Dropping cofinalities

Moti Gitik
School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Science
Tel Aviv University
Ramat Aviv 69978, Israel

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Abstract

Our aim is to present constructions in which some of the cofinalities drop down, i.e. the generators of PCF structure are far a part.

1 Some Preliminary Settings

Let $\lambda_0 < \kappa_0 < \lambda_1 < \kappa_1 < ... < \lambda_n < \kappa_n < ...., n < \omega$ be a sequence of cardinals such that for each $n < \omega$

- λ_n is $\lambda_n^{+\lambda_n^{+n+2}+2}$ - strong as witnessed by an extender E_{λ_n}
- κ_n is $\kappa_n^{+\kappa_n^{+n+2}+2}$ - strong as witnessed by an extender E_{κ_n}

Let $\kappa = \bigcup_{n<\omega} \kappa_n$. Fix some regular $\theta > \theta' \geq \kappa^+$.

Our aim will be to make $2^\kappa = \theta^+$, but so that each cofinality from the interval $[\kappa^{++}, \theta']$ is obtained using only indiscernibles related to λ_n's.

Let us force first with the preparation forcing $\mathcal{P}'$ of [6]. The assignment function of [6] is used here for models of cardinalities below θ' intersected with $H(\theta')$ but with range over λ_n's. We will use names of indiscernibles for λ_n's to define the assignment to κ_n's. Models of cardinalities in $[\kappa^+, \theta']$ will be assigned to those of cardinalities of this indiscernibles, so a way below κ_n's.

We deal first with the simplest case: $\theta = \kappa^{+3}$ and $\theta' = \kappa^+$. Such situation was considered in [3], but our approach here is different and generalizes to arbitrary θ, θ'.

2 Models and types

The main difference in present setting from those of [1], [4] and [6] will be due to the fact that the cardinalities of models in the range of a condition may be smaller than the number of existing types. So any such model may contain only a limited number of types. We would like to insure that it will be still sufficiently large.

Fix $n < \omega$. Set $\delta_n = \kappa_n^{+\kappa_n^{+n+2}+1}$. Denote by δ_n^- the immediate predecessor of δ_n, i.e. $\kappa_n^{+\kappa_n^{+n+2}}$. Fix using GCH an enumeration $\langle a_\alpha \mid \alpha < \kappa_n \rangle$ of $[\kappa_n]^{<\kappa_n}$ so that for every successor cardinal $\delta < \kappa_n$ the initial segment $\langle a_\alpha \mid \alpha < \delta \rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_n(\langle a_\alpha \mid \alpha < \kappa_n \rangle) = \langle a_\alpha \mid \alpha < j_n(\kappa_n) \rangle$ where j_n is the canonical embedding of the (κ_n, δ_n^+)-extender E_n. Then $\langle a_\alpha \mid \alpha < \delta_n^+ \rangle$ will enumerate $[\delta_n^+]^{\leq \delta_n}$ and we fix this enumeration. For each $k \leq \omega$ consider a structure

$$\mathfrak{A}_{n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, E_{\kappa_n}, E_{\lambda_n}, \lambda_n, \kappa_n, \delta_n, \delta_n^+,$$

$$\chi, \langle a_\alpha \mid \alpha < \delta_n^+ \rangle, 0, 1, \ldots, \alpha, \ldots \mid \alpha < \kappa_n^{+k} \rangle$$

in the appropriate language $\mathcal{L}_{n,k}$ with a large enough regular cardinal χ.

Remark 2.1 It is possible to use κ_n^{++} here (as well as in [1]) instead of κ_n^{+k}. The point is that there are only κ_n^{++} many ultrafilters over κ_n and we would like that equivalent conditions use the same ultrafilter. The only parameter that that need to vary is k in $H(\chi^{+k})$.

