Remarks on non-closure of the preparation forcing of [2] and an off-piste version of it.

Moti Gitik

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The preparation forcing \( P' \) of [2] Section 1 is \( \kappa^{++} \)-strategically closed by Lemma 1.1.19. We would like to examine the reasons for lack of closure and directed closure of this forcing.

1 The first reason for a non–closure.

Let us first point out that the forcing \( P' \) is \( \omega_1 \)-closed.

**Proposition 1.1** \( P' \) is \( \omega_1 \)-closed.

**Proof.** Let \( \langle p_n \mid n < \omega \rangle \) be an increasing sequence of conditions in \( P' \). Assume that for each \( n < \omega \) we have

\[
p_n = \langle \langle A_{n0}^{\kappa+}, A_{n1}^{\kappa+}, C_{n0}^{\kappa+} \rangle, A_{n1}^{\kappa++} \rangle.
\]

Arrange by induction that \( A_{n0}^{\kappa+} \in C_{n+1}^{\kappa+}(A_{n+1}^{\kappa+}) \), for every \( n < \omega \). Note that at each stage only finitely many switches are needed for this. Now we just take unions. Set

\[
B_{0}^{\kappa+} = \bigcup_{n < \omega} A_{n0}^{\kappa+},
\]

\[
B_{1}^{\kappa+} = \bigcup_{n < \omega} A_{n1}^{\kappa+} \cup \{B_0^{\kappa+}\},
\]

\[
D^{\kappa+} = \bigcup_{n < \omega} C_{n0}^{\kappa+} \cup \{\langle B_{0}^{\kappa+}, \{B_{0}^{\kappa+}\} \cup \{C_{n}^{\kappa+} \mid n < \omega \}\rangle\}
\]

and

\[
B_{1}^{\kappa++} = \bigcup_{n < \omega} B_{n0}^{\kappa++} \cup \{\sup_{n < \omega} \bigcup B_{n1}^{\kappa++}\}.
\]

Pick \( A_{\omega}^{\kappa+} \) to be a model of cardinality \( \kappa^+ \) such that
1. \( \kappa A^0_{\omega^+} \subseteq A^0_{\omega^+} \),
2. \( B^0_{\omega^+}, B^1_{\omega^+}, B^{1\omega^+}, D^{\kappa^+} \in A^0_{\omega^+} \).

Set
\[
A^1_{\omega^+} = B^1_{\omega^+} \cup \{A^0_{\omega^+}\}, \quad C^\omega_{\kappa^+} = D^{\kappa^+} \cup \{\langle A^0_{\omega^+}, D^{\kappa^+} (B^0_{\omega^+}) \cup \{A^0_{\omega^+}\} \rangle\}
\]
and
\[
A^{1\omega^+}_{\omega^+} = B^{1\omega^+} \cup \{\sup(A^0_{\omega^+} \cap \kappa^+)^3\}.
\]

Then
\[
p_\omega = \langle \langle A^0_{\omega^+}, A^1_{\omega^+}, C^\omega_{\kappa^+} \rangle, A^{1\omega^+}_{\omega^+} \rangle
\]
will a condition in \( P' \) stronger than every \( p_n \).

\[\square\]

The of the top models is formally required in the definition of \( P' \) in order to have the largest model of cardinality \( \kappa^+ \) to be closed under \( \kappa \)-sequences. It will be convenient, in the next proposition to deal with \( p_\omega \) having the top model removed. Let us denote \( B^0_{\omega^+} \) by \( A^0_{\omega^+}(p_\omega) \), \( B^1_{\omega^+} \) by \( A^1_{\omega^+}(p_\omega) \), \( D^{\kappa^+} \) by \( C^\omega_{\kappa^+}(p_\omega) \) and \( B^{1\omega^+} \) by \( A^{1\omega^+}_{\omega^+}(p_\omega) \).

**Proposition 1.2** \( P' \) is not \( \omega_2 \)-closed.

**Proof.** We construct an increasing sequence of conditions \( \langle p_\alpha \mid \alpha < \omega_1 \rangle \) of length \( \omega_1 \) without upper bound.

