Dropping cofinalities and gaps

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Abstract

Our aim is to show the consistency of $2^\kappa \geq \kappa^{+\theta}$, for any $\theta < \kappa$ starting with a singular cardinal κ of cofinality ω so that

$$\forall n < \omega \exists \alpha < \kappa (o(\alpha) = \alpha^{+n}).$$

Dropping cofinalities technics are used for this purpose.

1 $\aleph_1$-gap and infinite drops in cofinalities

Let κ be a singular cardinal of cofinality ω such that for each $\gamma < \kappa$ and $n < \omega$ there is $\alpha, \gamma < \alpha < \kappa$, such that $o(\alpha) = \alpha^{+n}$. We fix a sequence of cardinals $\kappa_0 < \kappa_1 < ... < \kappa_n < ..., n < \omega$ so that

- $\bigcup_{n<\omega} \kappa_n = \kappa$
- for every $n < \omega$, κ_n is κ_n^{+n+2} - strong as witnessed by an extender E_{κ_n}
- for every $n < \omega$, the normal measure of E_{κ_n} concentrates on

  τ's which are $\tau^{+n+2} + \omega_1$ - strong as witnessed by a coherent sequence of extenders $\langle E_\tau(\xi) | \xi < \omega_1 \rangle$

Fix also an increasing sequence $\langle \lambda_n | n < \omega \rangle$ such that

- $\lambda_0 < \kappa_0$

- $\kappa_{n-1} < \lambda_n < \kappa_n$, for every $n, 0 < n < \omega$
- for every $n < \omega$, λ_n is λ_n^{+n+2} - strong as witnessed by an extender E_{λ_n}

Our aim will be to make $2^\kappa = \kappa^{+\omega_1+1}$. There is nothing special here in choosing ω_1. The same construction will work if we replace everywhere ω_1 by an ordinal $\theta, \theta < \lambda_0$. Actually, replacing the original λ_0 by a bigger one, we can deal similar with any $\theta < \kappa$. Note that for finite θ's our assumption is not anymore optimal and for countable θ's the result was already known, see the detailed discussion in [3].

Let us force first with the iteration $\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})$ of the preparation forcing $\mathcal{P}'$ of [7].
We assign to κ^{++} at a level n the indiscernible η_n^{+n+2}, where η_n is the indiscernible for the normal measure of the extender E_{λ_n}. The correspondence between regular cardinals in the interval $[\kappa^{+3}, \kappa^{+\omega_1+1}]$ will be as follows: we assign to $\kappa^{+\omega_1+1}$ at a level n the indiscernible ρ_n^{+n+2}, where ρ_n is the indiscernible for the normal measure of the extender E_{κ_n}. Let $\langle \rho_{n\alpha} | \alpha < \omega_1 \rangle$ the Magidor sequence corresponding to the normal measures of E_{ρ_n} the one used in the extender based Magidor forcing to change cofinality of ρ_n to ω_1. For every $\alpha, 1 < \alpha < \omega_1$, we assign $\rho_{n\alpha}^{+n+2}$ to $\kappa^{+\alpha+1}$.

2 Level n

Let G be a generic subset of $\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})$ and $G(\mathcal{P}'(\kappa^{+\alpha+1}))$ be a generic subset of $\mathcal{P}'(\kappa^{+\alpha+1})$, which is the $\alpha+1$-th component of G.

Fix $n<\omega$. We describe the forcing used at the level n of the construction.

Definition 2.1 Let Q_{n0} be the set consisting of pairs of triples $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle$ so that:

1. f is partial function from κ^{+2} to λ_n of cardinality at most κ

2. a is a partial function of cardinality less than λ_n so that

   (a) There is $\langle\langle A^{0\kappa^+}(\kappa^{++}), A^{1\kappa^+}(\kappa^{++}), C^{\kappa^+}(\kappa^{++})\rangle\rangle \in G(\mathcal{P}'(\kappa^{++}))$ which we call it further **a background condition of** a, such that $\mathrm{dom}(a)$ consists of models appearing in $A^{1\kappa^+}(\kappa^{++})$, i.e. basically of ordinals below κ^{++}.

   Note that the third component $C^{\kappa^+}(\kappa^{++})$ of a condition is just the same as the second $A^{1\kappa^+}$. Also the inclusion is a linear order on $A^{1\kappa^+}(\kappa^{++})$ and this set is closed under unions.

   (b) for each $X \in \mathrm{dom}(a)$ there is $k \leq \omega$ so that $a(X) \subseteq H(\chi^{+k})$.

   Moreover,

   (i) $|a(X)| = \lambda_n^{+n+1}$ and $a(X) \cap \lambda_n^{+n+2} \in ORD$

   (iii) $A^{0\kappa^+}(\kappa^{++}) \in \mathrm{dom}(a)$.

   This way we arranged that λ_n^{+n+1} will correspond to κ^+ and λ_n^{+n+2} will correspond to κ^{++}.

   Further let us refer to $A^{0\kappa^+}(\kappa^{++})$ as **the maximal model of the domain of** a. Denote it as $\max(\mathrm{dom}(a))$.

   Later passing from Q_{0n} to $\mathcal{P}$ we will require that for every $k<\omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

   (c) (Models come from $A^{0\kappa^+}(\kappa^{++})$) If $X \in \mathrm{dom}(a)$ and $X \neq A^{0\kappa^+}(\kappa^{++})$ then $X \in A^{0\kappa^+}(\kappa^{++})$.

   The condition puts restriction on models in $\mathrm{dom}(a)$ and allows to control them via the maximal model of cardinality κ^+.

(d) If $X, Y \in \mathrm{dom}(a)$, $X \in Y$ (or $X \subseteq Y$) and k is the minimal so that $a(X) \subseteq H(\chi^{+k})$ or $a(Y) \subseteq H(\chi^{+k})$, then $a(X) \cap H(\chi^{+k}) \in a(Y) \cap H(\chi^{+k})$ (or $a(X) \cap H(\chi^{+k}) \subseteq a(Y) \cap H(\chi^{+k})$).

The intuitive meaning is that b is supposed to preserve membership and inclusion. But we cannot literally require this since $a(A)$ and $a(B)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(e) The image by a of $A^{0\kappa^+}$, i.e. $a(A^{0\kappa^+})$, intersected with λ_n^{+n+2} is above all the rest of $\mathrm{rng}(a)$ restricted to λ_n^{+n+2} in the ordering of the extender E_n (via some reasonable coding by ordinals).

Recall that the extender E_{λ_n} acts on λ_n^{+n+2} and our main interest is in Prikry sequences it will produce. So, parts of $\mathrm{rng}(a)$ restricted to δ_n^{+n+2} will play the central role.

3. $\{\alpha < \kappa^{+3} \mid \alpha \in \mathrm{dom}(a)\} \cap \mathrm{dom}(f) = \emptyset$

4. $A \in E_{\lambda_n, a(\max(a))}$

5. $\min(A) > |dom(a)| + |\mathrm{dom}(\underset{\sim}{b})|$

6. for every ordinals α, β, γ which are elements of $\mathrm{rng}(a)$ or actually the ordinals coding models in $\mathrm{rng}(a)$ we have

$$\alpha \geq_{E_{\lambda_n}} \beta \geq_{E_{\lambda_n}} \gamma \quad \text{implies}$$
$$\pi_{\lambda_n,\alpha,\gamma}(\rho) = \pi_{\lambda_n,\beta,\gamma}(\pi_{\lambda_n,\alpha,\beta}(\rho))$$

for every $\rho \in \pi''_{\lambda_n, \max \mathrm{rng}(a), \alpha}(A)$.

Let us turn now to the second component of a condition, i. e. to $\langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle$.

7. $\underset{\sim}{b} = \langle \underset{\sim}{b}_\alpha | 1 < \alpha \leq \omega_1\rangle$ is a name, depending on $\langle a, A\rangle$, of a partial functions b_α of cardinality less than λ_n. So, each choice of an element from A gives the actual function which is in V. Note that the relevant forcing is the One Element Prikry Forcing on Extender, which does not change V, i.e. it is trivial. The same holds for $\underset{\sim}{B} = \langle \underset{\sim}{B}_\alpha | 1 < \alpha \leq \omega_1\rangle$.

The following conditions are satisfied:

(a) (Domain)

the domain of each $\underset{\sim}{b}_\alpha \in V$, i.e. it is already decided in the sense that each choice of an element in A will give the same domain.

(b) (Background condition) There is

$$\langle\langle\langle A^{0\tau}(\kappa^{+\alpha+1}), A^{1\tau}(\kappa^{+\alpha+1}), C^\tau(\kappa^{+\alpha+1})\rangle | \tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]\rangle | 1 < \alpha \leq \omega_1\rangle \in G$$

which we call it further **a background condition of** $\underset{\sim}{b}$, such that $\mathrm{dom}(\underset{\sim}{b})$ consists of models appearing in $A^{1\tau}(\kappa^{+\alpha+1})$'s. We require that $A^{0\tau}(\kappa^{+\alpha+1}) = A^{0\tau}(\kappa^{+\omega+1}) \cap \kappa^{+\alpha+1}$ for each $1 < \alpha < \omega_1, \tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$. We do not require this property for arbitrary elements of $A^{1\tau}(\kappa^{+\alpha+1})$ neither for arbitrary members of $\mathrm{dom}(\underset{\sim}{b}_\alpha)$'s. The above type of projection property seems to be necessary once dealing with infinitely many drops. Just there should be one model that controls an infinite sequence of models of smaller cardinalities.

Let $\alpha \leq \omega_1$.

(c) for each $X \in \underset{\sim}{b}_\alpha$ and each $\nu \in A$ there is $k \leq \omega$ so that the interpretation according to ν of $\underset{\sim}{b}_\alpha(X)$ is a subset of $H(\chi^{+k})$.

