On density of old sets in Prikry type extensions.

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Abstract

Every set of ordinals of cardinality κ in a Prikry extension with a measure over κ contains an old set of arbitrary large cardinality below κ, and, actually, it can be split into countably many old sets. What about sets bigger cardinalities? Clearly, any set of ordinals in a forcing extension of a regular cardinality above the cardinality of the forcing used, contains an old set of the same cardinality. Still cardinals in the interval $(\kappa, 2^\kappa]$ remain. Here we would like to address this type of questions.

1 A situation under $2^\kappa = \kappa^+$.

Let us start with the following observation.

Proposition 1.1 *Suppose that $2^\kappa = \kappa^+$. Let U be a normal ultrafilter over κ and $\mathcal{P}_U$ be the Prikry forcing with U. Then in $V^{\mathcal{P}_U}$ there is a subset of κ^+ without old subsets of the same size.*

Proof. Work in V. Pick a generating sequence $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ for U such that for every $\alpha \leq \beta < \kappa^+, A_\beta \subseteq^* A_\alpha$ [1]. It is possible since $2^\kappa = \kappa^+$ and U is normal.
Define an other generating sequence $\langle A'_\alpha \mid \alpha < \kappa^+\rangle$ as follows.
Let $C \subseteq \kappa^+$ be a club such that $|[\alpha_\nu, \alpha_{\nu+1}]| = \kappa$, for every $\nu < \kappa^+$, where $\{\alpha_\nu \mid \nu < \kappa^+\}$ is an increasing enumeration of C.
We set $A'_{\alpha_\nu} = A_\nu$.
Pick a surjective map $h : \kappa \to [\kappa]^{<\omega}$. Let, for every $\nu < \kappa^+$, $g_\nu : (\alpha_\nu, \alpha_{\nu+1}) \longleftrightarrow \kappa$. Now if $\beta \in (\alpha_\nu, \alpha_{\nu+1})$, for some $\nu < \kappa^+$, then set $A'_\beta = A'_{\alpha_\nu} \cup h(g_\nu(\beta))$.
Clearly, $\langle A'_\alpha \mid \alpha < \kappa^+\rangle$ is a generating sequence for U and for every $\alpha \leq \beta < \kappa^+, A'_\beta \subseteq^* A'_\alpha$.

Now let $G \subseteq \mathcal{P}_U$ be generic and $\{\kappa_n \mid n < \omega\}$ be the corresponding Prikry sequence. Set

$$X = \{\alpha < \kappa^+ \mid A'_\alpha \supseteq \{\kappa_n \mid n < \omega\}\}.$$

[1] $A \subseteq^* B$ means that $|A \setminus B| < \kappa$

Lemma 1.2 $|X| = \kappa^+$.

Proof. Work in V. Let $\langle t, A\rangle \in \mathcal{P}_U$ and $\delta < \kappa^+$. Pick $\alpha_\nu \in C \setminus \delta + 1$ such that $A \supseteq^* A'_{\alpha_\nu}$. Find $\beta \in (\alpha_\nu, \alpha_{\nu+1})$ such that $A'_\beta = A'_{\alpha_\nu} \cup t$. Then

$$\langle t, A \cap A'_{\alpha_\nu}\rangle \Vdash \beta \in \underset{\sim}{X}.$$

So, we are done, using a density argument.
$\square$ of the lemma.

Suppose now that there is $X^* \subseteq X$,$|X^*| = \kappa^+$ and $X^* \in V$. Set

$$A = \bigcap_{\alpha \in X^*} A'_\alpha.$$

Then, clearly, $A \in V$ and $A \supseteq \{\kappa_n \mid n < \omega\}$. But then $A \in U$. Hence, there is $\alpha < \kappa^+$, $A'_\alpha \subseteq^* A$. Now, for every $\beta \in X, \beta \geq \alpha$, we have

$$A'_\beta \subseteq^* A'_\alpha \subseteq^* A.$$

But each such $A'_\beta \supseteq A$. Hence, for every $\beta \in X, \beta \geq \alpha$, we have

$$A'_\beta =^* A.$$

But this is impossible. Thus just split A in V into two disjoint sets each of cardinality κ. One of them should be in U, and so must almost contain on of such A'_β's. Contradiction.
$\square$

So there are sets of ordinals of cardinality κ^+ without old subsets of cardinality κ^+. But what about subsets of size κ? The next proposition provides an answer.

