A model with a measurable which does not carry a normal measure

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Abstract

We construct a model of ZF in which there is a measurable cardinal but there is no normal ultrafilter over it.

1 Introduction

Let U be a κ-complete non-trivial ultrafilter over κ and $f : \kappa \to \kappa$ a function that represents κ in the ultrapower by U. Then

$$\{X \subseteq \kappa \mid f^{-1}{}''X \in U\}$$

is a normal ultrafilter over κ, by a classical result of Scott, Keisler and Tarski. The axiom of choice is used to find such f. M. Spector [4] by forcing over AD type model showed in ZF alone the above need not be true.

Ralf Schindler asked us the following question:

(ZF) Suppose κ is a measurable cardinal. Is there a normal ultrafilter over κ?

According to Arthur Apter this question was asked in mid seventies.

The purpose of this paper is to provide a negative answer. We show the following:

Theorem 1.1 *Let V be a ZFC model of GCH+ there is a measurable cardinal κ. Then there is a symmetric submodel of a generic extension $\mathcal{N}$ which satisfies ZF+ κ is a measurable, but there is no normal ultrafilter over κ.*

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2 A basic model

We start with a ZFC model with a measurable cardinal κ. Assume GCH. Fix a normal ultrafilter U over κ. Let $j_U : V \to M_U$ be the corresponding elementary embedding.

Define an Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle$ by induction as follows. Suppose P_α is defined and $\alpha < \kappa$. Set Q_α to be a trivial forcing unless α is an inaccessible cardinal in which case Q_α will be $Q_{\alpha 1} * Q_{\alpha 2} * Q_{\alpha 3}$, where $Q_{\alpha 1}$ is the Cohen forcing over α which adds a function $f_\alpha : \alpha \to \omega$, i.e.

$$Cohen(\alpha) = \{t \mid |t| < \alpha, t \text{ is a partial function from } \alpha \text{ to } \omega\}.$$

Let G_α be a generic subset and set $f_\alpha = \bigcup G_\alpha$.
Set $S_{\alpha n} = \{\nu < \alpha \mid f_\alpha(\nu) = n\}$, for every $n < \omega$. Then each $S_{\alpha n}$ is a stationary subset of α.
$Q_{\alpha 2}$ will be an atomic forcing which picks $n_\alpha < \omega$
Finaly $Q_{\alpha 3}$ is a trivial forcing if $n_\alpha = 0$ and if $n_\alpha > 0$, then $Q_{\alpha 3}$ is the product of forcings $Q^n_\alpha, n < n_\alpha$, where for every $n < \omega$ Q^n_α is a forcing adding a club into $\alpha \setminus S_{\alpha n}$, i.e.

$$Q^n_\alpha = \{c \subseteq \alpha \mid |c| < \alpha, c \text{ is closed and } c \cap S_{\alpha n} = \emptyset\},$$

ordered by end-extension.

If $\alpha = \kappa$ then let Q_α be just $Cohen(\kappa)$.

We will consider further $S_{\kappa n}$, Q^n_κ, $n < \omega$, which are defined exactly as $S_{\alpha n}$ and Q^n_α above.

Let us point out the following:

Lemma 2.1 *For every $\alpha \leq \kappa + 1$, Q_α has a dense α-closed subset (in V^{P_α}).*

Proof. Set

$$D_\alpha = \{\langle t, m, c_0, ..., c_{m-1}\rangle \in Q_\alpha \mid t \in Cohen(\alpha), m < \omega, c_i \in Q^i_\alpha,$$

$$\text{for every } i < m, \max(c_0) = \max(c_1) = ... = \max(c_{m-1}) = \text{dom}(t)\}.$$

It is dense in Q_α and α-closed.
□

Let $G(P_{\kappa+1})$ be a generic subset of $P_{\kappa+1}$.

Lemma 2.2 *κ remains measurable in $V[G(P_{\kappa+1})]$. Moreover U and j_U extend, i.e. there is a normal ultrafilter $U^* \supseteq U$ over κ in $V[G(P_{\kappa+1})]$ such that $j_{U^*} \supseteq j_U$, where j_{U^*} is the corresponding elementary embedding.*

Proof. Consider $j_U(P_{\kappa+1})$ in M_U. At the stage κ the forcing $Q_\kappa = Q_{\kappa 1} * Q_{\kappa 2} * Q_{\kappa 3}$ should be used. Recall that $Q_{\kappa 1}$ is $Cohen(\kappa)$ and we forced with it on the V–side. Now, $Q_{\kappa 2}$ is an atomic forcing that picks a natural number n_κ. We are free to pick any value. Let us take $n_\kappa = 0$. Then $Q_{\kappa 3}$ will be a trivial forcing. It is easy now to produce M–generic object for the rest of $j_U(P_{\kappa+1})$ and then to extend the elementary embedding j_U.
$\square$.

A bit more general statement is true as well:

Lemma 2.3 *Let $n, 0 < n < \omega$. Consider $\prod_{i<n} Q^i_\kappa$ in $V[G(P_{\kappa+1})]$. Let H be its generic subset. Then κ remains measurable in $V[G(P_{\kappa+1})][H]$, moreover the ultrafilter U and its embedding j_U extend there.*

Proof. Proceed as in the previous lemma only set $n_\kappa = n$. Then $Q_{\kappa 3} = \prod_{i<n} Q^i_\kappa$ and we have a generic object H for this forcing. It is easy now to produce M–generic object for the rest of $j_U(P_{\kappa+1})$ and then to extend the elementary embedding j_U.
$\square$.

Now in $V[G(P_{\kappa+1})]$ consider the following (disastrous) forcing.

Let $\mathcal{P}$ will be a finite support product of Q^i_κ, $i < \omega$.

Clearly the forcing with $\mathcal{P}$ turns all $S_{\kappa n}$'s into non-stationary and so collapses κ to ω. Fortunately we are interested in a symmetric submodel.

3 Symmetric submodel

We refer to classical books T. Jech [3], [2] as excellent references for constructions of symmetric models. Define a group $\mathcal{G}$ of automorphisms of $\mathcal{P}$. It will be induced by a group of automorphisms $\mathcal{G}(Q^i_\kappa)$ of Q^i_κ. Let $c, d \in Q^i_\kappa$ with $\max(c) = \max(d)$. Define π_{cd} be the function such that $\pi_{cd}(e) = d \cup (e \setminus \max(c))$, for every $e \geq c$, $\pi_{cd}(e) = c \cup (e \setminus \max(c))$ for every $e \geq d$ and if e is incompatible with both c, d, then let $\pi_{cd}(e) = e$. Let $\mathcal{G}(Q^i_\kappa)$ be generated by such π_{cd} and let $\mathcal{G}$ be the group generated using elements of $\mathcal{G}(Q^i_\kappa)$'s for finitely many i's.

Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by fix(s), $s \subseteq \omega$ finite, where fix(s)= $\{\pi \in \mathcal{G} \mid \forall n \in s, \pi \restriction \{n\} = id\}$.

Pick a generic subset $G(\mathcal{P})$ of $\mathcal{P}$. Let $\mathcal{N}$ be the symmetric submodel of $V[G(P_{\kappa+1})][G(\mathcal{P})]$.[1]

Lemma 3.1 *κ remains a regular cardinal in $\mathcal{N}$.*

[1] $\mathcal{N} = \{i_{G(\mathcal{P})}(\underset{\sim}{x}) \mid \underset{\sim}{x} \in$HS$\}$, where HS is the class of all hereditarily symmetric $\mathcal{P}$-names and a name $\underset{\sim}{x}$ is symmetric if $\mathrm{sym}_{\mathcal{G}}(\underset{\sim}{x}) := \{\pi \in \mathcal{G} \mid \pi(\underset{\sim}{x}) = \underset{\sim}{x}\} \in \mathcal{F}$.

