BLOWING UP POWER OF A SINGULAR CARDINAL –WIDER GAPS

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Abstract

The paper is concerned with methods for blowing power of singular cardinals using short extenders. Thus, for example, starting with κ of cofinality ω with $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+n}\}$ cofinal in κ for every $n < \omega$ we construct a cardinal preserving extension having the same bounded subsets of κ and satisfying $2^\kappa = \kappa^{+\delta+1}$ for any $\delta < \aleph_1$.

0. Introduction

In Gitik-Mitchell [Git-Mit] the following was proved:

Theorem. *Suppose that there is no sharp for an inner model with a strong cardinal. Let κ be a strong limit cardinal of cofinality ω and $2^\kappa \geq \lambda > \kappa^+$, where λ is not the successor of a cardinal of cofinality less than κ. Then in the core model either*

(a) $o(\kappa) \geq \lambda$
or

(b) $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+n}\}$ *is cofinal in κ for each $n < \omega$.*

The forcing of Gitik-Magidor [Git-Mag1] provides the equiconsistency result if $\lambda < \kappa^{+\omega}$. Once $\lambda > \kappa^{+\omega}$ or κ is a singular cardinal in the core model the possibility (b) of the theorem comes into consideration.

In the present paper, we continue to develop methods for adding ω-sequences to cardinals κ satisfying the condition (b) of the theorem or conditions of similar flavor. The research in this direction was started in [Git1], then in [Git2] the power of κ satisfying (b) was blown up to κ^{++}.

The paper is based on forcing techniques of [Git-Mag1,2] and [Git2], but we do not assume the detailed knowledge of these articles. Rather, we present here the necessary apparatus in a simplified form. It is assumed only that the reader is familiar with the Prikry forcing and extenders. The book of A. Kanamori [Ka] is a good reference for both of them.

The paper is organized as follows: in Sections 1,2 we present simplified versions of forcings for blowing power of singular cardinals introduced in [Git-Mag2] and [Git2]. The next two sections are the main technical parts of the paper. In Section 3 it is shown how to make $2^\kappa = \kappa^{+3}$ starting with κ satisfying the condition (b). In Section 4, based on the ideas developed in Section 3, the method for obtaining $2^\kappa \geq \kappa^{+\delta}$ for any $\delta < \kappa$ is presented. Section 5 deals with generalizations based on the idea of Shelah [Sh1].

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1. A Simple Extender Based Forcing

In this section, we present a simplified version of extender-based forcing of Gitik-Magidor [Git-Mag1,2]. Such forcing will serve only as a motivation for one defined in the next section. Thus, the reader familiar with extender-based forcings may jump directly to Section 2.

Assume GCH. Let κ be a singular cardinal of cofinality ω and $\lambda \geq \kappa^+$ be a successor cardinal. Assume that $\kappa = \bigcup_{n<\omega} \kappa_n$, where $\langle \kappa_n \mid n < \omega \rangle$ is increasing and each κ_n is $\lambda + 1$-strong. This means that for every $n < \omega$ there is a (κ_n, λ^+)-extender E_n over κ_n which ultrapower contains $V_{\lambda+1}$. We fix such E_n and let $j_n : V \to M \simeq Ult(V, E_n)$. For every $\alpha < \lambda$ we define a κ_n − complete ultrafilter $U_{n\alpha}$ over κ_n by setting $X \in U_{n\alpha}$ iff $\alpha \in j_n(X)$. Notice that a lot of $U_{n\alpha}$'s will be comparable in the Rudin-Keisler order (further RK order). Recall that $U \leq_{RK} W$ iff there is $f : \cup W \to \cup U$ such that $X \in U$ iff $f^{-1''}(X) \in W$. Thus, for example, if $\alpha \geq \beta$, then $U_{n,\alpha+\beta} \geq_{RK} U_{n,\alpha}$ and $U_{n,\alpha+\beta} \geq_{RK} U_{n,\beta}$. We will need a strengthening of the Rudin-Kiesler order. Thus, for $\alpha, \beta < \lambda$ let $\alpha \leq_{E_n} \beta$ iff $\alpha \leq \beta$ and for some $f \in {}^{\kappa_n}\kappa_n \;\; j_n(f)(\beta) = \alpha$. Clearly then, $\alpha \leq_{E_n} \beta$ implies $U_{n\alpha} \leq_{RK} U_{n\beta}$ as witnessed by any $f \in {}^{\kappa_n}\kappa_n$ with $j_n(f)(\beta) = \alpha$. The partial order $\langle \lambda, \leq_{E_n} \rangle$ is

κ_n-directed in the RK-order. Actually it is κ_n^{++}-directed (see [Git-Mag1]), but for our purposes κ_n-directedness will be enough. Thus, using GCH, for some enumeration $\langle a_\alpha \mid \alpha < \kappa_n \rangle$ of $[\kappa_n]^{<\kappa_n}$ so that for every successor cardinal $\delta < \kappa_n \langle a_\alpha | \alpha < \delta \rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_n\ (\langle a_\alpha \mid \alpha < \kappa_n \rangle) = \langle a_\alpha \mid \alpha < j_n(\kappa_n) \rangle$. Then, $\langle a_\alpha \mid \alpha < \lambda \rangle$ will enumerate $[\lambda]^{<\lambda} \supseteq [\lambda]^{<\kappa_n}$. Let $\langle \alpha_i \mid i < \tau < \kappa_n \rangle$ be an increasing sequence of ordinals below λ. Find $\alpha < \lambda \backslash (\bigcup_{i<\tau} \alpha_i + 1)$ such that $a_\alpha = \{\alpha_i \mid i < \tau\}$. Then, it is easy to show that $\alpha >_{E_n} \alpha_i$ for every $i < \tau$. $V_{\lambda+1} \subseteq M_n$, so M_n-stationary subset of λ is really stationary. Hence we obtain the following:

Lemma 1.0 *For every set $a \subseteq \lambda$ of cardinality less than κ_n there are stationary many α's $< \lambda$ in every cofinality $< \lambda$ so that $\alpha >_{E_n} \beta$ for every $\beta \in a$.*

For every $\alpha, \beta < \lambda$ such that $\alpha \geq_{E_n} \beta$ we fix the projection $\pi_{\alpha\beta} : \kappa_n \to \kappa_n$ from the extender witnessing this. Let $\pi_{\alpha\alpha} = id$. The following lemma is routine:

Lemma 1.1 *Let $\alpha, \alpha_i < \lambda \; i < \tau < \kappa_n$. Assume that $\alpha \geq_{E_n} \alpha_i$ for every $i < \tau$. Then there is a set $A \in U_{n\alpha}$ so that for every $i, j < \tau$, $\nu \in A$*

*(1) if $\alpha_i \geq_{En} \alpha_j$ then $\pi_{\alpha\alpha_j}(\nu) = \pi_{\alpha_i\alpha_j}(\pi_{\alpha\alpha_i}(\nu))$;
and*

(2) if $\alpha_i > \alpha_j$ then $\pi_{\alpha\alpha_j}(\nu) < \pi_{\alpha\alpha_i}(\nu)$

Now we are ready to define our first forcing notion. We are aiming to blow up the power of κ to λ by adding λ Prikry sequences without adding new bounded subsets to κ. But now we will be much more modest. Fix some $n < \omega$.

Definition 1.2

Let $Q_{n1} = \{f \mid f$ is a partial function from λ to κ_n of cardinality at most $\kappa\}$. We order Q_{n1} by inclusion. Denote this order by $\leq_1$.

Thus, Q_{n1} is basically the usual Cohen forcing for blowing the power of κ^+ to λ. The only minor change is that the functions are taking values inside κ_n rather than 2 or κ^+.

Definition 1.3

Let Q_{n0} be the set of triples $\langle a, A, f\rangle$ so that

(1) $f \in Q_{n1}$

(2) $a \subseteq \lambda$ such that

(2)(i) $|a| < \kappa_n$

(2)(ii) $a \cap \ dom\ f = \emptyset$

(2)(iii) a has a $\leq_{E_n}$ maximal element.

(3) $A \in U_{nmax(a)}$

(4) for every $\alpha, \beta, \gamma \in a$, if $\alpha \geq_{E_n} \beta \geq_{E_n} \gamma$, then $\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$ for every $\rho \in \pi''_{max(a),\alpha}A$.

(5) for every $\alpha > \beta$ in a and every $\nu \in A$

$$\pi_{max(a),\alpha}(\nu) > \pi_{max(a),\beta}(\nu)\ .$$

The last two conditions require the full commutativity on A which is possible by Lemma 1.1.

Definition 1.4

Let $\langle a, A, f\rangle$, $\langle b, B, g\rangle \in Q_{n0}$. Then

$$\langle a, A, f\rangle \geq_0 \langle b, B, g\rangle$$

($\langle a, A, f\rangle$ is stronger than $\langle b, B, g\rangle$) iff

(1) $f \supseteq g$

(2) $a \supseteq b$

(3) $\pi''_{max(a),max(b)}A \subseteq B$

We now define a forcing notion Q_n which is an extender analog of the one element Prikry forcing.

Definition 1.5

$Q_n = Q_{n0} \cup Q_{n1}$.

Definition 1.6

The direct extension ordering $\leq^*$ on Q_n is defined to be $\leq_0 \cup \leq_1$.

Definition 1.7

Let $p, q \in Q_n$.

Then $p \leq q$ iff either

(1) $p \leq^* q$
or

(2) $p = \langle a, A, f\rangle \in Q_{n0}$, $q \in Q_{n1}$ and the following holds:

(2)(a) $q \supseteq f$

(2)(b) $dom\ q \supseteq a$

(2)(c) $q(max(a)) \in A$

(2)(d) for every $\beta \in a$ $q(\beta) = \pi_{max(a),\beta}(q(max(a)))$.

Clearly, the forcing $\langle Q_n, \leq\rangle$ is equivalent to $\langle Q_{n1}, \leq_1\rangle$, i.e. the Cohen forcing. However, the following basic facts relate it to the Prikry type forcing notion.

Lemma 1.8 *$\langle Q_n, \leq^*\rangle$ is κ_n-closed.*

Lemma 1.9 *$\langle Q_n, \leq, \leq^*\rangle$ satisfies the Prikry condition, i.e. for every $p \in Q_n$ and every statement σ of the forcing language there is $q \geq^* p$ deciding σ. Moreover, p and q have the same first coordinate.*

Proof. Let $p = \langle a, A, f\rangle$. Suppose otherwise. By induction on $\nu \in A$ define an increasing sequence $\langle p_\nu \mid \nu \in A\rangle$ of elements of Q_{n1} with $dom\ p_\nu \cap a = \emptyset$ as follows. Let $\langle p_\rho \mid \rho \in A \cap \nu\rangle$ be defined and $\nu \in A$. Define p_ν. Let $g = \bigcup_{\rho<\nu} p_\rho$. Then $g \in Q_{n1}$. Consider $q = \langle a, A, g\rangle$. Let $q^\frown\langle\nu\rangle = g \cup \{\langle\beta, \pi_{maxa,\beta}(\nu)\rangle \mid \beta \in a\}$. If there is $p \geq_1 q^\frown\langle\nu\rangle$ deciding σ, then let p_ν be some such p restricted to $\lambda\backslash a$. Otherwise, set $p_\nu = g$. Notice that here there will always be a condition deciding σ.

Finally, let $g = \bigcup_{\nu \in A} p_\nu$.

Shrink A to a set $B \in U_{nmax(a)}$ so that $p_\nu^\frown \langle \nu \rangle$ decides the same way or does not decide σ at all, for every $\nu \in B$. By our assumption $\langle a, B, g \rangle \nVdash \sigma$. However, pick some $h \geq \langle a, B, g \rangle$, $h \in Q_{n1}$ deciding on σ. Let $h(maxa) = \nu$. Then, $p_\nu^\frown \langle \nu \rangle$ decides σ. But this holds then for every $\nu \in B$. Hence, already $\langle a, B, g \rangle$ decides σ. Contradiction. □

Let us now define the main forcing of this section by putting the blocks Q_n together.

Definition 1.10

The set $\mathcal{P}$ consists of sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) for every $n < \omega$ $p_n \in Q_n$

(2) there is $\ell(p) < \omega$ so that for every $n < \ell(p)$ $p_n \in Q_{n1}$, for every $n \geq \ell(p)$ $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$ and $a_n \subseteq a_{n+1}$.

Definition 1.11

Let $p = \langle p_n \mid n < \omega \rangle$, $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$. We set $p \geq q \quad (p \geq^* q)$ iff for every $n < \omega$ $p_n \geq_{Q_n} q_n \quad (p_n \geq^*_{Q_n} q_n)$.

The proof of the next lemma is based on the argument 1.9.

Lemma 1.12 *$\langle \mathcal{P}, \leq \rangle$ satisfies κ^{++}-c.c.*

It follows by the usual Δ-system argument.

For $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$ we denote $p \restriction n = \langle p_m \mid m < n \rangle$ and $p \backslash n = \langle p_m \mid m \geq n \rangle$. Let $\mathcal{P} \restriction n = \{p \restriction n \mid p \in \mathcal{P}\}$ and $\mathcal{P} \backslash n = \{p \backslash n \mid p \in \mathcal{P}\}$. Then the following lemmas are obvious:

Lemma 1.13 *$\mathcal{P} \simeq \mathcal{P} \restriction n \times \mathcal{P} \backslash n$ for every $n < \omega$.*

Lemma 1.14 *$\langle \mathcal{P} \backslash n, \leq^* \rangle$ is κ_n-closed, moreover, if $\langle p^\alpha \mid \alpha < \delta < \kappa \rangle$ is a $\leq^*$-increasing sequence with $\kappa_{\ell(p_0)} > \delta$ then there is $p \geq^* p^\alpha$ for every $\alpha < \delta$.*

The proof of the next lemma is base on the argument 1.9.

Lemma 1.15 *$\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. Let $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$ and σ be a statement of the forcing language. Suppose that there is no $q \geq^* p$ deciding σ. Assume for simplicity that $\ell(p) = 0$. Let $p_n = \langle a_n, A_n, f_n \rangle (n < \omega)$. Assume also each A_n consists of limit ordinals. We define by induction on $\nu \in A_0$ a $\leq^*$-increasing sequence $\langle r^\nu \mid \nu \in A_0 \cup \{0\}$ or $\nu = \nu' + 1$ for some $\nu' \in A_0\rangle$ and a sequence $\langle q^{\nu+1} \mid \nu \in A_0\rangle$. Set $r^0 = p$.

Let $\nu \in A_0$ and assume that $\langle r^\mu \mid \mu \in A_0 \cap \nu$ or $\mu = \mu'$ for some $\mu' \in A_0 \cap \nu, \mu' < \nu\rangle$, $\langle q^{\mu+1} \mid \mu \in A_0 \cap \nu\rangle$ are defined. Assume, as an inductive assumption, that for each $\mu \in A_0 \cap \nu$ $a_0(r^\mu) = a_0$ and $A_0(r^\mu) = A_0$, where for $t \in \mathcal{P}$, $t = \langle t_n \mid n < \omega\rangle$ we denote by $a_n(t)$ the first coordinate of t_n, by $A_n(t)$ the second and by $f_n(t)$ the third coordinate of t. Define r^ν using 1.14 to be a $\leq^*$-extension of $\langle r^\mu \mid \mu \in A_0 \cap \nu\rangle$ with $a_0(r^\nu) = a_0$. Consider

$$r^{\nu\frown} < \nu >= \langle r_0^{\nu\frown} < \nu >, r_1^\nu, \dots, r_n^\nu, \dots \mid n < \omega\rangle$$

where $r_0^{\nu\frown} < \nu >= f_0(r^\nu) \cap \{\langle \beta, \pi_{max a_0, \beta}(\nu)\rangle \mid \beta \in a_0\}$. If there is no $q \geq^* r^{\nu\frown} < \nu >$ deciding σ then set $q^{\nu+1} = r^{\nu\frown} < \nu >$ and $r^{\nu+1} = r^\nu$. Otherwise, let $q^{\nu+1}$ be such a condition. Define $r^{\nu+1} = \langle r_n^{\nu+1} \mid n < \omega\rangle$ as follows. For every n, $1 \leq n < \omega$, set $r_n^{\nu+1} = q_n^{\nu+1}$ and let $r_0^{\nu+1} = \langle a_0, A_0, q_0^{\nu+1} \restriction (\lambda \backslash a_0)\rangle$.

This completes the construction of $\langle r^\nu \mid \nu \in A_0 \cup \{0\}$ or $\nu = \nu' + 1$ for some $\nu' \in A_0\rangle$ and $\langle q^{\nu+1} \mid \nu \in A_0\rangle$. Using 1.14 and the inductive assumption it is easy to find a $\leq^*$-extension r^* of r^ν's so that $a_0(r^*) = a_0$ and $A_0(r^*) = A_0$. Shrink the set A_0 to a set A_0^* so that for every $\nu \in A_0^*$ $r^{*\frown} < \nu > \not\parallel \sigma$. Since $r^* \geq^* p$ and we assumed that no $*$-extension of p decides σ, $A_0^* \in U_{0,max(a_0)}$. Let $p(0)$ be the condition obtained from r^* by replacing A_0 in it by A_0^*.

Now we should repeat the argument above with $p(0)$ replacing p and pairs $\langle \nu_0, \nu_1\rangle$ from $A_0^* \times A_1(p(0))$ replacing ν's from A_0. This will define $p(1)$. Continue in the same fashion for each $n < \omega$. Finally any extension deciding σ of a $*$-extension of $\langle p(n) \mid n < \omega\rangle$ will easily provide a contradiction. □

Combining these lemmas we obtain the following:

Proposition 1.16 *The forcing $\langle \mathcal{P}, \leq\rangle$ does not add new bounded subsets to κ and preserves all the cardinals above κ^+.*

Actually, it is not hard to show that κ^+ is preserved as well.

Finally, let us show that this forcing adds λ ω-sequences to κ. Thus, let $G \subseteq \mathcal{P}$ be generic. For every $n < \omega$ define a function $F_n : \lambda \to \kappa_n$ as follows:

$F_n(\alpha) = \nu$ if for some $p = \langle p_m \mid m < \omega\rangle \in G$ $\ell(p) > n$ and $p_n(\alpha) = \nu$.

Now for every $\alpha < \lambda$ set $t_\alpha = \langle F_n(\alpha) | n < \omega\rangle$. Let us show that the set $\{t_\alpha \mid \alpha < \lambda\}$ has cardinality λ.

Lemma 1.17 *For every $\beta < \lambda$ there is $\alpha, \beta < \alpha < \lambda$ such that t_α is different from every t_γ with $\gamma \leq \beta$.*

Proof. Suppose otherwise. Then there is $p = \langle p_n \mid n < \omega \rangle \in G$ and $\beta < \lambda$ such that

$$p \Vdash \forall \alpha (\beta < \alpha < \lambda \to \exists \gamma \leq \beta \underset{\sim}{t}_\alpha = \underset{\sim}{t}_\gamma) \ .$$

For every $n \geq \ell(p)$ let $p_n = \langle a_n, A_n, f_n \rangle$. Pick some $\alpha \in \lambda \setminus \Big(\bigcup_{n<\omega} a_n \cup \cup \ dom \ f_n \cup (\beta + 1) \Big)$. We extend p to a condition q so that $q \geq^* p$ and for every $n \geq \ell(q) = \ell(p) \quad \alpha \in b_n$, where $q_n = \langle b_n, B_n, g_n \rangle$. Then q will force that t_α dominates every t_γ with $\gamma < \alpha$. This leads to the contradiction. Thus, let $\gamma < \alpha$ and assume that q belongs to the generic subset of $\mathcal{P}$. Then either $t_\gamma \in V$ or it is a new ω-sequence. If $t_\gamma \in V$ then it is dominated by t_α by the usual density arguments. If t_γ is new, then for some $r \geq q$ in the generic set $\gamma \in c_n$ for every $n \geq \ell(r)$, where $r_n = \langle c_n, C_n, h_n \rangle$. Here we use the second part of 1.10(2). But also $\alpha \in c_n$ since $c_n \supseteq b_n$. This implies $F_n(\alpha) > F_n(\gamma)$ (see 1.3(5)) and we are done. □

2. Short Extenders Replacing Long Ones

In this section, we define basic tools which will be used in further forcing constructions. The material is simplified and adapted for further purposes from the version of [Git2].

We assume GCH. Let $\kappa = \bigcup_{n<\omega} \kappa_n$, $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ and for every $n < \omega$ κ_n is $\lambda_n + 1$-strong, where λ_n is a regular and not the successor of a singular cardinal satisfying $\kappa_n^{+n+2} \leq \lambda_n < \kappa_{n+1}$. Thus, instead of one λ above κ^+ in Section 1, we have different λ_n below κ. In this section, we will sketch the main result of [Git2] that even $\lambda_n = \kappa_n^{+n+2}$ $(n < \omega)$ will be enough for blowing the power of κ to κ^{++}. For each $n < \omega$ we fix an extender E_n witnessing $\lambda_n + 1$-strongness of κ_n. We define ultrafilters $U_{n\alpha}(\alpha < \lambda_n)$ as in Section 1 by setting $X \in U_{n\alpha}$ iff $\alpha \in j_n(X)$, where $j_n : V \to M \simeq Ult(V, E_n)$. Also the order $\leq_{E_n}$ over λ_n is defined as in Section 1. Let λ be a regular cardinal above κ. The first idea for blowing power of κ to λ is to simulate the forcing $\mathcal{P}$ of Section 1. It was built from blocks Q_n's. The essential part of Q_n is Q_{n0} which typical element has a form $\langle a, A, f \rangle$, where f is a Cohen condition, A is a set of measure one, but the main and problematic part $a \subseteq \lambda$ is actually a set of indexes of the extender E_n. E_n had length λ in Section 1 but now it is very short. Its length is $\lambda_n < \kappa_{n+1} < \kappa$. Here we take a to be an order preserving function from λ into the set of indexes of E_n, i.e. into λ_n. Formally:

Definition 2.1

Let Q_{n0} be the set of triples $\langle a, A, f \rangle$ so that

(1) $f \in Q_{n1}$, where Q_{n1} is defined in 1.2.

(2) a is a partial order preserving function from λ to λ_n such that

(2)(i) $|a| < \kappa_n$

(2)(ii) $dom\ a \cap\ dom\ f = \emptyset$

(2)(iii) $rnga$ has a $\leq_{E_n}$ – maximal element

(3) $A \in U_{nmax(rnga)}$

(4) for every $\alpha, \beta, \gamma \in rnga$, if $\alpha \geq_{E_n} \beta \geq_{E_n} \gamma$ then

$$\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$$

for every $\rho \in \pi''_{max(rnga),\alpha} A$

(5) for every $\alpha > \beta$ in $rnga$ and $\nu \in A$

$$\pi_{max(rnga),\alpha}(\nu) > \pi_{max(rnga),\beta}(\nu)\ .$$

The ordering $\leq_0$ of Q_{n0} is defined as in 1.4 only (b), (c) and (d) of 1.7(2) should by now formulated as follows:

(b) $dom\ q \supseteq\ dom\ a$

(c) $q(max(dom\ a)) \in A$

(d) for every $\beta \in\ dom\ a\ \ q(\beta) = \pi_{max(rnga),a(\beta)}(q(max(a)))$.

Lemmas 1.8, 1.9 are valid here with proofs requiring minor changes. The forcing $\mathcal{P}$ of 1.10 is defined here similarly:

Definition 2.2

The set $\mathcal{P}$ consists of sequences $p = \langle p_n \mid n < \omega\rangle$ so that

(1) for every $n < \omega$ $p_n \in Q_n$

(2) there is $\ell(p) < \omega$ so that for every $n < \omega$ $p_n \in Q_{n1}$, for every $n \geq \ell(p)$ $p_n = \langle a_n, A_n, f_n\rangle$ and $dom\ a_n \subseteq\ dom\ a_{n+1}$.