Moreover,

i. if $|X| = \kappa^{+\alpha+1}$, then it is forced that $|\underset{\sim}{b}_\alpha(X)| = \kappa_n^{+n+1}$ and $\underset{\sim}{b}_\alpha(X) \cap \kappa_n^{+n+2} \in ORD$, i.e. any choice of an element from A interprets $\underset{\sim}{b}_\alpha(X)$ in such a way.

ii. if $|X| = \kappa^+$, then for each $\nu \in A$ the interpretation of $\underset{\sim}{b}_\alpha(X)$ according to ν has cardinality $(\nu^0)^{+n+1}$, where ν^0 denotes the projection of ν to the normal measure of the extender E_{λ_n}.

iii. $\{A^{0\tau}(\kappa^{+\alpha+1}) | \tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]\} \subseteq \mathrm{dom}(\underset{\sim}{b}_\alpha)$.

iv. if $|X| = \kappa^{+\omega_1+1}$, then $\underset{\sim}{b}_{\omega_1}(X)$ has cardinality κ_n^{+n+1}.

v. if $|X| = \kappa^{+\beta+1}$, for some $\beta, 1 < \beta < \alpha$, then it is forced that $\underset{\sim}{b}_\alpha(X)$ is of cardinality $\underset{\sim}{\rho}_\beta^{+n+1}$, where $\underset{\sim}{\rho}_\beta$ is the β-th element of the generic Magidor sequence.

vi. if $|X| = \kappa^{+\beta+1}$, for some $\beta, 1 < \beta < \alpha$, then there is $Y \in \mathrm{dom}(\underset{\sim}{b}_{\omega_1})$ of the same cardinality such that $Y \cap \kappa^{+\alpha+1} = X$ and $\underset{\sim}{b}_{\omega_1}(Y)$, $\underset{\sim}{b}_\alpha(X)$ are the same up to the stage when the α-th element the Magidor sequence is picked.

Note that in the present context we shall require only that models X having Y as above will be addable to a condition. Actually, Y will be added first and then X's as its intersections with $\kappa^{+\alpha+1}$'s. Further it will be shown that this limitation does not effect the prove of κ^{++}-c.c. of the final forcing.

Also note that the strategic closure of the preparation forcing supplies plenty of Y's in the generic set for the preparation forcing $\mathcal{P}'(\kappa^{+\omega_1+1})$ such that $Y \cap \kappa^{+\alpha+1}$ is in the generic set for $\mathcal{P}'(\kappa^{+\alpha+1})$.

The main new point here is that in the final scale of functions we will have the following cardinal correspondence:

$\kappa^+ \mapsto (\nu^0)^{+n+1}, \kappa^{+\alpha+1} \mapsto \rho_\alpha^{+n+1}$, each $\alpha \leq \omega_1$, where ν_0 and ρ_α are as above. Note that $\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))$ is a model of relatively small cardinality. It may have as a members other models of cardinalities ρ_α^{+n+1} (say one corresponding to a model in dom($\underset{\sim}{b}_\alpha$ of cardinality $\kappa^{+\beta+1}$), for $\beta < \alpha$. But then ρ_β^{+n+1} and then also ρ_β should be in $\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))$. Remember that ρ_β is the β-th member of the Magidor sequence. This sequence is only supposed to be forced. So we should deal with names of its elements. But here we have κ_n^{+n+2}-many possibilities, namely we have a set of measure one for a certain measure of the extender over κ_n. So, what do we include in $\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))$?

We deal with the present situation as follows. Allow the images of models to change according to uncovering the elements of the Magidor sequence. Require the images increase once more elements of this sequence are uncovered. Only the images of the models of cardinality $\kappa^{+\omega_1+1}$ (basically the ordinals) with $otp_{\kappa^{+\omega_1+1}}$ of cofinality $\kappa^{+\omega_1+1}$ stay unchanged during this process. The above allows to define the final scale of functions. However the similarity of configurations over κ and κ_n suffers a bit here. Namely, adding new models is more complicated. Thus one should take care not only of the single configuration at level, but rather to deal with many possibilities (according to the Magidor sequence) at once. Models of cardinality κ_n^{+n+1} (both those which are images of the models of cardinality $\kappa^{+\omega_1+1}$ and others not in the range) will be used to take care of this difficulty.

$\underset{\sim}{b}_{\kappa^{+3}}$ will behave as in [8], but the values it takes over κ_n^{+n+2}, since the first member of the Magidor sequence is not decided yet (and once it is decided, a non direct extension will be taken over κ_n as well and so the most of the assignment will be irrelevant then).

We separated the cardinal κ^{++} on purpose, since the models of cardinality κ^+ play a special role in order to prove κ^{++}-c.c. of the final forcing.

So we have some new conditions.

(d) For each $X \in \mathrm{dom}(\underset{\sim}{b}_\alpha)$, if $|X| = \kappa^+$ (or $|X| = \kappa^{+\beta+1}$, for some $\beta, 1 < \beta < \alpha$), then for each $\nu \in A$, $\langle \nu_1, ..., \nu_k \rangle$ which is a part of the Magidor generic sequence (not necessarily for the normal measures of the extenders) (note also that the support of the condition in the Extender Based Magidor Forcing is κ_n, so it includes all the needed stuff) the interpretation of $\underset{\sim}{b}_\alpha(X)$, i.e. $\underset{\sim}{b}_\alpha(X)[\nu, \nu_1, ..., \nu_k]$ has cardinality $(\nu^0)^{+n+1}$ (or $(\nu_j^0)^{+n+1}$, where $j, 1 \leq j \leq k$, is so that ν_j is the β-th element of the Magidor sequence). Moreover we have an other sequence $\langle \rho_1, ..., \rho_l \rangle$ which extends $\langle \nu_1, ..., \nu_k \rangle$ (not necessary end extension, but rather like those in the Magidor forcing), then

$$\underset{\sim}{b}_\alpha(X)[\nu, \nu_1, ..., \nu_k] \subseteq \underset{\sim}{b}_\alpha(X)[\nu, \rho_1, ..., \rho_l].$$

Also the interpretations of models of cardinality $\kappa^{+\alpha+1}$ (actually the ordinals) which are inside do not change.

Further let us refer to $A^{0\kappa^+}(\kappa^{+\alpha+1})$ as **the maximal model of the domain of** $\underset{\sim}{b}_\alpha$. Denote it as $\max(\mathrm{dom}(\underset{\sim}{b}_\alpha))$.

Later passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

(e) $A^{0\kappa^+}(\kappa^{+\alpha+1})$ is a successor model and there are limit models of $A^{0\kappa^+\alpha+1}(\kappa^{+\alpha+1})$ in $A^{0\kappa^+}(\kappa^{+\alpha+1}) \cap \mathrm{dom}(\underset{\sim}{b}_\alpha)$. Moreover the last such model has $otp_{\kappa^{+\alpha+1}}$ of cofinality $\kappa^{+\alpha}$.

The reason is that we like to avoid changing of the models of the maximal cardinality (i.e. $\kappa^{+\alpha+1}$) at least below the maximal model of the condition ($A^{0\kappa^+}(\kappa^{+\alpha+1})$). Thus, if the cofinality of the $otp_{\kappa^{+\alpha+1}}$ drops below $\kappa^{+\alpha}$, then the image under $\underset{\sim}{b}_\alpha$ starts to depend on the Magidor sequence.

Let us deal now with names.

(f) Let $\nu \in A$. Consider $\underset{\sim}{b}(A^{0\kappa^+}(\kappa^{+\alpha+1}))$. By 7d it has cardinality $(\nu^0)^{+n+1}$. We require the following:

- for each two finite sequence $\vec{\rho}, \vec{\eta}$ come from the same places (sets of measure one) in the extender based Magidor forcing

$$\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu, \vec{\rho}] \text{ and } \underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu, \vec{\eta}]$$

realize the same type over $\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu]$.

- for each finite sequence $\vec{\sigma}, \vec{\rho}, \vec{\eta}$ such that $\vec{\sigma}^\frown\vec{\rho}, \vec{\sigma}^\frown\vec{\eta}$ come from the same places (sets of measure one) in the extender based Magidor forcing

$$\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu, \vec{\sigma}^\frown\vec{\rho}] \text{ and } \underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu, \vec{\sigma}^\frown\vec{\eta}]$$

$$\text{realize the same type over } \underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu, \vec{\sigma}].$$

  Note that the cardinality of the models involved is relatively small ($(\nu^0)^{+n+1} < \lambda_n$), hence the number of types using as parameters only elements of such models is small as well. Hence, shrinking sets of measure one (and forming diagonal intersections)insures the above properties.
  Recall that $\underset{\sim}{b}_\alpha(X)$ has cardinality κ_n^{+n+1} for each

$$X \in \underset{\sim}{b}_\alpha \cap A^{1\kappa^{+\alpha+1}}(\kappa^{+\alpha+1}).$$

(g) The same as 7f, but with $A^{0\kappa^+}(\kappa^{+\alpha+1})$ replaced by any

$$X \in \text{dom}(\underset{\sim}{b}_\alpha) \cap C^{\kappa^{+\beta+1}}(A^{0\kappa^{+\beta+1}}(\kappa^{+\alpha+1})),$$

for each $\beta, 1 < \beta < \alpha$.

This last two conditions allow us to add elements to $\text{dom}(\underset{\sim}{b}_\alpha)$ which are isomorphic (having the same $otp_{\kappa^{+\beta+1}}$) as those that are already inside. Thus, here we should deal simultaneously with many names of models and not with a single name as in [8]. The procedure will be like this:
add first a model realizing the same type as those of $\underset{\sim}{b}_\alpha(X)[\nu]$ and having the same types according to finite sequences for the extender based Magidor forcing (as in 7f, 7g). Now extensions of such models by finite sequences $\vec{\rho}$ will be realize the same type, and so will be isomorphic to $\underset{\sim}{b}_\alpha(X)[\nu, \vec{\rho}]$. Such extensions can be viewed just as applications of Skolem functions to indiscernibles.

(h) (Models come from $A^{0\kappa^+}(\kappa^{+\alpha+1})$) If $X \in \text{dom}(\underset{\sim}{b}_\alpha)$ and $X \neq A^{0\kappa^+}(\kappa^{+\alpha+1})$, then $X \in A^{0\kappa^+}(\kappa^{+\alpha+1})$.

(i) Let $E, F \in \text{dom}(\underset{\sim}{b}_\alpha)$, $E \in F$ (or $E \subseteq F$) and $\nu \in A$. If k is the minimal so that the interpretation of $\underset{\sim}{b}_\alpha(E)$ according to $\vec{\nu}$ is a subset of $H(\chi^{+k})$ or $\underset{\sim}{b}_\alpha(F)$ according to $\vec{\nu}$ is a subset of $H(\chi^{+k})$, then

$$\underset{\sim}{b}_\alpha(E)[\vec{\nu}] \cap H(\chi^{+k}) \in \underset{\sim}{b}_\alpha(F)[\vec{\nu}] \cap H(\chi^{+k})$$

$$(\text{or } \underset{\sim}{b}_\alpha(E)[\vec{\nu}] \cap H(\chi^{+k}) \subseteq \underset{\sim}{b}_\alpha(F)[\vec{\nu}] \cap H(\chi^{+k})),$$

where in the last two lines we mean the interpretations according to ν. Let us further deal with such interpretations without mentioning this explicitly.