Proof. Let $h \in \mathcal{N}$ be a function from some $\lambda < \kappa$ to κ. Fix a hereditarily symmetric $\mathcal{P}$-name $\underset{\sim}{h}$ of h. Let s be a finite subset of ω such that $\mathrm{sym}(\underset{\sim}{h}) \supseteq \mathrm{fix}(s)$. Pick a condition $\langle c_0, ..., c_{m-1}\rangle \in G(\mathcal{P})$ which forces "$\underset{\sim}{h} : \lambda \to \kappa$". Without loss of generality assume that $m > n$.

We claim now that $h \in V[G(P_{\kappa+1})][G(\mathcal{P}) \cap \prod_{i<m} Q^i_\kappa]$.

Let $p \geq \langle c_0, ..., c_{m-1}\rangle, \alpha < \lambda, \beta < \kappa$ and $p \Vdash \underset{\sim}{h}(\alpha) = \beta$.

Claim 1 $p \restriction m \Vdash \underset{\sim}{h}(\alpha) = \beta$.

Proof. Suppose otherwise. Then there are $q \in calP, q \restriction m \geq p \restriction m$ and $\beta' \neq \beta$ such that $q \Vdash \underset{\sim}{h}(\alpha) = \beta'$. Extend p, q to conditions $p^* = \langle e_0, ..., e_k\rangle, q^* = \langle d_0, ..., d_k\rangle$ such that

1. $p^* \restriction n = p \restriction n$,
2. $q^* \restriction n = q \restriction n$,
3. $\max(e_i) = \max(d_i)$, for every $i, n \leq i < k$.

We apply now the combination π of the automorphisms $\pi_{e_i d_i}$ for all $i, n \leq i < k$. Then $\pi \in\mathrm{fix}(s)$, and so $\pi(\underset{\sim}{h}) = \underset{\sim}{h}$. Hence $\pi(p^*) \Vdash \underset{\sim}{h}(\alpha) = \beta$, but $\pi(p^*)$ is compatible with $q^* \geq q$ and $q \Vdash \underset{\sim}{h}(\alpha) = \beta'$. Contradiction.

□ of the claim.

Now, granted $h \in V[G(P_{\kappa+1})][G(\mathcal{P}) \cap \prod_{i<m} Q^i_\kappa]$, we apply Lemma 2.3. It follows that $\mathrm{rng}(h)$ is bounded in κ, since κ is a measurable and so a regular in $V[G(P_{\kappa+1})][G(\mathcal{P}) \cap \prod_{i<m} Q^i_\kappa]$.

□

Lemma 3.2 *For every $n < \omega$, the set $S_{\kappa n}$ is non-stationary in $\mathcal{N}$.*

Proof. Let $n < \omega$. Consider $C_{\kappa n} := \bigcup(G(\mathcal{P}) \cap Q^n_\kappa)$. This is a club disjoint from $S_{\kappa n}$. It is easy to find a name $\underset{\sim}{C}_{\kappa n}$ of $C_{\kappa n}$ which is a hereditarily symmetric and $\mathrm{sym}(\underset{\sim}{C}_{\kappa n}) \supseteq\mathrm{fix}(\{n\})$. Hence $C_{\kappa n} \in \mathcal{N}$ and so $S_{\kappa n}$ is non-stationary there.

□

Lemma 3.3 *The filter Cub_κ is not σ-complete in $\mathcal{N}$.*

Proof. Follows from Lemma 3.2.

□

Remark 3.4 Note that intersection of less than κ many clubs in κ is always a club. The point here is that we cannot choose in $\mathcal{N}$ a sequence of clubs $\langle C_n \mid n < \omega\rangle$ such that $C_n \cap S_{\kappa n} = \emptyset$.

Lemma 3.5 *In $\mathcal{N}$ there are no normal filters on κ.*

Proof. The generic Cohen function $f_\kappa : \kappa \to \omega$ splits κ into ω non-stationary sets, by Lemma 3.2. So if W is a normal filter over κ in $\mathcal{N}$, then the regressive function f_κ should be constant on a W-positive set A. Then $A \subseteq S_{\kappa n}$ for some n. Hence $S_{\kappa n}$ is positive for W. But there is a club C_n which is disjoint to $S_{\kappa n}$. Define $g : S_{\kappa n} \setminus \min(C_n) + 1 \to \kappa$ by setting $g(\nu) = \max(C_n \cap \nu)$. Then g is a regressive function on a set in W which is constant only on bounded subsets of κ. Hence W cannot be normal.
□

Lemma 3.6 *If $X \in \mathcal{N}, X \subseteq \kappa$, then $X \in V[G(P_{\kappa+1})][\langle C_n \mid n \in fix(X)\rangle]$.*

Proof. The proof basically repeats the argument of Lemma 3.1.
□

4 Moving ultrafilters

We would like to explore Lemma 2.3 and produce various ultrafilters inside intermediate models. The main idea will be to use not only U, j_U and their extensions but rather extensions of U^n, j_{U^n} eventually increasing n.

In order to give more intuition let us consider the situation in the initial model $V[G(P_{\kappa+1})]$. We have the Cohen function f_κ there and in order to extend U the value $j(\underset{\sim}{f}_\kappa)(\kappa) = \underset{\sim}{f}_{j(\kappa)}(\kappa)$ should be decided, where $j := j_U$. Suppose that we set it to be 0. Let $U(0)$ be a resulting extension of U. Then $S_{\kappa 0}$ will be in $U(0)$.
At the next stage the forcing Q^0_κ is used. It adds a club disjoint with the set $S_{\kappa 0}$. How do we extend $U(0)$ now? Is possible at all? The problem is that $\kappa \in S_{j(\kappa)0}$, since $f_{j(\kappa)}(\kappa) = 0$. On the other hand, we forced a club disjoint with $S_{\kappa 0}$. So κ is its limit point of it, and hence κ should be in the image of the club under any elementary embedding. This means that there is no way to extend j in such forcing extension. However, it is still possible to extend $U(0)$ once giving up normality. Proceed as follows. Replace $j : V \to M$ by the embedding $j_2 := j_{U^2} : V \to M_2 \simeq V^{\kappa^2}/U^2$. Denote $j(\kappa)$ by κ_1 and $j_2(\kappa)$ by κ_2. There is an embedding of M to M_2 which moves κ to κ_1 and κ_1 to κ_2. It will be used in order to extend $U(0)$. We

move to κ_1 and set $j_2(\underset{\sim}{f}_\kappa)(\kappa_1)=0$. Now, in order to accumulate the club disjoint with $S_{\kappa 0}$, also set $j_2(\underset{\sim}{f}_\kappa)(\kappa)=1$. Note that then $\kappa\not\in S_{\kappa_2 0}$ which allows us to add κ to the club disjoint with $S_{\kappa_2 0}$. This way we will be able to extend j_2 and to produce an extension $U(1)$ of $U(0)$. At the next stage Q^1_κ is used resulting non stationarity of $S_{\kappa 1}$ which was in the normal ultrafilter Rudin-Keisler below $U(1)$. So again in order to extend $U(1)$ we move U^3, j_3 and replace κ,κ_1 by κ_1,κ_2, etc.
The problem will persists at the ω-th stage in ZFC setting, but here for ZF model there will be no ω-th stage.