Let \( \alpha < \omega_1 \) and suppose that \( \langle p_\beta \mid \beta < \alpha \rangle \) is defined. Define \( p_\alpha \). If \( \alpha \) is a limit of limit ordinals, then we use 1.1 to form \( p_\alpha \) for such limit \( \alpha \). Let for a successor \( \alpha \), \( p_\alpha \) be an extension of \( p_{\alpha - 1} \) which has at least \( \omega_1 \) many splitting points \( B \) from its central piste above \( \sup(A^0_{\omega^+}(p_{\alpha - 1})) \) such that if \( B_0, B_1 \) are the immediate predecessors of \( B \) with \( B_0 \in C^{\kappa^+}(p_\alpha)(A^0_{\omega^+}(p_\alpha)) \), then \( A^0_{\omega^+}(p_{\alpha - 1}) \) is in \( C^{\kappa^+}(p_\alpha)(B_1) \).

Assume now that \( \alpha \) is a limit of limit ordinals.

Let \( \langle \alpha_n \mid n < \omega \rangle \) a fixed in advance cofinal sequence in \( \alpha \) with \( \alpha_0 = 0 \) consisting of limit ordinals.

Define \( p'_\alpha \) to be the upper bound of \( \langle p_\beta \mid \beta < \alpha \rangle \) defined as in 1.1. Let us define \( p_\alpha \) by changing \( C^{\kappa^+}(p'_\alpha) \) as follows.

We leave all \( A^0_{\omega^+}(p_{\alpha_0}) \) inside \( C^{\kappa^+}(p_\alpha) \).

Pick a splitting point \( B \in C^{\kappa^+}(p_{\alpha_0})(A^0_{\omega^+}(p_{\alpha_0})) \). Let \( B_0, B_1 \) be its immediate predecessors with \( B_0 \in C^{\kappa^+}(p_{\alpha_0})(A^0_{\omega^+}(p_{\alpha_0})) \). Define \( C^{\kappa^+}(p_\alpha)(A^0_{\omega^+}(p_{\alpha_0})) \) by switching from \( B_0 \) to \( B_1 \).
Let now $n, 0 < n < \omega$. Consider $C^{n+}(p_{\alpha})(A^{0k+}(p_{\alpha}))$ in the interval between $A^{0k+}(p_{\alpha-1})$ and $A^{0k+}(p_{\alpha})$. Pick a splitting point $B \in C^{n+}(p_{\alpha})(A^{0k+}(p_{\alpha}))$ in this interval with immediate predecessors $B_0, B_1$ such that

1. $B_0 \in C^{n+}(p_{\alpha})(A^{0k+}(p_{\alpha}))$
2. $B_1 \notin C^{n+}(p_{\beta})(A^{0k+}(p_{\beta}))$, for every $\beta \leq \alpha_n$.

Note that this is possible since we required to have at least $\aleph_1$ many splitting points at each successor stage and $\alpha_n$ is countable.

Define $C^{n+}(p_{\alpha})(A^{0k+}(p_{\alpha}))$ by switching from $B_0$ to $B_1$.

This completes the definition of the sequence $\langle p_{\alpha} \mid \alpha < \omega_1 \rangle$.

Let us argue that there is no $p \in P'$ such that $p \geq p_{\alpha}$, for every $\alpha < \omega_1$.

Suppose otherwise. Let $p$ be such a condition. Set

$C := \{ \alpha < \omega_1 \mid \alpha \text{ is a limit of limit ordinals} \}$.

For every $\alpha \in C$ let $f(\alpha)$ be the least $\beta < \alpha$ such that $C^{n+}(p)(A^{0k+}(p))$ transforms into $C^{n+}(p_{\alpha})(A^{0k+}(p_{\alpha}))$ by switches below $A^{0k+}(p_{\beta})$. Recall that only finitely many switches are required to transform $C^{n+}(p)(A^{0k+}(p_{\alpha}))$ into $C^{n+}(p_{\alpha})(A^{0k+}(p_{\alpha}))$, by the definition of the order on $P'$, and hence there must be such $\beta$.

Find a stationary $S \subseteq C$ and $\beta^* < \omega_1$ such that $f(\alpha) = \beta^*$, for every $\beta \in S$. Pick $\alpha \in S$ which is a limit point of $S$. Let $\langle \gamma_n \mid n < \omega \rangle$ be cofinal in $\alpha$ sequence of elements of $S$. Then $C^{n+}(p_{\alpha})(A^{0k+}(p_{\gamma_n}))$ and $C^{n+}(p_{\gamma_n})(A^{0k+}(p_{\gamma_n}))$ agree on the final segment from $A^{0k+}(p_{\beta^*})$ up, for each $n < \omega$. But this contradicts the choice of $C^{n+}(p_{\alpha})(A^{0k+}(p_{\alpha}))$.