Proposition 1.3 *Every set of ordinals of size $> \kappa$ contains an old subset of size κ.*

Proof. Suppose otherwise. Pick $\langle t, A\rangle \in G$ and a name $\underset{\sim}{a}$ such that

$$\langle t, A\rangle \Vdash (|a| = \kappa^+ \text{ and } \underset{\sim}{a} \text{ does not contain an old subset of size } \kappa).$$

Let $\underset{\sim}{a} = \{\underset{\sim}{\alpha}_\nu \mid \nu < \kappa^+\}$. For every $\nu < \kappa^+$, pick $\langle t_\nu, A_\nu\rangle \in G, \langle t_\nu, A_\nu\rangle \geq \langle t, A\rangle$ such that

$$\langle t_\nu, A_\nu\rangle \parallel \underset{\sim}{\alpha}_\nu.$$

Find $S \subseteq \kappa^+, |S| = \kappa^+$ and t^* such that for every $\nu \in S$ we have $t_\nu = t^*$. Clearly, in order to derive a contradiction, it is enough to find $A^* \in U, A^* \subseteq A$ such that the condition

$\langle t^*, A^* \rangle$ decides simultaneously $\underset{\sim}{\alpha}_\nu$'s for $\kappa-$many ν's.
Work in V. Set

$$Z = \{\nu < \kappa^+ \mid \exists B \in U (B \subseteq A \wedge \langle t^*, B \rangle \parallel \underset{\sim}{\alpha}_\nu)\}.$$

Then $|Z| = \kappa^+$, since $Z \supseteq S$. Let us chose for every $\nu \in Z$ a set $B_\nu \in U, B_\nu \subseteq A$ such that $\langle t^*, B \rangle \parallel \underset{\sim}{\alpha}_\nu$.
Now let us use the following pretty observation of F. Galvin (see [1], but for a reader convenience let state the proof here):

Proposition 1.4 *(Galvin) Suppose that $2^{<\lambda} = \lambda$, I is a normal ideal on λ and $\{B_\nu \mid \nu < \lambda^+\} \subseteq I$. Then there is $X \subseteq \lambda^+, |X| = \lambda$ such that $\bigcup_{\nu \in X} B_\nu \in I$.*

Proof. Set

$$H_{\alpha\xi} = \{\beta < \lambda^+ \mid B_\alpha \cap \xi = B_\beta \cap \xi\},$$

for every $\alpha < \lambda^+$ and $\xi < \lambda$.

Lemma 1.5 *There is $\alpha < \lambda^+$ such that for every $\xi < \lambda$ we have $|H_{\alpha\xi}| = \lambda^+$.*

Proof. Suppose otherwise. Then for every $\alpha < \lambda^+$ there is $\xi_\alpha < \lambda$ such that $|H_{\alpha\xi}| \leq \lambda$. But for every $\xi < \lambda$ we have $2^\xi \leq \lambda$, so there are at most $\lambda-$possibilities for $B_\beta \cap \xi$'s. Hence,

$$|\bigcup_{\alpha < \lambda^+} H_{\alpha\xi_\alpha}| \leq \lambda.$$

But, clearly, $\alpha \in H_{\alpha\xi_\alpha}$, for every $\alpha < \lambda^+$. Contradiction.
$\square$ of the lemma.

Pick α such that for every $\xi < \lambda$,$|H_{\alpha\xi}| = \lambda^+$. Define a sequence $\langle \eta_\xi \mid \xi < \lambda \rangle$ by induction as follows.

$$\eta_\xi \in H_{\alpha\xi+1} \setminus \{\eta_{\xi'} \mid \xi' < \xi\}.$$

Set

$$B = \bigcup_{\xi < \lambda} B_{\eta_\xi}.$$

Then $B \in I$, since

$$B \setminus B_\alpha \subseteq \bigcup_{\xi < \lambda} (B_{\eta_\xi} \setminus \xi + 1),$$

and $\bigcup_{\xi<\lambda}(B_{\eta_\xi} \setminus \xi + 1) \in I$ due to normality of I.
$\square$

So, there will be $X \subseteq Z$ of cardinality κ and $B \in U$ such that

$$B \subseteq \bigcap_{\nu \in X} B_\nu.$$

Then $\langle t^*, B\rangle \geq^* \langle t^*, B_\nu\rangle$, for every $\nu \in X$, and so, $\langle t^*, B\rangle$ decides $\kappa-$many of $\underset{\sim}{\alpha}_\nu$'s.
□

2 A situation without $2^\kappa = \kappa^+$.

Let us show that the assumption $2^\kappa = \kappa^+$ cannot be dropped from 1.1.