Let $\mathcal{L}'_{n,k}$ be the expansion of $\mathcal{L}_{n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less or equal than δ_n let $\mathfrak{A}_{n,k,a}$ be the expansion of $\mathfrak{A}_{n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less or equal than δ_n. Denote by $tp_{n,k}(b)$ the $\mathcal{L}_{n,k}$-type realized by b in $\mathfrak{A}_{n,k}$. Further we identify it with the ordinal coding it and refer to it as the k-type of b. Let $tp_{n,k}(a, b)$ be a the $\mathcal{L}'_{n,k}$-type realized by b in $\mathfrak{A}_{n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [1].

Fix a sequence $\langle \mathfrak{U}_\nu | \nu < \lambda_n \rangle$ such that

1. $\mathfrak{U}_\nu \prec \mathfrak{A}_{n,\omega}$

2. $|\mathfrak{U}_\nu| \leq |\nu|$, once $\nu \geq \omega$

3. $\mathfrak{U}_\nu \in \mathfrak{U}_{\nu+1}$

4. $\mathfrak{U}_\nu \subset \mathfrak{U}_{\nu+1}$

5. ${}^{|\mathfrak{U}_\nu|>}\mathfrak{U}_\nu \subseteq \mathfrak{U}_\nu$

6. if ν is a limit, then $\mathfrak{U}_\nu = \bigcup_{\nu'<\nu} \mathfrak{U}_{\nu'}$.

Note that for each $k < \omega$ the set $\{tp_{n,k}(b) \mid b \in H(\chi^{+k})\}$ is in $\mathfrak{U}_0$. Just this set is definable in $\mathfrak{A}_{n,\omega}$.

For each $k < \omega$ and $\mathfrak{U} \prec \mathfrak{A}_{n,\omega}$ let us denote $\mathfrak{U} \cap \mathfrak{A}_{n,k}$ by $\mathfrak{U} \restriction k$.

The next lemma is obvious.

Lemma 2.2 *Suppose that for some $k < \omega, \nu < \lambda_{0n}$, $\mathfrak{U}_\nu \restriction k \prec \mathfrak{B} \prec \mathfrak{A}_{n,k}$. Let $X \in H(\chi^+ k')$, for some $k' \leq \omega$ be so that $tp_{n,k'}(X) \in \mathfrak{U}_\nu$. Then there is $Y \in B$ such that $tp_{n,\min\{k,k'\}}(X) = tp_{n,\min\{k,k'\}}(Y)$.*

Further we shall use models in the range of a condition such the interpretation X according to to a given $\nu < \lambda_n$ is so that

$$\mathfrak{U}_\nu \subseteq X \in \mathfrak{U}_{\nu+1}$$

or at least there is Y like this realizing the same type as X.
Note that the above may result the loss of closure of the forcing. Thus union of even countably many conditions can produce a type which is not in $\bigcup_{\nu<\lambda_n} \mathfrak{U}_\nu$. In order to overcome this we can either require that all models of the range are from $\bigcup_{\nu<\lambda_n} \mathfrak{U}_\nu$ and satisfy

$$\mathfrak{U}_\nu \subseteq X \in \mathfrak{U}_{\nu+1}$$

once ν is decided, or we can replace $\leq^*$ by $\rightarrow$ also for the closure arguments. Then each time a condition supposed to be replaced by an equivalent one inside $\bigcup_{\nu<\lambda_n} \mathfrak{U}_\nu$.

3 $\theta = \kappa^{+3}$ and $\theta' = \kappa^{+}$

In the present situation the preparation forcing $\mathcal{P}'(\kappa^{++})$ produces only a closed chain of models of cardinality κ^{+}. They are submodels of $H(\kappa^{++})$ and have intersections with κ^{++} just an ordinal. We assign them to those between λ_n^{+n+1} and λ_n^{+n+2}, for each $n < \omega$. The forcing at this part will basically the same as those used in [1].

