Some consistency results on density numbers.

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October 14, 2015

Abstract

We answer some questions of M. Kojman on density numbers.

1 Introduction

Menachem Kojman introduced and studied in [4],[5] the following natural notion.

Definition 1.1 (Kojman) Suppose $\theta \leq \mu$ are cardinals.

1. The $\theta-$density of μ, denoted by $D(\mu, \theta)$, is the least cardinality of a subset $D \subseteq [\mu]^\theta$ which is dense in $\langle [\mu]^\theta, \subseteq \rangle$ (i.e. every $X \subset \mu$ of cardinality θ contains an element of D).

2. The $\theta-$upper density of μ, denoted by $\overline{D}(\mu, \theta)$, is the least cardinality of a subset $D \subseteq [\mu]^\theta$ such that

    (a) for every $Z \in D$, for every $\alpha < \mu$, $|Z \cap \alpha| < \theta$,

    (b) for every $X \subset \mu$ of cardinality θ, such that for every $\alpha < \mu$, $|X \cap \alpha| < \theta$ contains an element of D.

3. The $\theta-$lower density of μ, denoted by $\underline{D}(\mu, \theta)$, is the least cardinality of a subset $D \subseteq \bigcup\{[\alpha]^\theta \mid \alpha < \mu\}$ which is dense in $\bigcup\{[\alpha]^\theta \mid \alpha < \mu\}$.

In [5], Kojman asked the following questions:

Question 1.

Is the negation of the following statement consistent:
There is κ such that for any two regular cardinals θ_1, θ_2 above κ, for every sufficiently large μ

$$\mu = \min(D(\mu, \theta_1), D(\mu, \theta_2))?$$

Question 2.
Is the negation of the following statement consistent:
For every κ there is a finite set F of regular cardinals above κ, for every sufficiently large μ

$$\mu = \min(D(\mu, \theta) \mid \theta \in F)?$$

Clearly the second statement is stronger and Kojman showed in [4], that it is impossible to replace finite by countable.

Our aim is to prove the following that answers both questions affirmatively:

Theorem 1.2 *Suppose that η is an inaccessible cardinal which is a limit of strong cardinals. Then there is a forcing extension $V[G]$ of V such that the model $V_\eta[G]$ satisfies the following:*

1. *ZFC,*
2. *for every finite set $\rho_1 < ... < \rho_n$ of regular cardinals, for every ξ, there are $\mu_1 < ... < \mu_n$ such that*
   (a) $\mu_1 > \xi$,
   (b) $\mathrm{cof}(\mu_1) = \rho_n, \mathrm{cof}(\mu_2) = \rho_{n-1}, ..., \mathrm{cof}(\mu_n) = \rho_1$,
   (c) $\mu_1^{\rho_n} = D(\mu_1, \rho_n) = \overline{D}(\mu_1, \rho_n) > \mu_2^{\rho_{n-1}} = D(\mu_2, \rho_{n-1}) = \overline{D}(\mu_2, \rho_{n-1}) > ... > \mu_n^{\rho_1} = D(\mu_n, \rho_1) = \overline{D}(\mu_n, \rho_1) > \mu_n$,
   (d) $\mu_n < \mu_n^{\rho_1} = D(\mu_n, \rho_1) < \mu_n^{\rho_2} = D(\mu_n, \rho_2) < ... < \mu_n^{\rho_n} = D(\mu_n, \rho_n)$,
3. *for every finite set of cardinals F (consisting not necessary of regular cardinals) there are arbitrary large cardinals μ such that $\mu \neq \min(\{D(\mu, \theta) \mid \theta \in F\})$.*

The idea of the construction goes back to [1], however we prefer to use more modern approach based on Extender Based Magidor forcings due to Merimovich [6], since it is more straightforward and allows to preform cardinal arithmetic calculations more easily.

2 Forcing constructions

Let η be an inaccessible cardinal which is a limit of strong cardinals.[1]
Assume GCH.

[1] Alternatively, it is possible to assume that that there are unboundedly many strongs and to work with classes instead of using η.