Start with the following general result.

Proposition 2.1 *Suppose that U is a normal ultrafilter over κ which has a generating family $\langle A_\nu \mid \nu < \chi\rangle$ such that $A_\beta \subseteq^* A_\alpha$, for every $\alpha < \beta < \chi$. Let E be a set of ordinals in $V^{\mathcal{P}_U}$. Then*

1. *if $|E| = \kappa$, then for every $\eta < |E|$, there is an old subset of E of cardinality η.*
2. *there is subset of κ of cardinality κ without an old subset of cardinality κ;*
3. *if $\omega < |E| < \kappa$, then for every $\eta \leq |E|$, there is an old subset of E of cardinality η;*
4. *there is a subset of κ of cardinality ω without infinite old subsets;*
5. *if $|E| > \mathrm{cof}(\chi)$, then for every $\eta \leq |E|$, there is an old subset of E of cardinality η;*
6. *there is a subset of χ of cardinality $\mathrm{cof}(\chi)$ without old subsets of the same cardinality;*
7. *for every cardinal $\mu < \chi$ with $(\mathrm{cof}(\mu))^V = \kappa$, there is a subset of μ without old subsets of the same cardinality;*
8. *if $\kappa < |E| \leq \chi$, then for every $\eta \leq |E|$, such that $|E| = \mathrm{cof}(\chi)$ or $(\mathrm{cof}(|E|))^V = \kappa$ imply $\eta < |E|$, there is an old subset of E of cardinality η.* [2]

Proof. The first four items are trivial. Item 5 is trivial as well, since $\mathcal{P}_U$ will have a dense subset of cardinality $\mathrm{cof}(\chi)$.

Item 6 follows from the argument of 1.1 only replacing κ^+ by $\mathrm{cof}(\chi)$.

Let us deal with Item 7.

[2]Remember that by König's theorem, $\mathrm{cof}(\chi) \geq \kappa^+$.

Suppose that $\kappa < |E| \le \chi$. Let $\eta \le |E|, \eta \ne \mathrm{cof}(\chi)$ and $\eta < |E|$, if $(\mathrm{cof}(|E|))^V = \kappa$. If $\eta = \kappa$, then the argument of 1.3 provides an old subset of E of cardinality κ. Assume that $\mathrm{cof}(\eta) > \kappa$.

Proceed as in 1.3. Let $\{\underset{\sim}{\alpha}_\xi \mid \xi < \eta\}$ be a set of η elements of E. Find $Z \subseteq \eta, |Z| = \eta$, $t^*, B_\nu \in U$, for each $\nu \in Z$ such that

$$\langle t^*, B_\nu \rangle \parallel \underset{\sim}{\alpha}_\nu.$$

For every $\nu \in Z$ there is β_ν such that $A_{\beta_\nu} \subseteq^* B_\nu$. Then there will be $\beta^* < \chi$ and $Z^* \subseteq Z, |Z^*| = \eta$, such that for every $\nu \in Z^*$ we have

$$A_{\beta^*} \subseteq^* A_{\beta_\nu} \subseteq^* B_\nu.$$

Then for every $\nu \in Z^*$ there is $\tau_\nu < \kappa$ such that

$$A_{\beta^*} \setminus \tau_\nu \subseteq B_\nu.$$

Pick $Z' \subseteq Z^*, |Z'| = \eta$ and $\tau' < \kappa$ such that for every $\nu \in Z'$, $\tau_\nu = \tau'$. It is possible since $\mathrm{cof}(\eta) > \kappa$. Now we will have

$$\langle t^*, B_\nu \setminus \tau' \rangle \le^* \langle t^*, A_{\beta^*} \setminus \tau' \rangle,$$

for every $\nu \in Z'$. Hence, the condition $\langle t^*, A_{\beta^*} \setminus \tau' \rangle$ decides simultaneously $\underset{\sim}{\alpha}_\nu$'s for $\eta-$many ν's. So we are done.
□

The existence of such generating families with $2^\kappa > \kappa^+$ follows from [4]:

Theorem 2.2 *Let κ be an almost huge cardinal with a measurable target λ.*[3] *Then for every cardinal $\chi, \kappa < \chi < \lambda, \mathrm{cof}(\chi) > \kappa$ there is a cofinalities preserving extension with a normal ultrafilter over κ which has a generating family $\langle A_\nu \mid \nu < \chi \rangle$ such that $A_\beta \subseteq^* A_\alpha$, for every $\alpha < \beta < \chi$.*

In particular, we can conclude the following:

Corollary 2.3 *It is consistent that in the Prikry forcing extension every set of ordinals of cardinality κ^+ contains an old subset of the same cardinality.*

Working a bit harder it is possible to show the following:

[3] I.e. there is $j : V \to M$ such that $\lambda = j(\kappa)$ is a measurable cardinal and ${}^{\lambda>}M \subseteq M$.