The forcing $\mathcal{P}'$ produces here models of cardinalities κ^{+} and κ^{++} only. They are submodels of $H(\kappa^{+3})$. Moreover intersections of such models of cardinality κ^{+} with κ^{++} will give ordinals below κ^{++} and the models of cardinality κ^{++} can be viewed as ordinals below κ^{+3}. The issue here is to arrange the correspondence to κ_n's. Thus κ^{+3} will correspond to κ_n^{+n+2}'s. Models of cardinality κ^{++} will be send to those of cardinality κ_n^{+n+1} which are basically ordinals below κ_n^{+n+2}. The delicate part will be to arrange images of models of cardinality κ^{+}. For those we will use names from the forcing over λ_n's. Thus the cardinality of corresponding models at a level n will be the indiscernible for λ_n^{+n+1}.

We may assume here that E_{λ_n} is a $(\lambda_n, \lambda_n^{+n+2})$-extender and E_{κ_n} is $(\kappa_n, \kappa_n^{+n+2})$-extender.

Let $G(\mathcal{P}'(\theta'))$ be a generic subset of $\mathcal{P}'(\theta')$ and $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$ over $V[G(\mathcal{P}'(\theta')]$. In the present case, i.e. $\theta' = \kappa^{++}$, the first forcing is just the forcing for adding a club subset to κ^{++} with conditions of cardinality κ^{+}. It is possible to proceed without it as well. Fix $n < \omega$.

Definition 3.1 Let Q_{n0} be the set consisting of pairs of triples $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle$ so that:

1. f is partial function from κ^{+2} to λ_n of cardinality at most κ
2. a is a partial function of cardinality less than λ_n so that
   (a) There is $\langle\langle A^{0\kappa^+}(\kappa^{++}), A^{1\kappa^+}(\kappa^{++}), C^{\kappa^+}(\kappa^{++})\rangle\rangle \in G(\mathcal{P}'(\kappa^{++}))$ which we call it further **a background condition of** a, such that $\mathrm{dom}(a)$ consists of models appearing in $A^{1\kappa^+}(\kappa^{++})$, i.e. basically of ordinals below κ^{++}.
   Note that the third component $C^{\kappa^+}(\kappa^{++})$ of a condition is just the same as the second $A^{1\kappa^+}$. Also the inclusion is a linear order on $A^{1\kappa^+}(\kappa^{++})$ and this set is closed under unions.
   (b) for each $X \in \mathrm{dom}(a)$ there is $k \leq \omega$ so that $a(X) \subseteq H(\chi^{+k})$.
   Moreover,
   (i) $|a(X)| = \lambda_n^{+n+1}$ and $a(X) \cap \lambda_n^{+n+2} \in ORD$
   (iii) $A^{0\kappa^+}(\kappa^{++}) \in \mathrm{dom}(a)$.

This way we arranged that λ_n^{+n+1} will correspond to κ^+ and λ_n^{+n+2} will correspond to κ^{++}.

Further let us refer to $A^{0\kappa^+}(\kappa^{++})$ as **the maximal model of the domain of** a. Denote it as $\max(\text{dom}(a))$.

Later passing from Q_{0n} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

(c) (Models come from $A^{0\kappa^+}(\kappa^{++})$) If $X \in \text{dom}(a)$ and $X \neq A^{0\kappa^+}(\kappa^{++})$ then $X \in A^{0\kappa^+}(\kappa^{++})$.

The condition puts restriction on models in $\text{dom}(a)$ and allows to control them via the maximal model of cardinality κ^+.

(d) If $X, Y \in \text{dom}(a)$, $X \in Y$ (or $X \subseteq Y$) and k is the minimal so that $a(X) \subseteq H(\chi^{+k})$ or $a(Y) \subseteq H(\chi^{+k})$, then $a(X) \cap H(\chi^{+k}) \in a(Y) \cap H(\chi^{+k})$ (or $a(X) \cap H(\chi^{+k}) \subseteq a(Y) \cap H(\chi^{+k})$).

The intuitive meaning is that b is supposed to preserve membership and inclusion. But we cannot literally require this since $a(A)$ and $a(B)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(e) The image by a of $A^{0\kappa^+}$, i.e. $a(A^{0\kappa^+})$, intersected with λ_n^{+n+2} is above all the rest of $\text{rng}(a)$ restricted to λ_n^{+n+2} in the ordering of the extender E_n (via some reasonable coding by ordinals).