Definition 2.3

Let $p = \langle p_n \mid n < \omega\rangle$, $q = \langle q_n \mid n < \omega\rangle \in \mathcal{P}$. We define $p \geq q \quad (p \geq^* q)$ iff for every $n < \omega$ $p_n \geq_{Q_n} q_n \quad (p_n \geq^*_{Q_n} q_n)$.

For $p = \langle p_n \mid n < \omega\rangle \in \mathcal{P}$ let $p \restriction n = \langle p_m \mid m < n\rangle$ and $p\backslash n = \langle p_m \mid m \geq n\rangle$. Set $\mathcal{P} \restriction n = \{p \restriction n \mid p \in \mathcal{P}\}$ and $\mathcal{P}\backslash n = \{p\backslash n \mid p \in \mathcal{P}\}$.

The following lemmas are obvious:

Lemma 2.4 $\mathcal{P} \simeq \mathcal{P} \restriction n \times \mathcal{P}\backslash n$ *for every* $n < \omega$.

Lemma 2.5 $\langle \mathcal{P}\backslash n, \leq^*\rangle$ *is* κ_n*-closed.*

The proof of the Prikry condition is the same as 1.15.

Lemma 2.6 $\langle \mathcal{P}, \leq, \leq^*\rangle$ *satisfies the Prikry condition.*

The ω-sequences $t_\alpha = \langle F_\alpha(n) \mid n < \omega\rangle$ defined as in Section 1 will witness that λ new ω-sequences are added by $\langle \mathcal{P}, \leq\rangle$. Thus we obtain the following:

Proposition 2.7 *The forcing* $\langle \mathcal{P}, \leq\rangle$ *does not add new bounded subsets to* κ *and it adds* λ *new* ω*-sequences to* κ.

The problem is that κ^{++}-c.c. fails badly. Thus, any two conditions p and q such that for infinitely many n's $rnga_n(p) = rnga_n(q)$ but $dom\ a_n(p) \neq\ dom\ a_n(q)$ are incompatible. Using this it is possible to show that $\langle \mathcal{P}, \leq\rangle$ collapses λ to κ^+. The rest of the section and actually of the paper will be devoted to the task of repairing the chain condition. Thus we shall identify various conditions in $\mathcal{P}$. The basic idea goes back to the problem raised in [Git-Mit,Q.1] on independence of the assignment function of precovering sets and its solution in [Git1]. Roughly speaking it is possible to arrange a situation where a Prikry sequence may correspond to various measures of extenders E_n*'s.*

Fix $n < \omega$. For every $k \leq n$ we consider a language $\mathcal{L}_{n,k}$ containing two relation symbols, a function symbol, a constant c_α for every $\alpha < \kappa_n^{+k}$ and constants c_{λ_n}, c. Consider a structure

$$\mathfrak{a}_{n,k} = \langle H(\chi^{+k}), \in, E_n, \quad \text{the enumeration of} \quad [\lambda_n]^{<\lambda n} \text{ (as in 1.0)}, \lambda_n, \chi, 0, 1, \ldots, \alpha \ldots \mid \alpha < \kappa_n^{+k}\rangle$$

in this language, where χ is a regular cardinal large enough. For an ordinal $\xi < \chi$ (usually ξ will be below λ_n) we denote by $tp_{n,k}(\xi)$ the $\mathcal{L}_{n,k}$-type realized by ξ in $\mathfrak{a}_{n,k}$.

Let $\mathcal{L}'_{n,k}$ be the language obtained from $\mathcal{L}_{n,k}$ by adding a new constant c'. For $\delta < \chi$ let $\mathfrak{a}_{n,k,\delta}$ be the $\mathcal{L}'_{n,k}$-structure obtained from $\mathfrak{a}_{n,k}$ by interpreting c' as δ. The type $tp_{n,k}(\delta, \xi)$ is the $\mathcal{L}'_{n,k}$–type realized by ξ in $\mathfrak{a}_{n,k,\delta}$. Further, we shall identify types with ordinals corresponding to them in some fixed well-ordering of the power sets of κ_n^{+k}'s.

Definition 2.8

Let $k \leq n$ and $\beta < \lambda_n$. β is called k-good iff

(1) for every $\gamma < \beta$ $tp_{n,k}(\gamma, \beta)$ is realized unboundedly many times below λ_n;

(2) for every $a \subseteq \beta$ if $|a| < \kappa_n$ then there is $\alpha < \beta$ corresponding to a in the enumeration of $[\lambda_n]^{<\lambda_n}$.

β is called good if it is k-good for some $k \leq n$.

Further we will be interested mainly in k-good ordinals for $k > 2$. If $\alpha, \beta < \lambda_n$ realize the same k-type for $k > 2$, then $U_{n\alpha} = U_{n\beta}$, since the number of different $U_{n\alpha}$'s is κ_n^{++}. Recall that we assume that each λ_n is a regular cardinal and is not the successor of a singular.

Lemma 2.9 *The set* $\{\beta < \lambda_n \mid \beta \quad is \quad n - good\} \cup \{\beta < \lambda_n \mid cf\beta < \kappa_n\}$ *contains a club.*

Proof. Let us show first that the set $\{\beta < \lambda_n \mid \forall \gamma < \beta \;\; tp_{n,n}(\gamma, \beta)$ is realized unboundedly often $\}$ contains a club. Suppose otherwise. Let S be a stationary set of β's such that there is $\gamma_\beta < \beta$ with $tp(\gamma_\beta, \beta)$ realized only boundedly many times below λ_n. Shrink S to a stationary S^* on which all γ_β's are the same. Let $\gamma_\beta = \gamma$ for every $\beta \in S^*$. The total number of n-types over γ, i.e. $tp_{n,n}(\gamma, -)$ is $\kappa_n^{+n+1} < \lambda_n$. Hence, there is a stationary $S^{**} \subseteq S^*$ such that for every $\alpha, \beta \in S^{**}$ $tp_{n,n}(\gamma, \alpha) = tp_{n,n}(\gamma, \beta)$. In particular the type $tp_{n,n}(\gamma, \beta)$ is realized unboundedly often below λ_n.

Contradiction.

Now, in order to finish the proof, notice that whenever $N \prec \mathfrak{a}_{n,n}$, $\beta = N \cap \lambda_n < \lambda_n$ and ${}^{\kappa_n >}N \subseteq N$ then β satisfies (2) of 2.8. $\square$

Lemma 2.10 *Suppose that* $n \geq k > 0$ *and* β *is* k*-good. Then there are arbitrarily large* $k-1$*-good ordinals below* β*.*

Proof. Let $\gamma < \beta$. Pick some $\alpha > \beta$ realizing $tp_{n,k}(\gamma, \beta)$. The facts that $\gamma < \beta < \alpha$ and β is $k-1$-good can be expressed in the language $\mathcal{L}'_{n,k}$. So the statement "$\exists y(\gamma < y < x) \wedge (y$ is $(k-1$-good)" belongs to $tp_{n,k}(\gamma, \alpha) = tp_{n,k}(\gamma, \beta)$. Hence, there is $\delta, \gamma < \delta < \beta$ which is $k-1$-good. $\square$

Let us now define a refinement of the forcing $\mathcal{P}$ of 2.2.

Definition 2.11

The set $\mathcal{P}^*$ is the subset of $\mathcal{P}$ consisting of sequences $p = \langle p_n \mid n < \omega \rangle$ so that for every n, $\ell(p) \leq n < \omega$ and $\beta \in dom\ a_n$ there is a nondecreasing converging to infinity sequence of natural numbers $\langle k_m \mid n \leq m < \omega \rangle$ so that for every $m \geq n$ $a_m(\beta)$ is k_m-good, where $p_m = \langle a_m, A_m, f_m \rangle$.

The orders on $\mathcal{P}^*$ are just the restrictions of $\leq$ and $\leq^*$ of $\mathcal{P}$. The following lemma is crucial for showing the Prikry property of $\langle \mathcal{P}^*, \leq, \leq^* \rangle$.

Lemma 2.12. *$\langle \mathcal{P}^*, \leq^* \rangle$ is κ_0-closed.*

Proof. Let $\langle p^\alpha \mid \alpha < \mu < \kappa_0 \rangle$ be a $\leq^*$-increasing sequence of elements of $\mathcal{P}^*$. Suppose for simplicity that $\ell(p^0) = 0$ and hence for every $\alpha < \mu$ $\ell(p^\alpha) = 0$. Let $p_n^\alpha = \langle a_n^\alpha, A_n^\alpha, f_n^\alpha \rangle$ for every $n < \omega$ and $\alpha < \mu$. For each $n < \omega$ set $f_n = \bigcup_{\alpha<\mu} f_n^\alpha$ and $a_n = \bigcup_{\alpha<\mu} a_n^\alpha$. Let β be a $\sup dom\ \bigcup_{n<\omega} a_n$. We like to extend a_n by corresponding to β an ordinal $\delta_n < \lambda_n$ which is above $\cup(rnga_n)$, RK-above every element of $rnga_n$ and also is n-good. Such δ_n exists by Lemmas 1.0 and 2.9. Set $b_n = a_n \bigcup \{\langle \beta, \delta_n \rangle\}$ and $B_n = \bigcap_{\alpha<\mu} \pi^{-1''}_{\delta_n max(rnga_\alpha)}(A_n^\alpha)$. We define $q_n = \langle b_n, B_n, f_n \rangle$ and $q = \langle q_n \mid n < \omega \rangle$. Then $q \geq^* p^\alpha$, for every $\alpha < \mu$ and $q \in \mathcal{P}^*$. Since the only new element added is β and for every $n < \omega$ $b_n(\beta) = \delta_n$ is n-good. $\square$

Now it is routine to show analogue of 2.4.7 for $\langle \mathcal{P}^*, \leq, \leq^* \rangle$.

Lemma 2.13. *$\langle \mathcal{P}^*, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Lemma 2.14. *For every $n < \omega$ $\quad \mathcal{P}^* \simeq \mathcal{P}^* \restriction n \times \mathcal{P}^* \backslash n$.*

Lemma 2.15. *$\langle \mathcal{P}^*, \leq \rangle$ does not add new bounded subsets to κ and it adds λ new ω-sequences to κ.*

Unfortunately, $\mathcal{P}^*$ still collapses λ to κ^+.

Let us now define an equivalence relation on $\mathcal{P}^*$.

Definition 2.16

Let $p = \langle p_n \mid n < \omega \rangle$, $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}^*$. We call p and q equivalent and denote this by $p \leftrightarrow q$ iff

(1) $\ell(p) = \ell(q)$

(2) for every $n < \ell(p)$ $p_n = q_n$

(3) there is a nondecreasing sequence $\langle k_n \mid \ell(p) \leq n < \omega\rangle$ with $\lim_{n\to\infty} k_n = \infty$ and $k_{\ell(p)} > 2$ such that for every n, $\ell(p) \leq n < \omega$ the following holds:

  (a) $f_n = g_n$

  (b) $dom\ a_n = dom\ b_n$

  (c) $rnga_n$ and $rngb_n$ are realizing the same k_n-type, (i.e. the least ordinals coding $rnga_n$ and $rngb_n$ are such)

  (d) $A_n = B_n$,
  where $p_n = \langle a_n, A_n, f_n\rangle$ and $q_n = \langle b_n, B_n, g_n\rangle$.

  Notice that, in particular the following is also true:

  (e) for every $\delta \in\ dom\ a_n = dom\ b_n \quad a_n(\delta)$ and $b_n(\delta)$ are realizing the same k_n-type

  (f) for every $\delta \in\ dom\ a_n = dom\ b_n$ and $\ell \leq k_n\ a_n(\delta)$ is ℓ-good if $b_n(\delta)$ is ℓ-good

  (g) for every $\delta \in\ dom\ a_n =\ dom\ b_n \quad max(rnga_n)$ projects to $a_n(\delta)$ the same way as $max(rngb_n)$ projects to $b_n(\delta)$, i.e. the projection functions $\pi_{max(nga_n),a_n(\delta)}$ and $\pi_{max(rngb_n),b_n(\delta)}$ are the same.

Let us also define a preordering $\rightarrow$ on $\mathcal{P}^*$.

Definition 2.17.

Let $p, q \in \mathcal{P}^*$.
Set $p \rightarrow q$ iff there is a sequence of conditions $\langle r_k \mid k < m < \omega\rangle$ so that

(1) $r_0 = p$

(2) $r_{m-1} = q$

(3) for every $k < m - 1$

$$r_k \leq r_{k+1} \quad \text{or} \quad r_k \leftrightarrow r_{k+1}\ .$$

The next two lemmas show that $\langle \mathcal{P}^*, \rightarrow\rangle$ is a nice subforcing of $\langle \mathcal{P}^*, \leq\rangle$.

Lemma 2.18. *Let $p, q, s \in \mathcal{P}^*$. Suppose that $p \leftrightarrow q$ and $s \geq p$. Then there are $s' \geq s$ and $t \geq q$ such that $s' \leftrightarrow t$.*

Proof. Let $\langle k_n \mid \ell(p) = \ell(q) \leq n < \omega \rangle$ be as in 2.16(3) witnessing $p \leftrightarrow q$. We need to define $s' = \langle s'_n \mid n < \omega \rangle$ and $t = \langle t_n \mid n < \omega \rangle$. Set $s'_n = t_n = s_n$ for every $n < \ell(p) = \ell(q)$. Set also $s'_n = s_n$ for every $n < \ell(s)$. Now let $\ell(p) \leq n < \ell(s)$. We show that $q_n = \langle b_n, B_n, g_n \rangle$ extends to s_n in the ordering of Q_n and then we'll set $t_n = s_n$. Let $p_n = \langle a_n, A_n, f_n \rangle$. By 2.16(3), $f_n = g_n$ and $A_n = B_n$.

We know that $s_n \geq \langle a_n, A_n, f_n \rangle$ (in the ordering of Q_n), hence $s_n(max(dom\ a_n)) \in A_n$ and for every $\beta \in\ dom\ a_n$ $s_n(\beta) = \pi_{max(rng\ a_n),a_n(\beta)}(s_n(max(dom\ a_n)))$. But by 2.16(3)

$$\pi_{max(rng\ (a_n)),a_n(\beta)} = \pi_{max(rng\ b_n),b_n(\beta)}$$

and $dom\ a_n = dom\ b_n$. Thus, $s_n \geq \langle b_n, A_n, f_n \rangle = q_n$.

Suppose now that $n \geq \ell(s)$. Let $p_n = \langle a_n, A_n, f_n \rangle$, $q_n = \langle b_n, A_n, f_n \rangle$ and $s_n = \langle c_n, C_n, h_n \rangle$.

Case 1.

$k_n = 3$.
Then we first extend s_n to a condition $s'_n \in Q_{n1}$ and proceed as above.

Case 2.

$k_n > 3$.
Set $s'_n = s_n$. Then $rnga_n$ and $rngb_n$ are realizing the same k_n-type.

Thus it is possible to find $\widetilde{d}_n$ realizing the same $k_n - 1$-type over $rngb_n$ as $rngc_n$ over $rnga_n$. Let d_n be the order preserving function from $dom\ a_n$ onto $\widetilde{d}_n$. Set $t_n = \langle d_n, C_n, h_n \rangle$.

This completes the construction. $s' = \langle s'_n \mid n < \omega \rangle$ and $t = \langle t_n \mid n < \omega \rangle$ are as desired.$\square$

Lemma 2.19 *For every $p, q \in \mathcal{P}^*$ such that $p \to q$ there is $s \geq p$ so that $q \to s$.*

The proof is an inductive application of the previous lemma. Thus, suppose for example that

$$\begin{array}{ccc} q & \longleftrightarrow & c \\ & & \vee| \\ a & \longleftrightarrow & b \\ \vee| & & \\ p & & \end{array}$$

i.e. a, b, c are witnessing $p \to q$. We apply Lemma 2.18 to a, b and c. It provides equivalent $c' \geq c$ and $a' \geq a$. But then $a' \geq p$ and $q \to a'$, since

$$\begin{array}{ccc} a' & \longleftrightarrow & c' \\ & & \vee \mid \\ q & \longleftrightarrow & c \end{array}$$

Lemma 2.20 *$\langle \mathcal{P}^*, \to \rangle$ satisfies λ-c.c.*

Proof. Let $\langle p^\alpha \mid \alpha < \lambda \rangle$ be a sequence of elements of $\mathcal{P}^*$. Using the Δ-system argument it is easy to find a stationary $S \subseteq \lambda$, $\delta < minS$, $\ell < \omega$ so that for every $\alpha, \beta \in S$, $\alpha < \beta$ the following holds

(a) $\ell(p^\alpha) = \ell$

(b) for every $n < \ell$ p^α_n and p^β_n are compatible

(c) for every $n \geq \ell$ let $p^\alpha_n = \langle a^\alpha_n, A^\alpha_n, f^\alpha_n \rangle$, then

(c)(i) $A^\alpha_n = A^\beta_n$

(c)(ii) f^α_n, f^β_n are compatible and $min(dom\ f^\beta_n \setminus \delta) \geq \beta > \sup(dom\ f^\alpha_n) + \sup(dom\ a^\alpha_n)$

(c)(iii) $a^\alpha_n \restriction \delta = a^\beta_n \restriction \delta$

(c)(iv) $min(dom\ a^\beta_n \setminus \delta) \geq \beta > \sup(dom\ f^\alpha_n) + \sup(dom\ a^\alpha_n)$

(c)(v) $rng\ a^\alpha_n = rng\ a^\beta_n$.

Let $\alpha < \beta$ be in S. We claim that p^α and p^β are compatible in $\langle \mathcal{P}^*, \to \rangle$. Define equivalent conditions $p \geq p^\alpha$ and $q \geq p^\beta$. First we set $p_n = q_n = p^\alpha_n \cup p^\beta_n$ for $n < \ell$. Let $\tau^\alpha = min\Big(\bigcup_{n \geq \ell} dom\ a^\alpha_n \setminus \delta\Big)$ and $\tau^\beta = min\Big(\bigcup_{n \geq \ell} dom\ a^\alpha_n \setminus \delta\Big)$. Assume for simplicity that $\tau^\alpha \in dom\ a^\alpha_\ell$ and $\tau^\beta \in\ dom\ a^\beta_\ell$. By 2.11 there is a nondecreasing converging to infinity sequences of natural numbers $\langle k_m \mid \ell \leq m < \omega \rangle$ so that for every $m \geq \ell$ $a^\alpha_m(\tau^\alpha) = a^\beta_m(\tau^\beta)$ is k_m-good. Let $n \geq \ell$.

Case 1.

$k_n \leq 4$.
Pick some $\nu \in A_n^\alpha = A_n^\beta$. Set $p_n = q_n = f_n^\alpha \cup f_n^\beta \cup \{\langle \gamma, \pi_{max(rnga_n^\alpha), a_n^\alpha(\gamma)}(\nu)\rangle \mid \gamma \in \ dom\ a_n^\alpha\} \cup \{\langle \gamma, \pi_{max(rng\ a_n^\beta), a_n^\beta(\gamma)}(\nu) \mid \gamma \in \ dom\ a_n^\beta\}$. The condition (c) above insures that this is a function in Q_{n1}.

Case 2.

$k_n > 4$.
Using Lemmas 2.9, 2.10 for $a_n^\alpha(\tau^\alpha) = a_n^\beta(\tau^\beta)$, we find t^α realizing the same $k_n - 1$-type over $rnga_n^\alpha \restriction \delta = rnga_n^\beta \restriction \delta$ as $rng(a_n^\alpha \backslash \delta) = rng(a_n^\beta \backslash \delta)$ does so that $mint^\alpha > max(rnga_n^\alpha)$. Set $a_n' = rnga_n^\alpha \cup t^\alpha$. $rng(a_n^\beta \backslash \delta)$ realizes over $rnga_n^\alpha \restriction \delta$ the same type as t^α. Hence there is t^β so that

$$min(rnga_n^\beta \backslash \delta) = a_n^\beta(\tau^\beta) > maxt^\beta$$

and if $b_n' = rnga_n^\beta \cup t^\beta$, then a_n' and b_n' are realizing the same $k_n - 1$-type. Now pick n-good ordinal ξ coding a_n'. Using the $k_n - 1$ equivalence of a_n', b_n' find $k_n - 2$-good ordinal ρ coding b_n' and so that ξ and ρ (and hence also $a_n' \cup \{\xi\}$ and $b_n' \cup \{\rho\}$ realize the same $k_n - 2$-type. Pick some $\gamma > \bigcup_{k<\omega} \left(dom\ f_k^\beta \cup \ dom\ a_k^\beta \right)$. Let a_n be the order isomorphism between $dom\ a_n^\alpha \cup dom\ a_n^\beta \cup \{\gamma\}$ and $a_n' \cup \{\xi\}$. Let b_n be the order isomorphism between $dom\ a_n^\alpha \cup \ dom\ a_n^\beta \cup \{\gamma\}$ and $b_n' \cup \{\rho\}$. We define $p_n = \langle a_n, A_n^\alpha, f_n^\alpha \cup f_n^\beta \rangle$ and $q_n = \langle b_n, A_n^\alpha, f_n^\alpha \cup f_n^\beta \rangle$.

By the construction such defined p and q are equivalent. So we are done. $\square$.

Thus the forcing with $\langle \mathcal{P}^*, \rightarrow \rangle$ preserves λ. However, it is not hard to see that all the cardinals (if any) in the interval (κ^+, λ) are collapsed to κ^+. In any case, starting with $\lambda = \kappa^{++}$ and $\lambda_n = \kappa_n^{+n+2}$ $(n < \omega)$ we obtain the main result of [Git2]:

The forcing with $\langle \mathcal{P}^*, \rightarrow \rangle$ preserves the cardinals, does not add new bounded subsets to κ and makes $2^\kappa = \kappa^{++}$.

3 The Gap Three Case

The goal of this section will be to get $2^\kappa = \kappa^{+++}$ preserving κ^{++} and κ^{+++}.

The problem with the straightforward generalization of the forcing $\langle \mathcal{P}^*, \rightarrow \rangle$ of the previous section is that the Δ-system argument of Lemma 2.20 breaks down, once replacing κ^{++} by κ^{+++}. The point is as follows. Suppose that at some level $n < \omega$ we have an $\alpha \in (\kappa^{++}, \kappa^{+++})$ corresponding to some α^*. Let $cf\alpha = cf\alpha^* = \aleph_0$. Then there is a cofinal in α^* sequence $\langle \alpha_m^* \mid m < \omega \rangle$ simply definable from α^* (say, for example, the least cofinal sequence in the canonical well ordering). But now there are κ^{++} (and not κ^+ as in Section 2) different cofinal ω-sequences in α that may correspond to $\langle \alpha^*_m \mid m < \omega \rangle$. Clearly, different choices will provide incompatible conditions.

In order to overcome this difficulty, we will pick an elementary submodel A of cardinality κ^+ with α inside and correspond it to an elementary submodel A^* on the level n with α^* inside. We allow only elements of A to correspond to elements of A^*. This will restrict the number of choices to κ^+. The problem now will be how to choose and put together such models for different α's. This matter is handled generically using a preparation forcing $\mathcal{P}$ which will be κ^{++}-strategically closed. It is desired on one hand to shrink generically $\mathcal{P}^*$ in order to get κ^{++}-c.c. and on the the other hand to keep a large enough part of $\mathcal{P}^*$ in order to insure the Prikry condition.