Fix an enumeration $\langle F_\nu \mid \nu < \eta\rangle$ of all finite sequences of regular cardinals below η. Assume that always $\nu \geq \max(F_\nu)$.
Split the set of strong cardinals $< \eta$ into $\eta-$disjoint sets $\langle S_\xi \mid \xi < \eta\rangle$ each of cardinality η.
Fix a function $f : \eta \to [\eta]^2$ such that for every $\xi, \nu < \eta$ we have

$$|\{\rho < \eta \mid f(\rho) = (\nu, \xi)\}| = \eta.$$

Define now by induction an Easton support iteration of Prikry type forcing notions (see [2] or [3])

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \eta, \beta < \eta\rangle.$$

Suppose that P_α is defined. Work in V^{P_α} and define Q_α.
Consider $f(\alpha)$. Let $f(\alpha) = (\nu_\alpha, \xi_\alpha)$. If some of the elements of F_{ν_α} is not regular anymore (i.e. it is singular in V^{P_α}, then let Q_α be the trivial forcing.
Suppose that all elements of F_{ν_α} remain regular in V^{P_α}. Let $\langle \rho_1, ..., \rho_n\rangle$ be an increasing enumeration of F_{ν_α}.
Pick some $\mu_1 < ... < \mu_n$ in S_{ξ_α} above $|P_\alpha|$. Clearly, they remain strong in V^{P_α}.
Define Q_α to be a finite iteration of forcing notions $Q_{\alpha,n} * ... * \underset{\sim}{Q}_{\alpha,1}$, where $Q_{\alpha,i}$'s are defined as follows.
Let $Q_{\alpha,n}$ be the extender based Magidor forcing (above μ_{n-1} or above $2^{|P_\alpha|}$, if $n - 1 = 0$) which changes the cofinality of μ_n to ρ_1 and blows up its power, to say, μ_n^{+7} (we will elaborate on this more below).
If $n > 1$, and assuming the right preparation was done below (see [1]), each $\mu_i, 1 \leq i < n$ remains strong. Define $Q_{\alpha,n-1}$ to be the extender based Magidor forcing (above μ_{n-2} or above $2^{|P_\alpha|}$, if $n - 2 = 0$) which changes the cofinality of μ_{n-1} to ρ_2 and blows up its power, to say, μ_n^{+14}.
If $n > 2$, then we continue and define $Q_{\alpha,n-2}$ in the same fashion, and so on.

This way the following cardinals configuration is arranged:

$$\mu_1^{\rho_n} > \mu_2^{\rho_{n-1}} > ... > \mu_n^{\rho_1} = \mu_n^{+7}.$$

Let us check this and accumulate more information on relevant cardinal arithmetic before turning to the density numbers.
Assume for simplification of the notation that $n = 2$. For the first forcing $Q_{\alpha,n}$, a coherent sequence $\vec{E}_2 = \langle E_2(\beta, \gamma) \mid \beta \in \mathrm{dom}(\vec{E}_2), \gamma < \rho_1\rangle$ of $(\beta, \beta^{+7})-$extenders is used with $\mathrm{dom}(\vec{E}_2) \subseteq \mu_2 + 1 \setminus \mu_1^{++}, \mu_2 \in \mathrm{dom}(\vec{E}_2)$.

The following was shown in [6]:

Lemma 2.1 *In a generic extension by $Q_{\alpha,2}$ the following hold:*

1. $\mathrm{cof}(\mu_2) = \rho_1$,

2. μ_2 *is a strong limit cardinal, in particular* $\mu_2^\tau = \mu_2$, *for every* $\tau < \rho_1$,

3. $\mu_2^{\rho_1} = 2^{\mu_2} = \mu_2^{+7}$,

4. $Q_{\alpha,2}$ *satisfies* $\mu_2^{++}-$*c.c. and preserves all cardinals,*

5. *Magidor sequences for measures of the extenders* $\langle E(\mu_2, \gamma), \gamma < \rho_1\rangle$ *form a scale mod bounded in the product of* $\langle \mu_{2i}^{+7} \mid i < \rho_1\rangle$ *of the length* μ_2^{+7}, *where* $\langle \mu_{2i} \mid i < \rho_1\rangle$ *is the Magidor sequence (a club in* μ_2*) for the normal measures.*