Turn now to a formal realization of the idea above.
Denote for every $n, 0<n<\omega$, by $j_n : V\to M_k\simeq V^{\kappa^n}/U^n$ the elementary embedding into n-th iterated ultrapower of V by U. Let $\kappa_n=j_n(\kappa)$.

First let us extend U to two normal ultrafilter $U(0,0), U(0,1)$ over κ in $V[G(P_{\kappa+1})]$. Proceed as follows. Let $i=0,1$. Pick in $V[G(P_{\kappa+1})]$ an M-generic subset $G^i(P_{j(\kappa)+1})$ of $j(P_{\kappa+1})$ such that

1. $G^i(P_{j(\kappa)+1})\restriction\kappa+1=G(P_{\kappa+1})$, where by $G^i(P_{j(\kappa)+1})\restriction\kappa+1$ we mean the restriction of $G^i(P_{j(\kappa)+1})$ to $P_{\kappa+1}$,
2. $f^i_{j(\kappa)}\restriction\kappa=f_\kappa$,
3. $f^i_{j(\kappa)}(\kappa)=i$,
4. $G^0(P_{j(\kappa)+1})\restriction j(\kappa)=G^1(P_{j(\kappa)+1})\restriction j(\kappa)$,
5. $f^0_{j(\kappa)}, f^1_{j(\kappa)}$ agree about all the values except the one of κ.

Then the embedding j extends to $j^{0,i}: V[G(P_{\kappa+1})]\to M[G^i(P_{j(\kappa)+1})]$. Set

$$U(0,i)=\{X\subseteq\kappa\mid\kappa\in j^{0,i}(X)\}.$$

It is not hard to see that $M[G^i(P_{j(\kappa)+1})]\simeq V[G(P_{\kappa+1})]^\kappa/U(0,i)$. Also, $M[G^0(P_{j(\kappa)+1})]=M[G^1(P_{j(\kappa)+1})]$, $U(0,0),U(0,1)$ are normal ultrafilters, $S_{\kappa 0}\in U(0,0)$,$S_{\kappa 1}\in U(0,1)$, and so $S_{\kappa 0}\not\in U(0,1)$.

Force now (over $V[G(P_{\kappa+1})]$) with Q^0_κ. Let $C_{\kappa 0}$ be a generic club. Then $C_{\kappa 0}$ is disjoint from $S_{\kappa 0}$. So $U(0,0)$ cannot be extended to a normal ultrafilter in $V[G(P_{\kappa+1})][C_{\kappa 0}]$. Let us give up the normality and extend it to a κ-complete ultrafilter.

Work in $V[G(P_{\kappa+1})]$. Consider $U(0,1)\times U(0,0)$. It is a κ-complete ultrafilter in $V[G(P_{\kappa+1})]$. Let $j^{10}: V[G(P_{\kappa+1})]\to M^*$ be its ultrapower embedding. It is obtained by first taking the ultrapower of $V[G(P_{\kappa+1})]$ by $U(0,1)$ and then taking the ultrapower by the image of $U(0,0)$. Then $j^{10} \supseteq j_2$ and $M^* = M_2[G(P_{j_2(\kappa)+1})]$ for M_2-generic subset $G(P_{j_2(\kappa)+1})$ of $P_{j_2(\kappa)+1}$. It is not hard to see that

1. $G(P_{j_2(\kappa)+1}) \upharpoonright \kappa_1 + 1 = G^1(P_{j(\kappa)+1})$,
2. $U(0,1) = \{X \subseteq \kappa \mid \kappa \in j^{10}(X)\}$,
3. $U(0,0) = \{X \subseteq \kappa \mid \kappa_1 \in j^{10}(X)\}$
4. $f_{j_2(\kappa)}(\kappa) = 1$,
5. $f_{j_2(\kappa)}(\kappa_1) = 0$.

Now given $f_{j_2(\kappa)}(\kappa) = 1$, it is easy to extend the embedding j^{10} to an embedding

$$j^1 : V[G(P_{\kappa+1})][C_{\kappa 0}] \to M[G(P_{j_2(\kappa)+1})][C_{j_2(\kappa)0}].$$

Just let $C_{j_2(\kappa)0} \upharpoonright \kappa + 1 = C_{\kappa 0} \cup \{\kappa\}$ and the first point of $C_{j_2(\kappa)0}$ above κ be above κ_1. This works because one needs to have that $\kappa \in C_{j_2(\kappa)0}$ and $\kappa_1 \notin C_{j_2(\kappa)0}$.
Set

$$U(1) = \{X \subseteq \kappa \mid \kappa_1 \in j^1(X)\}.$$

Then, clearly $U(1)$ will be a κ-complete ultrafilter in $V[G(P_{\kappa+1})][C_{\kappa 0}]$ which extends $U(0,0)$. Clearly $U(1)$ will not be normal and its projection (it will be the function $\xi \mapsto \max(C_{\kappa 0} \cap \xi)$) to the normal one will extend $U(0,1)$. Actually the ultrapower embedding by $U(1)$ will be j^1, since κ is definable from κ_1 now as the maximal element of the club $C_{j_2(\kappa)0}$ below κ_1 and $C_{j_2(\kappa)0}$ is in the range of the embedding j^1.

Now force with Q^1_κ over $V[G(P_{\kappa+1})][C_{\kappa 0}]$. Let $C_{\kappa 1}$ be a generic club disjoint from $S_{\kappa 1}$. The embedding j^1 cannot be extended anymore, but the filter $U(1)$ can. Use j_3. Namely define $U(0,2)$ as $U(0,0), U(0,1)$ but only set the value of $f_{j_3(\kappa)}(\kappa) = 2$. $U(0,2)$ extends smoothly to a normal ultrafilter in $V[G(P_{\kappa+1})][C_{\kappa 0}]$. Denote an extension by $U(1,2)$. Consider $U(1,2) \times U(1)$. The arguments as above allow to extend $U(1)$ in $V[G(P_{\kappa+1})][C_{\kappa 0}][C_{\kappa 1}]$ to a κ-complete ultrafilter. Its embedding will extend j_3 and the identity function will represent κ_2 in the ultrapower.

Continue in a similar fashion for each $n < \omega$.

The sequence $\langle U(n) \mid n < \omega \rangle$ is increasing (under the inclusion), but functions that represent κ form a decreasing sequence. Namely, id represents κ mod $U(0)$. But mod $U(1)$

it represents κ_1 and $\xi \mapsto \max(C_{\kappa 0} \cap \xi)$ represents κ. Next, mod $U(2)$, id represents κ_2, $\xi \mapsto \max(C_{\kappa 0} \cap \xi)$ represents κ_1 and $\xi \mapsto \max(C_{\kappa 1} \cap \xi)$ represents κ, etc.

We would like to have $\bigcup_{n<\omega} U(n)$ to be in a symmetric model. Unfortunately the group of automorphisms of the previous section does not allow this. In order to achieve the goal we will change the construction in the next section. Still it will be based on the same ideas.