The intuitive meaning is that b_α is supposed to preserve membership and inclusion. But we cannot literally require this since $b_\alpha(E)$ and $b_\alpha(F)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(j) The image by b_α of $A^{0\kappa^+}(\kappa^{+\alpha+1})$, i.e. $b(A^{0\kappa^+}(\kappa^{+\alpha+1}))$, intersected with κ_n^{+n+2} is above all (i.e. is forced by each $\vec{\nu}$ to be such) the rest of $\mathrm{rng}(b_\alpha)$ restricted to κ_n^{+n+2} in the ordering of the extender E_{κ_n} (via some reasonable coding by ordinals), but the models of cardinalities not mentioned in $\vec{\nu}$. Note that still we have mc the maximal coordinate of the condition which is above $b(A^{0\kappa^+}(\kappa^{+\alpha+1}))$ in the ordering of the extender.

Recall that the extender E_{κ_n} acts on κ_n^{+n+2} and our main interest is in Prikry sequences it will produce. So, parts of $\mathrm{rng}(b_\alpha)$ restricted to κ_n^{+n+2} will play the central role.

Let us, as in [7], denote by $otp_{\kappa^{+\beta}}(X)$ the order type of the maximal under inclusion chain of elements in $\mathcal{P}(X) \cap A^{1\kappa^{+\beta}}(\kappa^{+\alpha+1})$ which is just the order type of $C^{\kappa^{+\beta}}(X)$, for $X \in A^{1\kappa^{+\beta}}(\kappa^{+\alpha+1})$, for each $\beta \leq \alpha$. If $X \in C^{\kappa^{+\beta}}(A^{0\kappa^{+\beta}}(\kappa^{+\alpha+1}))$, then $C^{\kappa^{+\beta}}(X) = C^{\kappa^{+\beta}}(A^{0\kappa^{+\beta}}(\kappa^{+\alpha+1})) \cap (X \cup \{X\}) = C^{\kappa^{+\beta}}(A^{0\kappa^{+\beta}}) \restriction X+1$. Hence, in this case, $otp_{\kappa^{+\beta}}(X) = otp(C^{\kappa^{+\beta}}(A^{0\kappa^{+\beta}}(\kappa^{+\alpha+1})) \restriction X) + 1$. Note that $otp_{\kappa^{+\beta}}(X)$ is always a successor ordinal below $\kappa^{+\beta+1}$. Recall that by [7] we have for each $X \in A^{1\kappa^{+\beta}}(\kappa^{+\alpha+1})$ an element $Y \in C^{\kappa^+}(A^{0\kappa^{+\beta}}(\kappa^{+\alpha+1}))$ such that $otp_{\kappa^+}(X) = otp_{\kappa^+}(Y)$.

Next conditions deal with the connection between the structure over λ_n and those over κ_n.

(k) (Order types) If $X \in \mathrm{dom}(\underset{\sim}{b}_\alpha)$ and $|X| = \kappa^{+\beta}$, then $A^{0\kappa^{+\beta}}(\kappa^{+\beta+1}) \cap \kappa^{+\beta+1} \geq otp_{\kappa^{+\beta}}(X)$.

Let $|X| = \kappa^+$. Denote by $X(\lambda_n)$ the last element Z of $A^{1\kappa^+}(\kappa^{++})$ with $Z \cap \kappa^{++} < otp_{\kappa^+}(X)$. It will be the one corresponding to X at the level λ_n. Notice that the domain of a need not be an ordinal but rather a closed set of ordinals of cardinality less than λ_n. Hence, $otp_{\kappa^+}(X)$ itself or $otp_{\kappa^+}(X) - 1$ need not be in the domain of a. So, $X(\lambda_n)$ looks like a natural choice.

Similar, for each $\beta, 1 < \beta < \alpha$, denote by $X(\beta)$ the last element Z of $A^{0\kappa^\beta}(\kappa^{+\beta+1})$ with $Z \cap \kappa^{+\beta+1} < otp_{\kappa^+\beta}(X)$.

The next condition insures that the function $otp_{\kappa^+}(X) \to X(\lambda_n)$ is order preserving.

(l) (Order preservation) If $X, X' \in \mathrm{dom}(\underset{\sim}{b})$, then

- $otp_{\kappa^+}(X) = otp_{\kappa^+}(X')$ iff $X(\lambda_n) = X'(\lambda_n)$
- $otp_{\kappa^+}(X) < otp_{\kappa^+}(X')$ iff $X(\lambda_n) < X'(\lambda_n)$
- for each $\beta, 1 < \beta < \alpha$,
  - $otp_{\kappa^{+\beta}}(X) = otp_{\kappa^{+\beta}}(X')$ iff $X(\beta) = X'(\beta)$
  - $otp_{\kappa^{+\beta}}(X) < otp_{\kappa^{+\beta}}(X')$ iff $X(\beta) < X'(\beta)$

Let us deal first $\underset{\sim}{b}_{\kappa+3}$. The treatment is as in [8].

(m) (Dependence) Let $X \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap C^{\kappa^+}(A^{0\kappa^+}(\kappa^{+\alpha+1}))$. Then $\underset{\sim}{b}_\alpha(X)$ depends on the value of the one element Prikry forcing with the measure $a(X(\lambda_n))$ over λ_n (moreover $\underset{\sim}{b}_2$ depends only on it as the first element of the Magidor sequence). More precisely: let $A(X) = \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}{}''A$, then each choice of an element from $A(X)$ already decides $\underset{\sim}{b}(X)$, i.e. whenever $\nu_1, \nu_2 \in A$ and

$$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu_1) = \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu_2)$$

we have

$$\underset{\sim}{b}_\alpha(X)[\nu_1] = \underset{\sim}{b}_\alpha(X)[\nu_2].$$

Further let us denote, for $\nu \in A$, the projection of ν to $A(X)$, i.e. $\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu)$, by $\nu(X)$.

So $\underset{\sim}{b}_\alpha(X)$ depends on members of $A(X)$ rather than those of A.

The next condition is crucial for the κ^{++}-c.c. of the forcing.

(n) (Inclusion condition)

Let $\nu, \nu' \in A, \nu < \nu'$. Then

- $$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu') > \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu)$$
  implies
  $$\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu] \in \underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu'].$$

  This condition means that once $A^{0\kappa^+}(\lambda_n)$ -the set corresponding to $A^{0\kappa^+}$ at the level λ_n, is mapped by a according to ν' to a bigger set than those according to ν, then the same is true with corresponding models at the level κ_n.

- If $Y \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap C^{\kappa^+}(A^{0\kappa^+}(\kappa^{+\alpha+1}))$ and

$$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(Y(\lambda_n))}(\nu') > \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu),$$

  then

$$\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu] \in \underset{\sim}{b}_\alpha(Y)[\nu']$$

It is possible to have $Y \subset X$, but $\nu(X)$ smaller than $\nu'(Y)$ (note that $\nu(Y) < \nu(X)$ in this case by 7l). In such situation the interpretation will reverse the order. Note that given $\nu' \in A$ the number of possibilities for $\nu \in \nu' \cap A$ is bounded by $(\nu'^0)^{+n+1}$, as $\nu' < (\nu'^0)^{+n+2}$.

Let us turn to the general case when models depend on elements of the Magidor sequence. Consider $\underset{\sim}{b}_\alpha(X)$ with X of cardinality κ^+, or even, for simplicity $X = A^{0\kappa^+}(\kappa^{+\alpha+1})$. The difference here is that it depends not only on one element Prikry sequence for λ_n, but rather on finite sequences which are parts of the Magidor sequence. Let $\vec{\nu} = \langle \nu_1, ..., \nu_k \rangle$ be such a sequence and ν a one element Prikry sequence for λ_n. We need to deal with $\underset{\sim}{b}_\alpha(X)[\nu^\frown\vec{\nu}]$ -the interpretation of $\underset{\sim}{b}_\alpha(X)$ according to $\nu^\frown\vec{\nu}$. Suppose the sequence is changed. If only ν is replaced by some ν', then we can absorb this change as it was done above. But let now ν, ν_1 is replaced by some ν', ν'_1, say $\nu' > \nu$ and $\nu'_1 > \nu_1$. The number of possibilities for ν'_1 is too big in order to absorb all of them inside $\underset{\sim}{b}_\alpha(X)[\nu^\frown\vec{\nu}]$. So what do we do? Well, suppose first that ν_1 was the least member of the Magidor sequence. So the corresponding cardinal is κ^{+3} and models of this level depend only on λ_n. By 7n, we have

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))[\nu] \in \underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))[\nu'].$$

But remember that $\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))$ and $\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))$ are the same up to the stage where the first element of the Magidor sequence is decided, i.e. up to ν_1, ν'_1. Then

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu] \in \underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu'].$$

Also,

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))[\nu] \in \underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu']$$

and a function which projects

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))[\nu']$$

onto

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))[\nu]$$

is in

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu'].$$

Take, for example, an increasing enumeration of

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))[\nu']$$

and the place of

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+3}))[\nu]$$

in it. The number of possibilities is at most $(\nu')^{+n+1}$ (just the cardinality of the models involved). So, given

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu',\nu_1'],$$

we require that the models obtained by interpretations according to projection functions in $\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu']$ are in $\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu',\nu_1']$. In particular,

$$\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu,\nu_1],\ \underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu',\nu_1]$$

will be in $\underset{\sim}{b}_{\kappa^{+3}}(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu',\nu_1']$.

Let us now formulate the above formally.

(o) (General dependence) Let $X \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap C^{\kappa^+}(A^{0\kappa^+}(\kappa^{+\alpha+1}))$. Assume that $\vec{\nu} = \langle \nu_1, ..., \nu_k \rangle$ is a finite part of the Magidor sequence and ν a one element Prikry sequence for λ_n. Then

$$\underset{\sim}{b}_\alpha(X)[\nu,\nu_1,...,\nu_k]$$

depends only on the coordinate of the extender (over κ_n) which appear in it, i.e. the same model but for β's below α. Thus let $\beta < \alpha$ be above the measures used to produce $\langle \nu_1, ..., \nu_k \rangle$. Let

$$A(X)[\nu,\nu_1,...,\nu_k] =$$

$$\pi^{E_{\kappa_n,\beta}}_{\max(\underset{\sim}{b}_\beta(A^{0\kappa^+}(\kappa^{+\beta+1})[\nu,...,\nu_k],\underset{\sim}{b}_\beta(X)[\nu,\nu_1,...,\nu_k]}{}'' \text{ the corresponding set of measure one }.$$

Then, whenever $\mu_1, \mu_2 \in A(X)[\nu,\nu_1,...,\nu_k]$ and

$$\pi^{E_{\kappa_n,\beta}}_{\max(\underset{\sim}{b}_\beta(A^{0\kappa^+}(\kappa^{+\beta+1})[\nu,...,\nu_k],\underset{\sim}{b}_\beta(X)[\nu,\nu_1,...,\nu_k]}(\mu_1) = \pi^{E_{\kappa_n,\beta}}_{\max(\underset{\sim}{b}_\beta(A^{0\kappa^+}(\kappa^{+\beta+1})[\nu,...,\nu_k],\underset{\sim}{b}_\beta(X)[\nu,\nu_1,...,\nu_k]}(\mu_2),$$

we have

$$\underset{\sim}{b}_\alpha(X)[\nu,\nu_1,...,\nu_k,\mu_1] = \underset{\sim}{b}_\alpha(X)[\nu,\nu_1,...,\nu_k,\mu_2].$$

(p) (General inclusion condition 1) In the notation of the previous condition (7o), suppose that

$$Z \in \underset{\sim}{b}_\beta(A^{0\kappa^+}(\kappa^{+\beta+1}))[\nu, \nu_1, ..., \nu_k].$$

Let μ' be in the set of measure one for β (the measure codding the model $\underset{\sim}{b}_\beta(A^{0\kappa^+}(\kappa^{+\beta+1}))[\nu, \nu_1, ..., \nu_k]$) and μ will be the projection of μ' corresponding to Z. Then

$$\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu, \nu_1, ..., \nu_k, \mu] \in \underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\nu, \nu_1, ..., \nu_k, \mu'],$$

provided the set on the left side is defined.