Assume that the preparation for $Q_{\alpha,2}$ was done below μ_1 (or its strongness was indestructible under such forcings, as in [1]. [2]

Work in $V^{P_\alpha * Q_{\alpha,2}}$. Pick a coherent sequence of extenders for our next extender based Magidor forcing $Q_{\alpha,1}$. $\vec{E}_1 = \langle E_1(\beta, \gamma) \mid \beta \in \mathrm{dom}(\vec{E}_1), \gamma < \rho_2\rangle$ of $(\beta, g(\beta)^{+14})-$extenders is used with $\mathrm{dom}(\vec{E}_1) \subseteq \mu_1 + 1 \setminus |P_\alpha|^{++}, \mu_1 \in \mathrm{dom}(\vec{E}_1)$, $g : \mu_1 \to \mu_1$ represents μ_2 in the ultrapower by $E_1(\mu_1, \gamma)$, for every $\gamma < \rho_2$. In particular, over μ_1 itself, $E_1(\mu_1, \gamma)$'s are $(\mu_1, \mu_2^{+14})-$extenders.

Force with the extender based Magidor forcing with $\vec{E}_1$.

By [6], as in 2.1, we have the following:

Lemma 2.2 *In a generic extension by $Q_{\alpha,1}$ the following hold:*

1. $\mathrm{cof}(\mu_1) = \rho_2$,

2. μ_1 *is a strong limit cardinal, in particular* $\mu_1^\tau = \mu_1$, *for every* $\tau < \rho_2$,

3. $\mu_1^{\rho_2} = 2^{\mu_1} = \mu_2^{+14}$,

4. $Q_{\alpha,1}$ *satisfies* $\mu_1^{++}-$*c.c. and preserves all cardinals,*

5. *Magidor sequences for measures of the extenders* $\langle E(\mu_1, \gamma), \gamma < \rho_2\rangle$ *form a scale mod bounded in the product of* $\langle g(\mu_{1i})^{+14} \mid i < \rho_2\rangle$ *of the length* μ_2^{+14}, *where* $\langle \mu_{1i} \mid i < \rho_2\rangle$ *is the Magidor sequence (a club in* μ_1*) for the normal measures.*

[2] Actually we need it to be strong up to μ_2^{+15} only.

Note only that since the lengthes of the extenders are above $2^{\mu_2} = \mu_2^{+7}$, we still have μ_1-closure of the supports of the extenders used in the extender based Magidor forcing here. It would not be the case, if instead $(\mu_1, \delta)-$extenders were used with $\delta < \mu_2^{+7}$.

The next lemma provides an additional information on cardinal arithmetic in a generic extension by $Q_{\alpha,1}$.
Denote $V^{P_\alpha * Q_{\alpha,2}}$ by V_1.

Lemma 2.3 *In $V_1^{Q_{\alpha,1}}$ the following hold:*

1. $\mu_2^{\rho_1} = \mu_2^{+7}$,
2. *for every* $\zeta < \rho_1$, $\mu_2^{\zeta} = \mu_2$,
3. *for every* $\delta < \mu_2$, $\delta^{\rho_1} < \mu_2$.

Proof. Let us prove that $\mu_2^{\rho_1} = \mu_2^{+7}$. Two other claims are similar.
Note first that every set of ordinals X in $V_1^{Q_{\alpha,1}}$ can be covered by a set $Y \in V_1$ of cardinality $|X| + \mu_1$. It follows by $\mu_1^{++}-$c.c. of the forcing and the fact that $(\mu_1^+)^{V_1}$ is preserved, by 2.2(4).
By 2.2(2), $\mu_1^{\rho_1} = \mu_1$, in $V_1^{Q_{\alpha,1}}$.
Hence,

$$\mu_2^{+7} \leq \mu_2^{\rho_1} \leq (\mu_2^{\mu_1})^{V_1} \cdot \mu_1^{\rho_1} = (\mu_2^{\mu_1})^{V_1} \cdot \mu_1 = (\mu_2^{\rho_1})^{V_1} = \mu_2^{+7}.$$