Recall that the extender E_{λ_n} acts on λ_n^{+n+2} and our main interest is in Prikry sequences it will produce. So, parts of $\text{rng}(a)$ restricted to δ_n^{+n+2} will play the central role.

3. $\{\alpha < \kappa^{+3} \mid \alpha \in \text{dom}(a)\} \cap \text{dom}(f) = \emptyset$

4. $A \in E_{\lambda_n, a(\max(a))}$

5. $\min(A) > |dom(a)| + |\,\text{dom}(\underset{\sim}{b})|$

6. for every ordinals α, β, γ which are elements of $\text{rng}(a)$ or actually the ordinals coding

models in $\mathrm{rng}(a)$ we have

$$\alpha \geq_{E_{\lambda_n}} \beta \geq_{E_{\lambda_n}} \gamma \quad \text{implies}$$
$$\pi_{\lambda_n,\alpha,\gamma}(\rho) = \pi_{\lambda_n,\beta,\gamma}(\pi_{\lambda_n,\alpha,\beta}(\rho))$$

for every $\rho \in \pi''_{\lambda_n,\max \mathrm{rng}(a),\alpha}(A)$.

Let us turn now to the second component of a condition, i. e. to $\langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle$.

7. g is a function from κ^{+3} to κ_n of cardinality at most κ

8. $\underset{\sim}{b}$ is a name, depending on $\langle a, A\rangle$, of a partial function of cardinality less than λ_n. So, each choice of an element from A gives the actual function which is in V. Note that the relevant forcing is the One Element Prikry Forcing on Extender, which does not change V, i.e. it is trivial.

   The following conditions are satisfied:

   (a) (Domain)

   the domain of $\underset{\sim}{b} \in V$, i.e. it is already decided in the sense that each choice of an element in A will give the same domain.

   (b) (Background condition) There is $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, \langle A^{0\kappa^{++}}, A^{1\kappa^{++}}, C^{\kappa^{++}}\rangle\rangle \in G(\mathcal{P}'(\kappa^{+3}))$ which we call it further **a background condition of** $\underset{\sim}{b}$, such that $\mathrm{dom}(\underset{\sim}{b})$ consists of models appearing in $A^{1\kappa^{++}}$, i.e. basically of ordinals below κ^{++} and those of $A^{1\kappa^+}$.

   Note that for κ^{++} the third component $C^{\kappa^{++}}$ of a condition is just the same as the second $A^{1\kappa^{++}}$. Also the inclusion is a linear order on $A^{1\kappa^{++}}$ and this set is closed under unions.

   (c) for each $X \in \mathrm{dom}(\underset{\sim}{b})$ and each $\nu \in A$ there is $k \leq \omega$ so that the interpretation according to ν of $\underset{\sim}{b}(X)$ is a subset of $H(\chi^{+k})$.

   Moreover,

   i. if $|X| = \kappa^{++}$, then it is forced that $|\underset{\sim}{b}(X)| = \kappa_n^{+n+1}$ and $\underset{\sim}{b}(X) \cap \kappa_n^{+n+2} \in ORD$, i.e. any choice of an element from A interprets $\underset{\sim}{b}(X)$ in such a way.

   ii. if $|X| = \kappa^+$, then for each $\nu \in A$ the interpretation of $\underset{\sim}{b}(X)$ according to ν has cardinality $(\nu^0)^{+n+1}$, where ν^0 denotes the projection of ν to the normal measure of the extender E_{λ_n}.

iii. $A^{0\kappa^+}, A^{0\kappa^{++}} \in \mathrm{dom}(a)$.

Further let us refer to $A^{0\kappa^+}$ as **the maximal model of the domain of** $\underset{\sim}{b}$. Denote it as $\max(\mathrm{dom}(\underset{\sim}{b}))$.