(q) (General inclusion condition 2) In the notation of the previous condition (7o), suppose that $Y \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap C^{\kappa^+}(A^{0\kappa^+}(\kappa^{+\alpha+1}))$ and

$$\underset{\sim}{b}_\beta(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\xi, \xi_1, ..., \xi_i] \in \underset{\sim}{b}_\beta(Y)[\nu, \nu_1, ..., \nu_k],$$

with $\xi < \nu$. Let μ' be in the set of measure one for β (the measure codding the model $\underset{\sim}{b}_\beta(Y)[\nu, \nu_1, ..., \nu_k]$) and μ will be the projection of μ' corresponding to $\underset{\sim}{b}_\beta(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\xi, \xi_1, ..., \xi_i]$. Then

$$\underset{\sim}{b}_\alpha(A^{0\kappa^+}(\kappa^{+\alpha+1}))[\xi, \xi_1, ..., \xi_i, \mu] \in \underset{\sim}{b}_\alpha(Y)[\nu, \nu_1, ..., \nu_k, \mu'],$$

The continuation repeats the conditions of [8] with obvious adjustments.

(r) If $X \in \mathrm{dom}(\underset{\sim}{b}_\alpha)$ then $C^{|X|}(X) \cap \mathrm{dom}(\underset{\sim}{b}_\alpha)$ is a closed chain. Let $\langle X_i | i < j\rangle$ be its increasing continuous enumeration. For each $l < j$ consider the final segment $\langle X_i | l \le i < j\rangle$ and its image $\langle \underset{\sim}{b}_\alpha(X_i) | l \le i < j\rangle$. Find the minimal k so that

$$\underset{\sim}{b}_\alpha(X_i) \subseteq H(\chi^{+k}) \text{ for each } i, l \le i < j.$$

Then the sequence

$$\langle \underset{\sim}{b}_\alpha(X_i) \cap H(\chi^{+k}) | l \le i < j\rangle$$

is increasing and continuous. More precisely, all the interpretations are like this. Note that k here may depend on l, i.e. on the final segment.

(s) (The walk is in the domain) If $X \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap A^{1\xi}(\kappa^{+\alpha+1})$, for some $\xi \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$, then the general walk from $(A^{0\xi}(\kappa^{+\alpha+1}))^-$ to X is in $\mathrm{dom}(\underset{\sim}{b}_\alpha)$.

(t) If $X \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap A^{1\xi}(\kappa^{+\alpha+1})$, for some $\xi \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$ is a limit model and $\mathrm{cof}(otp_\xi(X) - 1) < \kappa_n$ (i.e. the cofinality of the sequence $C^\xi(X) \backslash \{X\}$ under the inclusion relation is less than κ_n) then a closed cofinal subsequence of $C^\xi(X) \backslash \{X\}$ is in $\mathrm{dom}(\underset{\sim}{b}_\alpha)$. The images of its members under b_α form a closed cofinal in $b_\alpha(X)$ sequence.

(u) (Minimal cover condition) Let $E \in A^{0\kappa^+}(\kappa^{+\alpha+1}) \cap \mathrm{dom}(\underset{\sim}{b}_\alpha)$, $X \in A^{0\kappa^{++}}(\kappa^{+\alpha+1}) \cap \mathrm{dom}(\underset{\sim}{b}_\alpha)$. Suppose that $E \not\subseteq X$. Then the smallest model of $E \cap C^{\kappa^+}(A^{0\kappa^+}(\kappa^{+\alpha+1}))$ including X is in $\mathrm{dom}(\underset{\sim}{b}_\alpha)$.

(v) (The first models condition) Suppose that $E \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap C^{\kappa^+}(A^{0\kappa^+}(\kappa^{+\alpha+1}))$, $F \in \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap C^{\kappa^{++}}(A^{0\kappa^{++}}(\kappa^{+\alpha+1}))$, $\sup(E) > \sup(F)$ and $F \notin E$. Then the first model $H \in A \cap C^{\kappa^{++}}(A^{0\kappa^{++}}(\kappa^{+\alpha+1}))$ which includes B is in $\mathrm{dom}(\underset{\sim}{b}_\alpha)$.

(w) (Models witnessing Δ-system type are in the domain) If $F_0, F_1, F \in A^{1\kappa^+} \cap \mathrm{dom}(\underset{\sim}{b})$ is a triple of a Δ - system type, then the corresponding models $G_0, G_0^*, G_1, G_1^*, G^*$, as in the definition of a Δ - system type (see [7]), are in $\mathrm{dom}(\underset{\sim}{b})$ as well and

$$\underset{\sim}{b}(F_0) \cap \underset{\sim}{b}(F_1) = \underset{\sim}{b}(F_0) \cap \underset{\sim}{b}(G_0) = \underset{\sim}{b}(F_1) \cap \underset{\sim}{b}(G_1).$$

(x) If $F_0, F_1, F \in A^{1\kappa^+}(\kappa^{+\alpha+1})$ is a triple of a Δ - system type and $F, F_0 \in \mathrm{dom}(\underset{\sim}{b})$ (or $F, F_1 \in \mathrm{dom}(\underset{\sim}{b})$), then $F_1 \in \mathrm{dom}(\underset{\sim}{b})$ (or $F_0 \in \mathrm{dom}(\underset{\sim}{b})$).

(y) (The isomorphism condition) Let $F_0, F_1, F \in A^{1\kappa^+} \cap \mathrm{dom}(\underset{\sim}{b}_\alpha)$ be a triple of a Δ - system type. Then

$$\langle \underset{\sim}{b}_\alpha(F_0) \cap H(\chi^{+k}), \in \rangle \simeq \langle \underset{\sim}{b}_\alpha(F_1) \cap H(\chi^{+k}), \in \rangle$$

where k is the minimal so that $\underset{\sim}{b}_\alpha(F_0) \subseteq H(\chi^{+k})$ or $\underset{\sim}{b}_\alpha(F_1) \subseteq H(\chi^{+k})$.
Note that it is possible to have for example $\underset{\sim}{b}_\alpha(F_0) \prec H(\chi^{+6})$ and $\underset{\sim}{b}_\alpha(F_1) \prec H(\chi^{+18})$. Then we take $k = 6$.
Let π be the isomorphism between

$$\langle \underset{\sim}{b}_\alpha(F_0) \cap H(\chi^{+k}), \in \rangle, \langle \underset{\sim}{b}_\alpha(F_1) \cap H(\chi^{+k}), \in \rangle$$

and $\pi_{F_0F_1}$ be the isomorphism between F_0 and F_1. Require that for each $Z \in F_0 \cap \mathrm{dom}(\underset{\sim}{b}_\alpha)$ we have $\pi_{F_0F_1}(Z) \in F_1 \cap \mathrm{dom}(\underset{\sim}{b}_\alpha)$ and

$$\pi(\underset{\sim}{b}_\alpha(Z) \cap H(\chi^{+k})) = \underset{\sim}{b}_\alpha(\pi_{F_0F_1}(Z)) \cap H(\chi^{+k}).$$

Let us turn to the component g of the condition.

8. $g = \langle g_\alpha | 1 < \alpha \leq \omega_1 \rangle$.
For each $\alpha, 1 < \alpha \leq \omega_1$, the following holds: g_α is function from $\kappa^{+\alpha+1}$ of cardinality at most κ such that for each $\xi \in \mathrm{dom}(g_\alpha)$ we have either

   - $g_\alpha(\xi) = \langle \tau, \mu \rangle$, for some $\mu < \tau < \kappa_n$
   or
   - $g_\alpha(\xi) = \langle \tau, \mu, \underset{\sim}{\nu} \rangle$, for some $\tau < \kappa_n, \mu < \tau^{+n+2}$ and $\underset{\sim}{\nu}$ a name in the extender based Magidor forcing over τ of the α-th member of the Magidor sequence for μ.

   We use here pairs or triples to be elements of the $\mathrm{rng}(g_\alpha)$. Intuitively, τ stands for the cardinal which possibly will change its cofinality to ω_1 via the extender based Magidor forcing over it. We do not require that it always be the case, moreover τ need not be a measurable at all. Specifically, let $g_\alpha(\xi) = \langle \tau, \mu, \underset{\sim}{\nu} \rangle$. If τ actually does not change its cofinality to ω_1 or it changes the cofinality to ω_1, but there is no Magidor sequence for μ, then $\underset{\sim}{\nu}$ may be viewed as void.
   Note that by [9], once τ change its cofinality to ω_1 (or any uncountable cardinal), then it is impossible to play anymore with the assignment function. In a sense a connection between μ and $\underset{\sim}{\nu}$ will be rigid now. But this happens only after τ, μ are picked and not over κ_n, where we do have a freedom to play with the measures of the extenders.
   Note also that that the use of names in g_α's will not effect the possibility to pick finally $\kappa^{+\omega_1+1}$-ω sequences. The reason basically would be that the sequences of τ's coming from g's will be old sequences.

9. For each $\alpha, 1 < \alpha \leq \omega_1$, the following holds:
$$\{\tau < \kappa^{+\alpha+1} \mid \tau \in \mathrm{dom}(\underset{\sim}{b}_\alpha)\} \cap \mathrm{dom}(g_\alpha) = \emptyset.$$

10. For each $\alpha, 1 < \alpha \leq \omega_1$, $\nu \in A$ we have $\underset{\sim}{B}_\alpha[\nu] \in E_{\kappa_n, \alpha, \underset{\sim}{b}_\alpha[\nu](\max(\underset{\sim}{b}_\alpha))}$.