Proposition 2.4 *It is consistent that in the Prikry forcing extension every set of ordinals of cardinality κ^+ is a countable union of old sets.*

Proof. We start with the model with $2^\kappa = \kappa^{++}$ and a normal ultrafilter U over κ with a generating family $\langle A_\nu \mid \nu < \kappa^{++}\rangle$ such that $A_\beta \subseteq^* A_\alpha$, for every $\alpha < \beta < \kappa^{++}$.
Let $G \subseteq \mathcal{P}_U$ be generic and $X = \{\alpha_\nu \mid \nu < \kappa^+\}$ be a set of ordinals of cardinality κ^+ in $V[G]$.
For every $\nu < \kappa^+$, pick $\langle t_\nu, B_\nu\rangle \in G$ which decides $\underset{\sim}{\alpha}_\nu$.
Let $\langle \kappa_n \mid n < \omega\rangle$ be the Prikry sequence. For every $n < \omega$, set

$$X_n = \{\alpha_\nu \mid \exists B \in U(\langle\langle\kappa_0, ..., \kappa_n\rangle, B\rangle \in G \wedge \langle\langle\kappa_0, ..., \kappa_n\rangle, B\rangle \parallel \underset{\sim}{\alpha}_\nu)\}.$$

It is enough to show that each X_n can be split into $\omega-$old sets.
Fix $n < \omega$. Set $t = \langle\kappa_0, ..., \kappa_n\rangle$. So, for every $\alpha_\nu \in X_n$, we have $\langle t, B_\nu\rangle \in G$ which decides $\underset{\sim}{\alpha}_\nu$.
Find $\xi < \kappa^{++}$ large enough such that

- $A_\xi \subseteq^* B_\nu$, for every $\alpha_\nu \in X_n$,
- there is $m, n \leq m < \omega$, $\langle\langle\kappa_0, ..., \kappa_m\rangle, A_\xi \setminus \kappa_m + 1\rangle \in G$.

Now, for every $k < \omega$, set

$$X_{nk} = \{\alpha_\nu \mid A_\xi \setminus \kappa_k \subseteq B_\nu\}.$$

Note that

$$B_\nu \supseteq (A_\xi \setminus \kappa_k) \cup \{\kappa_{n+1}, ..., \kappa_{\max(k,m)}\},$$

for every $\alpha_\nu \in X_{nk}$.
Then

$$\langle t, B_\nu\rangle \leq^* \langle t, (A_\xi \setminus \kappa_k) \cup \{\kappa_{n+1}, ..., \kappa_{\max(k,m)}\}\rangle$$

and

$$\langle t, (A_\xi \setminus \kappa_k) \cup \{\kappa_{n+1}, ..., \kappa_{\max(k,m)}\}\rangle \in G,$$

for every $\alpha_\nu \in X_{nk}$.
It follows now that

$$X_{nk} = \{\rho \mid \exists\nu < \kappa^+(\langle t, (A_\xi \setminus \kappa_k) \cup \{\kappa_{n+1}, ..., \kappa_{\max(k,m)}\}\rangle \Vdash \underset{\sim}{\alpha}_\nu = \check{\rho})\}.$$

Clearly, the set on the right is in V, and, hence $X_{nk} \in V$ as well, for every $k < \omega$. But also clear that

$$X_n = \bigcup_{k<\omega} X_{nk}.$$

So we are done.

□

Let us point out that $2^\kappa = \kappa^{++}$ does not imply the conclusion of 2.3.