5 New approach

Iterate with Easton support the atomic forcing that picks for each inaccessible $\nu < \kappa$ a natural number $n_\nu < \omega$. Let $\langle n_\nu \mid \nu \text{ is an inaccessible below } \kappa\rangle$ be a generic object. Set $V_1 = V[\langle n_\nu \mid \nu \text{ is an inaccessible below } \kappa\rangle]$. Over V_1 iterate with Easton support $Cohen(\nu)$ for each inaccessible $\nu \leq \kappa$. Denote by $f_\nu : \nu \to \omega$ a $Cohen(\nu)$–generic function produced by such iteration. Set $V_2 = V_1[\langle f_\nu \mid \nu \leq \kappa\rangle]$.
Define now forcings P^k_α over V_2, for each $k, 1 \leq k < \omega, \alpha \leq \kappa + 1$. P^k_α will Easton support iteration of the length α of forcings $Q_{\nu k}$ for $\nu < \alpha$, where $Q_{\nu k}$ is a trivial forcing unless ν is a Mahlo and $n_\nu \geq k$. In the later case $Q_{\nu k}$ is the forcing for adding a club of ν disjoint to $S_{\nu k-1} = \{\gamma < \nu \mid f_\nu(\gamma) = k - 1\}$ by initial segments.
Let P be defined over V_2 as $P = \prod_{1\leq k<\omega} P^k_\kappa$.

We will need some general facts about supports of conditions used here.
Let λ be an ordinal. A subset a of λ will be called *an Easton support set* or simply *a support* iff for every regular (or every inaccessible) $\delta \leq \lambda$, $|a \cap \delta| < \delta$.

Lemma 5.1 *Suppose that $\langle R_\alpha \mid \alpha \leq \lambda\rangle$ is an iteration such that for every $\zeta < \lambda$*

1. *R_ζ satisfies ζ^+-c.c.,*
2. *$R_\lambda/R_{\zeta+1}$ does not add new ζ-sequences of ordinals.*

Let G_λ be a generic subset of R_λ and $a \subseteq \lambda$ be a support in $V[G_\lambda]$. Then there is $b \supseteq a$ in V which is a support.

Proof. Let us show by induction on $\alpha \leq \lambda$ that $a \cap \alpha$ can be covered by a support set which is in V.

Suppose first that α is not a limit of regular cardinals. Then let $\delta < \alpha$ be the largest cardinal (unless $\alpha < \omega$ which is trivial). Apply the induction to $a \cap \delta$ and find a covering set $b' \in V$ such that $a \cap \delta \subseteq b' \subseteq \delta$. Now $b := b' \cup (\alpha \setminus \delta)$ will be as desired.

Let us assume now that α is an inaccessible cardinal. Then $\sup(a) < \alpha$, since a is a support. Apply the induction to $\sup(a)$.

Let us turn to the remaining and the principal case of a limit singular cardinal α. Denote the cofinality of α by η. Pick a cofinal sequence $\langle \alpha_i \mid i < \eta \rangle$. Apply induction and for each $i < \eta$ find a set $b_i \in V$ such that $b_i \subset \alpha_i, b_i \supseteq a \cap \alpha_i$ and b_i is a support. The sequence $\langle b_i \mid i < \eta \rangle$ belongs to $V[G_{\eta+1}]$ (where $G_{\eta+1}$ is the restriction of G_λ to $R_{\eta+1}$), by the first item of the statement of the lemma. Use η^+-c.c. of $R_{\eta+1}$ to find in V a sequence $\langle b'_i \mid i < \eta \rangle$ such that $b'_i \subseteq b_i$ and $|b'_i| = |b_i| + \eta$. Consider $a \cap \eta$. By induction there is a covering set $a^* \in V$ such that $a \cap \eta \subseteq a^* \subseteq \eta$. Set now $b = ((\bigcup_{i<\eta} b'_i) \setminus \eta) \cup a^*$. It is as desired.
$\square$

Lemma 5.2 *The forcing P preserves all the cardinals.*

Proof. Clearly P preserves all the cardinals above κ, since $|P| = \kappa$. κ is preserved since P satisfies κ-c.c.
Let $\eta < \kappa$ be a regular cardinal. Let us show that it is preserved. We would like to split P into a product of two forcings one below η which satisfies η-c.c. and an other above η which is η-closed. Note however that each of the components P^k_κ is an iteration so it does not break into a product. But let us replace P^k_κ by $P^k_\eta \times A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$, where $A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$ is a term space forcing, i.e.
$A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$ consists of canonical P^k_η-names of elements of $\underset{\sim}{P}^k_{>\eta}$ and the ordering given by

$$\underset{\sim}{\sigma} \leq_{A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})} \underset{\sim}{\tau} \Leftrightarrow \Vdash_{P^k_\eta} \underset{\sim}{\sigma} \leq_{\underset{\sim}{P}^k_{>\eta}} \underset{\sim}{\tau}.$$

We refer to Section 22 , pp.865-868 of J. Cummings chapter in Handbook of Set theory [1] for details on Termspace Forcing. The basic property of this forcing is the following:
Let G_η be P^k_η generic over V_2 and H be $A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$-generic over V_2. Set $I = \{i_{G_\eta}(\underset{\sim}{\tau}) \mid \underset{\sim}{\tau} \in H\}$. Then I is $i_{G_\eta}(\underset{\sim}{P}^k_{>\eta}) = P^k_{>\eta}$-generic over $V_2[G_>\eta]$.
So it is enough to show that the forcing $P_{>\eta} := \prod_{k<\omega} A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$ is nice enough to preserve η. We argue that the Cohen forcing above η combined with it will have a dense η-closed subset.
Denote by V'_2 the model $V_1[\langle f_\nu \mid \nu < \eta \rangle]$. Let $Cohen_{<\eta}$ denotes the iteration of the Cohen forcings which adds $\langle f_\nu \mid \nu < \eta \rangle$ and $Cohen_{\geq\eta}$ be the iteration of the Cohen forcings adding the rest of the sequence $\langle f_\nu \mid \kappa \geq \nu \geq \eta \rangle$. Clearly, $P_\eta := \prod_{k<\omega} P^k_\eta$ is defined in V'_2. Also P_η satisfies η-c.c.

Work in V'_2. Let p be in $Cohen_{\geq\eta}$ and $\underset{\sim}{p}$ be a $Cohen_{<\eta}$–name of it. By $\mathrm{supp}(p)$, a support of p, we mean a support set a of ordinals in the interval $[\eta, \kappa)$ such that $p_\alpha = \emptyset$, for every

$\alpha \in [\eta, \kappa) \setminus a$, where p_α is the α-th coordinate of p. Note that since $|Cohen_{<\eta}| \leq \eta$ the requirements of Lemma 5.1 are trivially satisfied and so it is possible always to extend $\underset{\sim}{p}$ to a $Cohen_{<\eta}$–name of a condition which has a support in V_1 (or even in V).
Let now $\underset{\sim}{\tau} \in A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$. Extend it to a condition $\underset{\sim}{\tau}' \in A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$ such that a support $\underset{\sim}{\tau}'$ (where again by a support of $\underset{\sim}{\tau}'$ we mean a support set a of ordinals $< \kappa$ such that on coordinates $\nu \in \kappa \setminus a$ we have $\Vdash_{P^k_\eta} \underset{\sim}{\tau}'(\nu) = \emptyset$) is in V_2' (or even in V_1 or further down in V). It is possible since $|P^k_\eta| \leq \eta$ and $Cohen_{\geq\eta}$ satisfies the requirements of Lemma 5.1.

Now we will proceed similar to Lemma 2.1 and define
a dense η-closed subset of $Cohen_{\geq\eta} * A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$.