11. For every $\alpha, 1 < \alpha \leq \omega_1$, $\nu \in A$ and every ordinals ξ, ρ, η which are elements of $\mathrm{rng}(\underset{\sim}{b}_\alpha)[\nu]$ or actually the ordinals coding models in $\mathrm{rng}(\underset{\sim}{b}_\alpha)[\nu]$ we have
$$\xi \geq_{E_{\kappa_n,\alpha}} \rho \geq_{E_{\kappa_n,\alpha}} \eta \quad \text{implies}$$
$$\pi_{\kappa_n,\xi,\eta}(\delta) = \pi_{\kappa_n,\rho,\eta}(\pi_{\kappa_n,\xi,\rho}(\delta))$$
for every $\delta \in \pi''_{\kappa_n, \max \mathrm{rng}(\underset{\sim}{b}_\alpha[\nu]), \xi}(\underset{\sim}{B}_\alpha[\nu])$.

We define now Q_{n1} and $\langle Q_n, \leq_n, \leq_n^* \rangle$ similar to [2, Sec.2].

Definition 2.2 Suppose that $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle$ and $\langle\langle a', A', f'\rangle, \langle \underset{\sim}{b}', \underset{\sim}{B}', g'\rangle\rangle$ are two elements of Q_{n0}. Define

$$\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle \geq_{Q_{n0}} \langle\langle a', A', f'\rangle, \langle \underset{\sim}{b}', \underset{\sim}{B}', g'\rangle\rangle$$

iff

1. $f \supseteq f'$.
2. For each $\alpha, 1 < \alpha \leq \omega_1$,
$$g_\alpha \supseteq g'_\alpha.$$
3. $a \supseteq a'$.
4. $\pi''_{\lambda_n, \max(a), \max(a')} A \subseteq A'$.
5. For each $\alpha, 1 < \alpha \leq \omega_1$,
$$\underset{\sim}{b}_\alpha \text{ extends } \underset{\sim}{b}'_\alpha,$$
according to the appropriate projections of measure one sets. This means just that the empty condition of (one element Prikry forcing*extender based Magidor) forces the inclusion.
6. For each $\alpha, 1 < \alpha \leq \omega_1$, $\nu \in A$ we have
$$\pi''_{\kappa_n, \max(\underset{\sim}{b}_\alpha[\nu]), \max(\underset{\sim}{b}'_\alpha[\pi_{\lambda_n, \max(a), \max(a')}(\nu)])} \underset{\sim}{B}_\alpha[\nu] \subseteq \underset{\sim}{B}'_\alpha[\pi_{\lambda_n, \max(a), \max(a')}(\nu)]$$

Definition 2.3 Q_{n1} consists of pairs $\langle f, g\rangle$ such that

1. f is a partial function from κ^{++} to λ_n of cardinality at most κ
2. $g = \langle g_\alpha | 1 < \alpha \leq \omega_1\rangle$.
For each $\alpha, 1 < \alpha \leq \omega_1$, the following holds: g_α is function from $\kappa^{+\alpha+1}$ of cardinality at most κ such that for each $\xi \in \mathrm{dom}(g_\alpha)$ we have either
   - $g_\alpha(\xi) = \langle \tau, \mu\rangle$, for some $\mu < \tau < \kappa_n$
   or

- $g_\alpha(\xi) = \langle \tau, \mu, \underset{\sim}{\nu} \rangle$, for some $\tau < \kappa_n, \mu < \tau^{+n+2}$ and $\underset{\sim}{\nu}$ a name in the extender based Magidor forcing over τ of the α-th member of the Magidor sequence for μ. Again, $\underset{\sim}{\nu}$ can be viewed as void if this forcing is undefined or does not have μ-th sequence.

Q_{n1} is ordered by extension. Denote this order by $\leq_1$.
So, it is basically the Cohen forcing for adding κ^{+3} Cohen subsets to κ^+.

Definition 2.4 Set $Q_n = Q_{n0} \cup Q_{n1}$. Define $\leq_n^* = \leq_{Q_{n0}} \cup \leq_{Q_{n1}}$.

Define now a natural projection to the first coordinate:

Definition 2.5 Let $p \in Q_n$. Set $(p)_0 = p$, if $p \in Q_{n1}$ and let $(p)_0 = \langle a, A, f \rangle$, if $p \in Q_{n0}$ is of the form $\langle \langle a, A, f \rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g \rangle \rangle$.

Let $(Q_n)_0 = \{(p)_0 \mid p \in Q_n\}$.

Definition 2.6 Let $p, q \in Q_n$. Then $p \leq_n q$ iff either

1. $p \leq_n^* q$

   or

2. $p = \langle \langle a, A, f \rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g \rangle \rangle \in Q_{n0}$, $q = \langle e, h \rangle \in Q_{n1}$ and the following hold:

   (a) $e \supseteq f$

   (b) $h = \langle h_\alpha | 1 < \alpha \leq \omega_1 \rangle$ and for each $\alpha, 1 < \alpha \leq \omega_1$ we have $h_\alpha \supseteq g_\alpha$

   (c) $\mathrm{dom}(e) \supseteq \mathrm{dom}(a)$

   (d) $e(\max(\mathrm{dom}(a))) \in A$

   (e) for every $\beta \in \mathrm{dom}(a), e(\beta) = \pi_{\lambda_n, a(\max(\mathrm{dom}(a))), a(\beta)}(e(\max(\mathrm{dom}(a)))$

   (f) for every $\alpha, 1 < \alpha \leq \omega_1$ we have $\mathrm{dom}(h_\alpha) \supseteq \mathrm{dom}(\underset{\sim}{b}_\alpha) \cap A^{1\kappa^{+\alpha+1}}(\kappa^{+\alpha+1})$

   (g) for every $\alpha, 1 < \alpha \leq \omega_1$ we require that
   $h(\max(\mathrm{dom}(\underset{\sim}{b}_\alpha)) = \langle \tau, \mu, \underset{\sim}{\nu} \rangle$, for some $\tau, \mu, \underset{\sim}{\nu}$ such that

   - $\tau \in \underset{\sim}{B}_{\omega_1}[e(\max(\mathrm{dom}(a))]$.
     I.e., we use $e(\max(\mathrm{dom}(a))$ in order to interpret $\underset{\sim}{B}_{\omega_1}$. Note that by 2d above, it is inside A and so the interpretation makes sense. We assume here for simplicity that elements of B are the same as well as the images under b_β's of the maximal models.

- $\mu < \tau^{+n+2}$ is the indiscernible corresponding to $\max(\text{rng}(\underset{\sim}{b}_\alpha))$.
- $\underset{\sim}{\nu}$ is a name of the α-th member of the Magidor sequence for μ.
- For every $\beta \in \text{dom}(\underset{\sim}{b}_\alpha) \cap A^{1\kappa^{+\alpha+1}}(\kappa^{+\alpha+1})$

$$h_\alpha(\beta) = \langle \tau, \pi_{\kappa_n, \max(\text{rng}(\underset{\sim}{b}_\alpha[\nu])), \underset{\sim}{b}_\alpha(\beta)[\nu]}(h_\alpha(\max(\text{dom}(\underset{\sim}{b}_\alpha)), \underset{\sim}{\rho}\rangle,$$

where $\nu = e(\max(\text{dom}(a)))$ and $\underset{\sim}{\rho}$ is a name of the α-th member of the Magidor sequence for $\pi_{\kappa_n, \max(\text{rng}(\underset{\sim}{b}_\alpha[\nu])), \underset{\sim}{b}_\alpha(\beta)[\nu]}(h_\alpha(\max(\text{dom}(\underset{\sim}{b}_\alpha))$ over τ.

Definition 2.7 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega\rangle$ so that

1. for every $n < \omega$, $p_n \in Q_n$
2. there is $\ell(p) < \omega$ such that
   (a) for every $n < \ell(p)$, $p_n \in Q_{n1}$
   (b) for every $n \geq \ell(p)$, $p_n = \langle\langle a_n, A_n, f_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle \in Q_{n0}$
   (c) for every $n, m \geq \ell(p)$, $\max(\text{dom}(a_n)) = \max(\text{dom}(a_m))$ and $\max(\text{dom}(\underset{\sim}{b}_n)) = \max(\text{dom}(\underset{\sim}{b}_m))$
   (d) for every $n \geq m \geq \ell(p)$, $\text{dom}(a_m) \subseteq \text{dom}(a_n)$ and $\text{dom}(\underset{\sim}{b}_m)) \subseteq \text{dom}(\underset{\sim}{b}_n)$
   (e) for every n, $\ell(p) \leq n < \omega$, and $X \in \text{dom}(a_n)$ the following holds:
   for each $k < \omega$ the set

$$\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$$

   is finite.
   (f) for every n, $\ell(p) \leq n < \omega$, and $X \in \text{dom}(\underset{\sim}{b}_n)$ the following holds:
   for each $k < \omega$ the set

$$\{m < \omega \mid \exists\vec{\nu}\exists\alpha, 1 < \alpha \leq \omega_1, \underset{\sim}{b}_{m\alpha}[\vec{\nu}] \text{ is defined, and } (\neg(\underset{\sim}{b}_{m\alpha}(X)[\vec{\nu}]\cap H(\chi^{+k}) \prec H(\chi^{+k})))\}$$

   is finite.

We define the orders $\leq, \leq^*$ as in [2].

Definition 2.8 Let $p = \langle p_n | n < \omega\rangle, q = \langle q_n | n < \omega\rangle$ be in $\mathcal{P}$. Define

1. $p \geq q$ iff for each $n < \omega, p_n \geq_n q_n$

2. $p \geq^* q$ iff for each $n < \omega, p_n \geq^*_n q_n$

Definition 2.9 Let $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$. Set $(p)_0 = \langle (p_n)_0 \mid n < \omega \rangle$.

Define $(\mathcal{P})_0 = \{(p)_0 \mid p \in \mathcal{P}\}$.