Later passing from $\underset{\sim}{Q}_{n0}$ to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

(d) (Models come from $A^{0\kappa^+}$) If $X \in \mathrm{dom}(\underset{\sim}{b})$ and $X \neq A^{0\kappa^+}$, then $X \in A^{0\kappa^+}$.

(e) Let $E, F \in \mathrm{dom}(\underset{\sim}{b})$, $E \in F$ (or $E \subseteq F$) and $\nu \in A$. If k is the minimal so that the interpretation of $\underset{\sim}{b}(E)$ according to ν is a subset of $H(\chi^{+k})$ or $\underset{\sim}{b}(F)$ according to ν is a subset of $H(\chi^{+k})$, then

$$\underset{\sim}{b}(E)[\nu] \cap H(\chi^{+k}) \in \underset{\sim}{b}(F)[\nu] \cap H(\chi^{+k})$$

$$(\text{or } \underset{\sim}{b}(E)[\nu] \cap H(\chi^{+k}) \subseteq \underset{\sim}{b}(F)[\nu] \cap H(\chi^{+k})),$$

where in the last two lines we mean the interpretations according to ν. Let us further deal with such interpretations without mentioning this explicitly.

The intuitive meaning is that b is supposed to preserve membership and inclusion. But we cannot literally require this since $b(E)$ and $b(F)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(f) The image by b of $A^{0\kappa^+}$, i.e. $b(A^{0\kappa^+})$, intersected with κ_n^{+n+2} is above all (i.e. is forced by each $\nu \in A$ to be such) the rest of $\mathrm{rng}(b)$ restricted to κ_n^{+n+2} in the ordering of the extender E_{κ_n} (via some reasonable coding by ordinals).

Recall that the extender E_{κ_n} acts on κ_n^{+n+2} and our main interest is in Prikry sequences it will produce. So, parts of $\mathrm{rng}(b)$ restricted to κ_n^{+n+2} will play the central role.

Let us, as in [6], denote by $otp_{\kappa^+}(X)$ the order type of the maximal under inclusion chain of elements in $\mathcal{P}(X) \cap A^{1\kappa^+}$ which is just the order type of $C^{\kappa^+}(X)$, for $X \in A^{1\kappa^+}$. If $X \in C^{\kappa^+}(A^{0\kappa^+})$, then $C^{\kappa^+}(X) = C^{\kappa^+}(A^{0\kappa^+}) \cap (X \cup \{X\}) = C^{\kappa^+}(A^{0\kappa^+}) \restriction X+1$. Hence, in this case, $otp_{\kappa^+}(X) = otp(C^{\kappa^+}(A^{0\kappa^+}) \restriction X)+1$. Note that $otp_{\kappa^+}(X)$ is always a successor ordinal below κ^{++}. Recall that by [6] we have for each $X \in A^{1\kappa^+}$ an element $Y \in C^{\kappa^+}(A^{0\kappa^+})$ such that $otp_{\kappa^+}(X) = otp_{\kappa^+}(Y)$.

Next conditions deal with the connection between the structure over λ_n and those over κ_n. Note that there were no similar structures in the previous papers [4], [6].

(g) (Order types) If $X \in \mathrm{dom}(\underset{\sim}{b})$, then $A^{0\kappa^+}(\kappa^{++}) \cap \kappa^{++} \geq otp_{\kappa^+}(X)$.

Denote by $X(\lambda_n)$ the last element Z of $A^{1\kappa^+}(\kappa^{++})$ with $Z \cap \kappa^{++} < otp_{\kappa^+}(X)$. It will be the one corresponding to X at the level λ_n. Notice that the domain of a need not be an ordinal but rather a closed set of ordinals of cardinality less than λ_n. Hence, $otp_{\kappa^+}(X)$ itself or $otp_{\kappa^+}(X) - 1$ need not be in the domain of a. So, $X(\lambda_n)$ looks like a natural choice.