Let D^k be a subset of $Cohen_{\geq\eta} * A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$ which consists of all pairs $(\underset{\sim}{p}, \underset{\sim}{\sigma})$ such that

1. $\mathrm{supp}(\underset{\sim}{p}) \in V$,
2. $\mathrm{supp}(\underset{\sim}{\sigma}) \in V$ and $\underset{\sim}{p}$ decides it,
3. $\mathrm{supp}(\underset{\sim}{\sigma}) \subseteq \mathrm{supp}(\underset{\sim}{p})$,
4. $\underset{\sim}{p}$ decides $\max(\underset{\sim}{\sigma}(\nu))$ for each $\nu \in \mathrm{supp}(\underset{\sim}{\sigma})$ and
   the decided value includes the set $\{\alpha \in \mathrm{dom}(\underset{\sim}{p}_\nu) \mid \underset{\sim}{p}_\nu(\alpha) = k-1\}$, where $\underset{\sim}{p}_\nu$ is the ν–th coordinate of $\underset{\sim}{p}$, i.e. the one that constructs a Cohen function f_ν.

It is not hard to see that such D^k is dense. Just start with any $(\underset{\sim}{q}, \underset{\sim}{\rho}) \in Cohen_{\geq\eta} * A(P^k_\eta, \underset{\sim}{P}^k_{>\eta})$ extend it first to some $(\underset{\sim}{q}', \underset{\sim}{\rho}')$ which satisfies items 1-3 above. In order to satisfy 4, let us recall that Q^k_α is trivial unless α is Mahlo and the iteration of Cohen forcings $Cohen_{<\alpha}$ up to a Mahlo cardinal α satisfies α–c.c. So we can proceed inductively dealing with coordinates ν in $\mathrm{supp}(\underset{\sim}{\rho}')$ extending first $\underset{\sim}{q}'$ (only on coordinates $\geq \nu$) in order to decide $\max(\underset{\sim}{\rho}'(\nu))$ and then, if necessary, to increase this maximum extending $\underset{\sim}{\rho}'(\nu)$. This procedure will construct a condition $(\underset{\sim}{p}, \underset{\sim}{\sigma}) \in D^k$ which is stronger than $(\underset{\sim}{q}, \underset{\sim}{\rho})$.
Now, as in Lemma 2.1, D^k is η-closed.

Finally let

$$D = \{\langle \underset{\sim}{p}, \langle \underset{\sim}{\sigma}_k \mid k < \omega \rangle\rangle \in Cohen_{\geq\eta} * P_{>\eta} \mid \forall k < \omega \quad (\underset{\sim}{p}, \underset{\sim}{\sigma}_k) \in D^k\}.$$

The arguments as above show that D is a dense η-closed subset of $Cohen_{\geq\eta} * P_{>\eta}$. Hence the forcing $Cohen_{\geq\eta} * P_{>\eta}$ does not collapse η and we are done.
□

The next two lemmas are analogs of Lemmas 2.2 and 2.3.

Lemma 5.3 *The forcing P preserves measurability of κ.*

Proof. Consider the image of the forcing in M_U. At the stage κ we need first to pick a natural number n_κ. There are no limitation of its choice. Let us take $n_\kappa = 0$. Then proceed as in 2.2 only apply the arguments of 5.2 above κ in order to construct a master condition sequence.

□

Given $n_\kappa > 0$, we can continue P naturally over κ just forcing n_κ–relevant clubs $C_{\kappa 0}, ..., C_{\kappa n_\kappa - 1}$. Namely we force with $(\prod_{1\leq k\leq n_\kappa} P^k_{\kappa+1}) \times (\prod_{n_\kappa<k<\omega} P^k_\kappa)$. $C_{\kappa k}$ will be $Q_{\kappa k}$–generic clubs disjoint to the sets $S_{\kappa k} = \{\gamma < \kappa \mid f_\kappa(\gamma) = k-1\}$, for every $k, 1 \leq k \leq n_\kappa$.

Lemma 5.4 *κ remains measurable in $V_2[G(P)][C_{\kappa 0}, ..., C_{\kappa n_\kappa -1}]$ for any $n = n_\kappa > 0$, where $G(P)$ is a generic subset of P.*

Proof. Construct a master conditions sequence using the arguments of Lemma 5.2. It is used here that for every l the forcing $P^l_{\kappa+1}$ is an iteration. Namely we need that $C_{\kappa,l}$ is an initial segment of $C_{j_l(\kappa),l}$, where j_l is the elementary embedding of U^l.
Let $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+\rangle$ be such a sequence of conditions in $Cohen_{>\kappa} * (\prod_{l<\omega} A(P^l_{\kappa+1}, \underset{\sim}{P}^l_{>\kappa+1}))$.
The ultrafilter $U(n)$ is defined as follows:
$X \in U(n)$ iff there are $\alpha < \kappa^+, p \in G(Cohen_{\leq\kappa})$, $q \in G(P)$ and $s_0 \in C_{\kappa 0}, ..., s_{n_\kappa -1} \in C_{\kappa n_\kappa -1}$ such that

$$(p, q, \langle s_0, ..., s_{n_\kappa-1}\rangle, \underset{\sim}{r}_\alpha) \Vdash \kappa_{n_k} \in j_{n_k}(\underset{\sim}{X}).$$

□

Let $U(n)$ be the ultrafilter on κ in $V_2[G(P)][C_{\kappa 0}, ..., C_{\kappa n_\kappa -1}]$ defined using Lemma 5.4 as in the previous section.
Note that a projection of $U(n)$ projects to a normal ultrafilter by the function $\nu \mapsto \max(C_{\kappa n} \cap \nu)$, since in the ultrapower κ is the largest point of $C_{j_n(\kappa),n}$ below $[id]_{U(n)} = \kappa_n$.
Denote $\max(C_{\kappa n} \cap \nu)$ by ν^0

Let $X \in U(n)$. Then, as in the previous lemma, we have $\alpha < \kappa^+, p \in G(Cohen_{\leq\kappa})$, $q \in G(P)$ and $s_0 \in C_{\kappa 0}, ..., s_{n_\kappa -1} \in C_{\kappa n_\kappa -1}$ such that

$$(p, q, \langle s_0, ..., s_{n_\kappa-1}\rangle, \underset{\sim}{r}_\alpha) \Vdash \kappa_{n_k} \in j_{n_k}(\underset{\sim}{X}).$$

Consider

$$X^* = \{\nu < \kappa \mid n_{\nu^0} = n_\kappa, p \in G(Cohen_{\leq \nu^0}), q \in G(P \restriction \nu^0),$$

$$s_0 \in G(Q_{\nu^0 1}), ..., s_{n_\kappa - 1} \in G(Q_{\nu^0 n_\kappa}), h_\alpha(\nu)_{G(P \restriction \nu^0 + 1)} \in G(P_{>\nu^0})\},$$

where h_α is a fixed function with $[h_\alpha]_{U(n)} = \alpha$ and for a forcing notion R, $G(R)$ denotes its generic subset, $P_{>\nu^0}$ denotes $P/G(P \restriction \nu^0 + 1)$.

Then X belongs to $U(n)$ iff X^* belongs to it.

6 Automorphisms

Let $k, 1 \leq k < \omega$. We arrange automorphisms not only of the component over κ but rather over all of $P^k_{\kappa+1}$. Thus let $c, d \in Q_{\kappa k}$. Assume that they are in V_2, otherwise just force with an initial segment of $P^k_{\kappa+1}$ to decide c, d. Assume also that $\max(c) = \max(d)$. Set $\pi_{cd}(c) = d$. Replace c by d in every condition stronger than c . Do this not only over κ but on a final segment below (in $Q_{\nu k}$'s).