Finally, the equivalence relation $\longleftrightarrow$ and the order $\rightarrow$ are defined on $(\mathcal{P})_0$ exactly as it was done in [1], [2] and [4]. We extend $\rightarrow$ to $\mathcal{P}$ as follows:

Definition 2.10 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$. Set $q \rightarrow p$ iff

1. $(q)_0 \rightarrow (p)_0$
2. $\ell(p) = \ell(q)$
3. for every $n < \ell(p), p_n$ extends q_n
4. for every $n \geq \ell(p)$, let $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$ and $q_n = \langle \langle a'_n, A'_n, f'_n \rangle, \langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n, g'_n \rangle \rangle$. Require the following:
   (a) $g_n \supseteq g'_n$
   (b) there is $\underset{\sim}{b}''_n$ such that for every $\nu \in A_n$ the following holds:
      i. $\mathrm{dom}(\underset{\sim}{b}'_n) = \mathrm{dom}(\underset{\sim}{b}''_n)$
      ii. $\pi''_{\kappa_n, \max(\underset{\sim}{b}_n[\nu]), \max(\underset{\sim}{b}'_n[\nu'])} \underset{\sim}{B}_n[\nu] \subseteq \underset{\sim}{B}'_n[\nu']$,
      where $\nu' = \pi_{\lambda_n, \max(\mathrm{rng}(a_n)), \xi}(\nu)$ and $\xi = a_n(\max(\mathrm{dom}(a'_n))$
      iii. $\underset{\sim}{b}_n[\nu]$ extends $\underset{\sim}{b}''_n[\nu']$ and for each $\vec{\nu}$ and its projection $\vec{\nu'}$ we have

$$\underset{\sim}{b}_n[\nu^\frown \vec{\nu}] \text{ extends } \underset{\sim}{b''}_n[\nu'^\frown \vec{\nu'}]$$

      iv. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'^\frown \vec{\nu'}] \longleftrightarrow_{k_n} \mathrm{rng}(\underset{\sim}{b}''_n)[\nu'^\frown \vec{\nu'}]$, where $\nu', \vec{\nu'}$ are as above and k_n is the k_n's member of a nondecreasing sequence converging to the infinity.
      v. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'] \restriction \kappa^{+n+1} = \mathrm{rng}(\underset{\sim}{b}''_n)[\nu'] \restriction \kappa^{+n+1}$

3 Basic Lemmas

In this section we study the properties of the forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$ defined in the previous section.

Lemma 3.1 *Let $p = \langle p_k \mid k < \omega \rangle \in \mathcal{P}$, $p_k = \langle \langle a_k, A_k, f_k \rangle, \langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k \rangle \rangle$ for $k \geq \ell(p)$ and X be a model appearing in an element of $G(\mathcal{P}'(\kappa^{++}))$. Suppose that*

(a) $X \notin \bigcup_{\ell(p) \leq k < \omega} \mathrm{dom}(a_k) \cup \mathrm{dom}(f_k)$

(b) *X is a successor model or if it is a limit one with* $\mathrm{cof}(otp_{\kappa^+}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega \rangle$, $q_k = \langle \langle a'_k, A'_k, f'_k \rangle, \langle \underset{\sim}{b}'_k, \underset{\sim}{B}'_k, g_k \rangle \rangle$ for $k \geq \ell(q)$, of p so that starting with some $n \geq \ell(q)$ we have $X \in \mathrm{dom}(a'_k)$ for each $k \geq n$. In addition the second part of the condition p, i.e. $\langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k \rangle$ remains basically unchanged (just names should be lifted to new A_k's).

The proof is the same as those of the corresponding lemma in [7].

A parallel lemma needed for adding elements of $G(\mathcal{P})$. Its proof is similar to the one of [7] once taking into the account the explanation given in 2.1(7g).

Lemma 3.2 *Let $p = \langle p_k \mid k < \omega \rangle \in \mathcal{P}$, $p_k = \langle \langle a_k, A_k, f_k \rangle, \langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k \rangle \rangle$ for $k \geq \ell(p)$ and X be a model appearing in an element of $G(\mathcal{P}'(\kappa^{+\omega_1+1}))$. Suppose that*

(a) $X \notin \bigcup_{\ell(p) \leq k < \omega} \mathrm{dom}(\underset{\sim}{b}_{k\omega_1}) \cup \mathrm{dom}(g_{k\omega_1})$

(b) *X is a successor model or if it is a limit one with* $\mathrm{cof}(otp_{|X|}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega \rangle$, $q_k = \langle \langle a'_k, A'_k, f'_k \rangle, \langle \underset{\sim}{b}'_k, \underset{\sim}{B}'_k, g_k \rangle \rangle$ for $k \geq \ell(q)$, of p so that starting with some $n \geq \ell(q)$ we have $X \in \mathrm{dom}(\underset{\sim}{b}'_{k\alpha})$ for each $k \geq n, 1 < \alpha \leq \omega_1$.

The ordering $\leq^*$ on $\mathcal{P}$ and $\leq_n$ on Q_{n0} is not closed in the present situation. Thus it is possible to find an increasing sequence of $\aleph_0$ conditions $\langle \langle a_{ni}, A_{ni}, f_{ni} \rangle \mid i < \omega \rangle$ in $(Q_{n0})_0$ with no upperbound. The reason is that the union of maximal models of these conditions, i.e. $\bigcup_{i<\omega} \max(\mathrm{dom} a_{ni})$ need not be in $A^{1\kappa^+}$ for any $A^{1\kappa^+}$ in $G(\mathcal{P}')$. The next lemma shows that still $\leq_n$ and so also $\leq^*$ share a kind of strategic closure. The proof is similar to those of [5, 3.5].

Lemma 3.3 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_0 \rangle$ does not add new sequences of ordinals of the length $< \lambda_n$, i.e. it is (λ_n, ∞) – distributive.*

Now as in [5] we obtain the following:

Lemma 3.4 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 3.5 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Let us turn now to the main lemma in the present context:

Lemma 3.6 *$\langle \mathcal{P}, \rightarrow \rangle$ satisfies κ^{++}-c.c.*

Proof. Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\zeta \mid \zeta < \kappa^{++}\rangle$ be a $\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})$- name of an antichain of the length κ^{++}. As in [7], using the κ^{++}-strategic closure of $\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})$ ([7, 1.6]) we find an increasing sequence

$$\langle\langle\langle\langle A^{0\tau}_\zeta(\kappa^{+\alpha+1}), A^{1\tau}_\zeta(\kappa^{+\alpha+1}), C^\tau_\zeta(\kappa^{+\alpha+1})\rangle | \tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]\rangle | \alpha \leq \omega_1\rangle \mid \zeta < \kappa^{++}\rangle$$

of elements of $\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})$ and a sequence $\langle p_\zeta \mid \zeta < \kappa^{++}\rangle$ so that for every $\zeta < \kappa^{++}$ the following holds:

1. $\langle\langle\langle A^{0\tau}_{\zeta+1}(\kappa^{+\alpha+1}), A^{1\tau}_{\zeta+1}(\kappa^{+\alpha+1}), C^\tau_{\zeta+1}(\kappa^{+\alpha+1})\rangle | \tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]\rangle | \alpha \leq \omega_1\rangle \Vdash \underset{\sim}{p}_\zeta = \check{p}_\zeta$

2. for every $\alpha, 1 < \alpha \leq \omega_1$ and $\tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$ we have $A^{0\tau}_\zeta(\kappa^{+\alpha+1}) = A^{0\tau}_\zeta(\kappa^{+\omega+1}) \cap \kappa^{+\alpha+1}$.

   This a new condition (relatively to [8]) which is easy to arrange using the strategic closure of $\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})$.

3. if ζ is a limit ordinal, then $\bigcup\{A^{0\tau}_\beta(\kappa^{+\alpha+1}) | \beta < \zeta\} = A^{0\tau}_\zeta(\kappa^{+\alpha+1})$, for each $\alpha \leq \omega_1$ and $\tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$

4. ${}^{\tau>}A^{0\tau}_{\zeta+1}(\kappa^{+\alpha+1}) \subseteq A^{0\tau}_{\zeta+1}(\kappa^{+\alpha+1})$, for each $\alpha \leq \omega_1$ and $\tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$

5. $A^{0\tau}_{\zeta+1}(\kappa^{+\alpha+1})$ is a successor model, for each $\alpha \leq \omega_1$ and $\tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$

6. $\langle\langle\langle A^{1\tau}_\beta(\kappa^{+\alpha'+1}) \mid \alpha' \leq \omega_1$ and $\tau \in Reg \cap [\kappa^+, \kappa^{+\alpha'+1}]\rangle\rangle \mid \beta \leq \zeta\rangle \in (A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))^-$ (i.e. the immediate predecessor over $C^{\kappa^+}_{\zeta+1}$),for each $\alpha \leq \omega_1$

7. for every $\zeta \leq \zeta' < \kappa^{++}, \alpha \leq \omega_1$ and $\tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$ we have

   $$A^{0\tau}_\zeta(\kappa^{+\alpha+1}) \in C^\beta(A^{0\tau}_{\zeta'}(\kappa^{+\alpha+1}))$$

8. $A^{0\tau}_{\zeta+2}(\kappa^{+\alpha+1})$ is not an immediate successor model of $A^{0\tau}_{\zeta+1}(\kappa^{+\alpha+1})$, for every $\zeta < \kappa^{++}, \alpha \leq \omega_1$ and $\tau \in Reg \cap [\kappa^+, \kappa^{+\alpha+1}]$

9. $p_\zeta = \langle p_{\zeta n} | n < \omega\rangle$

10. for every $n \geq \ell(p_\zeta)$ the maximal model of $\mathrm{dom}(a_{\zeta n})$ is $A^{0\kappa^+}_{\zeta+1}(\kappa^{++})$ and the maximal model of $\mathrm{dom}(\underset{\sim}{b}_{\zeta n\alpha})$ is $A^{0\kappa^+}(\kappa^{+\alpha+1})_{\zeta+1}$,

    where $p_{\zeta n} = \langle\langle a_{\zeta n}, A_{\zeta n}, f_{\zeta n}\rangle, \langle \underset{\sim}{b}_{\zeta n}, \underset{\sim}{B}_{\zeta n}, g_{\zeta n}\rangle\rangle$.

Let $p_{\zeta n} = \langle\langle a_{\zeta n}, A_{\zeta n}, f_{\zeta n}\rangle, \langle \underset{\sim}{b}_{\zeta n}, \underset{\sim}{B}_{\zeta n}, g_{\zeta n}\rangle\rangle$ for every $\zeta < \kappa^{++}$ and $n \geq \ell(p_\zeta)$. Extending by 3.2 if necessary, let us assume that $A^{0\kappa^+}_\zeta(\kappa^{++}) \in \mathrm{dom}(a_{\zeta n})$ and $A^{0\kappa^+}_\zeta(\kappa^{+\alpha+1}) \in \mathrm{dom}(\underset{\sim}{b}_{\zeta n\alpha})$, for every $n \geq \ell(p_\alpha)$ and $\alpha \leq \omega_1$. Shrinking if necessary, we assume that for all $\zeta, \eta < \kappa^{++}$ the following holds:

(a) $\ell = \ell(p_\zeta) = \ell(p_\eta)$

(b) for every $n < \ell$ $\quad p_{\zeta n}$ and $p_{\eta n}$ are compatible in Q_{n1}

(c) for every n, $\ell \leq n < \omega$ $\quad \langle \mathrm{dom}(a_{\zeta' n}), \mathrm{dom}(f_{\zeta' n}) \mid \zeta' < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0(\kappa^{++})$

(d) for every n, $\omega > n \geq \ell$ $\quad \mathrm{rng}(a_{\zeta n}) = \mathrm{rng}(a_{\eta n})$.