So, we are done.
□

Lemma 2.4 *In a generic extension by $Q_{\alpha,1}$ scales over μ_2 are preserved.*

Proof. It follows easily, since by 2.2(4), $Q_{\alpha,1}$ satisfies $\mu_1^{++}-$c.c.
□

Let us turn to the density numbers now.

Lemma 2.5 *In a generic extension by $Q_{\alpha,1}$ we have $D(\mu_1, \rho_2) = \overline{D}(\mu_1, \rho_2) = \mu_1^{\rho_2} = \mu_2^{+14}$.*

Proof. By 2.2, μ_1 is a strong limit cardinal of cofinality ρ_2 in a generic extension by $Q_{\alpha,1}$ and $\mu_1^{\rho_2} = \mu_1^{+14} = 2^{\mu_1}$. By [5], then $D(\mu_1, \rho_2) = \overline{D}(\mu_1, \rho_2)$. Clearly, $\overline{D}(\mu_1, \rho_2) \leq \mu_1^{\rho_2}$. But since, by 2.2(5), there is scale mod bounded of the length $\mu_1^{\rho_2}$, there must be an equality.
□

Lemma 2.6 *In a generic extension by $Q_{\alpha,1}$ we have $D(\mu_2, \rho_1) = \overline{D}(\mu_2, \rho_1) = \mu_2^{\rho_1} = \mu_2^{+7}$.*

Proof. First note that $D(\mu_2, \rho_1) = \overline{D}(\mu_2, \rho_1)$, since $\mathrm{cof}(\mu_2) = \rho_1$ and for every $\delta < \mu_2, \delta^{\rho_1} < \mu_2$, by 2.3. Now, due to the existence of a scale (2.1(5)), $\overline{D}(\mu_2, \rho_1) \geq \mu_2^{+7}$, but, by 2.3, μ_2^{+7} is $\mu_2^{\rho_1}$ of the extension. Clearly, $\overline{D}(\mu_2, \rho_1) \leq \mu_2^{\rho_1}$, and so we are done.
□

Lemma 2.7 *In a generic extension by $Q_{\alpha,1}$ we have*
$D(\mu_2, \rho_2) = \underline{D}(\mu_2, \rho_2) = \mu_2^{\rho_2} = \mu_1^{\rho_2} = \mu_2^{+14} = 2^{\mu_2}$.

Proof. By Lemmas 2.2,2.5 we have $\mu_2^{\rho_2} = \mu_1^{\rho_2} = \mu_2^{+14} = 2^{\mu_2}$. Clearly, $\mu_2^{\rho_2} \geq D(\mu_2, \rho_2) \geq \underline{D}(\mu_2, \rho_2) \geq D(\mu_1, \rho_2)$. Now, by Lemma 2.5, $D(\mu_1, \rho_2) = \mu_2^{+14}$, and so we are done.
□

This completes the definition of Q_α and the inductive construction.

Let now $G \subseteq P_\eta$ generic.

The next lemma follows from $\eta-$c.c. of the forcing (recall Easton support).

Lemma 2.8 *η remains an inaccessible cardinal in $V[G]$.*

Finally we combining everything together.