The main issue will be the definition of such forcing $\mathcal{P}$. We start with a definition of a poset $\mathcal{P}'$ which will serve as a part of $\mathcal{P}$ over κ.

Definition 3.1

The set $\mathcal{P}'$ consists of elements of the form $\langle \langle A^{00}, A^{10} \rangle, \langle A^{01}, A^{11} \rangle \rangle$ so that the following holds

(1) $A^{0i} (i \in 2)$ is an elementary submodel of $\langle H(\kappa^{+3}), \in, \kappa, \kappa^+, \kappa^{++} \rangle$ such that

- (a) $|A^{0i}| = \kappa^{+i+1}, A^{0i} \supseteq \kappa^{+i+1}$
- (b) ${}^{\kappa^{+i}}A^{0i} \subseteq A^{0i}$
- (c) $A^{01} \cap \kappa^{+3}$ is an ordinal

(2) $A^{00} \prec A^{01}$

(3) for every $i < 2$
A^{1i} is a set of at most κ^{+i+1} elementary submodels of A^{0i} so that

- (a) $A^{0i} \in A^{1i}$
- (b) for every $B \in A^{11}$ $\quad B \cap \kappa^{+3}$ is an ordinal
- (c) if $B, C \in A^{1i}$ and $B \subsetneqq C$ then $B \in C$
- (d) if $B, C \in A^{11}$ then either $B = C$, $B \in C$ or $C \in B$
- (e) if $B, C \in A^{10}$, $B \neq C$, $B \not\subset C$ and $C \not\subset B$ then
  (i) $otp(B \cap \kappa^{+3}) = otp(C \cap \kappa^{+3})$ implies that $B \cap \kappa^{++} = C \cap \kappa^{++}$ and there are $D_B \in A^{11} \cap A^{00}$ and $D_C \in A^{11} \cap A^{00}$ so that

$$B \cap C = D_B \cap B = D_C \cap C$$

(ii) $otp(B\cap\kappa^{+3}) < otp(C\cap\kappa^{+3})$ implies that there are $B', C'\in A^{10}$ such that $B\in B'$, $C'\in C$ $otp(B'\cap\kappa^{+3}) = otp(C\cap\kappa^{+3}), otp(B\cap\kappa^{+3}) = otp(C'\cap\kappa^{+3})$, both pairs (B, C') and (B', C) satisfy (i), $B\cap C = B\cap C' = B'\cap C'$, $\langle B', B, <\rangle$ and $\langle C, C', <\rangle$ are isomorphic over $B'\cap C$.

(f) if $B\in A^{10}$ is a successor point of A^{10} then B has at most two immediate predecessors (under the inclusion) and is closed under κ-sequences

(g) if $B\in A^{10}$ then either B is a successor point of A^{10} or B is a limit element and there is a closed chain of $B\cap A^{10}$ unbounded in $B\cap A^{10}$

(h) A^{11} is a closed chain of models with successor points closed under κ^{+}-sequences, in particular $\{B\cap\kappa^{+++}\mid B\in A^{11}\}$ is a closed set of κ^{++} ordinals

(4) for every $B\in A^{11}$ there is $B'\in(A^{00}\cap A^{11})\cup\{H(\kappa^{+3})\}$ so that $B\cap A^{00} = B'\cap A^{00}$.

Let A^{10}_{in} be the set $\{B\cap B'\mid B\in A^{10}$ and $B'\in A^{11}\}$. By (4), then $A^{10}_{in} = \{B\cap B'|B\in A^{10}$ and $B'\in(A^{00}\cap A^{11})\cup\{H(\kappa^{+3})\}\}$.

Definition 3.2

Let $x = \langle\langle A^{00}, A^{10}\rangle, \langle A^{01}, A^{11}\rangle\rangle$, $y = \langle\langle B^{00}, B^{10}\rangle, \langle B^{01}, B^{11}\rangle\rangle$ be elements of $\mathcal{P}'$. Then $x\geq y$ iff for every $i<2$

(1) $A^{1i}\supseteq B^{1i}$

(2) for every $A\in A^{11}$ $\quad A\cap B^{01}\in B^{11}$.

(3) for every $A\in A^{10}$ $\quad A\cap B^{00}\in B^{10}\cup B^{10}_{in}$.

Definition 3.3

We define $\mathcal{P}'_{\geq 1} = \{\langle A^{01}, A^{11}\rangle|$ for some $\langle A^{00}, A^{10}\rangle\quad\langle\langle A^{00}, A^{10}\rangle, \langle A^{01}, A^{11}\rangle\rangle\in\mathcal{P}'\}$.

For a generic $G(\mathcal{P}'_{\geq 1})\subseteq\mathcal{P}'_{\geq 1}$ we define $\mathcal{P}'_{<1} = \{\langle A^{00}, A^{10}\rangle|$ there is $\langle A^{01}, A^{11}\rangle\in G(\mathcal{P}'_{\geq 1})$ $\langle A^{00}, A^{10}\rangle$, $\langle A^{01}, A^{11}\rangle\rangle\in\mathcal{P}'\}$.

The following two lemmas are obvious.

Lemma 3.4 $\mathcal{P}'\simeq\mathcal{P}'_{\geq 1}*\mathcal{P}'_{<1}$.

Lemma 3.5 *$\mathcal{P}'_{\geq 1}$ is κ^{+3}-closed forcing.*

It is actually isomorphic to the Cohen forcing for adding a new subset to κ^{+3}.

Lemma 3.6 *$\mathcal{P}'$ is κ^{++}-closed.*

Proof. Let $x_\alpha = \langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha\rangle\rangle \in \mathcal{P}'$, $\alpha < \alpha^* < \kappa^{++}$. Suppose that $x_\alpha < x_{\alpha+1}$ for every $\alpha < \alpha^*$. We define $x_{\alpha^*} > x_\alpha$ for all $\alpha < \alpha^*$ as follows. Let $A^{00}_{\alpha^*}$ be the closure of $\bigcup_{\alpha<\alpha^*} A^{00}_\alpha \cup \{\langle A^{11}_\alpha \mid \alpha < \alpha^*\rangle\}$ under κ-sequences and Skolem functions. Set $A^{01}_{\alpha^*}$ to be an elementary submodel of $H(\kappa^{+3})$ including $\langle A^{01}_\alpha \mid \alpha < \alpha^*\rangle$, $A^{00}_{\alpha^*}$, closed under κ^+-sequences and having the intersection with κ^{+3} an ordinal.

Let $A^{1i}_{\alpha^*} = \bigcup_{\alpha<\alpha^*} A^{1i}_\alpha \cup \{A^{0i}_{\alpha^*}\} \cup \{\bigcup_{\alpha<\alpha^*} A^{0i}_\alpha\}$ for $i < 2$. We need to check that $x_{\alpha^*} = \langle\langle A^{00}_{\alpha^*}, A^{10}_{\alpha^*}\rangle, \langle A^{01}_{\alpha^*}, A^{11}_{\alpha^*}\rangle\rangle$ is in $\mathcal{P}'$ and is stronger than each x_α for $\alpha < \alpha^*$. Most of the conditions are trivial. Let us check only 3.1(4). Thus let $B \in A^{11}_{\alpha^*} \backslash A^{00}_{\alpha^*}$. We need to find $B' \in (A^{00}_{\alpha^*} \cap A^{11}_{\alpha^*}) \cup \{H(\kappa^{+3})\}$ so that $B \cap A^{00}_{\alpha^*} = B' \cap A^{00}_{\alpha^*}$. If $B = A^{01}_{\alpha^*}$ then we take $B' = H(\kappa^{+3})$. Now suppose that $B \in \bigcup_{\alpha<\alpha^*} A^{1i}_\alpha$. If $B \cap \kappa^{+3} \geq \sup(A^{00}_{\alpha^*} \cap \kappa^{+3})$ then we can take again $B' = H(\kappa^{+3})$. Suppose that $B \cap \kappa^{+3} < \sup(A^{00}_{\alpha^*} \cap \kappa^{+3})$. Let $\delta \in A^{00}_{\alpha^*} \cap \kappa^{+3}$ be the minimal above $B \cap \kappa^{+3}$. Recall that $E = \{D \cap \kappa^{+3} \mid D \in A^{11}_{\alpha^*}\}$ is a closed set of ordinals by 3.1(h). Also, $E_1 = E \backslash \{max(E)\} \in A^{00}_{\alpha^*}$. But then $A^{00}_{\alpha^*} \models (E_1$ is unbounded in $\delta)$. Hence $\delta \in E_1$. So there is $B' \in \bigcup_{\alpha<\alpha^*} A^{11}_\alpha \cap A^{00}_{\alpha^*}$ $B' \cap \kappa^{+3} = \delta$. Then $B' \cap A^{00}_{\alpha^*} = A^{00}_{\alpha^*} \cap \delta = A^{00}_{\alpha^*} \cap B$.
□

The following observation will be crucial for proving κ^{++}-c.c. of the final forcing.

Lemma 3.7 *Suppose that $\langle\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha\rangle\rangle \mid \alpha \leq \alpha^*\rangle$ is an increasing sequence of elements of $\mathcal{P}'$. Assume that $\bigcup_{\beta<\alpha} A^{00}_\beta \in A^{10}_\alpha$ for every $\alpha \leq \alpha^*$. Let $B \in A^{10}_{\alpha^*}$ and $otp(B \cap \kappa^{+3}) < otp(A^{00}_0 \cap \kappa^{+3})$. Then the set $\{B \cap A^{00}_\alpha \mid \alpha < \alpha^*\}$ is finite.*

Proof. Suppose otherwise. We pick the least $\rho \leq \alpha^*$ so that for some $B' \in A^{10}_\rho$ with $otp(B' \cap \kappa^{+3}) < otp(A^0_0 \cap \kappa^{+3})$ the set $\{B' \cap A^{00}_\alpha \mid \alpha < \rho\}$ is infinite. Notice that $B' \notin \bigcup_{\alpha<\rho} A^{00}_\alpha$. Since otherwise $B' \in A^{00}_{\rho'}$ for some $\rho' < \rho$ and then $B' \subseteq A^{00}_\alpha$ for all $\alpha, \rho' \leq \alpha < \rho$. So, $|\{B' \cap A^{00}_\alpha \mid \alpha < \rho'\}| \geq \aleph_0$ which contradicts the minimality of ρ. Let $C = \bigcup_{\alpha<\rho} A^{00}_\alpha$. Then $C \in A^{10}_\rho$. Now, both sets $C \backslash B'$ and $B' \backslash C$ are nonempty. The first one since $otp(C \cap \kappa^{+3}) > otp(B' \cap \kappa^{+3})$. The second one since $B' \subsetneqq C$ implies $B' \in C$ by 3.1(3(c)) but we just showed that $B' \notin C$. Then 3.1(e(ii)) applies. So there is $C' \in C \cap A^{10}_\rho$ with $otp(C' \cap \kappa^{+3}) = otp(B' \cap \kappa^{+3})$ such that $B' \cap C = B' \cap C'$ and the pair (B', C') satisfies 3.1(e(i)). Hence $B' \cap C \cap \kappa^{+3} = B' \cap C' \cap \kappa^{+3} = C' \cap D_{C'} \cap \kappa^{+3} = B' \cap D_{B'} \cap \kappa^{+3}$ where $D_{B'}, D_{C'}$ witness 3.1(e(i)). Now, for every $\alpha < \rho$ $B' \cap A^{00}_\alpha \cap \kappa^{+3} = (B' \cap C) \cap A^{00}_\alpha \cap \kappa^{+3} = B' \cap C' \cap A^{00}_\alpha \cap \kappa^{+3} = C' \cap D_{C'} \cap A^{00}_\alpha \cap \kappa^{+3} = (C' \cap A^{00}_\alpha) \cap D_{C'} \cap \kappa^{+3}$. This implies that the set $\{C' \cap A^{00}_\alpha \mid \alpha < \rho\}$ is infinite. But $C' \in C = \bigcup_{\alpha<\rho} A^{00}_\alpha$. So, as above we can now reduce ρ. Contradiction.

$\square$

Lemma 3.8 *Suppose that $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha\rangle\rangle \mid \alpha \leq \alpha^*\rangle$ is an increasing sequence of elements of $\mathcal{P}'$ with α^* a limit ordinal. Let $B \in A^{10}_{\alpha^*}$. Then there is $\overline{\alpha} < \alpha^*$ satisfying the following:*

(1) if $otp(B \cap \kappa^{+3}) < otp(A^{00}_\alpha \cap \kappa^{+3})$ for some $\alpha < \alpha^*$ and $\bigcup_{\beta<\alpha} A^{00}_\beta \in A^{10}_\alpha$ for every $\alpha \leq \alpha^*$ then $otp(B \cap \kappa^{+3}) < otp(A^{00}_{\overline{\alpha}} \cap \kappa^{+3})$ and there are $\overline{B} \in A^{00}_{\overline{\alpha}} \cap A^{10}_{\overline{\alpha}}$, $D_{\overline{B}} \in A^{00}_{\overline{\alpha}} \cap A^{11}_{\overline{\alpha}}$ so that for every $\alpha, \alpha^* > \alpha \geq \overline{\alpha}$

$$B \cap A^{00}_\alpha = \overline{B} \cap D_{\overline{B}}$$

(2) if for every $\alpha < \alpha^*$ $otp(B \cap \kappa^{+3}) > otp(A^{00}_\alpha \cap \kappa^{+3})$ then there is $D \in (A^{00}_{\overline{\alpha}} \cap A^{11}_{\overline{\alpha}}) \cup \{H(\kappa^{+3})\}$ so that for every α, $\alpha^* > \alpha \geq \overline{\alpha}$

$$B \cap A^{00}_\alpha = D \cap A^{00}_\alpha$$

Remark 3.8.1 Notice that for $B, C \in A^{10}_{\alpha^*}$ if $B \cap \kappa^{++} < C \cap \kappa^{++}$ then $otp(B \cap \kappa^{+3}) < otp(C \cap \kappa^{+3})$ by 3.1(e).

Proof.

(1) By Lemma 3.7, there is $\overline{\alpha} < \alpha^*$ so that for every $\alpha, \overline{\alpha} \leq \alpha < \alpha^*$ $\quad B \cap A^{00}_\alpha = B \cap A^{00}_{\overline{\alpha}}$. Using 3.2(3) for α^* and $\overline{\alpha}$ we find $\overline{B} \in A^{00}_{\overline{\alpha}} \cap A^{10}_{\overline{\alpha}}$ and $D_{\overline{B}} \in A^{00}_{\overline{\alpha}} \cap A^{11}_{\overline{\alpha}}$ so that $B \cap A^{00}_{\overline{\alpha}} = \overline{B} \cap D_{\overline{B}}$.

(2) If $B \supseteq \bigcup_{\alpha<\alpha^*} A^{00}_\alpha$, then use $D = H(\kappa^{+3})$. Suppose that $B \not\supseteq \bigcup_{\alpha<\alpha^*} A^{00}_\alpha$. Pick then the least $\delta \in \left(\bigcup_{\alpha<\alpha^*} A^{00}_\alpha \setminus B\right) \cap \kappa^{+3}$. Let $\overline{\alpha} < \alpha^*$ be the least such that $\delta \in A^{00}_{\overline{\alpha}}$. Clearly, $B \supseteq \bigcup_{\alpha<\alpha^*} A^{00}_\alpha \cap \delta$. Also, by 3.1(3e(ii)) $\delta > \kappa^{++}$.

Claim For every α, $\overline{\alpha} \leq \alpha < \alpha^*$ $B \cap A^{00}_\alpha \cap \kappa^{+3} = A^{00}_\alpha \cap \delta$.

Proof. Suppose otherwise. Then there are $\widetilde{\alpha}$, $\overline{\alpha} \leq \widetilde{\alpha} < \alpha^*$ and $\xi \in B \cap A^{00}_{\widetilde{\alpha}} \setminus \delta$. Consider the least $f_\xi : \kappa^{++} \leftrightarrow \xi$. It belongs to $B \cap A^{00}_{\widetilde{\alpha}}$. Now, $\delta \in A^{00}_{\widetilde{\alpha}} \cap \xi$, so there is $\nu \in A^{00}_{\widetilde{\alpha}} \cap \kappa^{++}$ such that $f_\tau(\nu) = \delta$. But $B \supseteq A^{00}_{\widetilde{\alpha}} \cap \kappa^{++}$, so $\nu \in B$ and hence also $\delta \in B$. Contradiction.

$\square$ of the claim.

Now, we apply 3.2(3) to $B \in A^{10}_{\alpha^*}$ and $A^{00}_{\overline{\alpha}}$. It implies that $B \cap A^{00}_{\overline{\alpha}} \in A^{10}_{\overline{\alpha}} \cup A^{10}_{in}$. Notice that $B \cap \kappa^{++} > A^{00}_{\overline{\alpha}} \cap \kappa^{++}$. Hence $B \cap A^{00}_{\overline{\alpha}} \notin A^{10}_{\overline{\alpha}}$. So $B \cap A^{00}_{\overline{\alpha}} \in A^{10}_{in}$. It implies that for some $D \in A^{00}_{\overline{\alpha}} \cap A^{11}_{\overline{\alpha}}$ $\quad B \cap A^{00}_{\overline{\alpha}} = A^{00}_{\overline{\alpha}} \cap D$.

But $D \cap \kappa^{+++}$ is an ordinal. Hence, it should be exactly δ and we are done. $\square$

Lemma 3.8.1. *Suppose that* $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha\rangle\rangle \mid \alpha \leq \alpha^*\rangle$ *be an increasing sequence of elements of* $\mathcal{P}'$ *with* α^* *a limit ordinal of uncountable cofinality. Let* $B \in A^{10}_{\alpha^*}$ *be closed under* ω*-sequence of its elements and* $otp(B \cap \kappa^{+3}) > otp(A^{00}_\alpha \cap \kappa^{+3})$ *then there are* $\overline{\alpha} < \alpha^*$ *and* $D \in (A^{00}_{\overline{\alpha}} \cap A^{11}_{\overline{\alpha}}) \cup \{H(\kappa^{+3}\}$ *so that*

(a) for every α, $\alpha^* > \alpha \geq \overline{\alpha}$

$$B \cap A^{00}_\alpha = D \cap A^{00}_\alpha$$

(b) for every $\beta \in B \cap \kappa^{+3} \backslash \sup(B \cap A^{00}_{\overline{\alpha}} \cap \kappa^{+3})$ *if there is* $\gamma \in \left(\bigcup_{\alpha<\alpha^*} A^{00}_\alpha \cap \kappa^{+3}\right) \backslash \beta$ *then the least such* γ *is in* $A^{00}_{\overline{\alpha}}$.

The proof is an easy application of 3.8 and fact that above $\sup(B \cap A^{00}_{\overline{\alpha}} \cap \kappa^{+3})$ only finitely many overlaps between $B \cap \kappa^{+3}$ and $\bigcup_{\alpha<\alpha^*} A^{00}_\alpha \cap \kappa^{+3}$ are possible since ${}^\omega B \subseteq B$.

We are not going to force with $\langle \mathcal{P}', \leq\rangle$, so the next lemma is not needed for the main results but we think that it contributes to the understanding of the main forcing and it will be used also in the proof for the main forcing.

Lemma 3.9 $\mathcal{P}'_{<1}$ *satisfies* κ^{+3}*-c.c. in* $V^{\mathcal{P}'_{\geq 1}}$.

Proof. Suppose otherwise. Let us assume that

$$\emptyset \Vdash_{\mathcal{P}'_{\geq 1}} \langle\langle \underset{\sim}{A}^{00}_\alpha, \underset{\sim}{A}^{10}_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle \quad \text{is an antichain in} \quad \mathcal{P}'_{<1}\ .$$

Define by induction an increasing sequence of conditions of $\mathcal{P}'_{\geq 1}$ $\langle\langle A^{01}_\alpha, A^{11}_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle$ and a sequence $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle$ so that for every $\alpha < \kappa^{+3}$ $\langle A^{01}_\alpha, A^{11}_\alpha\rangle \Vdash_{\mathcal{P}'_{\geq 1}}$ $''\langle \underset{\sim}{A}^{00}_\alpha, \underset{\sim}{A}^{10}_\alpha\rangle = \langle \check{A}^{00}_\alpha, \check{A}^{10}_\alpha\rangle''$. $\mathcal{P}'_{\geq 1}$ does not add new sets of size κ^{++}, so there is no problem with the induction.

Let $A^{10}_\alpha = \{X_{\alpha i} \mid i < \kappa^+\}$ for all $\alpha < \kappa^{+3}$.

We now form a Δ-system from $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle$. Thus we can insure that the following holds for some $\delta < \kappa^{+3}$, for every $\beta < \alpha < \kappa^{+3}$:

(a) $A^{00}_\alpha \cap \delta = A^{00}_\beta \cap \delta, min(A^{00}_\alpha \backslash \delta) \geq \alpha > A^{01}_\beta \cap \kappa^{+3} + \sup(A^{00}_\beta \cap \kappa^{+3})$.

(b) the function taking $X_{\alpha i}$ to $X_{\beta i}$ $(i < \kappa^+)$ is an isomorphism between the structures $\langle A^{01}_\alpha, \supset\rangle$ and $\langle A^{01}_\beta, \supset\rangle$

(c) $\langle A^{00}_\alpha, <\rangle \simeq \langle A^{00}_\beta, <\rangle$ by isomorphism π so that $\pi \restriction A^{00}_\alpha \cap \delta = id$ and $\pi'' X_{\alpha i} = X_{\beta i}$ for every $i < \kappa^+$.

Let us now show that $\langle A^{00}_\alpha, A^{10}_\alpha\rangle$ and $\langle A^{00}_\beta, A^{10}_\beta\rangle$ are compatible in $\mathcal{P}'_{<1}$ for $\alpha < \beta < \kappa^{+3}$. It is enough to prove compatibility of $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha\rangle\rangle$ and $\langle\langle A^{00}_\beta, A^{10}_\beta\rangle, \langle A^{01}_\beta, A^{11}_\beta\rangle\rangle$ in $\mathcal{P}'$. We define a condition $\langle\langle A^{00}, A^{10}\rangle, \langle A^{01}, A^{11}\rangle\rangle$ stronger than both of the conditions above. Take $A^{01} = A^{01}_\alpha$ and $A^{11} = A^{11}_\alpha$. Let A^{00} be an elementary submodel of cardinality κ^+ including $\{A^{00}_\alpha, A^{00}_\beta, A^{10}_\alpha, A^{10}_\beta, A^{01}_\alpha, A^{11}_\alpha\}$ and closed under κ-sequences. Set $A^{10} = A^{10}_\alpha \cup A^{10}_\beta \cup \{A^{00}\}$. Let $\nu_\alpha = min(A^{00}_\alpha \backslash \delta)$ and $\nu_\beta = min(A^{00}_\beta \backslash \delta)$.

Claim 3.9.1. There are $D_\alpha \in A^{11}_\alpha \cap A^{00}_\alpha$ and $D_\beta \in A^{11}_\beta \cap A^{00}_\beta$ so that $D_\alpha \cap \kappa^{+3} = \nu_\alpha$ and $D_\beta \cap \kappa^{+3} = \nu_\beta$.