Proposition 2.5 *It is consistent with $2^\kappa = \kappa^{++}$ that in the Prikry forcing extension there is a set of ordinals of cardinality κ^+ without old subsets of the same cardinality.*

Proof. Use the construction of [2]. We start with $V = L[\vec{E}]$ model with $o(\kappa) = \kappa^{++}$. Let U_0 denotes the normal measure on κ which concentrates on nonmeasurable cardinals and which extends to normal measure U in the final model V^* of $2^\kappa = \kappa^{++}$. The extension is of the form $V[G_{<\kappa}, G_\kappa]$, where $V[G_{<\kappa}]$ is an extension of V by a forcing of size κ and G_κ is the forcing over κ which consists of adding $\kappa^{++}-$Cohen subsets and may be some additional things which does not add new subsets to κ.
Let U be a normal ultrafilter over κ in $V^* = V[G_{<\kappa}, G_\kappa]$.
Force with $\mathcal{P}_U$ and let $\langle \kappa_n \mid n < \omega \rangle$ be the Prikry sequence. Then $\langle \kappa_n \mid n < \omega \rangle$ be a Prikry sequence also over V for U_0.
Pick in V a generating family $\langle A_\alpha \mid \alpha < \kappa^+ \rangle$ for U_0.
Proceed exactly as in 1.1 and define $\langle A'_\beta \mid \beta < \kappa^+ \rangle$ and

$$X = \{\beta < \kappa^+ \mid A'_\beta \supseteq \{\kappa_n \mid n < \omega\}\}.$$

As in 1.1, then $|X| = \kappa^+$.
We claim that X does not contain old (i.e. those in V^*) subsets of cardinality κ^+. Suppose otherwise. Let X^* witnesses this. Consider

$$A = \bigcap_{\beta \in X^*} A'_\beta.$$

Now, this A need not be in V, since X^* was picked in V^* and, so may not be in V. However

$$A \supseteq \{\kappa_n \mid n < \omega\},$$

and so $A \in U$. Consider

$$j_U : V^* \to M_U \simeq (V^*)^\kappa / U.$$

Then $j_U \upharpoonright V$ is an iterated ultrapower of V by its measures, but since U extends U_0, it follows that U_0 was applied first in this iteration process. In particular, it is not on the sequence of core model $\mathcal{K}_U$ of M_U and so not in M_U. But $A \in M_U$ and so be can define U_0 in M_U as follows:

$$\{Z \in \mathcal{P}(\kappa) \cap \mathcal{K}_U \mid A \subseteq^* Z\}.$$

Contradiction.
□

3 A remark on Extender based Prikry forcing.

Let E be an extender over κ of the length at least κ^+. Denote by $\mathcal{P}_E$ the Extender based forcing as defined in [3].[4]
Let $G \subseteq \mathcal{P}_E$ be a generic subset.

Proposition 3.1 *In $V[G]$ there is a subset of κ^+ without old subsets of cardinality κ.*

Proof. Let us denote by $b_\alpha : \omega \to \kappa$ the Prikry sequence of G for the $\alpha-$th measure E_α of E. Then, for each α, there is the least $n_\alpha < \omega$ such that for every $n, n_\alpha \leq n < \omega$,

$$\pi_{\alpha\kappa}(b_\alpha(n)) = b_\kappa(n),$$

where $\pi_{\alpha\kappa}$ denotes the canonical projection of E_α onto the normal measure E_κ of the extender.
There are $A^* \subseteq \kappa^+, |A^*| = \kappa^+$ and $n^* < \omega$ such that for every $\alpha \in A^*$, $n_\alpha = n^*$.
We claim that A^* does not contain old subsets of cardinality κ.
Suppose otherwise. Let B be such a subset.
Pick some $p = \langle p^\gamma \mid \gamma \in supp(p)\rangle^\frown\langle p^{mc}, T\rangle \in G$ forcing this and deciding B.
We can assume that each $\alpha \in B$ belongs to $supp(p)$, since otherwise we are completely free about b_α and can easily to make

$$\pi_{\alpha\kappa}(b_\alpha(n)) \neq b_\kappa(n),$$

for some $n \geq n^*$. Without loss of generality we can assume that for every $\alpha \in B$, $n^* \leq m^* := |p^\alpha|$ and $m^* = |p^{mc}|$ (and then $= p^0$). Pick now some $\nu \in Suc_T(p^{mc})$. Let ν^0, as usual be $\pi_{mc\kappa}(\nu)$. By the definition of the forcing, the set

$$\{\alpha \in supp(p) \mid \nu \text{ is permitted for } p^\alpha\}$$

[4]A very similar argument works for the Merimovich variations [5].

has cardinality $< \kappa$ (actually at most ν^0). Now, since $|B| = \kappa$, there is $\alpha \in B$ such that ν is not permitted for α. This means that in the extension p^ν of p by ν, p^α does not extend. But then,

$$p^\nu \Vdash \pi_{\alpha\kappa}(b_\alpha(m^*)) = b_\kappa(m^*).$$

Contradiction.

$\square$

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