The next condition insures that the function $otp_{\kappa^+}(X) \to X(\lambda_n)$ is order preserving.

(h) (Order preservation) If $X, X' \in \mathrm{dom}(\underset{\sim}{b})$, then

- $otp_{\kappa^+}(X) = otp_{\kappa^+}(X')$ iff $X(\lambda_n) = X'(\lambda_n)$
- $otp_{\kappa^+}(X) < otp_{\kappa^+}(X')$ iff $X(\lambda_n) < X'(\lambda_n)$

(i) (Dependence) Let $X \in \mathrm{dom}(\underset{\sim}{b}) \cap C^{\kappa^+}(A^{0\kappa^+})$. Then $\underset{\sim}{b}(X)$ depends on the value of the one element Prikry forcing with the measure $a(X(\lambda_n))$ over λ_n. More precisely: let $A(X) = \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}{}''A$, then each choice of an element from $A(X)$ already decides $\underset{\sim}{b}(X)$, i.e. whenever $\nu_1, \nu_2 \in A$ and

$$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu_1) = \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu_2)$$

we have

$$\underset{\sim}{b}(X)[\nu_1] = \underset{\sim}{b}(X)[\nu_2].$$

Further let us denote, for $\nu \in A$, the projection of ν to $A(X)$, i.e. $\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu)$, by $\nu(X)$.

So $\underset{\sim}{b}(X)$ depends only on members of $A(X)$ rather than those of A.

The next condition is crucial for the κ^{++}-c.c. of the forcing.

(j) (Inclusion condition)

Let $\nu, \nu' \in A, \nu < \nu'$. Then

- $$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu') > \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu)$$
  implies
  $$\underset{\sim}{b}(A^{0\kappa^+})[\nu] \in \underset{\sim}{b}(A^{0\kappa^+})[\nu'].$$

This condition means that once $A^{0\kappa^+}(\lambda_n)$ -the set corresponding to $A^{0\kappa^+}$ at the level λ_n, is mapped by a according to ν' to a bigger set than those according to ν, then the same is true with corresponding models at the level κ_n.

- If $Y \in \mathrm{dom}(\underset{\sim}{b}) \cap C^{\kappa^+}(A^{0\kappa^+})$ and

$$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(Y(\lambda_n))}(\nu') > \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu),$$

then either

$$\underset{\sim}{b}(A^{0\kappa^+})[\nu] \in \underset{\sim}{b}(Y)[\nu']$$

or
the k-type realized by $\underset{\sim}{b}(A^{0\kappa^+})[\nu] \cap H(\chi^{+k})$ is in $\underset{\sim}{b}(Y)[\nu']$, where $k < \omega$ is the least such that $\underset{\sim}{b}(Y)[\nu'] \subseteq H(\chi^{+k+1})$.
The same holds over any element of $\underset{\sim}{b}(Y)[\nu']$, i.e. $tp_k(z, \underset{\sim}{b}(A^{0\kappa^+})[\nu] \cap H(\chi^{+k})) \in \underset{\sim}{b}(Y)[\nu']$, for any $z \in \underset{\sim}{b}(Y)[\nu']$.
We require in addition that this $k > 2$.
Let us allow the above also if $\underset{\sim}{b}(Y)[\nu'] \subseteq H(\chi^{+\omega})$. In this case we take k to be any natural number above 2 and require that once we go up to the higher levels then corresponding k's increase (with n).

We cannot in general require only that

$$\underset{\sim}{b}(A^{0\kappa^+})[\nu] \in \underset{\sim}{b}(Y)[\nu']$$

since extending conditions the sequence C^{κ^+} of the maximal model of a new background condition may go not through the old maximal model. But still having the type inside Y will be enough for our purposes.

It is possible to have $Y \subset X$, but $\nu(X)$ smaller than $\nu'(Y)$ (note that $\nu(Y) < \nu(X)$ in this case by 8h). In such situation the interpretation will reverse the order.
Note that given $\nu' \in A$ the number of possibilities for $\nu \in \nu' \cap A$ is bounded by $(\nu'^0)^{+n+1}$, as $\nu' < (\nu'^0)^{+n+2}$.