Proof. Let us show this for α. The same argument will work also for β. We use 3.1(4) for $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha\rangle\rangle$. Thus $A^{01}_0 \in A^{11}_\alpha$, so $A^{01}_0 \cap A^{00}_\alpha = D_\alpha \cap A^{00}_\alpha$ for some $D_\alpha \in A^{11}_\alpha \cap A^{00}_\alpha$. But then $D_\alpha \cap \kappa^{+3}$ should be ν_α since $\delta < A^{01}_0 \cap \kappa^{+3} < \alpha \leq \nu_\alpha \in A^{00}_\alpha$ and $(A^{00}_\alpha \cap \kappa^{+3}) \cap (\delta, \nu_\alpha) = \emptyset$. $\square$ of the claim

Let us check that $\langle\langle A^{00}, A^{01}\rangle, \langle A^{01}, A^{11}\rangle\rangle$ is a condition. The only problematic cases are 3.1(3e), (4). Thus let B, C be as in 3.1(3e)(i). Then $B \in A^{00}_\alpha \cup \{A^{00}_\alpha\}$ and $C \in A^{00}_\beta \cup \{A^{00}_\beta\}$. If $B = A^{00}_\alpha$ (or $C = A^{00}_\beta$) then $C = A^{00}_\beta$ (or $B = A^{00}_\alpha$). So, $B \cap C = B \cap D_\alpha = C \cap D_\beta$ by the claim. Suppose otherwise. Find $i < \kappa^{+3}$ such that $B = X_{\alpha i}$. Consider $B' = X_{\beta i}$. Then $B' \in A^{10}_\beta$. We apply 3.1(3(e)(i)) to B' and C inside A^{10}_β. There are $D_{B'} \in A^{11}_\beta \cap A^{00}_\beta$ and $D_C \in A^{11}_\beta \cap A^{00}_\beta \cap A^{00}_\beta$ so that $B' \cap C = D_{B'} \cap B' = D_C \cap C$. Also $B' \cap \kappa^{++} = C \cap \kappa^{++}$. But $\delta > \kappa^+$ and the isomorphism π of (c) is the identity on δ. So, $B \cap \kappa^{++} = B' \cap \kappa^{++} = C \cap \kappa^{++} B \cap C \cap \kappa^{+3} = B \cap C \cap \delta = (B' \cap \delta) \cap C = (B' \cap \delta) \cap C \cap \delta = B' \cap C \cap \delta = C \cap D_C \cap \delta$. Now, if $D_C \cap \kappa^{+3} \geq \delta$, then $D_C \cap \delta = \delta$ and hence $C \cap D_C \cap \delta = C \cap D_\beta$. If $D_C \cap \kappa^{+3} < \delta$, then $C \cap D_C \cap \delta = C \cap D_C \cap \kappa^{+3}$. Hence $B \cap C = C \cap D_\beta$ or $B \cap C = C \cap D_C$. In order to find $D \in A^{11}_\alpha \cap A^{00}_\alpha$ so that $B \cap C = B \cap D$ we just repeat the argument first picking $C' = X_{\alpha j}$ and working inside A^{10}_α, where $C = X_{\beta j}$.

Let us now check 3.1(e)(ii). Assume that B, C are as above but $otp(B \cap \kappa^{+3}) < otp(C \cap \kappa^{+3})$. We find $i, j < \kappa^+$ so that $B = X_{\alpha i}$ and $C = X_{\beta j}$. Let $\widetilde{B} = X_{\alpha j}$. Then $\widetilde{B} \in A^{10}_\alpha$ and $otp(\widetilde{B} \cap \kappa^{+3}) = otp(C \cap \kappa^{+3}) > otp(B \cap \kappa^{+3})$. Apply 3.1(e(ii)) to B, $\widetilde{B}$. There will be $B', \tilde{B}' \in A^{10}_\alpha$, $B \in B', \tilde{B}' \in \widetilde{B}$ witnessing 3.1(e(ii)) for $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha\rangle\rangle$. Find i', j' such that $\overline{B}' = X_{\alpha i'}$ and $\tilde{B}' = X_{\alpha j'}$. Set $C' = X_{\beta j'}$. Let us show that such B', C' are as desired. First notice that $otp(B' \cap \kappa^{+3}) = otp(\widetilde{B} \cap \kappa^{+3}) = otp(X_{\alpha j} \cap \kappa^{+3}) = otp(X_{\beta j} \cap \kappa^{+3}) = otp(C \cap \kappa^{+3})$ by condition (b) on Δ-system. Similar, $otp(B \cap \kappa^{+3}) = otp(C' \cap \kappa^{+3})$. Now, $B \cap C = B \cap C \cap \delta = B \cap \widetilde{B} \cap \delta$. By (a) and since the isomorphism π of (c) is identity on δ. We continue, $B \cap \widetilde{B} \cap \delta = B \cap \tilde{B}' \cap \delta = \tilde{B}' \cap B' \cap \delta$ by the choice of $\tilde{B}', B'$. Again by the choice of π we obtain $B \cap \tilde{B}' \cap \delta = B \cap C' \cap \delta$ and $\tilde{B}' \cap B' \cap \delta = B' \cap C' \cap \delta$. By (a) we have $B \cap C' \cap \delta = B \cap C'$ and $B' \cap C' \cap \delta = B' \cap C'$. Hence $B \cap C = B \cap C' = B' \cap C'$. Also $\langle B', B, < \rangle$ and $\langle C, C', < \rangle$ are order isomorphic over $B' \cap C$. Since $\langle B', B, < \rangle, \langle \widetilde{B}, \tilde{B}', < \rangle$ are order isomorphic over$B' \cap \widetilde{B}, \langle \widetilde{B}, \tilde{B}', < \rangle$, $\langle C, C' \ < \rangle$ are order isomorphic over δ and $B' \cap C \cap \kappa^{+3} = B' \cap C \cap \delta$.

Let us now check 3.1(4). Suppose that $B \in A^{11} = A^{11}_\alpha$. Let $\nu = B \cap \kappa^{+3}$. If $\nu \geq \sup A^{00}$ then it is trivial. Suppose otherwise. Let $\rho = min A^{00} \backslash \nu$. We picked A^{00} so that $A^{11}_\alpha \in A^{00}$. Hence, $\{D \cap \kappa^{+3} \mid D \in A^{11}_\alpha\}$ is unbounded in ρ and so for some $D_\rho \in A^{11}_\alpha$ $D_\rho \cap \kappa^{+3} = \rho$. Then $D_\rho \in A^{00}$ (it is uniquely determined by its ordinal part). Hence $B \cap A^{00} = D_\rho \cap A^{00}$ and we are done.

Finally let us show that $\langle \langle A^{00}, A^{10} \rangle, \langle A^{00}, A^{11} \rangle \rangle$ is stronger than $\langle \langle A^{00}_\alpha, A^{10}_\alpha \rangle, \langle A^{01}_\alpha, A^{11}_\alpha \rangle \rangle$ and $\langle \langle A^{00}_\beta, A^{10}_\beta \rangle, \langle A^{01}_\beta, A^{11}_\beta \rangle \rangle$. Let us show this for α. We need only to check 3.2(3). Thus let $A \in A^{10} = A^{10}_\alpha \cup A^{10}_\beta \cup \{A^{00}\}$. If $A \in A_\alpha$ then $A \cap A^{00}_\alpha = A \in A^{10}_\alpha$. If $A = A^{00}$ then $A \cap A^{00}_\alpha = A^{00}_\alpha \in A^{10}_\alpha$. Suppose now that $A \in A^{10}_\beta$. Then $A \cap A^{00}_\alpha = A \cap \delta$. Let $A = X_{\beta i}$ for some $i < \kappa^+$. Consider $A^* = X_{\alpha i}$. Then $A \cap \delta = A^* \cap \delta$. Let D_α be as in the claim. It follows that $A \cap A^{00}_\alpha = A \cap \delta = A^* \cap \delta = A^* \cap D_\alpha$. But $D_\alpha \in A^{11}_\alpha$ and $A^* \in A^{10}$, so $A^* \cap D_\alpha \in A^{10}_{\alpha in}$. Hence we are done. □

Let us now define our main preparation forcing.

Definition 3.10

The set $\mathcal{P}$ consists of pairs of triples $\langle \langle A^{0\tau}, A^{1\tau}, F^\tau \rangle \mid \tau < 2 \rangle$ so that the following holds:

(0) $\langle \langle A^{00}, A^{10} \rangle, \langle A^{01}, A^{11} \rangle \rangle \in \mathcal{P}'$

(1) $F^0 \subseteq F^1 \subseteq \mathcal{P}^*$, where $\langle \mathcal{P}^*, \leq, \leq^* \rangle$ is as defined in Section 2

(2) for every $\tau < 2 \quad F^\tau$ is as follows:

  (a) $|F^\tau| = \kappa^{+\tau+1}$

  (b) for every $p = \langle p_n \mid n < \omega \rangle \in F^\tau$ if $n < \ell(p)$ then every α appearing in p_n is in $A^{0\tau} \cap \kappa^{+3}$; if $n \geq \ell(p)$, and $p_n = \langle a_n, A_n, f_n \rangle$, then every α appearing in f_n is in $A^{0\tau} \cap \kappa^{+3}$ and $dom\ a_n \subseteq (A^{01} \cap \kappa^{+3}) \cup A^{11} \cup \{B \prec A^{01} \mid |B| = \kappa^+\}$ if $\tau = 1$, $dom\ a_n \subseteq (A^{00} \cap \kappa^{+3}) \cup A^{10} \cup A^{10}_{in}$ if $\tau = 0$. We also require that every nonordinal member B of $dom\ a_n$ is closed under κ sequences if $|B| = \kappa^+$ and κ^+-sequences if $|B| = \kappa^{++}$.

  Let $\ell(p) \leq n < \omega$ and $p_n = \langle a_n, A_n, f_n \rangle$.

(c) there is an element of *dom* a_n, maximal under inclusion, it belongs to $A^{1\tau}$ or to $A^{1\tau}_{in} \cup A^{1\tau}$ if $\tau = 0$ and every other element of *dom* a_n maximal under inclusion belongs to it. Let us further denote this element as $max^1(a_n)$ or $max^1(p_n)$.

(d) if $B \in$ *dom* $a_n \backslash On$, then $a_n(B)$ is an elementary submodel of $\mathfrak{a}_{n,k_n}$ of Section 2 with $3 \leq k_n \leq n$. We require that $|a_n(B)| = \kappa_n^{+n+\tau'+1}$ and ${}^{\kappa_n^{+n+\tau'}}(a_n(B)) \subseteq a_n(B)$ whenever $|B| = \kappa^{+\tau'+1}$ ($\tau' < 2$).

(e) for every $B \in$ *dom* $a_n \backslash \kappa^{+3}$ and $\alpha \in$ *dom* $a_n \cap \kappa^{+3}$ $\quad \alpha \in B$ iff $a_n(\alpha) \in a_n(B)$

(f) for every $B, C \in$ *dom* $a_n \backslash \kappa^{+3}$

(f1) $B \in C$ iff $a_n(B) \in a_n(C)$

(f2) $B \backslash C \neq \emptyset$ and $C \backslash B \neq \emptyset$ iff $a_n(B) \backslash a_n(C) \neq \emptyset$ and $a_n(C) \backslash a_n(B) \neq \emptyset$. If this happens then the positions of B, C and $a_n(B), a_n(C)$ are the same, i.e. 3.1(e(i),(ii)) holds simultaneously for both of the pairs.

The next two conditions deal with cofinalities of correspondence:

(g)(i) if $\alpha \in$ *dom* $a_n \cap \kappa^{+3}$ and $cf\alpha \leq \kappa^+$ then $a_n(\alpha) < \kappa_n^{+n+3}$ and $cf\ a_n(\alpha) \leq \kappa_n^{+n+1}$

(g)(ii) if $\alpha \in$ *dom* $a_n \cap \kappa^{+3}$ and $cf\alpha = \kappa^{++}$ then $a_n(\alpha) < \kappa_n^{+n+3}$. and $cf a_n(\alpha) = \kappa_n^{+n+2}$.

(h) if $p \in F^\tau$ and $q \in \mathcal{P}^*$ is equivalent to p (i.e. $p \leftrightarrow q$ as in Section 2) with witnessing sequence $\langle k_n \mid n < \omega \rangle$ starting with $k_0 \geq 4$, then $q \in F^\tau$.

(i) if $p = \langle p_n \mid n < \omega \rangle \in F^\tau$ and $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}^*$ are such that

(i) $\ell(p) = \ell(q)$

(ii) for every $n < \ell(p)$ $p_n = q_n$

(iii) for every $n \geq \ell(p)$ $a_n = b_n$ and *dom* $g_n \subseteq A^{0\tau} \cap \kappa^{+3}$, where $p_n = \langle a_n, A_n, f_n \rangle$, $q_n = \langle b_n, B_n, g_n \rangle$

then $q \in F^\tau$.

The meaning of the last two conditions is that we are free to change (remaining inside $A^{0\tau}$) all the components of p except a_n's.

(k) for every $q \in F^\tau$ and $\alpha \in A^{0\tau} \cap \kappa^{+3}$ there is $p \in F^\tau$ $\quad p = \langle p_n \mid n < \omega \rangle$, $p_n = \langle a_n, A_n, f_n \rangle$ ($n \geq \ell(p)$) such that $p \geq^* q$ and $\alpha \in$ *dom* a_n starting with some $n_0 < \omega$.

(l) for every $q \in F^\tau$ and $B \in A^{11}, {}^{\kappa^+}B \subseteq B$ if $\tau = 1$ or $B \in A^{10} \cup A^{10}_{in}$ and ${}^{\kappa}B \subseteq B$ if $\tau = 0$, there is $p \in F^\tau$ $p = \langle p_n \mid n < \omega\rangle$, $p_n = \langle a_n, A_n, f_n\rangle$ $(n \geq \ell(p))$ such that $p \geq^* q$ $B \in$ *dom* a_n starting with some $n_0 < \omega$ and p is obtained from q by adding only B and the ordinals needed to be added after adding B.

(m) if $p \in F^0$, $B, C \in$ *dom* $a_n \backslash \kappa^{+3}$, $(n \geq \ell(p))$ and C is an initial segment of B then $a_n(C)$ is an initial segment of $a_n(B)$.

The next condition provides a degree of closedness needed for the proof of the Prikry condition of the main forcing.

(n) there is $F^{\tau *} \subseteq F^\tau$ dense in F^τ under $\leq^*$ such that every $\leq^*$-increasing sequence of elements of $F^{\tau *}$ having upperbound in $\mathcal{P}^*$ has it also in $F^{\tau *}$.

Our last conditions will be essential for proving κ^{++}-c.c. of the main forcing.

(o) let $p, q \in F^\tau$ be so that

- (i) $\ell(p) = \ell(q)$
- (ii) $max^1(p_n) = max^1(p_m)$, $max^1(q_n) = max^1(q_m)$ and $max^1(q_n) \in$ *dom* a_n, where $n, m \geq \ell(p)$, $p_n = \langle a_n, A_n, f_n\rangle$, $q_n = \langle b_n, B_n, g_n\rangle$
- (iii) $p_n = q_n$ for all $n < \ell(p)$
- (iv) f_n, g_n are compatible for every $n \geq \ell(p)$
- (v) $a_n \restriction max^1(q_n) \subseteq b_n$ for every $n \geq \ell(p)$, where $a_n \restriction B = \{\langle t \cap B, s \cap a_n(B)\rangle \mid \langle t, s\rangle \in a_n\}$

then the union of p and q is in F^τ, where the union is defined in obvious fashion taking $p_n \cup q_n$ for $n < \ell(p)$, $a_n \cup b_n$, $f_n \cup g_n$ etc. for $n \geq \ell(p)$.

(p) let $p = \langle p_n \mid n < \omega\rangle \in F^\tau$ and for every $n, \omega > n \geq \ell(p)$ let $B \in$ *dom* $a_n \backslash \kappa^{+3}$ where $p_n = \langle a_n, A_n, f_n\rangle$ then $p \restriction B \in F^\tau$, where $p \restriction B = \langle p_n \restriction B \mid n < \omega\rangle$ and for every $n < \ell(p)$ $p_n \restriction B$ is the usual restriction of the function p_n to B; if $n \geq \ell(p)$ then $p_n \restriction B = \langle a_n \restriction B, B_n, f_n \restriction B\rangle$, with $a_n \restriction B$ defined in (o)(v), $f_n \restriction B$ the usual restriction and B_n is the projection of A_n by $\pi_{max(p_n),B}$.

(q) let $p = \langle p_n \mid n < \omega\rangle \in F^\tau$, $p_n = \langle a_n, A_n, f_n\rangle$ and $A^{0\tau} \notin$ *dom* a_n $(\omega > n \geq \ell(p))$. Let $\langle \sigma_n \mid \omega > n \geq \ell(p)\rangle$ be so that

- (i) $\sigma_n \prec \mathfrak{a}_{n,k_n}$ for every $n \geq \ell(p)$

(ii) $\langle k_n \mid n \geq \ell(p)\rangle$ is increasing

(iii) $k_0 \geq 5$

(iv) ${}^{\kappa_n}\sigma_n \subseteq \sigma_n$ for every $n \geq \ell(p)$

(v) $rnga_n \in \sigma_n$ for every $n \geq \ell(p)$.

Then the condition obtained from p by adding $\langle A^{00}, \sigma_n\rangle$ to each p_n with $n \geq \ell(p)$ belongs to F^τ.

Definition 3.11

Let $\langle A^{0\tau}, A^{1\tau}, F^\tau \mid \tau \leq 1\rangle$ and $\langle\langle B^{0\tau}, B^{1\tau}, G^\tau\rangle \mid \tau \leq 1\rangle$ be in $\mathcal{P}$. We define $\langle\langle A^{0\tau}, A^{1\tau}, F^\tau\rangle \mid \tau \leq 1\rangle\rangle > \langle\langle B^{0\tau}, B^{1\tau}, G^\tau \mid \tau \leq 1\rangle$ iff

(1) $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq 1\rangle\rangle > \langle\langle B^{0\tau}, B^{1\tau} \mid \tau \leq 1\rangle$ in $\mathcal{P}'$

(2) for every $\tau \leq 1$

(a) $F^\tau \supseteq G^\tau$

(b) for every $p \in F^\tau$ and $B \in B^{11}$ (if $\tau = 1$) or $B \in B^{10} \cup B^{10}_{in}$ (if $\tau = 0$), if for every $n \geq \ell(p)$ $B \in$ $dom\ a_n$ then $p \restriction B \in G^\tau$, where the restriction is defined as in 3.10(p), $p = \langle p_n \mid n < \omega\rangle$ $p_n = \langle a_n, A_n, f_n\rangle$ for $n \geq \ell(p)$.

Definition 3.12

Set $\mathcal{P}_{\geq 1} = \{\langle A^{01}, A^{10}, F^1\rangle \mid \exists < A^{00}, A^{10}, F^0\rangle\ \langle\langle A^{00}, A^{10}, F^0\rangle, \langle A^{01}, A^{11}, F^1\rangle\rangle \in \mathcal{P}\}$.

Let $G(\mathcal{P}_{\geq 1}) \subseteq \mathcal{P}_{\geq 1}$ be generic. Define $\mathcal{P}_{<1} = \{\langle A^{00}, A^{10}, F^0\rangle \mid \exists \langle A^{01}, A^{11}, F^1\rangle \in G(\mathcal{P}_{\geq 1})\langle\langle A^{00}, A^{10}, F^0\rangle, \langle A^{01}, A^{11}, F^1\rangle\rangle \in \mathcal{P}\}$.

The following lemma is obvious.

Lemma 3.13 $\mathcal{P} \simeq \mathcal{P}_{\geq 1} * \underset{\sim}{\mathcal{P}}_{<1}$.

Let μ be a cardinal. Consider the following game $\mathcal{G}_\mu$

$$\begin{array}{ccccccccc} I & p_0 & & p_2 & \cdots & p_{2\alpha} & & \cdots \\ & |\wedge & \vee| & & & \wedge| & & \\ II & & p_1 & & \cdots & & p_{2\alpha+1} & \cdots \end{array}$$

where $\alpha < \mu$ and the players are picking an increasing sequence of elements of $\mathcal{P}$. The first plays at even stages (including the limit ones) and the second at odd stages. The second player wins if at some stage $\alpha < \mu$ there is no legal move for I. Otherwise I wins.

If there is a winning strategy for I in the game $\mathcal{G}_\mu$, then we say that $\mathcal{P}$ is μ-strategically closed.

Lemma 3.14 *$\mathcal{P}$ is κ^{++}-strategically closed.*

Proof. Let us describe a winning strategy for Player I, i.e. those who plays at even stages. Our main concern will be with limit stages. For successor one a similar and simpler argument will work.

Thus, let $\alpha < \kappa^{++}$ be a limit ordinal and $\langle x_\beta \mid \beta < \alpha \rangle$ be a play in which Player I uses the desired strategy. We are supposed to define his next move $x_\alpha = \langle \langle A^{00}_\alpha, A^{10}_\alpha, F^0_\alpha \rangle, \langle A^{01}_\alpha, A^{11}_\alpha, F^1_\alpha \rangle \rangle$. Set A^{00}_α to be the closure of $\bigcup_{\beta<\alpha} A^{00}_\beta \cup \{\langle A^{11}_\beta \mid \beta < \alpha \rangle\} \cup \{\langle A^{10}_\beta \mid \beta < \alpha \rangle\}$ under κ-sequences and Skolem functions, where $x_\beta = \langle \langle A^{00}_\beta, A^{10}_\beta, F^0_\beta \rangle, \langle A^{01}_\beta, A^{11}_\beta, F^1_\beta \rangle \rangle$. Set A^{01}_α to be an elementary submodel of $H(\kappa^{+3})$ including $\langle A^{01}_\beta \mid \beta < \alpha \rangle$, $\langle A^{00}_\beta \mid \beta \leq \alpha \rangle$ closed under κ^+-sequences and having the intersection with κ^{+3} an ordinal. Let $A^{1i}_\alpha = \bigcup_{\beta<\alpha} A^{1i}_\beta \cup \{A^{0i}_\alpha\} \cup \{\bigcup_{\beta<\alpha} A^{0i}_\beta\}$ for $i < 2$. By Lemma 3.6 (actually its proof) $\langle \langle A^{00}_\alpha, A^{10}_\alpha \rangle, \langle A^{10}_\alpha, A^{11}_\alpha \rangle \rangle \in \mathcal{P}'$. Let us turn to definitions of F^0_α and F^1_α. First we put $\bigcup_{\beta<\alpha} F^0_\beta$ inside F^0_α and $\bigcup_{\beta<\alpha} F^1_\beta$ inside F^1_α. Then we jump to definitions of dense closed subsets F^{0*}_α and F^{1*}_α of F^0_α and F^1_α. Final sets F^0_α and F^1_α will be defined from F^{0*}_α and F^{1*}_α in a direct fashion satisfying 3.10.

We assume by induction that for every even $\beta < \alpha$, $i < 2$ there is a dense closed $F^{i*}_\beta \subseteq F^\beta$ such that for every $p = \langle p_n \mid n < \omega \rangle \in F^{i*}_\beta$ the following holds:

(1) $A^{0i}_\beta \in$ *dom* a_n for all $n \geq \ell(p)$ (where as usual $p_n = \langle a_n, A_n, f_n \rangle$)

(2) if $\gamma < \beta$ is even and $A^{0i}_\gamma \in$ *dom* a_n for every $n \geq \ell(p)$ then $p \restriction A^{0i}_\gamma \in F^{i*}_\gamma$.

Also we assume that for every $p \in F^{0*}_\beta$ there is $q \in F^{1*}_\beta$ such that $A^{00}_\beta \in$ *dom* b_n for all $n \geq \ell(q) = \ell(p)$ and $q \restriction A^{00}_\beta = p$, where $q_n = \langle b_n, B_n, g_n \rangle$.