(e) for every n, $\omega > n \geq \ell$ $\quad A_{\zeta n} = A_{\eta n}$

(f) for every n, $\ell \leq n < \omega$ $\quad \langle \mathrm{dom}(\underset{\sim}{b}_{\zeta' n}), \mathrm{dom}(g_{\zeta' n}) \mid \zeta' < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$.

(g) for every n, $\omega > n \geq \ell, \alpha, 1 < \alpha \leq \omega_1$ $\quad \mathrm{rng}(\underset{\sim}{b}_{\zeta n\alpha}) = \mathrm{rng}(\underset{\sim}{b}_{\eta n\alpha})$, i.e. it is just the same name.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let ζ be in S. For each $n, \ell \leq n < \omega$, there will be $\eta(\zeta, n) < \zeta$ such that

- $\mathrm{dom}(a_{\zeta n}) \cap A^{0\kappa^+}_\zeta(\kappa^{++}) \subseteq A^{0\kappa^+}_{\eta(\zeta,n)}(\kappa^{++})$

  and

- for every $\alpha, 1 < \alpha \leq \omega_1$,

$$\mathrm{dom}(\underset{\sim}{b}_{\zeta n\alpha}) \cap A^{0\kappa^+}_\zeta(\kappa^{+\alpha+1}) \subseteq A^{0\kappa^+}_{\eta(\zeta,n)}(\kappa^{+\alpha+1}).$$

Just recall that $|a_{\zeta n}| < \lambda_n$ and $|\mathrm{dom}(\underset{\sim}{b}_{\zeta n\alpha})| < \lambda_n$. Shrink S to a stationary subset S^* so that for some $\zeta^* < \min S^*$ of cofinality κ^+ we will have $\eta(\zeta, n) < \zeta^*$, whenever $\zeta \in S^*, \ell \leq n < \omega$. Now, the cardinality of both $A^{0\kappa^+}_{\zeta^*}$ and $A^{0\kappa^+}_{\zeta^*}(\kappa^{++})$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\zeta, \eta \in S^*, \ell \leq n < \omega, \alpha, 1 < \alpha \leq \omega_1$ the following hold:

- $\mathrm{dom}(a_{\zeta n}) \cap A^{0\kappa^+}_{\zeta}(\kappa^{++}) = \mathrm{dom}(a_{\eta n}) \cap A^{0\kappa^+}_{\eta}(\kappa^{++})$

  and

- $\mathrm{dom}(\underset{\sim}{b}_{\zeta n\alpha}) \cap A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1}) = \mathrm{dom}(\underset{\sim}{b}_{\eta n\alpha}) \cap A^{0\kappa^+}_{\eta}(\kappa^{+\alpha+1})$.

Let us add $A^{0\kappa^+}_{\zeta^*}(\kappa^{++})$ and all $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$, for $1 < \alpha < \omega_1$, to each $p_\alpha, \alpha \in S^*$. Note that $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$'s satisfy the projection condition (2) above. By 3.2, it is possible to do this without adding other additional models except the images of $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$ under isomorphisms. Thus, $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1}) \in C^{\kappa^+}(A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1}))$ and $A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1}) \in \mathrm{dom}(\underset{\sim}{b}_{\zeta n\alpha}) \cap C^{\kappa^+}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))$. So, 2.1(**??**) was already satisfied after adding $A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1})$. The rest of 2.1 does not require adding additional models in the present situation.

Denote the result for simplicity by p_ζ as well. Note that (again by 3.2 and the argument above) any $A^{0\kappa^+}_{\gamma}(\kappa^{+\alpha+1})$ for $\gamma \in S^* \cap (\zeta^*, \zeta)$ or, actually any other successor or limit model $X \in C^{\kappa^+}(A^{0\kappa}_{\zeta}(\kappa^{+\alpha+1}))$ with $\mathrm{cof}(otp_{\kappa^+}(X)) = \kappa^+$, which is between $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$ and $A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1})$, with $\alpha \leq \omega_1$, can be added without adding other additional models or ordinals except the images of it under isomorphisms.

Let now $\eta < \zeta$ be ordinals in S^*. We claim that p_η and p_ζ are compatible in $\langle \mathcal{P}, \rightarrow \rangle$. First extend p_ζ by adding $A^{0\kappa^+}_{\eta+2}(\kappa^{+\alpha+1})$, for each $\alpha \leq \omega_1$. As it was remarked above, this will not add other additional models or ordinals except the images of $A^{0\kappa^+}_{\eta+2}(\kappa^{+\alpha+1})$ under isomorphisms to p_ζ. Let p be the resulting extension. Denote p_η by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$, $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$ and $q_n = \langle \langle a'_n, A_n, f'_n \rangle, \langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n, g'_n \rangle \rangle$. Note that by (5) above the sets of measure one of p_n, q_n are the same. Without loss of generality we may assume that $a_n(A^{0\kappa^+}_{\eta+2}(\kappa^{++}))$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $\mathrm{rng}(a'_n)$ inside $a_n(A^{0\kappa^+}_{\eta+2}(\kappa^{++}))$ over the common parts $\mathrm{dom}(a'_n)$ and $\mathrm{dom}(a_n)$. This will produce $\langle a''_n, A'_n, f'_n \rangle$ which is $k_n - 1$-equivalent to $\langle a'_n, A'_n, f'_n \rangle$ and with $\mathrm{rng}(a''_n) \subseteq a_n(A^{0\kappa^+}_{\eta+2}(\kappa^{++}))$. Doing the above for all $n \geq \ell(p)$ we will obtain $\langle \langle a''_n, A'_n, f'_n \rangle \mid n < \omega \rangle$ equivalent to $\langle \langle a'_n, A'_n, f'_n \rangle \mid n < \omega \rangle$ (i.e. $\langle \langle a''_n, A''_n, f''_n \rangle \mid n < \omega \rangle \longleftrightarrow \langle \langle a'_n, A'_n, f'_n \rangle \mid n < \omega \rangle$).

Let $t = \langle \langle \langle a''_n, A'_n, f'_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle \mid n < \omega \rangle$. Extend t to t' by adding to it

$$\langle A^{0\kappa^+}_{\eta+2}(\kappa^{++}), a_n(A^{0\kappa^+}_{\eta+2}(\kappa^{++})) \rangle$$

as the maximal set for every $n \geq \ell(p)$. Recall that $A^{0\kappa^+}_{\eta+1}(\kappa^{++})$ was its maximal model. So we are adding a top model, also, by the condition (8) above $A^{0\kappa^+}_{\eta+2}(\kappa^{++})$ is not an immediate successor of $A^{0\kappa^+}_{\eta+1}(\kappa^{++})$. Hence no additional models or ordinals are added at all.

Let $t'_n = \langle\langle a'''_n, A'''_n, f'_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle$, for every $n \geq \ell(p)$.

Combine now the first coordinates of p and t' together, i.e. $\langle a_n, A_n, f_n\rangle$'s with those of t'. Thus for each $n \geq \ell(p)$ we add a'''_n to a_n. Add if necessary a new top model to insure 2.1(2(d)). Let $r = \langle r_n | n < \omega\rangle$ be the result, where $r_n = \langle\langle c_n, C_n, h_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle$, for $n \geq \ell(p)$.

Claim 3.6.1 $r \in \mathcal{P}$ and $r \geq p$.

Proof. Fix $n \geq \ell(p)$. The main points here are that a'''_n and a_n agree on the common part and adding of a'''_n to a_n does not require other additions of models except the images of a'''_n under isomorphisms.

The check of the rest of conditions of 2.1 is routine. We refer to [2] or [5] for similar arguments.

$\square$ of the claim.

Now let us turn to the second coordinates of q and r. Recall that for a condition $x \in Q_{n0}$ we denote by $(x)_0$ its first coordinate, i.e. the first triple. If $y = \langle y_n | n < \omega\rangle \in \mathcal{P}$, then $(y)_0$ denotes $\langle (y_n)_0 | n < \omega\rangle$. So, we have $(q)_0 \rightarrow (r)_0$. Shrinking if necessary A_n's (the sets of measure one of $(q_n)_0$'s), we can assume that for each $n \geq \ell(p) = \ell(r) = \ell(q)$ the set of measure one for $(r_n)_0$, i.e. C_n projects exactly to A_n by $\pi_{\lambda_n, \max(\mathrm{rng}((r_n)_0), \max(\mathrm{rng}((q_n)_0)}$. Remember that the interpretations of both $\langle \underset{\sim}{b}_n, \underset{\sim}{B}_n\rangle$ and $\langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n\rangle$ depend at the level of λ_n only on a choice of elements of A_n.

Our tusk will be extend r to r^* so that $q \rightarrow r^*$. This will show that p and q are compatible. Which provides the desired contradiction.
The way of doing this here will be first to deal with interpretations of models according to the one element extender based Prikry forcing which done at levels λ_n's. In [8] no further interpretations were needed, but here there is an additional forcing- the extender based Magidor forcing. So, we will need to continue and make further interpretations according to the values of the Magidor sequences. Note that for the least member of the Magidor sequence no interpretation beyond those with λ_n's is required.

Fix n, $\omega > n \geq \ell(p)$, large enough. Let σ be the maximal coordinate of $(r_n)_0$ (i.e. the ordinal coding $\max(\mathrm{rng}(c_n))$, θ those of $(p_n)_0$ (which is the same for $(q_n)_0$, since (4) above) and μ the one corresponding to θ (of $(q_n)_0$) under $(q_n)_0 \rightarrow (r_n)_0$. Denote $\pi''_{\lambda_n,\sigma,\mu} C_n$ by D_n. Assuming that $n > 2$, it follows from the definitions of the equivalence relation $\longleftrightarrow$ and of the order $\rightarrow$, that $E_{\lambda_n}(\mu)$ (the μ's measure of the extender) is the same as $E_{\lambda_n}(\theta)$. Also, $D_n \subseteq A_n$.

