension. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$ and $p_n = \langle a_n, A_n, f_n \rangle$. Let $q_n = \langle b_n, B_n, g_n \rangle$. Without loss of generality we may assume that $a_n(A^{0\kappa^+}_{\beta+2})$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $\mathrm{rng}(b_n)$ inside $a_n(A^{0\kappa^+}_{\beta+2})$ over the common parts $\mathrm{dom}(b_n)$ and $\mathrm{dom}(a_n)$. This will produce $q'_n = \langle b'_n, B_n, g_n \rangle$ which is $k_n - 1$-equivalent to q_n and with $\mathrm{rng}(b'_n) \subseteq a_n(A^{0\kappa^+}_{\beta+2})$. Doing the above for all $n \geq \ell(p)$ we will obtain $q' = \langle q'_n \mid n < \omega \rangle$ equivalent to q (i.e. $q' \longleftrightarrow q$).

Extend q' to q'' by adding to it $\langle A^{0\kappa^+}_{\beta+2}, a_n(A^{0\kappa^+}_{\beta+2}) \rangle$ as the maximal set for every $n \geq \ell(p)$. Recall that $A^{0\kappa^+}_{\beta+1}$ was its maximal model. So we are adding a top model, also, by the condition (h) above $A^{0\kappa^+}_{\beta+2}$ is not an immediate successor of $A^{0\kappa^+}_{\beta+1}$. Hence no additional models or ordinals are added at all. Let $q''_n = \langle b''_n, B_n, g_n \rangle$, for every $n \geq \ell(p)$.

Combine now p and q'' together. Thus for each $n \geq \ell(p)$ we add b''_n to a_n as well as all of its isomorphic images by $\pi_{A^{0\kappa^+}_{\beta+2} X}$, for every X in $\mathrm{dom}(a_n)$ which is isomorphic to $A^{0\kappa^+}_{\beta+2}$. The rest of the parts are combined in the obvious fashion (we put together the functions and intersect sets of measure one moving first to the same measure). Add if necessary a new top model to insure 3.1(2(d)). Let $r = \langle r_n | n < \omega \rangle$ be the result, where $r_n = \langle c_n, C_n, h_n \rangle$, for $n \geq \ell(p)$.

Claim 3.7.1 $r \in \mathcal{P}$ and $r \geq p$.

Proof. Fix $n \geq \ell(p)$. The main points here are that b''_n and a_n agree on the common part and adding of b''_n to a_n does not require other additions of models except the images of b''_n under isomorphisms. Thus $A^{0\kappa^+}_\alpha$ is in $\mathrm{dom}(a_n)$. So, for each model $X \in \mathrm{dom}(a_n) \cap C^\xi((A^{0\xi})_{\alpha+1})^-)$, with $\xi > \kappa^+$, if $A^{0\kappa^+}_\alpha \notin X$, then we will have the first model $E \in A^{0\kappa^+}_\alpha \cap C^\eta((A^{0\eta}_{\alpha+1})^-)$ inside $\mathrm{dom}(a_n)$, by 3.1(2p), where $\eta = \min((A^{0\kappa^+}_\alpha \cap s) \backslash \xi)$. Then such E must be in $A^{0\kappa^+}_{\alpha^*}$ and hence it is eventually in $\mathrm{dom}(a_{\beta n})$. This means that walks down to models like X will not require adding elements to $\mathrm{dom}(a_{\beta n})$. Dealing with X's in $\mathrm{dom}(a_{\beta n})$ is simpler, since $A^{0\kappa^+}_\alpha \supset A^{0\kappa^+}_{\beta+1}$

which is the maximal model of $\mathrm{dom}(a_{\beta n})$. Thus we have

$$X \in A^{0\kappa^+}_{\alpha+1} \in C^{\kappa^+}(A^{0\kappa^+}_{\alpha}), A^{0\kappa^+}_{\alpha} \in C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1}).$$

Models relevant for 3.1(2p) with such X will be already inside $\mathrm{dom}(a_{\beta n})$.

The check of the rest of conditions of 3.1 is routine. We refer to [**?**] for similar detailed arguments.

$\square$ of the claim.

Now we have $r \geq p, q''$. Hence, $p \to r$ and $q \to r$. Contradiction.

$\square$

4 Concluding remarks.

(1) It is possible to use the above construction in order to obtain models with the power of the first fixed point of the $\aleph$ function as large as one likes with GCH below. It was done first in [3]. An additional element needed here are collapses which can be added exactly as in [3](sec. 2). The initial assumption here is a bit stronger, thus we used a sequence of cardinals κ_n's which are $\kappa_n^{+\kappa_n+1}$ -strong instead of the assumption of [3]: there is κ of cofinality ω such that for every $\tau < \kappa$ the set $\{\alpha < \kappa | o(\alpha) \geq \alpha^{+\tau}\}$ is unbounded in κ.

It looks possible working with names on the $\mathrm{rng}(a_n)$ to reduce the present assumption to those of [3].

(2) Building on analogy of gap 3 here and the Velleman simplified morass with linear limits of the gap 1, [5], the preparation forcing $\mathcal{P}'$ of the first section produces an object that may be viewed as a simplified morass with linear limits of arbitrary gap.

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