A typical element of F^{i*}_α ($i < 2$) is obtained as follows: let $\langle p^\nu \mid \nu < \rho \rangle$ be a $\leq^*$-increasing sequence with union in $\mathcal{P}^*$, $p^\nu \in F^{i*}_{\beta_\nu}$ for every $\nu < \rho$ and $\langle \beta_\nu \mid \nu < \rho \rangle$ is an increasing sequence of even ordinals below α. Let p^ρ be the union of $\langle p^\nu \mid \nu < \rho \rangle$. Extend p^ρ to p by adding to it A^{0i}_α (i.e. we add it to *dom* a_n for each $n \geq \ell(p^\rho)$) and if $i = 1$ then also A^{00}_α provided that *dom* $a^\rho_n \subseteq A^{00}_\alpha$. Put this p into F^{i*}_α.

Let us show that such defined F^{i*}_α is really closed. First notice that $\langle p^\nu \mid \nu < \rho \rangle$ as above can always be reorganized as follows. Set $\tilde{p}^\nu = \bigcup_{\rho > \nu' \geq \nu} p^{\nu'} \restriction A^{0i}_{\beta_\nu}$ for every $\nu < \rho$.

Then $p^\nu \leq^* p^{\nu'}$ $(\nu' \geq \nu)$, by (1) above $A^{0i}_{\beta_\nu} \in$ *dom* $a^{\nu'}_n$ for every $n \geq \ell(p^\nu) = \ell(p^{\nu'})$, and by (2) $p^{\nu'} \restriction A^{i0}_{\beta_\nu} \in F^{i*}_{\beta_\nu}$. Hence also $\tilde{p}^\nu \in F^{i*}_{\beta_\nu}$. So we obtain a new sequence $\langle \tilde{p}^\nu \mid \nu < \rho \rangle$ with the same limit but in addition $\tilde{p}^{\nu'} \restriction A^{i0}_{\beta_\nu} = \tilde{p}^\nu$ for every $\nu \leq \nu' < \rho$. Now suppose that we have such an additional sequence $\langle q^\xi \mid \xi < \mu \rangle$ corresponding to the increasing sequence $\langle \gamma_\xi \mid \xi < \mu \rangle$ of ordinals below α. Assume that $\bigcup_{\nu<\rho} \tilde{p}^\nu \leq^* \bigcup_{\xi<\mu} q^\xi$. Consider A^{0i}_β. There is $\xi_0 < \mu$ and $n_0 \geq \ell(q^{\xi_0}) = \ell(q^0)$ such that $A^{0i}_{\beta_0}$ belongs to the domain of the first coordinate of $q^{\xi_0}_{n_0}$. Then the same is true for every $n \geq n_0$. So, it remains only finitely many places between $\ell(q^0)$ and n_0. Thus, there is $\widetilde{\xi}_0$, $\mu > \widetilde{\xi}_0 \geq \xi_0$ such that $A^{0i}_{\beta_0}$ appears in $q^{\widetilde{\xi}_0}$ for every $n \geq \ell(q^0)$. But then $q^{\xi_0} \restriction A^{i0}_{\beta_0} \in F^{i*}_{\beta_0}$ and, since $q^\xi \restriction A^{i0}_{\widetilde{\xi}_0} = q^{\widetilde{\xi}_0}$ by our assumption, $q^{\widetilde{\xi}_0} \restriction A^{i0}_{\beta_0} \geq^* \tilde{p}^0$. Continuing in the same fashion, we find a nondecreasing sequence $\langle \widetilde{\xi}_\nu \mid \nu < \rho \rangle$ such that $A^{i0}_{\beta_\nu}$ is in the domain of the first coordinate of $q^{\widetilde{\xi}_\nu}_n$ for every $n \geq \ell(q^0)$ and $q^{\widetilde{\xi}_\nu} \restriction A^{i0}_{\beta_\nu} \geq^* \tilde{p}^\nu (\nu < \rho)$. Also, $q^{\widetilde{\xi}_\nu} \restriction A^{i0}_{\beta_\nu} \in F^{i*}_{\beta_\nu}$. We deal with infinite increasing sequences from F^{i*}_α exactly in the same way. Thus we put together everything below $A^{i0}_{\beta_0}$ first, then below $A^{i0}_{\beta_1}$ and so on going over all A^{i0}_β's with even β appearing in the elements of the sequences.

The point that prevents us from obtaining κ^{++}-closureness instead of only strategic κ^{++}-closureness is 3.10(2(l)). Thus let $B \in A^{10}_\alpha \cup A^{10}_{\alpha in}$ and $q \in F^0_\alpha$. We like to add B to q. If $B = A^{00}_\alpha$ or it is an initial segment of A^{00}_α then this is clear since there is no problem to add the largest set. It remains the case when $B \in \bigcup_{\beta<\alpha} A^{00}_\beta$. Thus let $B \in A^{00}_\beta$ with $\beta < \alpha$. If $q \in \bigcup_{\gamma<\alpha} F^0_\gamma$, then $q \in F^0_\gamma$ for some γ, $\beta \leq \gamma < \alpha$ and 3.10(2)(l) is satisfied by $\langle A^{00}_\gamma, A^{10}_\gamma, F^0_\gamma \rangle$. Then in this case B is addable to q. But now suppose that q is really new. Then by the construction of F^0_α (or actually F^{0*}_α) it is a union or is $\leq^*$ below a union of some sequence $\langle q^\nu \mid \nu < \rho \rangle$ such that A^{00}_γ appears in it, for some $\gamma \geq \beta$ and $q^\nu \restriction A^{00}_\gamma = q^{\nu'} \restriction A^{00}$ starting with some $\nu_0 < \rho$. Then B is addable to $q^{\nu_0} \restriction A^{00}_\gamma$ and hence to all the rest of the above. $\square$

The proof of 3.14 actually provides more. Thus the following holds:

Lemma 3.15 *Let $\langle D_\alpha \mid \alpha < \kappa^{++} \rangle$ be a list of dense open subsets of $\langle \mathcal{P}, \leq \rangle$. Then there is an increasing sequence $\langle \langle A^{0i}_\alpha, A^{1i}_\alpha, F^{i\alpha}_\alpha \rangle \mid i < 2 \quad \alpha < \kappa^{++} \rangle$ of elements of $\mathcal{P}$ and an increasing under inclusion sequence $\langle F^{i*}_\alpha \mid \alpha < \kappa^{++} \rangle$ $i < 2$ so that for every $\alpha < \kappa^{++} (i < 2)$ the following hold*

(1) $\langle \langle A^{00}_{\alpha+1}, A^{10}_{\alpha+1}, F^0_{\alpha+1} \rangle, \langle A^{01}_{\alpha+1} A^{11}_{\alpha+1}, F^1_{\alpha+1} \rangle \rangle \in D_\alpha$

(2) $F^{i*}_\alpha \subseteq F^i_\alpha$ *is dense and closed*

(3) if α *is limit then* $\bigcup_{\beta<\alpha} A^{0i}_\beta \in A^{1i}_\alpha$.

If we restrict ourselves to $\mathcal{P}_{\geq 1}$ then the proof of 3.14 gives more closure:

Lemma 3.16 *$\mathcal{P}_{\geq 1}$ is κ^{+3}-strategically closed.*

Let $G \subseteq \mathcal{P}$ be generic. We define our main forcing $\mathcal{P}^{**}$ to be $\cup\{F^0 \mid \exists A^{00}, A^{01}, A^{10}, A^{11}, F^1$ $\langle\langle A^{00}, A^{10}, F^0\rangle, \langle A^{01}, A^{11}, F^1\rangle\rangle \in G\}$. The orderings $\leq$ and $\leq^*$ on $\mathcal{P}^{**}$ are just the restrictions of those of $\mathcal{P}^*$. Notice that $\langle\mathcal{P}^{**}, \leq^*\rangle$ is not κ_0-closed anymore. But 3.15 provides a replacement which is sufficient for showing that $\langle\mathcal{P}^{**}, \leq\rangle$ does not add new bounded subsets to κ and $\langle\mathcal{P}^{**}, \leq, \leq^*\rangle$ satisfies the Prikry condition.

Lemma 3.17 *Let N be an elementary submodel of $H(\chi)$ (in V) for χ big enough having cardinality κ^+ and closed under κ-sequences of its elements. Then there are an increasing sequence $\langle\langle\langle A^{00}_\alpha, A^{10}_\alpha, F^0_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha, F^1_\alpha\rangle\rangle \mid \alpha \leq \kappa^+\rangle$ of elements of $\mathcal{P}$ and an increasing under inclusion sequence $\langle F^{0*}_\alpha \mid \alpha \leq \kappa^+\rangle$ so that*

(1) $\{\langle\langle A^{00}_\alpha, A^{10}_\alpha, F_\alpha\rangle, \langle A^{01}_\alpha, A^{11}_\alpha, F^1_\alpha\rangle\rangle, \mid \alpha < \kappa^+\}$ is N-generic

(2) $F^{0*}_\alpha \subseteq F_\alpha$ is dense and closed for every $\alpha \leq \kappa^+$.

Our next subject will be chain conditions. First we need to show that κ^{+3} is preserved in $V^{\mathcal{P}}$. By 3.16, $\mathcal{P}_{\geq 1}$ is κ^{+3}-strategically closed. Thus the following analogue of 3.9 will be enough.

Lemma 3.18 *$\mathcal{P}_{<1}$ satisfies κ^{+3}-c.c. in $V^{\mathcal{P}_{\geq 1}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \Vdash_{\mathcal{P}'_{\geq 1}} {}''\langle\langle \underset{\sim}{A}{}^{00}_\alpha, \underset{\sim}{A}{}^{10}_\alpha, F^0_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle \quad \text{is an antichain in} \quad \mathcal{P}_{<1}{}'' .$$

Using 3.16, we define by induction an increasing sequence of conditions in $\mathcal{P}_{\geq 1}$ $\langle\langle A^{01}_\alpha, A^{11}_\alpha, F^1_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle$ and a sequence $\langle\langle A^{00}_\alpha, A^{10}_\alpha, F^0_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle$ so that for every $\alpha < \kappa^{+3}$

$$\langle A^{01}_\alpha, A^{11}_\alpha, F^1_\alpha\rangle \Vdash_{\mathcal{P}_{\geq 1}} {}''\langle \underset{\sim}{A}{}^{00}_\alpha, \underset{\sim}{A}{}^{10}_\alpha, F^0_\alpha\rangle = \langle \check{A}^{00}_\alpha, \check{A}^{10}_\alpha, \check{F}^0_\alpha\rangle''$$

and $\bigcup_{\beta<\alpha} A^{01}_\beta \in A^{11}_\alpha$. Let $A^{10}_\alpha = \{X_{\alpha_i} \mid i < \kappa^+\}$, $F^0_\alpha = \{p^{\alpha i} \mid i < \kappa^+\}$, $p^{\alpha i} = \langle p^{\alpha i}_n \mid n < \omega\rangle$ $(i < \kappa^+)$ and for every $i < \kappa^+$, $n \geq \ell(p^{\alpha i})$ $p^{\alpha i}_n = \langle a^{\alpha i}_n, A^{\alpha i}_n, f^{\alpha i}_n\rangle$ for all $\alpha < \kappa^{+3}$. As in the proof of 3.9, we now form Δ-system also including F^0_α's. Thus we can assume that for some $\delta < \kappa^{+3}$, for every $\beta < \alpha < \kappa^{+3}$ (a), (b), (c) of 3.9 and in addition:

(d) for every $i, j < \kappa^+, n < \omega$ the following holds:

  (i) $\ell(p^{\alpha i}) = \ell(p^{\beta i})$

(ii) if $n < \ell(p^{\alpha i})$ then $p_n^{\alpha i}$ and $p_n^{\beta i}$ are compatible in Q_{n1}, i.e. $p_n^{\alpha i} \cup p_n^{\beta i}$ is a function.

(iii) if $n \geq \ell(p^{\alpha i})$ then $A_n^{\alpha i} = A_n^{\beta i}$, $f_n^{\alpha i} \cup f_n^{\beta i}$ is a function, $dom\ a_n^{\alpha i}$ and $dom\ a_n^{\beta i}$ are isomorphic over δ, $a_n^{\alpha i} \restriction \delta = a_n^{\beta i} \restriction \delta$, $X^{\alpha j} \in\ dom\ a_n^{\alpha i}$ iff $X^{\beta j} \in\ dom\ a_n^{\beta i}$ and $a_n^{\alpha i}(X^{\alpha j}) = a_n^{\beta i}(X^{\beta j})$.

(e) $(A_\alpha^{00} \cap \bigcup_{\gamma<\alpha} A_\gamma^{01}) \cap \kappa^{+3} = (A_\beta^{00} \cap \bigcup_{\gamma<\beta} A_\gamma^{01}) \cap \kappa^{+3} = A_\alpha^{00} \cap A_\beta^{00} \cap \kappa^{+3}$

Let us show that $\langle A_\alpha^{00}, A_\alpha^{10}, F_\alpha^0 \rangle$ and $\langle A_\beta^{00}, A_\beta^{10}, F_\beta^0 \rangle$ are compatible in $\mathcal{P}_{<1}$ for $\alpha > \beta > 0$, $cf\alpha = cf\beta = \kappa^{++}$. It is enough to prove compatibility of $\langle\langle A_\alpha^{00}, A_\alpha^{10}, F_\alpha^0 \rangle, \langle A_\alpha^{01}, A_\alpha^{11}, F_\alpha^1 \rangle\rangle$ and $\langle\langle A_\beta^{00}, A_\beta^{10}, F_\beta^0 \rangle, \langle A_\beta^{01}, A_\beta^{11}, F_\beta^1 \rangle\rangle$ in $\mathcal{P}$. We define a condition $\langle\langle A^{00}, A^{10}, F^0 \rangle, \langle A^{01}, A^{11}, F^1 \rangle\rangle$ stronger than both of these conditions. Let $\langle A^{00}, A^{10} \rangle$, $\langle A^{01}, A^{11} \rangle$ be as in the proof of Lemma 3.9. Set $F^1 = F_\alpha^1$. We left with definition of F^0. First we include both F_α^0 and F_β^0 into F^0.

Now let $p^0 = \langle p_n^0 \mid n < \omega \rangle \in F_\alpha^0$ and $p^1 = \langle p_n^1 \mid n < \omega \rangle \in F_\beta^0$ be so that

(1) $\ell(p^0) = \ell(p^1)$

(2) p_n^0, p_n^1 are compatible for every $n < \ell(p^0)$

(3) for every $n \geq \ell(p^0)$ D_α, $A_\alpha^{00} \in\ dom\ a_n^0, D_\beta, A_\beta^{00} \in\ dom\ a_n^1$, where $D_i = A_i^{00} \cap \bigcup_{\gamma<i} A_\gamma^{01} (i = \alpha, \beta)$, $p_n^0 = \langle a_n^0, A_n(0), f_n^0 \rangle$ and $p_n^1 = \langle a_n^1, A_n(1), f_n^1 \rangle$

(4) $a_n^0 \restriction D_\alpha = a_n^1 \restriction D_\beta$

(5) p^0 and p^1 are compatible in $\mathcal{P}^*$, i.e., basically $rng(a_n^0 \backslash D_\alpha)$ sits above everything in $rng\ a_n^1$

Notice that 3.10(1) and 3.10(2(l)) used with D_α and D_β insure that above every condition in F_α^0 will be one including D_α and above every condition in F_β^0 will be one including D_β.

In order to guarantee (5) above, we will need to find n big enough such that $a_n^0(A_\alpha^{00} \cap D_\alpha) \prec \mathfrak{a}_{n,5}$ see 3.10(2(d)). Now the argument of 2.20 (Case 2) can be used to generate $rnga_n^1$. Finally, by 3.10(2(h)) F_β is closed under "$\leftrightarrow$", so one obtains a_n^1 as desired.

Extend p^0 to a condition $p \in F_\alpha^1$ by adding A^{00} and A_β^{00} to $dom\ a_n^0$. It is possible by 3.10 (2(l). Then, we use 3.10(2(0)) for p and p^1. Let $r \in F_\alpha^1$ be the resulting condition. We include such r into F^0.

We need to check that $\langle\langle A^{00}, A^{10}, F^0 \rangle, \langle A^{01}, A^{11}, F^1 \rangle\rangle$ is as desired. Lemma 3.9 provides the argument for A-part, i.e. $\langle\langle A^{00}, A^{10} \rangle, \langle A^{01} A^{11} \rangle\rangle$. So we concentrate on F-part. Most of the checking is straightforward. Let us check 3.10(2(n)). Thus, we need to find a dense closed subset F^{0*} of F^0. Let F_α^{0*} and F_β^{0*} be such subsets of F_α^0 and F_β^0. Let $\overline{F}_i^{0*} =$

$\{p \in F_i^{0*}|$ for every $n, \omega > n \geq \ell(p) A_i^{00} \in$ *dom* $a_n\}$, where $i \in \{\alpha, \beta\}$. Then $\overline{F}_\alpha^{0*}$ and $\overline{F}_\beta^{0*}$ are again dense closed subsets of F_α^0 and F_β^0, since it is always possible to add the maximal set. We include $\overline{F}_\alpha^{0*}$ and $\overline{F}_\beta^{0*}$ into F^{0*}. Suppose now that $p \in F$. If $p \in F_\alpha^0 \cup F_\beta^0$ then we use $\overline{F}_\alpha^{*} \cup \overline{F}_\beta^{0*}$. Assume that $p \notin F_\alpha^0 \cup F_\beta^0$. Then it is obtained as a union of conditions satisfying (1)-(5) above. In particular, $A_\alpha^{00}, A_\beta^{00}$ and $A_\alpha^{00} \cap A_\beta^{00}$ belong to *dom* a_n for every $n \geq \ell(p)$. Notice that $D_\alpha \cap A_\alpha^{00} = D_\beta \cap A_\beta^{00} = A_\alpha^{00} \cap A_\beta^{00}$. Consider $p \upharpoonright A_\alpha^{00}$. Pick some $p(0) \in F_\alpha^{0*}$ above it. Then $p(0) \upharpoonright (A_\alpha^{00} \cap A_\beta^{00})$ is addable to $p \upharpoonright A_\beta^{00}$ in F_β^0 by 3.10(2)(0)). Let $q \in F_\beta^0$ be the result of combining $p \upharpoonright A_\beta^{00}$ and $p(0) \upharpoonright (A_\alpha^{00} \cap A_\beta^{00})$ together. Pick $p(1) \in F_\beta^{0*}$ above it. Consider $p(1) \upharpoonright (A_\alpha^{00} \cap A_\beta^{00})$. It belongs to F_α^0 and is addable to $p(0)$. Put them together and pick $p(2) \in F_\alpha^{0*}$ to be a stronger condition. Continue in the same fashion. Finally, $p^0 = \bigcup_{n<\omega} p(2n) \in F_\alpha^{0*}$ and $p^1 = \bigcup_{n<\omega} p(2n+1) \in F_\alpha^{1*}$ will have the same restriction to $A_\alpha^{00} \cap A_\beta^{00}$. Thus, the combination of p^0 and p^1 will be in F^0. Hence we can define F^{0*} to be $\overline{F}_\alpha^{0*} \cup \overline{F}_\beta^{0*}$ together with the set of combinations of all $p^0 \in F^{0*}$ and $p^1 \in F^{1*}$ satisfying the conditions (1)-(5) above and the result will be dense and closed. □

Hence the forcing $\langle \mathcal{P}, \leq \rangle$ preserves the cardinals. We now turn to our main forcing $\langle \mathcal{P}^{**}, \rightarrow \rangle$ where $\rightarrow$ is the restriction of $\rightarrow$ on $\mathcal{P}^*$ to $\mathcal{P}^{**}$.

Lemma 3.19 *In $V^{\mathcal{P}}$, $\langle \mathcal{P}^{**}, \rightarrow \rangle$ satisfies κ^{++}-c.c.*

Proof. Suppose otherwise. Let us work in V and let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++} \rangle$ be a name of an antichain of the length κ^{++}. Using 3.15 we find an increasing sequence $\langle \langle \langle A_\alpha^{00}, A_\alpha^{10}, F_\alpha^0 \rangle, \langle A_\alpha^{01}, A_\alpha^{11}, F_\alpha^1 \rangle \rangle \mid \alpha < \kappa^{++} \rangle$ of elements of $\mathcal{P}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ so that for every $\alpha < \kappa^{++}$ the following holds:

(a) $\langle \langle A_{\alpha+1}^{00}, A_{\alpha+1}^{10}, F_{\alpha+1}^0 \rangle, \langle A_{\alpha+1}^{01}, A_{\alpha+1}^{11}, F_{\alpha+1}^1 \rangle \rangle \Vdash \underset{\sim}{p}_\alpha = \check{p}_\alpha$

(b) $\bigcup_{\beta<\alpha} A_\beta^{00} \in A_\alpha^{10}$

(c) ${}^\kappa A_\alpha^{00} \subseteq A_\alpha^{00}$

(d) $p_\alpha \in F_{\alpha+1}^0$ and for every $n \geq \ell(p_\alpha)$ $A_{\alpha+1}^{00} \in$ *dom* $a_{n\alpha}$ where $p_{n\alpha} = \langle a_{n\alpha}, A_{n\alpha}, f_{n\alpha} \rangle$

(e) $\langle A_\beta^{11} \mid \beta < \alpha \rangle \in A_\alpha^{00}$.

Let $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n} \rangle$ for every $\alpha < \kappa^{++}$ and $n \geq \ell(p_\alpha)$. Extending if necessary, let us assume that $A_\alpha^{00} \in$ *dom* $a_{\alpha n}$ for every $n \geq \ell(p_\alpha)$. Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^+$ the following holds:

(1) $\ell = \ell(p_\alpha) = \ell(p_\beta)$

(2) for every $n < \ell \quad p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1}

(3) for every n, $\ell \le n < \omega \quad \langle dom\ a_{\alpha n}, dom\ f_{\alpha n} \mid \alpha < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in A_0^{00}

(4) for every n, $\omega > n \ge \ell \quad rnga_{\alpha n} = rnga_{\beta n}$.

Without loss of generality we can assume that $A_\alpha^{00} \cap \kappa^{++} \ge \alpha$ and if $cf\alpha = \kappa^+$ then $A_\beta^{00} \cap \kappa^{++} < \alpha$ for every $\beta < \alpha$. Let us split elements of $dom\ a_{\alpha n} \backslash \kappa^{+3}$ into two groups. The first will consist of B's such that $B \cap \kappa^{++} \ge \alpha$ and the second, of B's with $B \cap \kappa^{++} < \alpha$. Again, shrinking if necessary, we can assume that for every $\alpha < \kappa^{++}$ and B from the second group of $dom\ a_{\alpha n} \quad B \cap \kappa^{++} < A_0^{00} \cap \kappa^{++}$. Now we can use 3.8(1) and shrink again to obtain that for every $\beta < \alpha$ $B \cap A_\beta^{00} = \overline{B} \cap D_{\overline{B}}$ for some $\overline{B} \in A_0^{00} \cap A_0^{10}$ and $D_{\overline{B}} \in A_0^{00} \cap A_0^{11}$. Hence we can assume one more condition:

(5) for every limit $\alpha < \kappa^{++}$, $n \ge \ell$ and $B \in (dom\ a_{\alpha n}) \backslash \kappa^{+3}$ if $B \cap \kappa^{++} < \alpha$ then $B \cap \left(\bigcup_{\beta<\alpha} A_\beta^{00}\right) = B \cap A_0^{00} \in A_0^{00}$.