(k) If $X \in \mathrm{dom}(\underset{\sim}{b})$ then $C^{|X|}(X) \cap \mathrm{dom}(\underset{\sim}{b})$ is a closed chain. Let $\langle X_i | i < j\rangle$ be its increasing continuous enumeration. For each $l < j$ consider the final segment $\langle X_i | l \leq i < j\rangle$ and its image $\langle \underset{\sim}{b}(X_i) | l \leq i < j\rangle$. Find the minimal k so that

$$\underset{\sim}{b}(X_i) \subseteq H(\chi^{+k}) \text{ for each } i, l \leq i < j.$$

Then the sequence

$$\langle \underset{\sim}{b}(X_i) \cap H(\chi^{+k}) | l \leq i < j \rangle$$

is increasing and continuous. More precisely, each $\nu \in A$ forces this.

Note that k here may depend on l, i.e. on the final segment.

(l) (The walk is in the domain) If $X \in \mathrm{dom}(\underset{\sim}{b}) \cap A^{1\xi}$, for some $\xi \in \{\kappa^+, \kappa^{++}\}$, then the general walk from $(A^{0\xi})^-$ to X is forced by each $\nu \in A$ to be in $\mathrm{dom}(\underset{\sim}{b})$.

(m) If $X \in \mathrm{dom}(\underset{\sim}{b}) \cap A^{1\xi}$, for some $\xi \in \{\kappa^+, \kappa^{++}\}$ is a limit model and $\mathrm{cof}(otp_\xi(X) - 1) < \kappa_n$ (i.e. the cofinality of the sequence $C^\xi(X) \backslash \{X\}$ under the inclusion relation is less than κ_n) then a closed cofinal subsequence of $C^\xi(X) \backslash \{X\}$ is in $\mathrm{dom}(\underset{\sim}{b})$. The images of its members under b form a closed cofinal in $b(X)$ sequence.

(n) (Minimal cover condition) Let $E \in A^{0\kappa^+} \cap \mathrm{dom}(\underset{\sim}{b}), X \in A^{0\kappa^{++}} \cap \mathrm{dom}(\underset{\sim}{b})$. Suppose that $E \not\subseteq X$. Then the smallest model of $E \cap C^{\kappa^+}(A^{0\kappa^+})$ including X is in $\mathrm{dom}(\underset{\sim}{b})$

(o) (The first models condition) Suppose that $E \in \mathrm{dom}(\underset{\sim}{b}) \cap C^{\kappa^+}(A^{0\kappa^+}), F \in \mathrm{dom}(\underset{\sim}{b}) \cap C^{\kappa^{++}}(A^{0\kappa^{++}})$, $\sup(E) > \sup(F)$ and $F \notin E$. Then the first model $H \in A \cap C^{\kappa^{++}}(A^{0\kappa^{++}})$ which includes B is in $\mathrm{dom}(\underset{\sim}{b})$.

(p) (Models witnessing Δ-system type are in the domain) If $F_0, F_1, F \in A^{1\kappa^+} \cap \mathrm{dom}(\underset{\sim}{b})$ is a triple of a Δ - system type, then the corresponding models $G_0, G_0^*, G_1, G_1^*, G^*$, as in the definition of a Δ - system type (see [6]), are in $\mathrm{dom}(\underset{\sim}{b})$ as well and

$$\underset{\sim}{b}(F_0) \cap \underset{\sim}{b}(F_1) = \underset{\sim}{b}(F_0) \cap \underset{\sim}{b}(G_0) = \underset{\sim}{b}(F_1) \cap \underset{\sim}{b}(G_1).$$

(q) If $F_0, F_1, F \in A^{1\kappa^+}$ is a triple of a Δ - system type and $F, F_0 \in \mathrm{dom}(\underset{\sim}{b})$ (or $F, F_1 \in \mathrm{dom}(\underset{\sim}{b})$), then $F_1 \in \mathrm{dom}(\underset{\sim}{b})$ (or $F_0 \in \mathrm{dom}(\underset{\sim}{b})$).