Turn now to a formal definition of $\mathcal{G}^k$, for a fixed $k, 1 \leq k < \omega$.
Let us first specify a dense subset of $P^k_{\kappa+1}$ over which automorphisms will be defined. Set $D = \{\langle \underset{\sim}{p}_\nu \mid \nu \leq \kappa \rangle \in P^k_{\kappa+1} \mid \langle \max(\underset{\sim}{p}_\nu) \mid \nu \leq \kappa, \underset{\sim}{p}_\nu \neq 0_{Q_{\nu k}} \rangle$ and $\mathrm{otp}(\underset{\sim}{p}_\kappa)$ are decided and are in $V_2\}$.[2]

Lemma 6.1 *The set D is dense in $P^k_{\kappa+1}$.*

Proof. Let $q = \langle \underset{\sim}{q}_\nu \mid \nu \leq \kappa \rangle \in P^k_{\kappa+1}$. The forcing P^k_ν satisfies ν–c.c. for every $\nu \leq \kappa$. Hence we can find $\alpha_\nu < \nu$ such that $f_\nu(\alpha_\nu) \neq k - 1$ and $0_{P^k_\nu} \Vdash \alpha_\nu > \max(\underset{\sim}{q}_\nu)$. Set $\underset{\sim}{p}_\nu = \underset{\sim}{q}_\nu \cup \{\alpha_\nu\}$. Then $p = \langle \underset{\sim}{p}_\nu \mid \nu \leq \kappa \rangle$ will be a condition stronger than q and in D.
□

Let $p = \langle \underset{\sim}{p}_\nu \mid \nu \leq \kappa \rangle, q = \langle \underset{\sim}{q}_\nu \mid \nu \leq \kappa \rangle \in D$ be so that

1. $p \restriction \kappa := \langle \underset{\sim}{p}_\nu \mid \nu < \kappa \rangle = q \restriction \kappa := \langle \underset{\sim}{q}_\nu \mid \nu < \kappa \rangle$,
2. $\underset{\sim}{p}_\kappa$ is not empty,
3. $\underset{\sim}{q}_\kappa$ is not empty,
4. $\mathrm{otp}(\underset{\sim}{p}_\kappa) = \mathrm{otp}(\underset{\sim}{q}_\kappa)$
5. $\max(\underset{\sim}{p}_\kappa) = \max(\underset{\sim}{q}_\kappa)$.

[2]Recall that we work over V_2 and $\underset{\sim}{p}_\nu$'s are P^k_ν-names.

Denote by η the least ordinal below κ such that for every $\nu, \eta \leq \nu < \kappa$ the ν-th coordinate $\underset{\sim}{p}_\nu$ of p (and hence also of q) is $0_{Q_{\nu k}}$, i.e. supports of $p \restriction \kappa = \langle \underset{\sim}{p}_\nu \mid \nu < \kappa \rangle$ and $q \restriction \kappa = \langle \underset{\sim}{q}_\nu \mid \nu < \kappa \rangle$ are contained in η. Recall that the direct limit is used over κ, and so such η exists. We have $p \restriction \eta = q \restriction \eta$. Let $\delta = \max(\underset{\sim}{p}_\kappa) = \max(\underset{\sim}{q}_\kappa)$. We can assume, using the κ-c.c. of P^k_κ, that $\underset{\sim}{p}_\kappa, \underset{\sim}{q}_\kappa$ are P^k_η–names. Denote them by $\underset{\sim}{c}$ and $\underset{\sim}{d}$ respectively. We can assume by increasing η if necessary that $\delta < \eta$.

Define an automorphism π_{pq} for such p and q.
Consider a Mahlo cardinal ν such that $\eta \leq \nu \leq \kappa$ and $f_\kappa \restriction \nu = f_\nu$. Let $G(P^k_\nu)$ be a generic subset of P^k_ν with $p \restriction \eta \in G(P^k_\nu) \restriction \eta$. Define an automorphism $\pi_{pq\nu}$ of a dense subset of $Q_{\nu k}$ which consists of r_ν with $\max(r_\nu) \geq \max(c)$. If $r_\nu \geq_{Q_{\nu k}} c$ then let $\pi_{pq\nu}(r_\nu) = (r_\nu \setminus c) \cup d$. If $r_\nu \geq_{Q_{\nu k}} d$ then let $\pi_{pq\nu}(r_\nu) = (r_\nu \setminus d) \cup c$. Note that the assumption $f_\kappa \restriction \nu = f_\nu$ implies that both c and d are in $Q_{\nu k}$, and so it makes sense to compare them with other elements of $Q_{\nu k}$. If r_ν is incompatible with c and with d, then set $\pi_{pq\nu}(r_\nu) = r_\nu$.
Let $\underset{\sim}{\pi}_{pq\nu}$ be a P^k_ν-name of such $\pi_{pq\nu}$.
We are ready now to define π_{pq}. Let $r = \langle \underset{\sim}{r}_\nu \mid \nu \leq \kappa \rangle \in D$. Suppose that for each $\nu, \eta \leq \nu \leq \kappa$ we have $0_{P^k_\nu} \Vdash \underset{\sim}{r}_\nu \in \underset{\sim}{Q}_{\nu k}$. If $r \geq p$ or $r \geq q$, then set $\pi_{pq}(r) = s$ iff $s = \langle \underset{\sim}{s}_\nu \mid \nu \leq \kappa \rangle$ is such that

1. $\underset{\sim}{s}_\nu = \underset{\sim}{r}_\nu$, for every $\nu < \eta$,

2. if $\nu, \eta \leq \nu \leq \kappa$, then $\underset{\sim}{s}_\nu$ is a canonical name of $\pi_{pq\nu}(r_\nu)$.

If r is incompatible with p and with q, then let $\pi_{pq}(r) = r$.

Let us check that such defined automorphism π_{pq} does what is intended.

Let G be a generic subset of $P^k_{\kappa+1}$ with $p \in G$. Denote $G \restriction Q_{\nu k}$ by $G(\nu)$. Then π_{pq} transforms G into another $P^k_{\kappa+1}$ generic set H. Let $H(\nu) = H \restriction Q_{\nu k}$.

Lemma 6.2 *$c \in G(\nu)$ iff $d \in H(\nu)$, for every $\nu, \eta \leq \nu \leq \kappa$.*

Proof. Induction on ν. $G(\nu)$ remains generic over $V_2[H \restriction \nu]$, since $V_2[H \restriction \nu] = V_2[G \restriction \nu]$ and this is because $\pi_{pq} \restriction \nu$ is an automorphism of P^k_ν. Now $\pi_{pq\nu}$ moves $G(\nu)$ to $H(\nu)$, by its definition.
□

Denote by $\mathcal{G}^k$ the group of automorphisms of generated by such π_{pq}'s.

Let $\mathcal{G}$ be the group of automorphisms of $\prod_{1 \leq k < \omega} P^k_{\kappa+1} = \prod_{1 \leq k < \omega} (P^k_\kappa * Q_{\kappa k})$ generated by $\langle \mathcal{G}^k \mid k < \omega \rangle$, i.e. $\pi \in \mathcal{G}$ iff $\pi = \langle \pi_0, ..., \pi_k, ... \mid k < \omega \rangle$ and there is a finite $a \subseteq \omega$ such that for every $k \in \omega \setminus a$ the automorphism $\pi_k \in \mathcal{G}^k$ is the identity.

For every finite $a \subseteq \omega$ consider $\text{fix}(a) := \{\pi \in \mathcal{G} \mid \forall k \in a, \pi_k$ is the identity $\}$. Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by $\{\text{fix}(a) \mid a \subseteq \omega$ finite $\}$.

Let $\mathcal{N}$ be a corresponding symmetric submodel.