Also we can assume the following since $|dom\ a_{\alpha n}| < \kappa_n$

(6) for every $\alpha < \kappa^{++}, n \ge \ell$ $\left(dom\ a_{\alpha n} \cap \bigcup_{\beta<\alpha} A_\beta^{00}\right) \cap \kappa^{+3} \subseteq A_0^{00} \cap \kappa^{+3}$.

Now let us assume also that

(7) for every $\alpha < \kappa^{++}$, $n \ge \ell$

(i) if $\gamma \in\ dom\ a_{\alpha n} \cap \kappa^{+3}$ and there is $\gamma' \in \bigcup_{\beta<\alpha}(A_\beta^{00} \cap \kappa^{+3})$ above it, then the least such γ' is in A_0^{00}.

(ii) if $B \in\ dom\ a_{\alpha n} \backslash \kappa^{+3}$ and there is $\gamma' \in \bigcup_{\beta<\alpha} A_\beta^{00}$ above $\cup(B \cap \kappa^{+++})$ then the least such γ' is in A_0^{00}.

For B's in the first group we can use 3.8(2) and insure the following:

(8) for every $\alpha < \kappa^{++}$ of cofinality κ^+, $n \ge \ell$ if $B \in\ dom\ a_{\alpha n} \backslash \kappa^{+3}$ and $B \cap \kappa^{++} \ge \alpha$ then there is $D_B \in (A_0^{00} \cap A_0^{10}) \cup \{H(\kappa^{+3})\}$ so that for every β, $0 \le \beta < \alpha \quad B \cap A_\beta^{00} = D_B \cap A_\beta^{00}$.

Notice, that 3.8(2) together with 3.1(2(e)) imply that for $\alpha < \kappa^{++}$ of cofinality κ^+ and $B \in\ dom\ a_{\alpha n} \backslash \kappa^{+3}$ with $B \cap \kappa^{++} \ge \alpha$ if there is no γ' as in (7)(ii) then $B \supseteq \bigcup_{\beta<\alpha} A^{0\beta}$.

Now we add A_0^{00} to each p_α ($\alpha < \kappa^{++}$) and $\bigcup_{\beta<\alpha} A_\beta^{00}$ but only for $cf\alpha = \kappa^+$. Shrinking again if necessary we assume that the conditions (1)-(8) are valid.

Now let $\beta < \alpha < \kappa^{++}$ be ordinals of cofinality κ^+. We claim that p_β and p_α are compatible in $\langle \mathcal{P}^{**}, \rightarrow \rangle$. First extend p_α by adding $A_{\beta+2}^{00}$. Let p be the resulting condition. Denote p_β by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$ and $p_n = \langle a_n, A_n, f_n \rangle$. Let $q_n = \langle b_n, B_n, g_n \rangle$. W.l. of g we may assume that $a_n(A_{\beta+2}^{00})$ is an elementary submodel of $\mathfrak{a}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $rngb_n$ inside $a_n(A_{\beta+2}^{00})$ over the common parts $dom\ b_n$ and $dom\ a_n$. This will produce $q'_n = \langle b'_n, B_n, g_n \rangle$ $k_n - 1$-equivalent to q_n and with $rngb'_n \subseteq a_n(A_{\beta+2}^{00})$. Doing the above for all $n \geq \ell(p)$ we will obtain $q' = \langle q'_n \mid n < \omega \rangle$ equivalent to q (i.e. $q' \longleftrightarrow q$). Extend q' to q'' by adding to it $\langle A_{\beta+2}^{00}, a_n(A_{\beta+2}^{00}) \rangle$ as the maximal set for every $n \geq \ell(p)$. By 3.10(2(q)), $q'' \in F_{\beta+2}^{00}$. Then $max^1(q''_n) = A_{\beta+2}^{00} \in\ dom\ a_n$. But q'' is addable to p. Now, by 3.10(2(o)) there is a condition $r \in F_{\alpha+1}^0$ $r \geq p, q''$. Hence, $p \rightarrow r$ and $q \rightarrow r$. Contradiction.

$\square$

4 Wider Gaps

In this section we present a generalization of the forcing $\mathcal{P}$ of the previous section which allows to make $2^\kappa \geq \kappa^{+\delta+2}$ for every δ, $1 < \delta < \kappa_0$. The length of the extender over κ_n will be here $\kappa_n^{+n+\delta+1}$. Such an assumption is optimal when $\delta < \omega$ and κ is a singular cardinal in the core model see [Git-Mit].

The difference here from the gap 3 case is that we cannot at once restrict the choices to sets of cardinality κ^+, as it was done in the previous section. The total number of possible choices is now $\kappa^{+\delta}$ and not κ^{++} as in the previous case. As a result of this the straightforward generalization of $\mathcal{P}$ will fail to satisfy κ^{++}-c.c. and actually will collapse $\kappa^{+\delta}$ to κ^{++}. In order to overcome this difficulty we define the preparation forcing restricting the number of possibilities gradually to κ^+.

As in Section 3, define first the part of the preparation forcing over κ.

Definition 4.1

The set $\mathcal{P}'$ consists of sequences of pairs $\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \leq \delta \rangle$ so that the following holds

(1) for every $\tau \leq \delta$ $A^{0\tau}$ is an elementary submodel of $\langle H(\kappa^{+\delta+2}), \in, \langle \kappa^{+i} \mid i \leq \delta+2 \rangle$ such that

(a) $|A^{0\tau}| = \kappa^{+\tau+1}$ and $A^{0\tau} \supseteq \kappa^{+\tau+1}$

(b) ${}^{\kappa^{+\tau}}A^{0\tau} \subseteq A^{0\tau}$

(2) for every $\tau < \tau' \leq \delta$ $A^{0\tau} \subseteq A^{0\tau'}$

(3) for every $\tau \leq \delta$ $A^{1\tau}$ is a set of at most $\kappa^{+\tau+1}$ elementary submodels of $A^{0\tau}$ so that

(a) $A^{0\tau} \in A^{1\tau}$

(b) if $B, C \in A^{1\tau}$ and $B \subsetneqq C$ then $B \in C$

(c) if $B, C \in A^{1\tau}, B \neq C$, $B \not\subset C$ and $C \not\subset B$ then

(i) $otp(C \cap On) = otp(B \cap On)$ implies that there are $D_B, D_C \in A^{0\tau} \cap A^{1\rho}$ $B \cap C = B \cap D_B = C \cap D_C$, for some ρ, $\tau < \rho \leq \delta$

(ii) if $otp(C \cap On) > otp(B \cap On)$, then there is $C' \in C \cap A^{1\tau}$ such that $otp(C' \cap On) = otp(B \cap On)$, $B \cap C = B \cap C'$ and B, C' are isomorphic over $B \cap C'$.

(d) if $B \in A^{1\tau}$ is a successor point of $A^{1\tau}$ then B has at most two immediate predecessors under the inclusion and is closed under $\kappa^{+\tau}$-sequences.

(e) let $B \in A^{1\tau}$ then either B is a successor point of $A^{1\tau}$ or B is a limit element and then there is a closed chain of elements of $B \cap A^{1\tau}$ unbounded in $B \cap A^{1\tau}$ and with limit B.

Let for $\tau \leq \delta$ $A^{1\tau}_{in}$ be the set $\{B \cap B_1 \cap \cdots \cap B_n \mid B \in A^{1\tau}, n < \omega \text{ and } B_i \in A^{1\rho_i} \text{ for some } \rho_i, \tau < \rho_i \leq \delta \text{ for every } i, 1 \leq i \leq n\}$.

Definition 4.2

Let $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle$ and $\langle\langle B^{0\tau}, B^{1\tau}\rangle \mid \tau \leq \delta\rangle$ be elements of $\mathcal{P}'$. Then $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle \geq \langle\langle B^{0\tau}, B^{1\tau}\rangle \mid \tau \leq \delta\rangle$ iff for every $\tau \leq \delta$

(1) $A^{1\tau} \supseteq B^{1\tau}$

(2) for every $A \in A^{1\tau}$ $A \cap B^{0\tau} \in B^{1\tau} \cup B^{1\tau}_{in}$.

Definition 4.3

Let $\tau \leq \delta$. Set $\mathcal{P}'_{\geq\tau} = \{\langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \mid \exists\langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle\ \langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle^\frown\langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}'\}$.

Let $G(\mathcal{P}'_{\geq\tau}) \subseteq \mathcal{P}'_{\geq\tau}$ be generic. Define $\mathcal{P}'_{<\tau} = \{\langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle \mid \exists\langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in G(\mathcal{P}'_{\geq\tau})\ \langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu < \tau\rangle \frown \langle\langle A^{0\rho}, A^{1\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}'\}$.

The following lemma is obvious

Lemma 4.4 $\mathcal{P}' \simeq \mathcal{P}'_{\geq\tau} * \underset{\sim}{\mathcal{P}'}_{<\tau} \quad (\tau \leq \delta)$.

At this point we assume the existence of box sequences. They either can be added generically in advance or V can be taken of the form $L[\vec{E}]$.

Lemma 4.5 *Suppose that A, B are two elementary submodels, ${}^{\omega}A \subseteq A$, ${}^{\omega}B \subseteq B, \kappa, \delta, \tau \in A \cap B$, $\kappa^{+\tau+1} = |A| \leq |B| (\tau \leq \delta)$. Let $\mu \in (\kappa^{+\tau+1}, \kappa^{+\delta+1}]$, $cf\mu > \aleph_0$ be a limit of elements of $A \cap B$. Then $otp\Big(C_\mu{}^{|\mu|}\Big)$ is a limit of elements of $A \cap B$, where $C_\mu{}^{|\mu|}$ is the μ-th element of the canonical box sequence for μ^+.*

Proof. Denote $C_\mu{}^{|\mu|}$ by C_μ and $otpC_\mu$ by ρ. Let $\xi < \rho$ be a limit ordinal. We find $\xi' \in A \cap B$ $\xi < \xi' < \rho$. Let $\widetilde{\xi}$ be ξ-th point of C_μ. Consider $A \cap B \cap \mu$. It is an ω-closed unbounded in the μ set. Hence, also $C_\mu \cap A \cap B$ is such a set. Pick some limit $\rho' \in C_\mu \cap A \cap B$ above $\widetilde{\xi}$. Then $otpC_{\rho'} > otpC_{\widetilde{\xi}} = \xi$. Clearly, by elementarity, $otpC_{\rho'} \in A \cap B \cap \rho$. So $\xi' = otpC_{\rho'}$ is as desired.

$\square$

Lemma 4.6 *Let A, B, μ be as in 4.5. Assume that $\mu \notin A$ and $\nu = min(A \backslash \mu) < \mu^+$. Then*

(a) $\mu \in C_\nu{}^{|\mu|}$. Moreover, for every club $C \in A$ of ν $\mu \in C$.

(b) $otpC_\nu{}^{|\mu|} = min(A \backslash otpC_\mu{}^{|\mu|})$

(c) $otpC_\mu{}^{|\mu|} \notin A$

Proof. (a) is easy since A is an elementary submodel closed under ω-sequences with $\sup(A \cap \nu) = \mu$.

(b) Let us denote $otpC_\mu{}^{|\mu|}$ by μ^* and $otpC_\nu{}^{|\mu|}$ by ν^*. Also, we drop the upper index $\square_{|\mu|}$. Clearly, by (a), $\nu^* > \mu^*$. If there is $\gamma \in A$, $\nu^* > \gamma > \mu^*$, then there is such a limit γ. But then the γ-th element of C_ν is in A and is above μ. Since μ is μ^*-th element of C_ν. But this contradicts the minimality of ν.

(c) If $\mu^* = otpC_\mu{}^{|\mu|} \in A$, then $\mu \in A$. Since μ is μ^*-th element of C_ν.

$\square$

Lemma 4.7. *Let A, B, μ and ν be as in 4.6. Assume that for some τ^*, $\tau \le \tau^* \le \delta$ the following holds:*

(*) for every τ', $\tau^* \le \tau' \le \delta$

$$min(A \setminus \sup(A \cap B \cap \kappa^{+\tau'+1})) < \kappa^{+\tau'+1}$$

Then $cf\nu \le \kappa^{+\tau*}$.

Proof. By 4.6(b) $otpC_\nu{}^{|\mu|} = min(A \setminus otpC_\mu{}^{|\mu|})$. If $|\mu| \le \kappa^{+\tau^*}$, then $otpC_\nu{}^{|\mu|} \le |\mu| \le \kappa^{+\tau*}$. Also $cf\nu = cf(otpC_\nu{}^{|\mu|})$. Hence $cf\nu \le \kappa^{+\tau^*}$. If $|\mu| \ge \kappa^{+\tau^*+1}$, then by 4.4.1 and (*) we can replace μ and ν by $otpC_\mu{}^{|\mu|}$ and $otpC_\nu{}^{|\mu|}$ which are smaller ordinals of the same cofinalities. After finitely many steps $\kappa^{+\tau^*}$ will be reached.

$\square$

In particular, 4.7 implies the following:

Lemma 4.8. *Let A, B be as in 4.5. For every ν, $\tau \le \nu \le \delta$ let $\delta_\nu = min\ (A \setminus \sup(A \cap B \cap \kappa^{+\nu+1}))$. Suppose that starting with some τ^*, $\tau \le \tau^* \le \delta$ each $\delta_\nu < \kappa^{+\nu+1}$. Then $cf\delta_\nu \le \kappa^{+\tau^*}$.*

Lemma 4.9. *Suppose that A, B are elementary submodels, ${}^\omega A \subseteq A, {}^\omega B \subseteq B, \kappa, \delta, \tau \in A \cap B$, $\delta \subseteq A$, $\kappa^{+\tau+1} = |A| \le \kappa^{+\rho+1} = |B|$ and $\kappa^{+\rho+1} \subseteq B$. Assume that for some $E \in A$ of cardinality $\kappa^{+\rho+1}$, $E \supseteq B$. For every ν, $\nu \le \delta$ let $\delta_\nu = min(A \setminus \sup(A \cap B \cap \kappa^{+\nu+1}))$. Then for every $\nu \le \delta$ the following holds:*

(a) if $\nu \le \rho$ then $\delta_\nu = \kappa^{+\nu+1}$ and if $\nu > \rho$ then $\delta_\nu < \kappa^{+\nu+1}$

(b) $cf\delta_\nu \le \kappa^{+\rho+1}$

(c) if $\nu \ge \rho + 1$, then there is $C \in A$, so that C is a club in δ_ν and $C \cap A \subseteq B$.

Proof. (a) is obvious. Let us show (b). Let $\nu \ge \rho + 1$. Then, $\cup(B \cap \kappa^{+\nu+1}) < \kappa^{+\nu+1}$. Let $E \in A$ be of cardinality $\kappa^{+\rho+1}$ and $E \supseteq B$. Then, $\kappa^{+\nu+1} > \cup(E \cap \kappa^{+\nu+1}) \ge \cup(B \cap \kappa^{+\nu+1})$. So, $\delta_\nu < \kappa^{+\nu+1}$. Since $\cup(E \cap \kappa^{+\nu+1}) \in A$. Then by 4.8. $cf\delta_\nu \le \kappa^{+\rho+1}$. This proves (b).

In order to show (c) we use the canonical box sequences for ordinals of cofinality $\le \kappa^{+\rho+1}$. Let C_{δ_ν} be such a sequence for δ_ν. Then $C_{\delta_\nu} \in A$, since $\delta_\nu \in A$. $otpC_{\delta_\nu} \le \kappa^{+\rho+1}$. Set $\mu = \sup(B \cap A \cap \kappa^{+\nu+1})$. Then, either $\mu = \delta_\nu$ or by 4.6(a) $\mu \in C_{\delta_\nu}$ and it is a limit point there and hence $C_\mu = C_{\delta_\nu} \cap \mu$. So, $|C_\mu| \le \kappa^{+\rho+1}$, if $\mu = \delta_\nu$ and $|C_\mu| = \kappa^{+\rho}$ if $\mu < \delta_\nu$.

Claim 4.9.1.

$C_\mu \subseteq B$.

Proof. For every $\gamma < \mu$ there is γ', $\gamma < \gamma' < \mu$ which is a limit point of $C_\mu \cap B$, since $C_\mu \cap B$ is ω-closed unbounded subset of μ. Then $C_{\gamma'} = C_\mu \cap \gamma'$. But also $C_{\gamma'} \subseteq B$ since $B \supseteq \kappa^{+\rho}$, $C_{\gamma'} \in B$ and $otpC_{\gamma'} < otpC_\mu \leq \kappa^{+\rho+1}$. Hence for every $\gamma < \mu$ $B \supseteq C_\mu \cap \gamma$. Clearly then $B \supseteq C_\mu$.

$\square$ of the claim.

Now (c) follows, since $C_{\delta_\nu} \cap A = C_{\delta_\nu} \cap \mu \cap A = C_\mu \cap A \subseteq C_\mu \subseteq B$.

$\square$

Lemma 4.10. *Suppose that A, B are as 4.5. Let $|B| = \kappa^{+\rho+1}$, $B \supseteq \kappa^{+\rho+1}$ and for every $\nu \leq \delta$ let $\delta_\nu = min(A \backslash \sup(A \cap B \cap \kappa^{+\nu+1}))$. Assume that $A \supseteq \delta$ and for some $E \in A$ of cardinality $\kappa^{+\rho+1}$, $E \supseteq B$. Then for every $\nu \leq \delta$ the following holds:*

(a) for every $\nu \leq \rho$ $\delta_\nu = \kappa^{+\nu+1}$

(b) for every $\nu > \rho$ $\delta_\nu < \kappa^{+\nu+1}$

(c) if $\nu \leq \rho$ then $B \cap A \cap \kappa^{+\nu+1} = A \cap \kappa^{+\nu+1}$

(d) if $\nu > \rho$ then $B \cap A \cap (\kappa^{+\nu}, \kappa^{+\nu+1}) = \{\alpha \in (\kappa^{+\nu}, \kappa^{+\nu+1}) \mid$ for some $\beta \in A \cap C(\nu)$, $\beta > \alpha$ there is $\alpha' \in B \cap A \cap \kappa^{+\nu}$ $f_\beta(\alpha') = \alpha\}$ where $f_\beta : |\beta| \leftrightarrow \beta$ is fixed canonical mapping and $C(\nu) \in A$ is unbounded subset of δ_ν so that $B \supseteq A \cap C(\nu)$. In particular $C(\nu)$ can be taken C_{δ_ν} as in 4.9 and then it is definable from δ_ν.

Proof. (a),(b),(c) are immediate since $B \supseteq \kappa^{+\rho+1}$. For every $\nu > \rho$, $\sup B \cap \kappa^{+\nu+1} < \sup E \cap \kappa^{+\nu+1} < \kappa^{+\nu+1}$ and $E \cap \kappa^{+\nu+1} \in A$. Let us show (d).

If $\alpha \in B \cap A \cap (\kappa^{+\nu}, \kappa^{+\nu+1})$ then pick some $\beta \in A \cap C(\nu)$ above α. Let $\alpha' = f_\beta^{-1}(\alpha)$. $A \cap C(\nu) \subseteq B$, so $\beta \in A \cap B$. Hence also f_β and α' are in $A \cap B$. If $\alpha = f_\beta(\alpha')$ for some $\alpha' \in A \cap B \cap \kappa^{+\nu}$ and $\beta \in A \cap C(\nu)$, $\beta > \alpha$, then $\alpha \in A \cap B \cap (\kappa^{+\nu}, \kappa^{+\nu+1})$, since $\beta \in B$.

$\square$

Lemma 4.11. *Let A be a model closed under ω-sequences of cardinality $\kappa^{+\tau+1}$ and with $\delta \subseteq A$. Then the set $\{A \cap B \mid B$ is model closed under ω-sequences, of cardinality $\kappa^{+\rho+1}$ for some $\rho \leq \delta$, $B \supseteq \kappa^{+\rho+1}$, there is $E \in A$ $E \supseteq B$ and $|E| = |B|\}$ has cardinality at most $\kappa^{+\tau+1}$.*

Proof. $(\kappa^{+\tau+1})^{\kappa^{+\tau}} = \kappa^{+\tau+1}$ (we are assuming GCH), so it is enough to deal with B's of cardinality $\geq \kappa^{+\tau+1}$. By 4.9,4.10, $B \cap A$ can be uniquely defined from the sequences $\langle \delta_\nu \mid \nu \leq \delta \rangle$. The total number of such sequences is $(\kappa^{+\tau+1})^\delta = \kappa^{+\tau+1}$. So we are done.

□

Lemma 4.12. *Suppose that $A_i = \bigcup_{j<i} A_j$ is an increasing union of elementary submodels of cardinality $\kappa^{+\tau+1}$, closed under $\kappa^{+\tau}$ sequence and $cfi = \kappa^{+\tau+1}$. Let B be an elementary submodel of cardinality $\kappa^{+\rho+1}$, $B \supseteq \kappa^{+\rho+1}$ and closed under ω sequences for some $\tau \leq \rho \leq \delta$. Assume that there is $E \in A_i$ of cardinality $\kappa^{+\rho+1}$, $E \supseteq B$. Then there are $\widetilde{i} < i$ and a sequence of ordinals $\langle \delta_\nu \mid \nu \leq \delta \rangle \in A_{\widetilde{i}}$ so that*

(1) for every $\nu \leq \delta$, $\kappa^{+\nu} < \delta_\nu \leq \kappa^{+\nu+1}$

(2) for every $\nu \leq \delta$, if $\delta_\nu = \kappa^{+\nu+1}$, then for every $\nu' < \nu$ $\delta_{\nu'} = \kappa^{+\nu'+1}$

(3) for every $\nu \leq \delta$ and j, $\widetilde{i} \leq j \leq i$ $\quad \delta_\nu = min(A_j \setminus \sup(B \cap A_j \cap \kappa^{+\nu+1}))$.

Proof. Set $\delta_\nu = min(A_i \setminus \sup(B \cap A_i \cap \kappa^{+\nu+1}))$, for every $\nu \leq \delta$. Find $\widetilde{i} < i$ so that $\langle \delta_\nu \mid \nu \leq \delta \rangle \in A_\tau$ and for every $\nu \geq \rho + 1$ a set $C \in A_i$ as in 4.9(c) is in $A_{\widetilde{i}}$. Lemma 4.9(a), (b) implies (1), (2). Let us show (3). Let $j \in [\widetilde{i}, i]$ and $\nu \leq \delta$. If $\nu \leq \rho$, then $B \supseteq \kappa^{+\nu+1} = \delta_\nu$. Hence, $min(A_j \setminus \sup(B \cap A_j \cap \kappa^{\nu+1})) = \kappa^{+\nu+1} = \delta_\nu$.

Now let $\nu > \rho$. By 4.9(c) and the choice of $\widetilde{i}$ there is a club $C \subset \delta_\nu$, $C \in A_{\widetilde{i}}$ and $C \cap A_i \subseteq B$. Now, $C \cap A_j \subseteq C \cap A_i \subseteq B$ and $C \cap A_j$ is unbounded in $A_j \cap \delta_\nu$, since $C \in A_j$. Hence $\sup(A_j \cap \delta_\nu) = \sup(C \cap A_j) = \sup(B \cap C \cap A_j)$. So $\delta_\nu = min(A_j \setminus \sup(B \cap A_j \cap \kappa^{+\nu+1}))$.