Theorem 2.9 *The model $V_\eta[G]$ satisfies the following:*

1. *ZFC,*
2. *for every finite set $\rho_1 < ... < \rho_n$ of regular cardinals, for every ξ, there are $\mu_1 < ... < \mu_n$ such that*
   - *(a)* $\mu_1 > \xi$,
   - *(b)* $\mathrm{cof}(\mu_1) = \rho_n, \mathrm{cof}(\mu_2) = \rho_{n-1}, ..., \mathrm{cof}(\mu_n) = \rho_1$,
   - *(c)* $\mu_1^{\rho_n} = D(\mu_1, \rho_n) = \overline{D}(\mu_1, \rho_n) > \mu_2^{\rho_{n-1}} = D(\mu_2, \rho_{n-1}) = \overline{D}(\mu_2, \rho_{n-1}) > ... > \mu_n^{\rho_1} = D(\mu_n, \rho_1) = \overline{D}(\mu_n, \rho_1) > \mu_n$,
   - *(d)* $\mu_n < \mu_n^{\rho_1} = D(\mu_n, \rho_1) < \mu_n^{\rho_2} = D(\mu_n, \rho_2) < ... < \mu_n^{\rho_n} = D(\mu_n, \rho_n)$.

Proof. Follows from the construction using the previous lemmas.
□

3 Further analysis

Let us continue to analyze the cardinal arithmetic of $V[G]$ in order to compute $D(\mu_2, \mu)$'s for singular μ's as well.
We return to the stage α of the construction and continue to deal with the forcings $Q_{\alpha,2}$ followed by $Q_{\alpha,1}$ in V^{P_α}.

Lemma 3.1 *In a generic extension by $Q_{\alpha,2}$, we have $D(\mu_2, \rho) = \mu_2$, for every $\rho < \mu_2$ such that* $\mathrm{cof}(\rho) \neq \rho_1$.

Proof. Suppose that $\rho < \mu_2$ is such that $\mathrm{cof}(\rho) \neq \rho_1$. Then $D(\mu_2, \rho) = \underline{D}(\mu_2, \rho)$, since by [5], $D(\mu_2, \rho) = \underline{D}(\mu_2, \rho) + \overline{D}(\mu_2, \rho)$ and $\overline{D}(\mu_2, \rho) = 0$, as $\mathrm{cof}(\rho) \neq \rho_1 = \mathrm{cof}(\mu_2)$. Now, since μ_2 is a strong limit cardinal in $V^{P_\alpha * Q_{\alpha,2}}$, we must have $\underline{D}(\mu_2, \rho) = \mu_2$.
□

Let us deal now with singular ρ's of cofinality ρ_1.

Lemma 3.2 *In a generic extension by $Q_{\alpha,2}$, we have $D(\mu_2, \rho) = \mu_2^{\rho_1} = \mu_2^{+7}$, for every $\rho < \mu_2$ of cofinality ρ_1.*

Proof. Suppose that $\rho < \mu_2$ has cofinality ρ_1. By [5],

$$D(\mu_2, \rho) = \underline{D}(\mu_2, \rho) + \overline{D}(\mu_2, \rho).$$

μ_2 is a strong limit cardinal in $V^{P_\alpha * Q_{\alpha,2}}$, hence $\underline{D}(\mu_2, \rho) = \mu_2$.
Let us argue that $\overline{D}(\mu_2, \rho) = \mu_2^{\rho_1}$.
Consider the Magidor sequence $\langle \mu_{2i} \mid i < \rho_1 \rangle$. It is a club in μ_2. We have

$$D(\mu_{2i}, \xi) \leq 2^{\mu_{2i}} = \mu_{2i}^{+7} < \mu_2,$$

for every $i < \rho_1, \xi \leq \mu_{2i}$.
Claim 3.2.1. $\overline{D}(\mu_2, \rho) \leq \mu_2^{+7}$.
Proof. Let $\mathcal{P}(\mu_{2i}) = \langle Z_{i,\nu} \mid \nu < \mu_{2i}^{+7} \rangle$.

Set

$$E = \{X \in [\mu_2]^\rho \mid \exists h \in \prod_{i<\rho_1} \mu_{2i}^{+7} (X = \bigcup_{i<\rho_1} Z_{i,h(i)})\}.$$

Clearly, $|E| = 2^{\mu_2} = \mu_2^{+7}$ and E is dense in $\langle [\mu_2]^\rho, \subseteq \rangle$.
□ of the claim.