The lemmas 3.1, 3.2, 3.3 are still valued here.

Such defined group of automorphisms will insure eventually that the set (filter)

$$\{X^* \mid X \in \bigcup_{n<\omega} U(n)\}$$

is in $\mathcal{N}$, where X^*'s are as defined at the end of the previous section.

Note that if $X \in U(n)$ there is no need in elements of the master sequence $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+\rangle$ to witness this, i.e. there are $p \in G(Cohen_{\leq\kappa})$, $q \in G(P)$ and $s_0 \in C_{\kappa 0}, ..., s_{n_\kappa - 1} \in C_{\kappa n_\kappa - 1}$ such that

$$(p, q, \langle s_0, ..., s_{n_\kappa-1}\rangle, \emptyset) \Vdash \kappa_{n_k} \in j_{n_k}(\underset{\sim}{X}),$$

then images of corresponding $\underset{\sim}{X}^*$ under automorphisms of $\mathcal{G}$ will include final segments X^* and so will remain inside $U(n)$.

Unfortunately sets which correspond to the master conditions sequence are still problematic. Namely, let $\alpha < \kappa^+$ and h_α be a function which represents $\underset{\sim}{r}_\alpha$ in the ultrapower. Then the set

$$\{\nu \mid h_\alpha(\nu)_{G(P \restriction \nu^0+1)} \in G(P_{>\nu^0})\}$$

is in $\bigcup_{n<\omega} U(n)$, but images of it under automorphisms of $\mathcal{G}$, which change $G(Q_{\nu k})$'s for unboundedly many ν's, may take it out.

In what follows we deal with this problem by redefining the master condition sequence and making it more symmetric.

Assume for simplicity that $n = 0$ and redefine a master condition sequence for $U(0)$. In a general case only notation are more complicated.

Fix some $k, 1 \leq k < \omega$. Work in $M'[G(P^k_\kappa)]$, where M' is the ultrapower of V_2 by a normal ultrafilter which extends U and which embedding extends j_U. We have there the forcing $Q_{\kappa k} * \underset{\sim}{R}$, where R is the continuation of $j(P^k_\kappa)$ above κ.

Fix a list $\langle \underset{\sim}{D}_\alpha \mid \alpha < \kappa^+\rangle$ of all dense open subsets of $\underset{\sim}{R}$ which are in M'. Assume that already the weakest condition of $P^k_\kappa * Q_{\kappa k}$ forces this.

Let us construct first $\underset{\sim}{r}_0$ an element of $\underset{\sim}{D}_0$ which will be the first element of the the master condition $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+\rangle$ sequence chosen in a special way.

Pick first a $Q_{\kappa k}$-name $\underset{\sim}{q}$ such that $\emptyset \Vdash_{Q_{\kappa k}} \underset{\sim}{q} \in \underset{\sim}{D}_0$.

Consider the following maximal antichain in $Q_{\kappa k}$:

$$A = \{\{\beta\} \mid \beta < \kappa, \{\beta\} \in Q_{\kappa k}\}.$$

Let $\alpha = \min(A)$, i.e. $\alpha = \min\{\nu < \kappa \mid f_\kappa(\nu) \neq k-1\}$. Set

$\underset{\sim}{q}^{\alpha 0} = \{(c, \underset{\sim}{a}) \in \underset{\sim}{q} \mid c$ is compatible with (i.e. an end extension of or an initial segment of) $\{\alpha\}\}$.

Let β is the least in $A \setminus \{\alpha\}$. Consider an automorphism $\pi_{\alpha\beta}$ of $Q_{\kappa k}$ which switches $\{\alpha\}$ and $\{\beta\}$. Actually the corresponding $\pi_{\{\alpha\}\{\beta\}}$ will be a member of $\mathcal{G}^k$, but let us deal only with its main relevant here part which acts over $Q_{\kappa k}$. Thus let $\pi_{\alpha\beta}(\{\alpha\}) = \{\beta\}$, $\pi_{\alpha\beta}(\{\beta\}) = \{\alpha\}$ and then for every $c \in Q_{\kappa k}$ stronger than $\{\alpha\}$ (or $\{\beta\}$) with $\max(c) > \alpha, \beta$ set $\pi_{\alpha\beta}(c) = \{\beta\} \cup c \setminus (\max(\alpha,\beta)+1)$ (or $\pi_{\alpha\beta}(c) = \{\alpha\} \cup c \setminus (\max(\alpha,\beta)+1)$). On elements incompatible with both $\{\alpha\}, \{\beta\}$ let it be the identity.
$\pi_{\alpha\beta}$ defines naturally the automorphism of $\underset{\sim}{R}$ which is a set of $Q_{\kappa k}$-names. Denote it by $\pi_{\alpha\beta}$ as well in order not to overcomplicate the notation. Consider $\pi_{\alpha\beta}(\underset{\sim}{q}^{\alpha 0})$. In general $\{\beta\}$ does not force "$\pi_{\alpha\beta}(\underset{\sim}{q}^{\alpha 0}) \in \underset{\sim}{D}_0$". Extend $\pi_{\alpha\beta}(\underset{\sim}{q}^{\alpha 0})$ to a condition $\underset{\sim}{q}^{\beta 1}$ such that $\{\beta\} \Vdash_{Q_{\kappa k}} \underset{\sim}{q}^{\beta 1} \in \underset{\sim}{D}_0$. Now move $\underset{\sim}{q}^{\beta 1}$ back to α using $\pi^{-1}_{\alpha\beta}$. Denote the result by $\underset{\sim}{q}^{\alpha 1}$. Note that $\{\alpha\} \Vdash_{Q_{\kappa k}} \underset{\sim}{q}^{\alpha 0} \leq \underset{\sim}{q}^{\alpha 1}$, since $\pi_{\alpha\gamma}$ is an automorphism on $\underset{\sim}{R}$ as well.
Let now γ be the least in $A \setminus \{\alpha, \beta\}$. Consider an automorphism $\pi_{\alpha\gamma}$ of $Q_{\kappa k}$ which switches $\{\alpha\}$ and $\{\gamma\}$. Extend $\pi_{\alpha\gamma}(\underset{\sim}{q}^{\alpha 1})$ to a condition $\underset{\sim}{q}^{\gamma 2}$ such that $\{\gamma\} \Vdash_{Q_{\kappa k}} \underset{\sim}{q}^{\gamma 2} \in \underset{\sim}{D}_0$. Move $\underset{\sim}{q}^{\gamma 2}$ back to α using $\pi^{-1}_{\alpha\gamma}$. Denote the result by $\underset{\sim}{q}^{\alpha 2}$. Set $\underset{\sim}{q}^{\beta 2}$ to be $\pi_{\alpha\beta}(\underset{\sim}{q}^{\alpha 2})$.

Continue further in the same fashion. At limit stages we use the Cohen part to provide a needed degree of completeness, as it was done in Lemma 5.2. At the final stage we will have $\underset{\sim}{q}^{\delta\kappa}$'s for every $\delta < \kappa$ with $\{\delta\} \in A$. Set $\underset{\sim}{q}^{\delta} = \underset{\sim}{q}^{\delta\kappa}$. The following will hold:

1. $\{\delta\} \Vdash_{Q_{\kappa k}} \underset{\sim}{q}^{\delta} \in \underset{\sim}{D}_0$
2. $\pi_{\delta\delta'}(\underset{\sim}{q}^{\delta}) = \underset{\sim}{q}^{\delta'}$.