□

Lemma 4.13. *Let $\tau \leq \delta$. Suppose that $\langle \langle A_i^{0\rho}, A_i^{1\rho} \rangle \mid \tau \leq \rho \leq \delta, i < \kappa^{+\tau+2} \rangle$ is an increasing sequence of elements of $\mathcal{P}'_{\geq \tau}$ satisfying the following:*

for every $i < \kappa^{+\tau+2}$ of cofinality $\kappa^{+\tau+1}$

(a) $A_i^{0\tau} = \bigcup_{j<i} A_j^{0\tau}$

(b) $\bigcup_{j<i} A_j^{0\rho} \in A_i^{0\rho}$ for every ρ, $\tau < \rho \leq \delta$

(c) if $B \in A_i^{1\rho}$ then either $B \in \bigcup_{j<i} A_j^{1\rho}$ or $B \supseteq A_i^{0\tau}$,

(d) $\{A_j^{1\rho} \mid j < i, \tau \leq \rho \leq \delta\} \subseteq A_i^{0\tau}$.

Then for every $i < \kappa^{+\tau+2}$ of cofinality $\kappa^{+\tau+1}$ and $B \in \cup\{A^{1\rho}_j \mid j \geq i, \tau \leq \rho \leq \delta\}$ ${}^{|B|>}B \subseteq B$ there are $\widetilde{i} < i$ and a sequence of ordinals $\langle \delta_\nu \mid \nu \leq \delta \rangle \in A^{0\tau}_{\widetilde{i}}$ so that

(1) for every $\nu \leq \delta \quad \kappa^{+\nu} < \delta_\nu \leq \kappa^{+\nu+1}$

(2) for every $\nu \leq \delta$, if $\delta_\nu = \kappa^{+\nu+1}$, then for every $\nu' < \nu$ $\delta_{\nu'} = \kappa^{+\nu'+1}$

(3) for every $\nu \leq \delta$ and j, $\widetilde{i} \leq j \leq i \quad \delta_\nu = min(A^{0\tau}_j \backslash (B \cap A^{0\tau}_j \cap \kappa^{+\nu+1}))$.

Proof. First let us deal with $B \in \cup\{A^{1\rho}_i \mid \tau \leq \rho \leq \delta\}$. Let $B \in A^{1\rho}_i$ for some ρ, $\tau \leq \rho \leq \delta$. If $B \notin \bigcup_{j<i} A^{1\rho}_j$, then $B \supseteq A^{0\tau}_i$, by (c). So we can take $\widetilde{i} = 0$ and $\langle \delta_\nu \mid \nu \leq \delta \rangle = \langle \kappa^{+\nu+1} \mid \nu \leq \delta \rangle$. If $B \in \bigcup_{j<i} A^{1\rho}_j$, then we can use the previous lemma. Notice that by (d) we can use $A^{0\rho}_j$ as E there.

Now suppose that $B \in A^{1\rho}_j$ for some j, $i < j < \kappa^{+\tau+2}$ and ρ, $\tau \leq \rho \leq \delta$. Then $B \cap A^{0\rho}_i \in A^{1\rho}_i \cup A^{1\rho}_{iin}$, since $\langle \langle A^{0\rho'}_j, A^{1\rho'}_j \rangle \mid \tau \leq \rho' \leq \delta \rangle \geq \langle \langle A^{0\rho'}_i, A^{1\rho'}_i \rangle \mid \tau \leq \rho' \leq \delta \rangle$. If $B \cap A^{0\rho}_i \in A^{1\rho}_i$, then we are in the situation considered above. Otherwise $B \cap A^{0\rho}_i = B_0 \cap B_1 \cap \cdots \cap B_n$ for some $B_k \in A^{1\rho_k}_i$ and $\rho \leq \rho_0 < \rho_1 < \cdots < \rho_n \leq \delta$, $n < \omega$. If for some $k \leq n$ $B_k \notin \bigcup_{j<i} A^{1\rho_k}_j$, then it will contain $A^{0\tau}_i$ and hence its influence on $B \cap A^{0\tau}_i$ will be trivial. Thus, removing such B_k's if necessary, we assume that for every $k \leq n$ $B_k \in \bigcup_{j<i} A^{1\rho_k}_j$. Pick $j_0 < i$ so that $B_k \in A^{1\rho_k}_{j_0}$ for all $k \leq n$. Let $\widetilde{B} = B_0 \cap B_1 \cap \cdots \cap B_n$. Then $|\widetilde{B}| = |B_0| = \kappa^{+\rho_0+1}$ and ${}^{\kappa^{+\rho_0}}\widetilde{B} \subseteq \widetilde{B}$, since each model in the intersection is closed under sequences of the length least $\kappa^{+\rho_0}$. Also $\widetilde{B} \subseteq A^{0\rho_0}_{j_0} \in A^{0\tau}_i$ and hence for every ν, $\rho_0 < \nu \leq \delta$, $\sup(\widetilde{B} \cap \kappa^{+\nu+1}) < \sup(A^{0\rho_0}_{j_0} \cap \kappa^{+\nu+1}) \in A^{0\tau}_i$.

Now the argument above applies to $\widetilde{B}$.

□

Let us define now the main preparation forcing $\mathcal{P}$.

Definition 4.14

The set $\mathcal{P}$ consists of sequences of triples $\langle \langle A^{0\tau}, A^{1\tau}, F^\tau \rangle \mid \tau \leq \delta \rangle$ so that the following holds:

(0) $\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \leq \delta \rangle \in \mathcal{P}'$

(1) for every $\tau_1 \leq \tau_2 \leq \delta$ $F^{\tau_1} \subseteq F^{\tau_2} \subseteq \mathcal{P}^*$

(2) for every $\tau \leq \delta$ F^τ is as follows:

  (a) $|F^\tau| = \kappa^{+\tau+1}$

  (b) for every $p = \langle p_n \mid n < \omega \rangle \in F^\tau$ if $n < \ell(p)$ then every α appearing in p_n is in $A^{0\tau}$; if $n \geq \ell(p)$ and $p_n = \langle a_n, A_n, f_n \rangle$ then every α appearing in f_n is in $A^{0\tau}$ and

(i) $dom\ a_n \cap On \subseteq A^{0\tau} \cap \kappa^{+\delta+2}$

(ii) $dom\ a_n \backslash On$ consists of elements of the following sets: $\{B \subseteq A^{0\tau} \mid \kappa^+ \leq |B| \leq \kappa^{+\tau}\}$, $A^{1\tau}$ and $A^{1\tau}_{in}$ such that the elements of the last two sets are closed under $\kappa^{+\tau}$ sequences of its elements. If $\tau = 0$, then the first set is empty.

(c) there is the largest element of $dom\ a_n$, it belongs to $A^{1\tau}$ an every other element of $dom\ a_n$ belongs to it.

Let us further denote this element as $max^1(p_n)$ or $max^1(a_n)$

(d) if $B \in\ dom\ a_n \backslash On$, then $a_n(B)$ is an elementary submodel of $\mathfrak{a}_{n,k_n}$ of Section 2 with $3 \leq k_n \leq n$, including also δ as a constant. We also require that $|a_n(B)| = \kappa_n^{+n+\tau'+1}$ and ${}^{\kappa_n^{+n+\tau'}}(a_n(B)) \subseteq a_n(B)$, whenever $|B| = \kappa_n^{+\tau'+1}$

(e) if $B \in\ dom\ a_n \backslash On$ and $\alpha \in\ dom\ a_n \cap A^{0\tau}$ then $a_n(\alpha) \in a_n(B)$ iff $\alpha \in B$

(f) If $B, C \in\ dom\ a_n \backslash On$ then

(f1) $B \in C$ iff $a_n(B) \in a_n(C)$

(f2) $B \subset C$ iff $a_n(B) \subset a_n(C)$.

(g) The next two conditions deal with the cofinalities correspondence

(g)(i) if $\alpha \in\ dom\ a_n$ and $cf\alpha \leq \kappa^+$ then $cf a_n(\alpha) \leq \kappa_n^{+n+1}$

(g)(ii) if $\alpha \in\ dom\ a_n$ and $cf\alpha = \kappa^{+\rho}$ then $cf a_n(\alpha) = \kappa_n^{+n+\rho}$ for every $1 \leq \rho \leq \delta + 1$.

(h) if $p \in F^\tau$ and $q \in \mathcal{P}^*$ is equivalent to p $(q \leftrightarrow p)$ with witnessing sequence $\langle k_n \mid n < \omega\rangle$ starting with $k_0 \geq 4$ then $q \in F^\tau$.

(i) if $p = \langle p_n \mid n < \omega\rangle \in F^\tau$ and $q = \langle q_n \mid n < \omega\rangle \in \mathcal{P}^*$ are such that

(i) $\ell(p) = \ell(q)$

(ii) for every $n < \ell(p)$ $p_n = q_n$

(iii) for every $n \geq \ell(p)$ $a_n = b_n$ and $dom\ g_n \subseteq A^{0\tau}$ where $p_n = \langle a_n, A_n, f_n\rangle$, $q_n = \langle b_n, B_n, g_n\rangle$

then $q \in F^\tau$.

(k) if $p = \langle p_n \mid n < \omega\rangle \in F^\tau$ $q = \langle q_n \mid n < \omega\rangle \in \mathcal{P}^*$ are such that

(i) $\ell(q) \geq \ell(p)$

(ii) for every $n \geq \ell(q)$ $p_n = q_n$

(iii) every α appearing in q_n for $n < \ell(q)$ is in $A^{0\tau}$

then $q \in F^\tau$.

The meaning of the last two conditions is that we are free to change inside $A^{0\tau}$ all the components of p except a_n's.

(l) for every $q \in F^\tau$ and $\alpha \in A^{0\tau}$ there is $p \in F^\tau$ $p = \langle p_n \mid n < \omega\rangle$, $p_n = \langle a_n, A_n, f_n\rangle$ $(n \geq \ell(p))$ such that $p \geq^* q$ and $\alpha \in$ *dom* a_n starting with some $n_0 < \omega$.

(m) for every $q \in F^\tau$ and $B \in A^{1\tau} \cup A^{1\tau}_{in}$ as in (b)(ii), there is $p \in F^\tau$ $p = \langle p_n \mid n < \omega\rangle$, $p_n = \langle a_n, A_n, f_n\rangle$ $(n \geq \ell(p))$ such that $p \geq^* q$ and $B \in$ *dom* a_n starting with some $n_0 < \omega$. Also, this p is obtained from q by adding only B and the ordinals needed to be added after adding B.

(n) Let $p, q \in F^\tau$ be so that

(i) $\ell(p) = \ell(q)$

(ii) $max^1(p_n) = max^1(p_n)$, $max^1(q_n) = max^1(q_m)$ and $max^1(q_n) \in$ *dom* a_n, where $n, m \geq \ell(p)$, $p_n = \langle a_n, A_n, f_n\rangle$, $q_n = \langle b_n, B_n, g_n\rangle$

(iii) $p_n = q_n$ for every $n < \ell(p)$

(iv) f_n, g_n are compatible for every $n \geq \ell(p)$

(v) $a_n \restriction max^1(q_n) \subseteq b_n$ for every $n \geq \ell(p)$, where

$$a_n \restriction B = \{\langle t \cap B, s \cap a_n(B)\rangle \mid \langle t, s\rangle \in a_n\}$$

then the union of p and q is in F^τ where the union is defined in obvious fashion taking $p_n \cup q_n$ for $n < \ell(p)$, we take at each $n \geq \ell(p)$ $a_n \cup b_n$, $f_n \cup g_n$ etc.

(o) there is $F^{\tau *} \subseteq F^\tau$ dense in F^τ under $\leq^*$ such that every $\leq^*$-increasing sequence of elements of $F^{\tau *}$ having the union in $\mathcal{P}^*$ has it also in F^τ. We require that $F^{\tau *}$ will be closed under the equivalence relation $\leftrightarrow$.

(p) let $p = \langle p_n \mid n < \omega\rangle \in F^\tau$ and $p_n = \langle a_n, A_n, f_n\rangle$ $(\ell(p) \leq n < \omega)$. If for every n, $\omega > n \geq \ell(p)$ $B \in$ $dom\ a_n \backslash On$, $|B| = \kappa^{+\tau+1}$ or $B \in A^{1\tau'}$ for some $\tau' \leq \tau$, *then* $p \restriction B \in F^{\tau'}$, where $p \restriction B = \langle p_n \restriction B \mid n < \omega\rangle$ and for every $n < \ell(p)$ $p_n \restriction B$ is the usual restriction of the function p_n to B; if $n \geq \ell(p)$ then $p_n \restriction B = \langle a_n \restriction B, B_n, f_n \restriction B\rangle$ with $a_n \restriction B$ defined in (n)(v), $f_n \restriction B$ is the usual restriction and B_n is the projection of A_n by $\pi_{maxp_n,B}$.

(q) let $p = \langle p_n \mid n < \omega\rangle \in F^\tau$, $p_n = \langle a_n, A_n, f_n\rangle$ and $A^{0\tau} \notin$ $dom\ a_n$ $(\omega > n \geq \ell(p))$. Let $\langle \sigma_n \mid \omega > n \geq \ell(p)\rangle$ be so that

(i) $\sigma_n \prec \mathfrak{a}_{n,k_n}$ for every $n \geq \ell(p)$

(ii) $\langle k_n \mid n \geq \ell(p)\rangle$ is increasing

(iii) $k_0 \geq 5$

(iv) ${}^{\kappa_n^{+n+\tau}}\sigma_n \subseteq \sigma_n$ and $|\sigma_n| = \kappa_n^{+n+\tau+1}$ for every $n \geq \ell(p)$

(v) $rnga_n \in \sigma_n$ for every $n \geq \ell(p)$.

Then the condition obtained from p by adding $\langle A^{0\tau}, \sigma_n\rangle$ to each p_n with $n \geq \ell(p)$ belongs to F^τ.

(r) if A is an elementary submodel of $H(\kappa^{+\delta+2})$ of cardinality $\kappa^{+\rho+1}$, closed under $\kappa^{+\rho}$-sequences and including $\langle\langle A^{0\tau'}, A^{1\tau'}\rangle \mid \tau' \leq \delta\rangle$ for some $\rho < \tau$, then A is addable to any $p \in F^\tau \cap A$, with the maximal element of *dom* a_n's $A^{0\tau}$. I.e. $A \cap A^{0\tau}$ can be added to p remaining in F^τ.

Definition 4.15

Let $\langle\langle A^{0\tau}, A^{1\tau}, F^\tau\rangle \mid \tau \leq \delta\rangle$ and $\langle\langle B^{0\tau}, B^{1\tau}, G^\tau\rangle \mid \tau \leq \delta\rangle$ be in $\mathcal{P}$. We define

$$\langle\langle A^{0\tau}, A^{1\tau}, F^\tau\rangle \mid \tau \leq \delta\rangle > \langle\langle B^{0\tau}, B^{1\tau}, G^\tau\rangle \mid \tau \leq \delta\rangle$$

iff

(1) $\langle\langle A^{0\tau}, A^{1\tau}\rangle \mid \tau \leq \delta\rangle > \langle\langle B^{0\tau}, B^{1\tau}\rangle \mid \tau \leq \delta\rangle$ in $\mathcal{P}'$

(2) for every $\tau \leq \delta$

(a) $F^\tau \supseteq G^\tau$

(b) for every $p \in F^\tau$ and $B \in B^{1\tau} \cup B^{1\tau}_{in}$ if for every $n \geq \ell(p)$ $B \in$ $dom\ a_n$ then $p \restriction B \in G^\tau$, where the restriction is as defined in 4.14 (p), $p = \langle p_n \mid n < \omega\rangle$, $p_n = \langle a_n, A_n, f_n\rangle$ for $n \geq \ell(p)$.

Definition 4.16

Let $\tau \leq \delta$. Set $\mathcal{P}_{\geq\tau} = \{\langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \mid \exists\langle\langle A^{0\nu}, A^{1\nu}, F^{\nu}\rangle \mid \nu < \tau\rangle$ $\langle\langle A^{0\nu}, A^{1\nu}, F^{\nu}\rangle \mid \nu < \tau\rangle \frown \langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}\}$.

Let $G(\mathcal{P}_{\geq\tau}) \subseteq \mathcal{P}_{\geq\tau}$ be generic. Define $\mathcal{P}_{<\tau} = \{\langle\langle A^{0\nu}, A^{1\nu}, F^{\nu}\rangle \mid \nu < \tau\rangle, \mid \exists\langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in G(\mathcal{P}_{\geq\tau})\ \langle\langle A^{0\nu}, A^{1\nu}, F^{\nu}\rangle \mid \nu < \tau\rangle \frown \langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \in \mathcal{P}\}$.

The following lemma is obvious

Lemma 4.17 *$\mathcal{P} \simeq \mathcal{P}_{\geq\tau} * \underset{\sim}{\mathcal{P}}_{<\tau}$ for every $\tau \leq \delta$.*

Lemma 4.18 *For every $\tau \leq \delta$ $\mathcal{P}_{\geq\tau}$ is $\kappa^{+\tau+2}$-strategically closed.*

Proof. Fix $\tau \leq \delta$. Let $\langle\langle A_i^{0\rho}, A_i^{1\rho}, F_i^{\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle \mid i < i^* < \kappa^{+\tau+2}\rangle$ be an increasing sequence of conditions in $\mathcal{P}_{\geq\tau}$ already generated by playing the game. We need to proceed and define the move $\langle\langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho}, F_{i^*}^{\rho}\rangle \mid \tau \leq \rho \leq \delta\rangle$ of Player I at stage i^*.

Suppose first that i^* is a successor ordinal. Then $i^* - 1$ exists. We proceed by induction on ρ.

Set $\tilde{A}_{i^*}^{0\rho}$ to be the closure under the Skolem functions and $\kappa^{+\rho}$-sequences of $\langle\langle A_i^{j\rho'} | i < i^*\rangle | \tau \leq \rho' \leq \delta\rangle$ $(j \in 2)$ and $\langle A_{i^*}^{1\rho'} \mid \tau \leq \rho' < \rho\rangle$. In the previous section we took $A_{i^*}^{0\rho} = \tilde{A}_{i^*}^{0\rho}$, but here there is a complication. Thus, we need to take care of intersections of the form $B \cap \tilde{A}_{i^*}^{0\rho}$ for B in $A_{i^*-1}^{1\rho'}$ with $\rho' > \rho$. Such elements are supposed to be in $A_{i^* in}^{1\rho}$. So, a dense closed set $F_{i^*}^{\rho *} \subseteq F_{i^*}^{\rho}$ should deal with them also. Thus we need to insure that in particular, restrictions to $A_{i^*-1}^{0\rho'}$ of unions from $F_{i^*}^{\rho *}$ are in $F_{i^*-1}^{\rho'}$. In order to achieve this, let us define $A_{i^*}^{0\rho}$ to be an increasing union of length $\kappa^{+\rho+1}$. For each ρ', $\rho \leq \rho' \leq \delta$ we fix in advance a dense closed $F_{i^*-1}^{\rho' *} \subseteq F_{i^*-1}^{\rho'}$. Let $A_{i^*0}^{0\rho}$ be the closure under the Skolem functions and $\kappa^{+\rho}$-sequences of $\tilde{A}_{i^*}^{0\rho}$, $\langle F_{i^*-1}^{\rho'} \mid \rho \leq \rho' \leq \delta\rangle$ and $\langle F_{i^*-1}^{\rho' *} \mid \rho \leq \rho' \leq \delta\rangle$. For a limit α, $0 < \alpha < \kappa^{+\rho+1}$ let $A_{i^*\alpha}^{0\rho} = \bigcup_{\alpha'<\alpha} A_{i^*\alpha'}^{0\rho}$. Let $A_{i^*\alpha+1}^{0\rho}$ be the closure of $A_{i^*\alpha}^{0\rho} \cup \{A_{i^*\alpha}^{0\rho}\}$ under the Skolem functions and $\kappa^{+\rho}$-sequences, for every $\alpha < \kappa^{+\rho+1}$. Finally we set $A_{i^*}^{0\rho} = \bigcup_{\alpha<\kappa^{+\rho+1}} A_{i^*\alpha}^{0\rho}$. Let $A_{i^*}^{1\rho} = \bigcup_{i<i^*} A_i^{1\rho} \cup \{A_{i^*\alpha}^{0\rho} \mid \alpha < \kappa^{+\rho+1}\} \cup \{A_{i^*}^{0\rho}\}$. Now we turn to definitions of $F_{i^*}^{\rho}$ and its dense closed subset $F_{i^*}^{\rho *}$. First we should to obey the inclusions $F_{i^*-1}^{\rho} \subseteq F_{i^*}^{\rho}$ and $F_{i^*}^{\rho'} \subseteq F_{i^*}^{\rho}$ for every $\rho' < \rho$. Then let us generate new elements in the following fashion.

Let $q \in F_{i^*-1}^{\rho} \cup \cup\{F_{i^*}^{\rho'} \mid \rho' < \rho\}$. We extend it first to a condition q' by adding $A_{i^*-1}^{0\delta}$. If $q \in F_{i^*}^{\rho'}$ for some $\rho' < \rho$ then we assume that this is possible as an inductive assumption. Now, we extend q' to q_0 by adding $A_{i^*0}^{0\rho}$. This is possible by 4.14(2(r)). Let $r \in F_{i^*-1}^{\delta}$ be an extension of $q_0 \restriction A_{i^*-1}^{0\delta}$. Consider $r \restriction A_{i^*0}^{0\rho}$ and its combination with q'. By Lemma 4.11

both of them are in $A^{0\rho}_{i^*1}$, since $F^{\delta}_{i^*-1}, A^{0\rho}_{i^*0}, q \in A^{0\rho}_{i^*1}$ and $A^{0\rho}_{i^*1} \supseteq \kappa^{+\rho+1}$. Let us assume for notational simplicity that q' does not contain parts from $F^{\rho'}_{i^*}$ for $\rho' < \rho$. Otherwise we will need only to add them all the time. Pick $\widetilde{q}_{01} \geq^* r \restriction A^{0\rho}_{i^*0}$ in $A^{0\rho}_{i^*1} \cap F^{\delta *}_{i^*-1}$. It is possible since both $F^{\delta *}_{i^*-1}$ and $r \restriction A^{0\rho}_{i^*0}$ are in $A^{0\rho}_{i^*1}$. Extend it to q_1 by adding $A^{0\rho}_{i^*1}$. Define $F^{\rho}_{i^*1}$ to be the set of all such q_1's.

Now consider some $q_1 \in F^{\rho}_{i^*1}$. $q_1 \restriction A^{0\delta}_{i^*-1}$ is in $F^{\delta}_{i^*-1}$ by 4.14(2(r)). Let $q_{11} \geq^* q_1 \restriction A^{0\delta}_{i^*-1}$ be in $A^{0\rho}_{i^*2} \cap F^{\delta *}_{i^*-1}$. Extend it to q_2 by adding q_1 and $A^{0\rho}_{i^*2}$. Let $F^{\rho}_{i^*2}$ be the set of all such q_2's. We continue in the same fashion and define q_α's for every $\alpha < \kappa^{+\rho+1}$. There is no problem at limit stages since always $q_\alpha \restriction A^{0\delta}_{i^*-1} \in F^{\delta *}_{i^*-1}$. Thus, let α be a limit ordinal and $\langle q_\gamma \mid \gamma < \beta \rangle \in \bigcup_{\alpha' < \alpha} F^{\rho}_{i^*\alpha'}$ be $\leq^*$-increasing and the union $q \in \mathcal{P}^*$. Then, $\langle q_\gamma \restriction A^{0\delta}_{i^*-1} \mid \gamma < \beta \rangle$ is $\leq^*$-increasing sequence inside $F^{\delta *}_{i^*-1}$. So, $q \restriction A^{0\delta}_{i^*-1} \in F^{\delta *}_{i^*-1}$. Hence we can define $F^{\rho}_{i^*\alpha}$ to be the set of all such q's.