□

We would like to use now the above reason of non-closure in order to construct a square like principle which is inconsistent with a supercompact cardinal.

**Theorem 1.3** The gap 3 preparation forcing $P'$ of [2], chapter 1 adds a weak form of $\square_{\kappa^{++}}^{\kappa^+}$.

Let $G(P')$ be a generic subset of $P'$.

Introduce few notions.

**Definition 1.4** A limit ordinal $\xi < \kappa^{+3}$ is called good iff there is $\langle A^{0k+}, A^{1k+}, C^{n+}, A^{1k+} \rangle \in G(P')$ such that

1. $\xi \in A^{0k+}$,
2. \( \xi \in A^{1\kappa^+} \),

3. \( \text{cof}(\xi) \leq \kappa^+ \),

4. there is \( A \in A^{1\kappa^+} \) such that
   
   (a) \( \xi \in A \),
   
   (b) \( A \) is an immediate successor of a limit model in \( C^{\kappa^+}(A) \).
   
   Denote this model by \( A^- \).
   
   (c) For every \( E \in C^{\kappa^+}(A) \setminus \{A\} \), \( \xi \notin A \),
   
   (d) \( A^- \cap \xi \) is unbounded in \( \xi \),
   
   (e) \( E \cap \xi \) is bounded in \( \xi \), for every \( E \in C^{\kappa^+}(A) \setminus \{A, A^-\} \).

**Lemma 1.5** Let \( \xi \) be a good ordinal and \( \langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle \in G(P') \) be a condition witnessing this. Let \( A \in A^{1\kappa^+} \) be such that

1. \( \xi \in A \),

2. for every \( B \in A^{1\kappa^+} \) with \( B \not\subset A \), \( \xi \notin B \).

Then \( A \) satisfies (4) of Definition 1.4. In addition, the sequence \( \langle \xi \cap E \mid E \in C^{\kappa^+}(A) \setminus \{A, A^-\} \rangle \) does not depend on \( A \).

**Proof.** Clearly, \( A \) is a successor model. Let \( A^* \) be a model witnessing (4) of Definition 1.4.

**Claim 1** There is no \( B \in A^{1\kappa^+} \cap A^* \) with \( \xi \in B \).

**Proof.** Suppose otherwise. Then there is a piste from \( A^* \) to \( B \). But \( A^* \) is the immediate successor of \( A^+ \) in \( C^{\kappa^+}(A^*) \). Hence it should go via \( A^+ \). Which is impossible since \( \xi \in B \setminus A^+ \).

\( \Box \) of the claim.

Use now the intersection property for \( A, A^* \). Then, by the claim and the property (2) of \( A \), for some \( \eta \in A, \eta^* \in A^* \),

\[ A \cap A^* = A \cap \eta = A^* \cap \eta^* \, . \]

Then \( otp(A) = otp(A^*) \) and hence \( C^{\kappa^+}(A) \) and \( C^{\kappa^+}(A^*) \) have the same order type. In particular, \( A \) is an immediate successor of a limit model. Also structures

\[ \langle A, C^{\kappa^+}(A), \eta, \in, \subseteq \rangle, \langle A^*, C^{\kappa^+}(A^*), \eta^*, \in, \subseteq \rangle \]
are isomorphic with the isomorphism which is identity over the common part. Then $A$ satisfies (4) of Definition 1.4. In addition we obtain that the sequences $\langle E \cap \xi \mid E \in C^{\kappa^+}(A) \setminus \{A\} \rangle$ and $\langle E \cap \xi \mid E \in C^{\kappa^+}(A^*) \setminus \{A^*\} \rangle$ are the same.

\[ \square \]

