An application of the Silver theorem on decomposability

February 10, 2021

Our aim is to prove the following:

**Theorem 0.1** Suppose that $\aleph_\omega$ is a strong limit. Let $U$ be a uniform ultrafilter over a cardinal $\eta > \aleph_\omega$. Suppose that for some $n^* < \omega$, $U$ is $\aleph_n$–indecomposable, for all $\aleph_n \in [\aleph_n^*, 2^{\aleph_n^*}]$.

Let $K^U$ be a subset of $\eta$ which consists of regular cardinals $\rho$ such that

1. $\sup(j^U''\rho)$ exists.
   Note that $M_U$ is not well-founded, so it need not be the case always.

2. $\sup(j^U''\rho) < j_U(\rho)$.
   This means that $U$ is $\rho$–decomposable, i.e. $U_\rho = \{X \subseteq \rho \mid \sup(j^U''\rho) \in j_U(X)\}$ is a uniform ultrafilter over $\rho$ which is Rudin-Keisler below $U$.

3. $M_U \models \text{cof}(\sup(j^U''\rho)) < j_U(\aleph_\omega)$.
   Equivalently, $U_\rho$ concentrates on ordinals of cofinality less than $\aleph_\omega$.

Then $|K^U| < (2^{\aleph_n^*})^+$. In particular, if $n^* = 1$, then $|K^U| < (2^{\omega})^+$.

**Remark 0.2** Note that by Kunen-Prikry theorem [3], $U$ is $\aleph_n$–indecomposable for every $n, n^* \leq n < \omega$.

*Proof*. Suppose otherwise. Fix $\langle \rho_i \mid i < (2^{\omega_n^*})^+ \rangle$ an increasing sequence of consisting of elements of $K^U$.

Then by the theorem of Silver, see [2], there is an ultrafilter $D$ over some $\aleph_m, m < n^*$ such that $j_D(\omega) = j_U(\omega)$. Note that $j_D(\omega)$ is the first infinite cardinal in sense of $M_U$.

Denote it further by $\tilde{\omega}$. Its real cardinality (i.e. the cardinality of the set $\tilde{\omega}$ in $V$ is $\leq 2^{\aleph_m} < \aleph_\omega$. Denote it by $\delta$. 

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Consider \( j_\mathbb{U} (\aleph_\omega) \). By elementarity, \( M_\mathbb{U} \models j_\mathbb{U} (\aleph_\omega) = \aleph_\omega \).

Then the number in \( V \) of \( M_\mathbb{U} \)-cardinals below \( \aleph_\omega \) is \( \delta \). We have

\[
M_\mathbb{U} \models \text{cof}(\sup(j''_\mathbb{U} \rho_i)) < \aleph_\omega
\]

\( i < (2^\omega)^+ \). Hence, there will be \( i < i' < (2^\omega)^+ \), such that

\[
M_\mathbb{U} \models \text{cof}(\sup(j''_\mathbb{U} \rho_i)) = \text{cof}(\sup(j''_\mathbb{U} \rho_{i'})).
\]

Pick then in \( M_\mathbb{U} \) a function \( f \) such that

\[
M_\mathbb{U} \models f \text{ is an increasing function which maps a cofinal subset of } \sup(j''_\mathbb{U} \rho_i)
\]

onto a cofinal subset of \( \sup(j''_\mathbb{U} \rho_{i'}) \).

Let us now define in \( V \) an order preserving function \( g \) from \( \rho_{i'} \) to a subset of \( \rho_i \). The existence of such function is clearly impossible and, so, will provide the desired contradiction. Proceed by induction. Suppose that \( \nu < \rho' \) and \( g \mid \nu \) is defined. By the inductive assumption, there is \( \alpha_\nu < \rho \) such that \( g'' \nu \subseteq \alpha_\nu \).

There exists some \( x_\nu \) such that

\[
M_\mathbb{U} \models j_\mathbb{U} (\alpha_\nu) < x_\nu < \sup(j''_\mathbb{U} \rho_i), x_\nu \in \text{dom}(f), f(x_\nu) > j_\mathbb{U} (\nu) \text{ and it is the least like this.}
\]

Pick some \( \beta_\nu, \alpha_\nu < \beta_\nu < \rho \) such that

\[
M \models x_\nu < j_\mathbb{U} (\beta_\nu).
\]

Set \( g(\nu) = \beta_\nu \).

This completes the construction of \( g \), and so the proof of the theorem.

\( \Box \)

**Theorem 0.3** Indecomposable ultrafilters of Ben David -Magidor [1] satisfy the assumptions of 0.1.

**Proof.** Let \( U \) over \( P_\kappa (\lambda) \) be an indecomposable ultrafilter constructed as in Ben David -Magidor [1]. Note that the function \( P \mapsto \sup(P) \) is one to one on a set in \( U \), by Solovay, since \( U \) extends a normal ultrafilter in the ground model.

Use the Prikry condition argument similar to [4] in order to show that for every function \( f : P_\kappa (\lambda) \to \aleph_{\omega+k} \) in \( V[(\kappa_n \mid n < \omega), \{F_n \mid n < \omega\}] \), if \( f(P) < \sup(P \cap \aleph_{\omega+k}) \), then for some \( \alpha < \aleph_{\omega+k} \) and \( A \in U \), \( f(P) < \alpha \), for all \( \alpha \in A \).
References


[5] D. Raghavan and S. Shelah, A SMALL ULTRAFILTER NUMBER AT SMALLER CARDINALS,

[6] R. Solovay,