Claim 3.2.2. $\overline{D}(\mu_2, \rho) \geq \mu_2^{+7}$.
Proof. Suppose otherwise. Fix some D dense in $\langle [\mu_2]^\rho, \subseteq \rangle$ of cardinality less than μ_2^{+7}. Let $\langle h_j \mid j < \mu_1^{+7} \rangle$ be a scale in $\prod_{i<\rho_1} \mu_{2i}^{+7}$ (mod bounded), which exists by 2.1(5).
Define, for every $X \in D$, a function $\chi_X \in \prod_{i<\rho_1} \mu_{2i}^{+7}$ as follows:
$\chi_X(i) = \sup(X \cap \mu_{2i}^{+7})$, if $\rho < \mu_{2i}^{+7}$ and 0 otherwise.
There is $j^* < \mu_1^{+7}$ such that for every $j, j^* \leq j < \mu_1^{+7}$ and for every $X \in D$ we have $h_j(i) > \chi_X(i)$, for all but boundedly many i's. Without loss of generality we can assume that $h_{j^*}(i) \geq \mu_{2i}$, for every $i < \rho_1$
Recall that $\mathrm{cof}(\rho) = \rho_1$. Fix a witnessing cofinal sequence $\langle \rho(i) \mid i < \rho_1 \rangle$.
Define a set Y to be the union of disjoint intervals $[h_{j^*}(i), h_{j^*}(i) + \rho(i)], i < \rho_1$. Then $Y \in [\mu_2]^\rho$, but there is no $X \in D$ which is a subset of Y. Thus, if $X \subseteq Y, |X| = \rho$, then $X \cap [h_{j^*}(i), h_{j^*}(i) + \rho(i)] \neq \emptyset$ for ρ_1 many i's, but once $X \cap [h_{j^*}(i), h_{j^*}(i) + \rho(i)] \neq \emptyset$, we must to have $\chi_X(i) \geq h_{j^*}(i)$. Which is possible to have only for less than ρ_1-many i's. Contradiction.
$\square$ of the claim.
$\square$

Not that actually, by Claim 3.2.2 above, $\overline{D}(\mu_2, \rho) \geq \mu_2^{+7}$ whenever $\langle h_j \mid j < \mu_1^{+7} \rangle$ is a scale in $\prod_{i<\rho_1} \mu_{2i}^{+7}$ (mod bounded).
Hence, $\mu_1^{++}-$c.c. of $Q_{\alpha,2}$ implies the following:

Lemma 3.3 *In $V^{P_\alpha * Q_{\alpha,2} * Q_{\alpha,1}}$, $\overline{D}(\mu_2, \rho) \geq \mu_2^{+7}$,*
for every $\rho < \mu_2$ of cofinality ρ_1.

The following lemma is completely analogues to 3.2

Lemma 3.4 *In $V^{P_\alpha * Q_{\alpha,2} * Q_{\alpha,1}}$, we have $D(\mu_1, \rho) = \mu_1^{\rho_2} = \mu_2^{+17}$,*
for every $\rho < \mu_1$ of cofinality ρ_2.

Return to the main theorem 2.9. We can add now an additional property that $V_\eta[G]$ satisfies:

For every finite set of cardinals F (not necessary regular) there are arbitrary large cardinals $\mu \neq \min(\{D(\mu, \theta) \mid \theta \in F\})$.

Just given finite set of cardinals $F = \{\theta_1, ..., \theta_m\}$ below η. Consider the finite set of regular cardinals $F' := \{\mathrm{cof}(\theta_1), ..., \mathrm{cof}(\theta_1)\}$. Let $F' = F_\nu$, for some $\nu < \eta$. Now pick some $\alpha < \eta$, such that

1. $|P_\alpha| > \max(F)$,

2. $f(\alpha) = (\nu, \xi_\alpha)$, for some $\xi_\alpha < \eta$.

Then all members of the finite sequence of strongs used in the definition of Q_α will be above $\max(F)$. Let μ be the largest strong used there. By the construction (namely 2.9(d)) and 3.4, we will have $\mu \neq \min(\{D(\mu, \theta) \mid \theta \in F\})$.

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