**Lemma 1.6** A limit of $\leq \kappa^+$ good ordinals is a good ordinal.

**Proof.** Let $\langle \xi_i \mid i < \delta \leq \kappa^+ \rangle$ be an increasing sequence of good ordinals and $\xi = \bigcup_{i<\delta} \xi_i$. Consider a piste from $A^{0\kappa^+}$ to $\xi$. Let $A$ be the terminal model of this piste. Then $A$ cannot be a limit model and also it cannot be an immediate successor of a non-limit model by the previous lemma, as $\xi$ is a limit of good ordinals. Denote by $A^-$ the immediate predecessor of $A$. Consider $C^{\kappa^+}(A) \setminus \{A, A^\rightarrow\}$. Then $\xi$ is not a member of any of the elements of this set. Moreover, if $E \in C^{\kappa^+}(A) \setminus \{A, A^\rightarrow\}$, then $E \cap \xi$ is bounded in $\xi$. Otherwise let $E \cap \xi$ is unbounded in $\xi$. Let $E^+$ be the immediate successor of $E$ in $C^{\kappa^+}(A)$. We have $\xi \notin E^+$ but there are ordinals $\geq \xi$ in $E^+$, for example $\sup(E)$. Let $\eta$ be the least such ordinal. Then $\text{cof}(\eta) > \kappa^+$, by elementarity of $E^+$. So $E \cap \eta \subseteq \xi$. But $E \cap \eta \in E^+$, hence also $\xi = \sup(E \cap \eta) \in E^+$. Contradiction.

Now, $C^{\kappa^+}(A) \setminus \{A, A^\rightarrow\}$ witness goodness of $\xi$.

\[ \square \]

**Corollary 1.7** The set of good ordinals is a $\kappa^+$–club.

Now we are ready to prove the theorem. Denote by

$$C := \{ \alpha < \kappa^{+3} \mid \alpha \text{ is a good ordinal } \}.$$  

We will define a partial square sequence $\langle C_\alpha \mid \alpha \in C \rangle$ over $C$. This by standard argument allows to extend it to

$$\{ \alpha < \kappa^{+3} \mid \text{cof}(\alpha) < \kappa^{++} \}.$$  

Proceed as follows. If $\alpha$ is a good ordinal then pick a model $A$ witnessing this and set

$$C_\alpha(p) = \{ \sup(E \cap \alpha) \mid E \in C^{\kappa^+}(A^-) \setminus \{A^-\} \},$$

where $p \in G(P')$ and $A \in A^{1\kappa^+}(p)$.

Now if we have $p, q \in G(P')$ with $A \in A^{1\kappa^+}(p), A^{1\kappa^+}(q)$ then $C^{\kappa^+}(p)(A^-)$ and $C^{\kappa^+}(q)(A^-)$ may differ only on an initial segment and both sets have the same order type, since we can move from $C^{\kappa^+}(p)$ to $C^{\kappa^+}(q)$ using finitely many switches.
Let us pick for every good $\alpha$ a condition $p_\alpha \in G(P')$ with a witnessing $A_\alpha \in A^{1\kappa^+}(p_\alpha)$ and set

$$C_\alpha := C_\alpha(p_\alpha).$$

**Lemma 1.8** Let $\alpha$ be a good ordinal and $\beta$ is a limit point of $C_\alpha$, then we will have a following type of coherency:

1. $C_\alpha \cap \beta$ and $C_\beta$ have a common final segment,
2. $\text{otp}(C_\alpha \cap \beta) = \text{otp}(C_\beta)$.

**Proof.** It follows since $C_\beta(p_\alpha) = C_\alpha \cap \beta$ (the coherency for good $\alpha$’s with same $p$ follows by Lemma 1.5) and $C_\beta(p_\alpha), C_\beta(p_\beta) = C_\beta$ have a common final segment and the same order type.

$\square$

Using ideas from Cummings, Foreman, Magidor [1] it is possible to show that this type of a square is weaker than $\square^+\kappa$ (at least assuming the consistency of a supercompact cardinal).

If $\text{cof}(\alpha) = \kappa^{++}$ and $\alpha$ is a limit point of $A^{1\kappa^{++}}$, for an element of $G(P')$ then set

$$C_\alpha = \{\sup(E \cap \alpha) \mid E \in C^{\kappa^+}(A), A \in A^{1\kappa^+}, \alpha \in A \text{ for some } \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(P')\}.$$

Carmi Merimovich [4] showed that such defined $C_\alpha$’s provide a partial $\square^{Cof\kappa^{++}}\kappa$. This type of a square lives well with a supercompact cardinals.