Finally, we define $F^{\rho *}_{i^*}$ to be the set obtained by adding $A^{0\rho}_{i^*}$ as the maximal element to members of $\cup \{ F^{\rho}_{i^*\alpha} \mid \alpha < \kappa^{+\rho+1} \}$.

Claim 4.18.1

$F^{\rho *}_{i^*}$ is $\leq^*$-closed and for every $q \in F^{\rho^*}_{i^*}$ $q \restriction A^{0\delta}_{i^*-1} \in F^{\delta}_{i^*-1}$.

Proof. The addition of $A^{0\rho}_{i^*}$ on the top does not effect the closure. If $q \in F^{\rho *}_{i^*}$, then $q \restriction A^{0\delta}_{i^*-1}$ with $A^{0\rho}_{i^*}$ removed is in $F^{\delta *}_{i^*-1}$. By 4.14(2(o)), it is possible to add $A^{0\rho}_{i^*}$ remaining in $F^{\delta}_{i^*-1}$ (but probably no more in $F^{\delta *}_{i^*-1}$).

$\square$ of the claim.

$F^{\rho}_{i^*}$ is obtained in obvious fashion including in addition to $F^{\rho *}_{i^*}$ all the necessary stuff in order to satisfy 4.14. We need to check that $F^{\rho *}_{i^*}$ is still dense. Actually the only problem is with conditions p containing $A^{0\rho}_{i^*}$. Thus, $p \restriction A^{0\delta}_{i^*-1}$ was not probably taken into account during the construction of $F^{\rho *}_{i^*}$ because of the set $A^{0\rho}_{i^*} \cap A^{0\delta}_{i^*-1}$ in it or because of $A^{0\rho}_{i^*} \cap B$'s for $B \in A^{1\rho'}_{i^*-1}$ and B is the domain of p.

Let us rule out this possibility using Lemma 4.13. Thus its conclusion (3) applied κ many times implies that there will be $\widetilde{\alpha} < \alpha$ for every B as mentioned above

$$min(A^{0\rho}_{i^*\widetilde{\alpha}} \setminus \sup(B \cap A^{0\rho}_{i^*\widetilde{\alpha}} \cap \kappa^{+\nu+1})) = min(A^{0\rho}_{i^*} \setminus \sup(B \cap A^{0\rho}_{i^*} \cap \kappa^{+\nu+1}))$$

for every $\nu \leq \delta$. Now we extend p to q by adding $A^{0\rho}_{i^*\widetilde{\alpha}}$ and $A^{0\rho}_{i^*\widetilde{\alpha}+1}$ to it. Let r be obtained from q by removing $A^{0\rho}_{i^*}$. Then $q \restriction A^{0\rho}_{i^*\widetilde{\alpha}+1} = r$. Let r' be obtained from r by removing $A^{0\rho}_{i^*\widetilde{\alpha}+1}$. r' was taken into account in the definition of $F^{\rho}_{i^*\widetilde{\alpha}+1}$. So there is some $r^* \geq^* r'$ in $F^{\rho}_{i^*\widetilde{\alpha}+1}$.

The maximal set of r^*, $A^{0\rho}_{i^*\tilde{\alpha}+1}$ is added back. Then $r^* \geq^* r = q \restriction A^{0\rho}_{i^*\tilde{\alpha}+1}$. Let s be obtained from r^* by adding $A^{0\rho}_{i^*}$. By the definition of $F^{\rho *}_{i^*}$, $s \in F^{\rho *}_{i^*}$. Clearly then $s \geq^* p$.

Now let i^* be a limit ordinal. If $cfi^* = \kappa^{+\tau+1}$, then the treatment of τ-th coordinate will be a bit simpler than the treatment of the rest due to the fact that $A^{0\tau}_{i^*}$ should be of cardinality $\kappa^{\tau+1}$ and closed under $\kappa^{+\tau}$-sequences. Thus, in this case we take $A^{0\tau}_{i^*}$ to be the union of $A^{0\tau}_i$ with $i < i^*$. Set $A^{1\tau}_{i^*} = \bigcup_{i<i^*} A^{1\tau}_i \cup \{A^{0\tau}_{i^*}\}$. $F^{\tau *}_{i^*}$ is defined as the set of all elements of $\cup\{F^{\tau *}_i \mid i < i^*, i \text{ is even }\}$ with $A^{0\tau}_{i^*}$ added as the maximal element.

Suppose now that ($cfi^* = \kappa^{+\tau+1}$ and $\tau < \rho \leq \delta$) or ($cfi^* < \kappa^{+\tau+1}$ and $\tau \leq \rho \leq \delta$). Assume that for all ρ', $\tau \leq \rho' < \rho$ $\langle A^{0\rho'}_{i^*}, A^{1\rho'}_{i^*}, F^{\rho'}_{i^*}\rangle$ is defined. Let us define $\langle A^{0\rho}_{i^*}, A^{1\rho}_{i^*}, F^{\rho}_{i^*}\rangle$. The treatment of this case is very similar to the case of the successor ordinal. We define $\tilde{A}^{0\rho}_{i^*}, A^{0\rho}_{i^*\alpha}(\alpha < \kappa^{+\rho+1})$, $A^{0\rho}_{i^*}$ and $A^{1\rho}_{i^*}$ as they were defined at a successor stage. Now suppose that $q \in F^{\rho}_{i^{**}}$ for some even $i^{**} < i^*$. We first extend q to condition q' by adding $A^{0\delta}_{i^{**}}$ to it. Now we proceed as at a successor stage replacing $i^* - 1$ by i^{**}. This will define $F^{\rho^*}_{i^*}(i^{**})$ the part of $F^{\rho *}_{i^*}$ depending on i^{**}. In order to obtain $F^{\rho *}_{i^*}$ let us take the union of $F^{\rho *}_{i^*}$ (i^{**}) over all even $i^{**} < i^*$. Using the appropriate inductive assumption, it is easy to insure that for every $q \in F^{\rho *}_{i^*}$ if for some even $i^{**} < i^*$ $A^{0\rho}_{i^{**}}$ appears in q, then $A^{0\delta}_{i^{**}}$ appears as well and $q \restriction A^{0\delta}_{i^{**}} \in F^{\delta *}_{i^{**}}$.

The rest of the proof is routine.

$\square$

Lemma 4.19 *For every $\tau \leq \delta$ $\mathcal{P}_{<\tau}$ satisfies $\kappa^{+\tau+2}$-c.c. in $V^{\mathcal{P}_{\geq\tau}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \underset{\mathcal{P}_{\geq\tau}}{\Vdash} (\langle\langle\langle \underset{\sim}{A}^{0\nu}_{\alpha}, \underset{\sim}{A}^{1\nu}_{\alpha}, \underset{\sim}{F}^{0}_{\alpha}\rangle \mid \nu < \tau\rangle \mid \alpha < \kappa^{+\tau+2}\rangle \quad \text{is an antichain in} \quad \underset{\sim}{\mathcal{P}}_{<\tau}) .$$

We use the winning strategy of the player II defined in 4.18 in order to decide the names of the elements of the antichain. Thus let $\langle\langle A^{0\rho}_{\alpha}, A^{1\rho}_{\alpha}, F^{\rho}_{\alpha}\rangle \mid \rho \leq \delta, \alpha < \kappa^{+\tau+2}\rangle$ be so that for every $\alpha < \kappa^{+\tau+2}$ $\langle\langle A^{0\rho}_{\alpha+1}, A^{1\rho}_{\alpha+1}, F^{\rho}_{\alpha+1}\rangle \mid \tau \leq \rho \leq \delta\rangle \Vdash'' {}_{\mathcal{P}_{\geq\tau}} \forall_{\alpha' \leq \alpha+1} \langle \underset{\sim}{A}^{0\nu}_{\alpha'}, \underset{\sim}{A}^{1\nu}_{\alpha'}, \underset{\sim}{F}^{\nu}_{\alpha'}\rangle \mid \nu < \tau\rangle$ $= \langle\langle \check{A}^{0\nu}_{\alpha'}, \check{A}^{1\nu}_{\alpha'}, \check{F}^{\nu}_{\alpha'} \mid \nu < \tau\rangle''$ and for every $\alpha < \kappa^{+\tau+2}$ of cofinality $\kappa^{+\tau+1}$ $A^{0\tau}_{\alpha} = \bigcup_{\beta<\alpha} A^{0\tau}_{\beta}$. Using 4.11 we form a Δ-system satisfying the same conditions as in 3.18 with 1 replaced by τ and 0 by any $\nu < \tau$. Thus the condition (e) of 3.18 in the present situation is as follows: for every $\nu < \tau$,

$$(A^{0\nu}_{\alpha+1} \cap \bigcup_{\gamma<\alpha} A^{0\tau}_{\gamma}) \cap \kappa^{+\delta+1} = (A^{0\nu}_{\beta+1} \cap \bigcup_{\gamma<\beta} A^{0\tau}_{\gamma}) \cap \kappa^{+\delta+1} = A^{0\nu}_{\alpha+1} \cap A^{0\nu}_{\beta+1} \cap \kappa^{+\delta+1} .$$

Now suppose that $\alpha < \beta < \kappa^{+\tau+2}$, $cf\alpha = cf\beta = \kappa^{+\tau+1}$. We like to show that $\langle\langle A^{0\rho}_{\alpha+1}, A^{1\rho}_{\alpha+1}, F^{\rho}_{\alpha+1}\rangle \mid \rho \leq \delta\rangle$ and $\langle\langle A^{0\rho}_{\beta+1}, A^{1\rho}_{\beta+1}, F^{\rho}_{\beta+1}\rangle \mid \rho \leq \delta\rangle$ are compatible. Clearly, there is no problem with ρ's above τ. Define a stronger condition $\langle\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle \mid \rho \leq \delta\rangle$. For $\rho \geq \tau$ let $\langle A^{0\rho}, A^{1\rho}, F^{\rho}\rangle = \langle A^{0\rho}_{\beta+1}, A^{1\rho}_{\beta+1}, F^{\rho}_{\beta+1}\rangle$. If $\rho < \tau$, then we proceed by induction on ρ. So suppose that for every $\rho' < \rho < \tau$ $A^{0\rho'}, A^{1\rho'}$ and $F^{\rho'}$ are already defined. Defined $A^{0\rho}, A^{1\rho}, F^{\rho}$. First set $\tilde{A}^{0\rho} = A^{0\tau}_{\beta+1}\cap$ the closure under the Skolem functions and $\kappa^{+\rho}$-sequences of $\langle\langle A^{0\rho'}_{\gamma}, A^{1\rho'}_{\gamma}, F^{\rho'}_{\gamma}\rangle \mid \gamma \leq \beta+1, \rho' \leq \delta\rangle$, $\langle F^{\rho' *}_{\gamma} \mid \gamma \leq \beta+1, \rho' \leq \delta\rangle$, $\{\alpha+1\}$, $\langle A^{1\rho'} \mid \rho' < \rho\rangle$. Define $A^{0\rho}$ to be the union of the increasing continuous chain $\langle A^{0\rho}(\gamma) \mid \gamma < \kappa^{+\rho+1}\rangle$ where $A^{0\rho}(0) = \tilde{A}^{0\rho}$ and $A^{0\rho}(\gamma+1)$ is the closure of $A^{0\rho}(\gamma)\cup\{A^{0\rho}(\gamma)\}$ under the Skolem functions and $\kappa^{+\rho}$-sequences. Let $A^{1\rho} = A^{1\rho}_{\alpha+1}\cup A^{1\rho}_{\beta+1}\cup\{A^{0\rho}(\gamma) \mid \gamma < \kappa^{+\rho+1}\}\cup\{A^{0\rho}\}$. Let us turn to definitions of F^{ρ} and $F^{\rho *}$. Let $p^0 = \langle p^0_n \mid n < \omega\rangle \in F^{\rho}_{\alpha+1}$ and $p^1 = \langle p^1_n \mid n < \omega\rangle \in F^{\rho}_{\beta+1}$ be so that

(1) $\ell(p^0) = \ell(p^1)$

(2) p^0_n, p^1_n are compatible for every $n < \ell(p^0)$

(3) for every $n \geq \ell(p^0)$ $A^{0\tau}_{\alpha}, A^{0\rho}_{\alpha+1} \in$ *dom* a^0_n and $A^{0\tau}_{\beta}, A^{0\rho}_{\beta+1} \in$ *dom* a^1_n, where, as usual $p^i_n = \langle a^i_n, A_n(i), f^i_n\rangle$ $(i = 0, 1)$.

(4) $a^0_n \restriction A^{0\tau}_{\alpha} = a^1_n \restriction A^{0\tau}_{\beta}$

(5) p^0 and p^1 are compatible in $\mathcal{P}^*$, i.e. they can be combined together without destroying the preservation of order (both "$\in$" and "$\subseteq$").

Notice that by 4.14(2)(r) each $A^{0\rho}(\gamma)$ is addable to p^0 and p^1.

Now, $F^{\nu}_{\alpha+1} \subseteq F^{\tau}_{\alpha+1} \subseteq F^{\tau}_{\beta} \subseteq F^{\tau}_{\beta+1}$ and $F^{\nu}_{\beta+1} \subseteq F^{\tau}_{\beta+1}$. Hence, $p^0, p^1 \in F^{\tau}_{\beta+1} \subseteq F^{\delta}_{\beta+1}$. Using 4.14(2(n) and 2(q)), as in the proof of 3.18, they can be combined together into a condition $q \in F^{\delta}_{\beta+1}$ with $A^{0\delta}_{\beta+1}$ as the maximal set. Thus, we add to $p^0 \in F^{\tau}_{\beta}$ as the maximal element $A^{0\tau}_{\beta}$. Let $\tilde{p}^0$ be the resulting condition. Let $\tilde{p}^1$ be obtained from p^1 by adding $A^{0\tau}_{\beta+1}$ as the maximal element. By (4) above and 4.14((2(n)) the combination q of $\tilde{p}^0$ and $\tilde{p}^1$ is in $F^{\tau}_{\beta+1}$.

Let $F^{\rho *}_{\alpha+1}$ and $F^{\rho *}_{\beta+1}$ be the fixed dense closed subsets of $F^{\rho}_{\alpha+1}$ and $F^{\rho}_{\beta+1}$ respectively. As in 3.18 for each q as above we can find $q^* \in F^{\tau}_{\beta+1}$ such that $q \leq^* q^*$, $q^* \restriction A^{0\rho}_{\alpha+1} \in F^{\rho *}_{\alpha+1}$ and $q^* \restriction A^{0\rho}_{\beta+1} \in F^{\rho *}_{\beta+1}$. Now we add $A^{0\rho}(0)$ to q^* and extend $q^* \restriction A^{0\tau}_{\beta+1}$ to an element q^{**} of $F^{\tau *}_{\beta+1} \cap A^{0\rho}(1)$, where $F^{\tau *}_{\beta+1}$ is the fixed dense closed subset of $F^{\tau}_{\beta+1}$. At the next stage we extend the combination of q^* and q^{**} by adding $A^{0\rho}(1)$. We continue to extend in the same fashion for every $\gamma < \kappa^{+\rho+1}$, exactly as in Lemma 4.18. Finally, we add $A^{0\rho}$

as the maximal coordinate. $F^{\rho *}$ is generated by this process. F^ρ is obtained from $F^{\rho *}$ by adding everything necessary in order to satisfy the requirements of 4.14. Thus, let us check 4.14(2(n)), which contains a small new point. Let $p \in F^\rho$ which includes both $A^{0\rho}_{\alpha+1}$ and $A^{0\rho}_{\beta+1}$, and $F^\rho_{\alpha+1} \ni q \geq^* p \restriction A^{0\rho}_{\alpha+1}$. We need to show that then the combination of p and q is in F^ρ. By the choice of $F^{\rho *}$ and then F^ρ, $A^{0\tau}_\beta$ is in p. Then, the choice of the Δ-system implies that $p \restriction A^{0\tau}_\beta$ with $A^{0\tau}_\beta$ removed is exactly $p \restriction A^{0\rho}_{\alpha+1}$. Since everything inside $A^{0\rho}_{\beta+1}$ intersected with $A^{0\tau}_\beta$ is already inside the kernel, i.e. $A^{0\rho}_0$. Let $\widetilde{q}$ be obtained from q by adding $A^{0\tau}_\beta$ as the maximal element. Then $\widetilde{q} \in F^\tau_\beta \subseteq F^\tau_{\beta+1}$. Now, both $\widetilde{q}$ and p are in $F^\tau_{\beta+1}$ and $p \restriction A^{0\tau}_\beta \leq^* \widetilde{q}$. So, by 4.14(2(n)) for $F^\tau_{\beta+1}$, the combination of p and $\widetilde{q}$ is in $F^\tau_{\beta+1}$. Clearly, it is the same as the combination of p and q. So the combination of p and q is in $F^\tau_{\beta+1}$ and hence also in F^ρ.

This completes the inductive definition of $\langle A^{0\rho}, A^{1\rho}, F^\rho \rangle$ and hence also $\langle \langle A^{0\rho}, A^{1\rho}, F^\rho \rangle \mid \rho \leq \delta \rangle$. Which leads to the contradiction since the last condition is stronger than both $\langle \langle A^{0\rho}_{\alpha+1}, A^{1\rho}_{\alpha+1}, F^\rho_{\alpha+1} \rangle \mid \rho \leq \delta \rangle$ and $\langle \langle A^{0\rho}_{\beta+1}, A^{1\rho}_{\beta+1}, F^\rho_{\beta+1} \rangle \mid \rho \leq \delta \rangle$.

□

Let $G \subseteq \mathcal{P}$ be generic. We define as in Section 3 our main forcing $\mathcal{P}^{**}$ to be $\cup \{ F^0 \mid \exists A^{00}, A^{10}, \langle \langle A^{0\tau}, A^{1\tau}, F^\tau \rangle \mid 0 < \tau \leq \delta \rangle \ \langle \langle A^{0\nu}, A^{1\nu}, F^\nu \rangle \mid \nu \leq \delta \rangle \in G \}$. Then such forcing $\langle \mathcal{P}^{**}, \leq \rangle$ shares the properties of those of Section 3.

The proof of the next lemma repeats the proof of 3.19 with 4.13 replacing 3.8.

Lemma 4.20 *In $V^{\mathcal{P}}$, $\langle \mathcal{P}^{**}, \rightarrow \rangle$ satisfies κ^{++}-c.c.*

5 Wide Gaps With Shorter Extenders

In this section we shall implement Shelah's idea [Sh1] which allows us here to use shorter extenders while making wider gaps than those of the previous section. Unfortunately we are unable to break completely the linkage between the number of cardinals in between κ and 2^κ and the lengths of extenders used over κ_n's, and by Shelah *pcf* theory [Sh2] for good reasons.

Let us first deal with countable gaps. Our aim will be to show the following:

Theorem 5.1 *Let $\alpha < \omega_1$. Suppose that κ is a cardinal of cofinality ω and for every $n < \omega$ $\{\beta < \kappa \mid o(\beta) = \beta^{+n}\}$ is unbounded in κ. Then there is a cofinality preserving extension satisfying $2^\kappa \geq \kappa^{+\alpha}$.*

Remark. By [Git-Mit] this provides the equiconsistency for infinite α's. Or if κ is singular in the core model.

Proof. Let us use the notation of previous sections. Assume now that each κ_n has an extender of the strength κ^{+n+2+n}. Fix a countable ordinal α. Let $\alpha\backslash\{0,1,2\} = \{\alpha_k \mid k<\omega\}$. We set $D_1 = \{\kappa^{++}, \kappa^{+\alpha+1}\}$, $D_2 = D_1 \cup \{\kappa^{+\alpha_0+1}\}$ and $D_{k+1} = D_k \cup \{\kappa^{+\alpha_{k-1}+1}\}$ for every $k<\omega$.

We like to use the forcing of the type $\mathcal{P}$ of the previous section with corrections on the number of cardinals reserved at each level $n<\omega$.

First let us assign cardinals below κ to the cardinals $\{\kappa^{+\beta+1} \mid 1 \leq \beta \leq \alpha\}$. At level 1 let κ^{++} corresponds to κ_1^{+1+2} and $\kappa^{+\alpha+1}$ corresponds to κ_1^{+1+2+1}. At level 2 let κ^{++} corresponds to κ_2^{+2+2}, $\kappa^{+\alpha+1}$ to κ_2^{+2+2+2} and $\kappa^{+\alpha_0+1}$ to κ^{+2+2+1}. At level n let $\kappa^{+\nu_i}$ corresponds to κ_n^{+n+2+i}, where $\{\kappa^{+\nu_i} \mid i \leq n\}$ is an increasing enumeration of D_n.

Definition 5.2. The forcing notion $\mathcal{P}(\alpha)$ consists of all sequences $\langle\langle A^{0\nu}, A^{1\nu}, F^\nu\rangle \mid \nu \leq \alpha\rangle$ so that

(1) $\langle\langle A^{0\nu}, A^{1\nu}\rangle \mid \nu \leq \alpha\rangle$ is as in 4.14

(2) for every $\nu \leq \alpha$ F^ν consists of $p = \langle p_n \mid n<\omega\rangle$ and every $n \geq \ell(p)$, $p_n = \langle a_n, A_n, f_n\rangle$ as in 4.14 with the following changes related only to a_n:

(i) $a_n(\kappa^{+\nu_i}) = \kappa_n^{+n+2+i}$ for every $i \leq n$, where $\{\kappa^{+\nu_i} \mid i \leq n\}$ is the increasing enumeration of D_n.

(ii) $\kappa^{+\nu+1}, \kappa^{+\nu+2} \in D_{\ell(p)}$ and hence are in each D_m with $m \geq \ell(p)$

(iii) only models of cardinalities in D_n can appear in *dom* a_n.

The definition of the order on $\mathcal{P}(\alpha)$ is as in the previous sections. Also $(\mathcal{P}(\alpha))_{\geq\tau}$ and $(\mathcal{P}(\alpha))_{<\tau}$ are defined as in Section 4.

The basic lemmas of the previous section hold here (almost) without changes.

Lemma 5.3 *For every $\tau \leq \alpha$ $(\mathcal{P}(\alpha))_{\geq\tau}$ is $\kappa^{+\tau+2}$-strategically closed.*

Lemma 5.4 *For every $\tau \leq \alpha$ $(\mathcal{P}(\alpha))_{<\tau}$ satisfies $\kappa^{+\tau+1}$-c.c. inside $V^{(\mathcal{P}(\alpha))_{\geq\tau}}$.*

Finally, we define $\langle\mathcal{P}^*(\alpha), \rightarrow\rangle$ in $V^{\mathcal{P}(\alpha)}$, as in Section 4.

Lemma 5.5 *$\langle\mathcal{P}^*(\alpha), \rightarrow\rangle$ satisfies κ^{++}-c.c. in $V^{\mathcal{P}(\alpha)}$.*

Using similar ideas it is not hard to show the following

Theorem 5.6 *Suppose that $\kappa = \bigcup_{n<\omega}\kappa_n$, $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ Let α be a countable ordinal and $\delta < \kappa_0$. Assume that for every $n<\omega$ $o(\kappa_n) = \kappa_n^{+\delta+m_n}+1$, where $\langle m_n \mid n<\omega\rangle$ is* *a converging to infinity sequence of natural numbers. Then there is a cofinality preserving extension satisfying* $2^\kappa \geq \kappa^{+\delta+\alpha}$.

In a further paper we plan to extend the present techniques in order to handle arbitrary gaps between κ and 2^κ. The subject of consistency strength here was almost completely ignored. We hope to deal with this matter in a further paper as well.

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