Now we combine $\underset{\sim}{q}^{\delta}$'s together into a single $Q_{\kappa k}$–name of an element of $\underset{\sim}{R}$. Set $\underset{\sim}{q}(1) = \{\underset{\sim}{q}^{\delta} \mid \{\delta\} \in A\}$. Then, by the construction, the following holds:

$(*)_1$ $\emptyset \Vdash_{Q_{\kappa k}} \underset{\sim}{q}(1) \in \underset{\sim}{D}_0$ and for every $\delta, \delta' \in A$, $\quad \pi_{\delta\delta'}(\underset{\sim}{q}(1)) = \underset{\sim}{q}(1)$.

Proceed by induction and define $\underset{\sim}{q}(\tau)$ for every $\tau, 1 < \tau \leq \kappa$.

If τ is a limit ordinal, then combine first the names $\langle \underset{\sim}{q}(\mu) \mid \mu < \tau \rangle$ together into one name

$$\underset{\sim}{q}'(\tau) = \{\underset{\sim}{t} \mid \exists \mu < \tau, \underset{\sim}{t} \in \underset{\sim}{q}(\mu)\}.$$

Then $\underset{\sim}{q}(\tau)$ is obtained from $\underset{\sim}{q}'(\tau)$ by adding maximums whenever it is necessary (thus, for example it may happen that for some $c \in Q_{\kappa k}$ and an increasing sequence $\langle \rho_i \mid i < \xi \rangle$ the pairs $(c, \check{\rho}_i)$ appear in $\underset{\sim}{q}(\mu)$'s, we add (setting the right value of the corresponding Cohen function first) $(c, \check{\rho}_\xi)$, where $\rho_\xi = \bigcup_{i<\xi} \rho_i$).

If $\tau < \kappa$, then we continue and define $\underset{\sim}{q}(\tau + 1)$, in order to take care of automorphisms which involve switching between closed subsets of κ of order type $\tau + 1$.

Set

$$A_{\tau+1} = \{c \in Q_{\kappa k} \mid \text{ the order type of } c \text{ is } \tau + 1\}.$$

It is a maximal antichain. Fix some well ordering on $A_{\tau+1}$. Let c be the least element of $A_{\tau+1}$. Set $\underset{\sim}{q}^{c0}$ to be

$$\{(e, \underset{\sim}{a}) \in \underset{\sim}{q}(\tau) \mid e \text{ is compatible with (i.e. an end extension of or an initial segment of) } c\}.$$

Let d be the least member of $A_{\tau+1} \setminus \{c\}$ in the fixed well ordering. Consider $\pi_{cd}(\underset{\sim}{q}^{c0})$. Let $\underset{\sim}{q}^{d1}$ be its extension such that $\{d\} \Vdash_{Q_{\kappa k}} \underset{\sim}{q}^{d1} \in \underset{\sim}{D}_0$. Now move $\underset{\sim}{q}^{d1}$ back to c using π_{cd}^{-1}. Denote the result by $\underset{\sim}{q}^{c1}$.

Continue in the same fashion by induction and go through all elements of $A_{\tau+1}$. At the final stage we will have $\underset{\sim}{q}^{e\kappa}$'s for every $e \in A_{\tau+1}$. Set $\underset{\sim}{q}^{e} = \underset{\sim}{q}^{e\kappa}$. The following will hold:

1. $e \Vdash_{Q_{\kappa k}} \underset{\sim}{q}^{e} \in \underset{\sim}{D}_0$
2. for every $e, e' \in A_{\tau+1}$, $\quad \pi_{ee'}(\underset{\sim}{q}^{e}) = \underset{\sim}{q}^{e'}$.

Combine $\underset{\sim}{q}^{e}$'s together. Set $\underset{\sim}{q}(\tau + 1) = \{\underset{\sim}{q}^{e} \mid e \in A_{\tau+1}\}$.

Then, by the construction, the following holds:

$$(*)_{\tau+1} \quad \emptyset \Vdash_{Q_{\kappa k}} \underset{\sim}{q}(1) \in \underset{\sim}{D}_0 \text{ and for every } e, e' \in A_{\tau+1}, \quad \pi_{ee'}(\underset{\sim}{q}(\tau + 1)) = \underset{\sim}{q}(\tau + 1).$$

We obtain the desired condition $\underset{\sim}{r}_0$ at the final stage κ. Just set $\underset{\sim}{r}_0 = \underset{\sim}{q}(\kappa)$.

Then we will have the following, since $(*)_{\tau+1}$ holds, for $\tau < \kappa$:

$$(*)_\kappa \quad \emptyset \Vdash_{Q_{\kappa k}} \underset{\sim}{r}_0 \in \underset{\sim}{D}_0 \text{ and for every } e, e' \in Q_{\kappa k} \text{ of a same order type}, \quad \pi_{ee'}(\underset{\sim}{r}_0) = \underset{\sim}{r}_0.$$

The next lemma states a crucial property of the above construction.

Lemma 6.3 *Let $C \subseteq Q_{\kappa k}$ be generic, $c \in C$, $c' \in Q_{\kappa k}$ have the same order type. Let $C' = \pi''_{cc'} C$. Assume that $G(\underset{\sim}{R})$ is R-generic (interpretations of $\underset{\sim}{R}$ with C or with C' are the same) with $(\underset{\sim}{r}_0)_C$ inside and let $G'(\underset{\sim}{R}) = \pi_{cd}$"$G(\underset{\sim}{R})$. Then $(\underset{\sim}{r}_0)_{C'} \in G'(\underset{\sim}{R})$.*

Proof. It was arranged at the stage $otp(c)$ that $\pi_{cc'}(\underset{\sim}{q}^{c}) = \underset{\sim}{q}^{c'}$. Hence $\pi_{cc'}(\{\underset{\sim}{q}^{c}, \underset{\sim}{q}^{c'}\}) = \{\underset{\sim}{q}^{c'}, \underset{\sim}{q}^{c}\}$. Clearly, $\{\underset{\sim}{q}^{c}, \underset{\sim}{q}^{c'}\}_C = (\underset{\sim}{q}^{c})_C$ and $\{\underset{\sim}{q}^{c}, \underset{\sim}{q}^{c'}\}_{C'} = (\underset{\sim}{q}^{c'})_{C'}$.
Further in the process of construction of $\underset{\sim}{r}_0$ this symmetry above c and c' remains which guarantees $(\underset{\sim}{r}_0)_{C'} \in G'(\underset{\sim}{R})$.
□

Now we define $\underset{\sim}{r}_1$ starting with $\underset{\sim}{r}_0$ and using $\underset{\sim}{D}_1$ instead of $\underset{\sim}{D}_0$. Continue in the same fashion and define $\underset{\sim}{r}_\alpha$, for every $\alpha < \kappa^+$. This defines a desired master condition sequence.

Finally use this new master condition sequences to redefine $U(n)$'n. The next lemma concludes the argument.

Lemma 6.4 *$\bigcup_{n<\omega} \underset{\sim}{U}(n)$ is a symmetric name, and hence $\bigcup_{n<\omega} U(n) \in \mathcal{N}$.*

Proof. Let $n < \omega$ and $\underset{\sim}{X}$ be a name of an element of $U(n)$. Replace X by X^* defined at the end of the previous section. Then automorphisms of $\mathcal{G}$ do not effect final segments of X^*. This means that $U(n)$ remains unchanged.
□.

7 A related question.

Note that $V_\kappa^{\mathcal{N}}$ does not satisfy AC. So the following question looks natural:

Question. Is it possible to have a model of ZF with a measurable cardinal κ without a normal measure but such that V_κ satisfies AC?

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