## 2 New definition.

Let us define a new partial order (actually a pre-order) on $P'$ which will allows to eliminate the first reason of non-closure.

**Definition 2.1** Let $p = \langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), C^{\kappa^+}(p)\rangle, A^{1\kappa^{++}}(p)\rangle,$

$q = \langle\langle A^{0\kappa^+}(q), A^{1\kappa^+}(q), C^{\kappa^+}(q)\rangle, A^{1\kappa^{++}}(q)\rangle$ be conditions in $P'$. Define $p \geq_{\text{new}} q$ iff there is $D$ such that

1. $\langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), D\rangle, A^{1\kappa^{++}}(p)\rangle \in P'$,
2. $A^{0\kappa^+}(q) \in D(A^{0\kappa^+}(p))$,
3. $D(A^{0\kappa^+}(q)) = C^{\kappa^+}(q)(A^{0\kappa^+}(q)).$
Remark 2.2 Note that any two conditions \( \langle A^{0c^+}, A^{1c^+}, C^{κ^+}, A^{1κ^+} \rangle \) and 
\( \langle A^{0c^+}, A^{1c^+}, D^{κ^+}, A^{1κ^+} \rangle \) are \( \leq_{\text{new}} \)-equivalent. They were equivalent according to the order (pre-order) \( \leq \) only if it was possible to change \( C^{κ^+} \) to \( D^{κ^+} \) by finitely many switches. With \( \leq_{\text{new}} \) infinitely many of them may be applied.

Proposition 2.3 Let \( η < κ^{++} \) and \( \langle p_{α} \mid α ≤ η < κ^{++} \rangle \) be a \( \leq_{\text{new}} \)-increasing sequence of elements of \( P' \). Suppose that for each limit \( α < η \) the set \( \bigcup_{β < α} A^{0c^+}(p_β) \) is in \( A^{1c^+}(p_α) \). Then there is \( p ∈ P' \), \( p ≥_{\text{new}} p_α \), for every \( α < η \).

Proof. Use \( \langle A^{0c^+}(p_α) \mid α < η \rangle \) together with \( \langle \bigcup_{β < α} A^{0c^+}(p_β) \mid α < η, α \text{ is a limit ordinal} \rangle \) in order to form \( C^{κ^+}(p) \), where \( p \) is the obvious upper bound of \( p_α \)'s without the pistes.

\[ \square \]

3 Additional reason for a non-closure.

There is one more reason for non-closure. It has to do with chains of models inside a condition with their union not inside.
Let us describe this type of situation.

Let \( \langle p_n \mid n < ω \rangle \) be an increasing sequence of conditions of \( P' \). There are potentially two ways to extend it. The first (and one which is always available, and which was used above in 1.1) is to take the union of \( A^{0c^+}(p_n) \)'s and then to extend this to a condition. The second (which is not always possible) is like this: there is \( p ∈ P' \) such that

1. \( p ≥ p_n \), for all \( n < ω \),
2. \( \bigcup_{n < ω} A^{0c^+}(p_n) \not∈ A^{1c^+}(p) \).

Proposition 3.1 Let \( \langle p_α \mid α < ω_1 \rangle \) be an increasing sequence of elements of \( P' \) such that for every limit \( α < ω_1 \), \( \bigcup_{β < α} A^{0c^+}(p_β) \not∈ A^{1c^+}(p_α) \). Then there is no \( p ∈ P' \) with \( p ≥ p_α \), for every \( α < ω_1 \) and \( \bigcup_{α < ω_1} A^{0c^+}(p_α) \in A^{1c^+}(p) \).

Proof. Suppose otherwise. Let \( p \) be an upper bound and \( \bigcup_{α < ω_1} A^{0c^+}(p_α) \in A^{1c^+}(p) \). Denote \( \bigcup_{α < ω_1} A^{0c^+}(p_α) \) by \( A \). Let us argue that for every \( X ∈ A ∩ A^{1c^+}(p) \) there is \( α < ω_1 \) with \( X ∈ A^{0c^+}(p_α) \). Consider \( η = \sup(X ∩ κ^{+3}) \). Then \( η \in A \), and hence for some \( α, η ∈ A^{0c^+}(p_α) \). But \( \text{cof}(η) ≤ κ^+ \). So \( A^{0c^+}(p_α) \) is unbounded in \( η \). By intersection property, then \( X ⊆ A^{0c^+}(p_α) \). But \( η ∈ A^{0c^+}(p_α) \setminus X \), hence \( X ∈ A^{0c^+}(p_α) \).
It follows that \( \{ \bigcup_{\beta<\gamma} A^{0\kappa^+}(p_\beta) \mid \gamma < \omega_1 \} \) is club in \( A \cap A^{1\kappa^+}(p) \). So it must intersect \( C^{\kappa^+}(p) \). Contradiction.