Define now a condition

$$r_n^* = \langle\langle c_n, C_n, h_n\rangle, \langle \underset{\sim}{e}_n, \underset{\sim}{E}_n, g_n\rangle\rangle \in Q_{n0}$$

which extends

$$r_n = \langle\langle c_n, C_n, h_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle.$$

The addition will depend only on the coordinate μ of E_{λ_n}. So we need to deal with each $\nu \in D_n$. Set $\mathrm{dom}(\underset{\sim}{e}_{n\alpha}) = \mathrm{dom}(\underset{\sim}{b}_{n\alpha}) \cup \mathrm{dom}(\underset{\sim}{b}'_{n\alpha})$, for each $\alpha, 1 < \alpha \leq \omega_1$. Let $X \in \mathrm{dom}(\underset{\sim}{e}_{n\alpha})$. If $X \in \mathrm{dom}(\underset{\sim}{b}_{n\alpha})$, then set

$$\underset{\sim}{e}_{n\alpha}(X)[\rho] = \underset{\sim}{b}_{n\alpha}(X)[\rho],$$

for each $\rho \in C_n$. Now, if X is new, i.e. $X \in \mathrm{dom}(\underset{\sim}{b}'_{n\alpha}) \backslash \mathrm{dom}(\underset{\sim}{b}_{n\alpha})$, then we consider X_ζ the model that corresponds to X in p_ζ under the Δ-system.

By Definition 2.1(7n), we have

$$\underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1})[\nu] \in \underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta})[\rho].$$

Recall that

$$\underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1})[\nu] = \underset{\sim}{b}'_{n\alpha}(A^{0\kappa^+}_{\eta+1})[\nu]$$

and

$$\underset{\sim}{b}_{n\alpha}(X_\zeta)[\nu] = \underset{\sim}{b}'_{n\alpha}(X)[\nu].$$

Set now $\underset{\sim}{e}_{n\alpha}(X)[\rho]$ to be $\underset{\sim}{b}'_{n\alpha}(X)[\nu]$,
for each $\rho \in C_n$ and $\nu = \pi_{\lambda_n,\sigma,\mu}(\rho)$.

The following claim suffice in order to complete the argument:

Claim 3.6.2 $r_n^* \in Q_{n0}$, $r_n^* \geq_0 r_n$ and $q_n \to r_n^*$.

Proof. Let us check first that q_n, r_n or basically $\underset{\sim}{b}'_n$ and $\underset{\sim}{c}_n$ agree about the values of models in $\mathrm{dom}(\underset{\sim}{b}'_{n\alpha}) \cap \mathrm{dom}(\underset{\sim}{c}_{n\alpha})$, for any $\alpha, 1 < \alpha \leq \omega_1$.

Suppose that X is such a model. Then, by the assumptions we made on the Δ-system, $X \in A^{0\kappa^+}_{\zeta^*}$. Also,

$$A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1}) \in \mathrm{dom}(\underset{\sim}{b}'_{n\alpha}) \cap \mathrm{dom}(\underset{\sim}{c}_{n\alpha}),$$

$$otp_{\kappa^+}(A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})) = A^{0\kappa^+}_{\zeta^*}(\kappa^{++}) \cap \kappa^{++}$$

and

$$A^{0\kappa^+}_{\zeta^*}(\kappa^{++}) \in \mathrm{dom}(c_n).$$

We deal first with interpretations according to λ_n. By 2.1, $\underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1}))$ depends only on the measure indexed by the code of

$$c_n(A^{0\kappa^+}_{\zeta^*}(\kappa^{++})) = a_n(A^{0\kappa^+}_{\zeta^*}(\kappa^{++})) = a'_n(A^{0\kappa^+}_{\zeta^*}(\kappa^{++})).$$

Let δ denotes the index of this measure (or its code). Then for each $\rho \in C_n$ we will have

$$\pi_{\lambda_n,\sigma,\delta}(\rho) = \pi_{\lambda_n,\mu,\delta}(\pi_{\lambda_n,\sigma,\mu}(\rho)).$$

Hence, restricting $(q_n)_0$ to D_n, i.e. by replacing A_n in $(q_n)_0$ with D_n, we can insure that $\underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1}))$ and $\underset{\sim}{b}'_{n\alpha}(A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1}))$ agree. The same applies to any $X \in A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$ which is in the common domain, since its value too will depend on the δ-th measure of the extender only.

Consider now the maximal model of q_n. By 10, above, it is $A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1})$ and the one of p_n is $A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1})$. Now, for each $\nu \in A_n$, by the condition (g) on the Δ-system above we have

$$\underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))[\nu] = \underset{\sim}{b}'_{n\alpha}(A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1}))[\nu].$$

Pick $\rho \in C_n$. Let $\nu = \pi_{\lambda_n,\sigma,\mu}(\rho)$ and $\varepsilon = \pi_{\lambda_n,\sigma,\theta}(\rho)$. Then

$$\underset{\sim}{e}_{n\alpha}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))[\rho] = \underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))[\varepsilon]$$

and

$$\underset{\sim}{e}_{n\alpha}(A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1}))[\rho] = \underset{\sim}{b}'_{n\alpha}(A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1}))[\nu].$$

The first equality holds since e_n extends b_n and the second by the same reason as e_n was defined this way above.

The crucial observation is that $\varepsilon, \nu \in A_n$ (just $D_n \subseteq A_n$) and $\varepsilon > \nu$, so by Definition 2.1(7n),

$$\underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))[\nu] \subseteq \underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))[\varepsilon].$$

Hence, also,

$$\underset{\sim}{b}'_{n\alpha}(A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1}))[\nu] \subseteq \underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1})[\varepsilon],$$

since

$$\underset{\sim}{e}_{n\alpha}(A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1}))[\rho] = \underset{\sim}{b}'_{n\alpha}(A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1}))[\nu].$$

The same inclusion holds, by Definition 2.1(7n), if we replace $A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1})$ with any $Y \in \mathrm{dom}(\underset{\sim}{b}_{n\alpha}) \cap C^{\kappa^+}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))$ such that $\varepsilon(Y) > \nu$, where $\varepsilon(Y)$ is the measure corresponding to Y. Thus

$$\underset{\sim}{b}'_{n\alpha}(A^{0\kappa^+}_{\eta+1}(\kappa^{+\alpha+1}))[\nu] = \underset{\sim}{b}_{n\alpha}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))[\nu] \subseteq \underset{\sim}{b}_{n\alpha}(Y)[\varepsilon].$$

In the present case we have the least such Y. It is $A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1})$. Just below it everything falls into $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$ the kernel of the Δ-system. Consider now Y's in $\mathrm{dom}(\underset{\sim}{b}_{n\alpha})\backslash C^{\kappa^+}(A^{0\kappa^+}_{\zeta+1}(\kappa^{+\alpha+1}))$. If such Y is in $A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1})$, then it belongs to $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$ the kernel of the Δ-system. Hence as it was observed in the beginning of the proof of this claim, we have the agreement. Suppose now that $Y \not\in A^{0\kappa^+}_{\zeta}$. By the basic properties of $G(\mathcal{P}')$ there will be $Z \in A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1})$ such that

$$Y \cap A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1}) = Z \cap A^{0\kappa^+}_{\zeta}(\kappa^{+\alpha+1}).$$

Then again this Z falls into $A^{0\kappa^+}_{\zeta^*}(\kappa^{+\alpha+1})$ and into the kernel of the Δ-system on which we have the agreement.

We deal similar with further interpretations, i.e. those according to finite sequences from the extender based Magidor forcing. Thus, given $\rho^\frown\vec{\rho}$ and its projection $\nu^\frown\vec{\nu}$, we may assume by induction that for each $k < |\vec{\rho}|$ the interpretations according to $\rho^\frown\vec{\rho} \restriction k$ and to $\nu^\frown\vec{\nu} \restriction k$ fit together nicely. Now run the argument above for the last element of the sequence $\vec{\rho}$ and its projection - the last element of the sequence $\vec{\nu}$. Note only that the projection function is inside the larger models involved.

This completes the proof of the claim.

$\square$ of the claim.

$\square$

Force with $\langle \mathcal{P}, \rightarrow \rangle$. Let $G(\mathcal{P})$ be a generic set. By the lemmas above no cardinals are collapsed. Let $\langle \nu_n \mid n < \omega \rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\lambda_n} \mid n < \omega \rangle$ and $\langle \rho_{n\alpha} \mid 1 < \alpha \leq \omega_1 \rangle$, for each $n < \omega$, the Magidor sequence for the normal measures of E_{κ_n}. We can deduce now the following conclusion:

Theorem 3.7 *The following hold in* $V[G(\mathcal{P}'(\kappa^+) * ... * \mathcal{P}'(\kappa^{+\alpha+1}) * ... * \mathcal{P}'(\kappa^{+\omega_1+1})), G(\mathcal{P})]$*:*

(1) $\mathrm{cof}(\prod_{n<\omega} \nu_n^{+n+2} / \textit{ finite }) = \kappa^{++}$

(2) for each $\alpha, 1 < \alpha \leq \omega_1$*,*

$$\mathrm{cof}(\prod_{n<\omega} \rho_{n\alpha}^{+n+2} / \textit{ finite }) = \kappa^{+\alpha+1}$$

The proof follows easily from the construction.

4 Some generalizations

1. It is possible to use the previous method in order to make $2^\kappa \geq \kappa^{+\kappa} = \aleph_\kappa$. Just instead of a fixed value $\theta < \kappa$ we change cofinalities in each interval $[\kappa_n, \kappa_{n+1})$ to some $\theta_n > \kappa_n$.

2. Similar any $\aleph_\alpha$, for $\kappa \leq \alpha < \kappa^+$, can be reached. Just we need to split the cardinals into omega many groups each of cardinality less than κ.

3. The following allows us to proceed up to $\aleph_{\aleph_\kappa}$. Thus, for example we would like to reach $\aleph_{\kappa^{+\omega_1}}$. Then at each level n (i.e. between κ_n and κ_{n+1}) we split into two blocks. The first will be as in Section 1. It will take care of all the cardinals between κ and $\kappa^{+\omega_1}$. The second block will be responsible for the cardinals in the interval $[\kappa^{++}, \aleph_{\kappa^{+\omega_1}})$. We will use the dropping in cofinality between the two blocks. This will insure that the models corresponding to those of cardinalities $\kappa^+, ..., \kappa^{+\omega_1}$ will not include all the relevant cardinalities.

4. Repeating the process of 3, but using finite (increasing) number of blocks instead of just two, allows to reach the α-th repeat point of the $\aleph$ function above κ, for every $\alpha < \omega_1$.

5. Working a bit harder, for any $\alpha < (2^\omega)^+$ the α-th repeat point above κ can be reached.

   Let $\langle f_i | i < \alpha \rangle$ be an increasing (mod finite) sequence in ${}^\omega\omega$. We define the correspondence between cardinals above κ and those below such that for every $i < j < \alpha$, below the least n with $f_i(m) < f_j(m)$, for each $m \geq n$, we allow cardinals from the intervals [i-th repeat point above κ, $i+1$-th repeat point above κ) and [j-th repeat point above κ, $j+1$-th repeat point above κ) to correspond to the cardinals of the same blocks (still in the order preserving fashion). But at the level n and above the corresponding blocks should be one above another.

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