\[ \square \]

4 Absence of directed closure.

If we have countably many conditions such that any finite family of them is compatible, then the \( \omega_1 \)-closure (1.1) implies the existence of an upper bound.

But suppose now that we have \( \omega_1 \) many conditions \( \langle p_n \alpha \mid n < \omega, \alpha < \omega_1 \rangle \) such that for every \( \alpha \),

1. \( A^{0\kappa^+}(p_{\alpha n+1}) \supset A^{0\kappa^+}(p_{\alpha n}) \),

2. \( A^{0\kappa^+}(p_{\alpha+1n+1}) \supset A^{0\kappa^+}(p_{\alpha n}) \),

3. \( \bigcup_{n<\omega} A^{0\kappa^+}(p_{\alpha n+1}) \not\supset \bigcup_{n<\omega} A^{0\kappa^+}(p_{\alpha n}) \).

It is impossible to find an upper bound for \( \langle p_n \alpha \mid n < \omega, \alpha < \omega_1 \rangle \) without adding \( \bigcup_{n<\omega} A^{0\kappa^+}(p_{\alpha n}) \) to the central piste for unboundedly many \( \alpha \)'s, which is not allowed.

5 Off-piste version of the preparation forcing.

In [?] Merimovich used a variation of the Velleman simplified morass forcing [5] as the preparation forcing for gap 3. The advantage of using it is a directed closure of this forcing.

Here we would like to present off-piste version of the preparation forcing for higher gaps. Absence of pistes will provide a directed closure. Unfortunately the resulting structure lacks of the intersection property for gaps 4 and above, and so it is unclear how to implement it into the final forcing.

Let us deal with gap 4 case the treatment of higher gaps is similar.

We will have three types of models - of size \( \kappa^+ \), of size \( \kappa^{++} \) and of size \( \kappa^{+3} \) (ordinals). Denote as usual the corresponding sets accumulating this models inside a condition by \( A^{1\kappa^+}, A^{1\kappa^{++}}, A^{1\kappa^{+3}} \). \( A^{1\kappa^{+3}} \) is a closed set of ordinals of size at most \( \kappa^{+3} \). \( A^{1\kappa^{++}} \) is defined as in [?]. Let us define \( A^{1\kappa^+} \).

**Definition 5.1** \( A^{1\kappa^+} \) is a set of at most \( \kappa^+ \) models of size \( \kappa^+ \) such that the following holds:

1. there is the largest model \( A^{0\kappa^+} \),
2. For every \( A \in A^{1\kappa^+} \) the following holds:

(a) either there is a largest \( \alpha \in A \cap A^{1\kappa^+} \) or \( \text{sup}(A \cap \kappa^+) \) is a limit point of \( A^{1\kappa^+} \),

(b) either there is a largest (under the inclusion) model \((A)_{\kappa^+} \in A \cap A^{1\kappa^+} \) or \( A \cap A^{1\kappa^+} \) is directed,

(c) either \( A \cap A^{1\kappa^+} \) is directed or \( A \) has immediate predecessors and if \( A' \) is an immediate predecessor of \( A \) then either

i. there is an immediate predecessor \( A_0 \) of \( A \) and \( A, A_0, A' \) form a \( \Delta \)-system triple;

or

ii. there is \( A_0 \in A \cap A^{1\kappa^+} \) (which need not be an immediate predecessor of \( A \)) and \( A' = \pi_{F_0 F_1}[A_0] \), where \( F_0, F_1 \in A \cap A^{1\kappa^+} \) are of a same order type.

Let us argue that the intersection property even in its weakest form can break down in the present setting.

**Example.**

Suppose that \( A \in A^{1\kappa^+} \), the largest model \((A)_{\kappa^+} \in A \cap A^{1\kappa^+} \) exists and it is a limit point of \( A^{1\kappa^+} \) (limit means here that \((A)_{\kappa^+} \cap A^{1\kappa^+} \) is directed or equivalently of a limit rank).

Assume that there is no increasing sequence of elements of \((A)_{\kappa^+} \cap A^{1\kappa^+} \) which union is \((A)_{\kappa^+} \).

Note that existence of a limit model \( X \in (A)_{\kappa^+} \) which is not a limit of an increasing sequence of elements of \((A)_{\kappa^+} \) is forced upon us, as in 4, if we like to have directed closure and not only a closure.

Assume that the rank of \((A)_{\kappa^+} \) is some \( \mu < \kappa^{+3} \) of cofinality \( \kappa^{+3} \).

Suppose that the only model in \( A \cap A^{1\kappa^+} \) that includes \( B \) is \((A)_{\kappa^+} \).

Consider

\[
A \cap (A)_{\kappa^+} = \bigcup (A \cap A^{1\kappa^+}) \setminus \{(A)_{\kappa^+}\}.
\]

Set \( Z = \bigcup (A \cap A^{1\kappa^+}). \) If \( Z \in A^{1\kappa^+} \), then use the intersection property between \( B \) and \( Z \). Suppose \( Z \notin A^{1\kappa^+} \). Pick \( Y \in (A)_{\kappa^+} \in A \cap A^{1\kappa^+}, Y \supset Z \) of the smallest rank (it must be \( \text{sup}(A \cap \mu) \)).

If \( B \in (A)_{\kappa^+} \in A \cap A^{1\kappa^+} \) is a model of rank \( \delta \), for some \( \delta, \mu > \delta \geq \text{sup}(A \cap \mu) \). Consider \( Y \cap B. \) Then there is \( \xi \in Y \cap A^{1\kappa^{+3}} \) such that

\[
Y \cap B = Y \cap \xi.
\]
Hence

\[ A \cap B = A \cap Y \cap B = A \cap (A)_{\kappa^+} \cap \xi.\]

But if the rank of \( B \) is small, then \( B \) can be an element of \( Y \setminus Z \) which does not include \( Z \), and even the rank of \( B \) may be in \( A \). If in addition no element of \( Z \) includes \( B \cap Z \), then the intersection property between \( A \) and \( B \) will break down.

Let us construct an example having such \( B \). Fix a continuous chain of elementary submodels \( \langle M_i \mid i \leq \kappa^+ + 1 \rangle \) each of size \( \kappa^{+3} \), \( M_i \cap \kappa^{+4} = \mu_i \) and \( \kappa^{++} \mid M_{i+1} \subseteq M_{i+1} \). Let \( X \) be an elementary submodel of \( M_{\kappa^{++}+1} \) of size \( \kappa^{++} \) such that \( \langle \mu_i \mid i \leq \kappa^{++} \rangle \in X \).

For each \( i < \kappa^{++} \) let \( X_i \in M_{i+1} \) be a reflection of \( X \) to \( M_{i+1} \) over \( X \cap M_i \).

Add models of size \( \kappa^{++} \) which include \( X_0, X \) and then reflect it down to every \( i, 0 < i < \kappa^{++} \). Continue in a similar fashion and extend the family into directed one. Then pick a model which includes it and again reflect down. Proceed \( \kappa^{++} + 1 \) many stages. Let \( E \) be the final model of the rank \( \kappa^{++} \). Through all the models from the constructed family of models of size \( \kappa^{++} \) which have ordinals \( \geq \mu_{\kappa^{++}} \) but keep \( E \) (in particular, \( X \) is out).

The resulting family will be our \( A_{1^{\kappa^{++}}} \). Let \( A^{1\kappa^{+3}} \) be the set \( \{ M_i \mid i \leq \kappa^{++} \} \). Let \( A = A^{0\kappa^{+}} \) be an elementary submodel of \( M_{\kappa^{++}+1} \) of size \( \kappa^+ \) with \( A^{1\kappa^{++}} \) and \( A^{1\kappa^{+3}} \) inside. Let \( A \cap \kappa^{++} = \eta \).

Then \( \sup (A \cap \mu_{\kappa^{++}}) = \mu_\eta \). Set \( B = X_\eta \). The pair \( A, B \) will fail to have the intersection property as it was explained above.

**References**


[3] C. Merimovich,