(r) (The isomorphism condition) Let $F_0, F_1, F \in A^{1\kappa^+} \cap \mathrm{dom}(\underset{\sim}{b})$ be a triple of a Δ - system type. Then

$$\langle \underset{\sim}{b}(F_0) \cap H(\chi^{+k}), \in \rangle \simeq \langle \underset{\sim}{b}(F_1) \cap H(\chi^{+k}), \in \rangle$$

where k is the minimal so that $\underset{\sim}{b}(F_0) \subseteq H(\chi^{+k})$ or $\underset{\sim}{b}(F_1) \subseteq H(\chi^{+k})$.
Note that it is possible to have for example $\underset{\sim}{b}(F_0) \prec H(\chi^{+6})$ and $\underset{\sim}{b}(F_1) \prec H(\chi^{+18})$. Then we take $k = 6$.
Let π be the isomorphism between

$$\langle \underset{\sim}{b}(F_0) \cap H(\chi^{+k}), \in \rangle, \langle \underset{\sim}{b}(F_1) \cap H(\chi^{+k}), \in \rangle$$

and $\pi_{F_0 F_1}$ be the isomorphism between F_0 and F_1. Require that for each $Z \in F_0 \cap \mathrm{dom}(\underset{\sim}{b})$ we have $\pi_{F_0 F_1}(Z) \in F_1 \cap \mathrm{dom}(\underset{\sim}{b})$ and

$$\pi(\underset{\sim}{b}(Z) \cap H(\chi^{+k})) = \underset{\sim}{b}(\pi_{F_0 F_1}(Z)) \cap H(\chi^{+k}).$$

(s) $\{\alpha < \kappa^{+3} \mid \alpha \in \mathrm{dom}(\underset{\sim}{b})\} \cap \mathrm{dom}(g) = \emptyset$.

(t) For each $\nu \in A$ we have $\underset{\sim}{B}[\nu] \in E_{\kappa_n, \underset{\sim}{b}[\nu](\max(\underset{\sim}{b}))}$.

(u) for every $\nu \in A$ and every ordinals α, β, γ which are elements of $\mathrm{rng}(\underset{\sim}{b})[\nu]$ or actually the ordinals coding models in $\mathrm{rng}(\underset{\sim}{b})[\nu]$ we have

$$\begin{aligned} &\alpha \geq_{E_{\kappa_n}} \beta \geq_{E_{\kappa_n}} \gamma \quad \text{implies} \\ &\pi_{\kappa_n,\alpha,\gamma}(\rho) = \pi_{\kappa_n,\beta,\gamma}(\pi_{\kappa_n,\alpha,\beta}(\rho)) \end{aligned}$$

for every $\rho \in \pi''_{\kappa_n, \max \mathrm{rng}(\underset{\sim}{b}[\nu]), \alpha}(\underset{\sim}{B}[\nu])$.

We define now Q_{n1} and $\langle Q_n, \leq_n, \leq_n^* \rangle$ similar to [2, Sec.2].

Definition 3.2 Suppose that $\langle \langle a, A, f \rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g \rangle \rangle$ and $\langle \langle a', A', f' \rangle, \langle \underset{\sim}{b}', \underset{\sim}{B}', g' \rangle \rangle$ are two elements of Q_{n0}. Define

$$\langle \langle a, A, f \rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g \rangle \rangle \geq_{Q_{n0}} \langle \langle a', A', f' \rangle, \langle \underset{\sim}{b}', \underset{\sim}{B}', g' \rangle \rangle$$

iff

1. $f \supseteq f'$
2. $g \supseteq g'$
3. $a \supseteq a'$
4. $\pi''_{\lambda_n, \max(a), \max(a')} A \subseteq A'$
5. for every $\nu \in A$ we have

$$\underset{\sim}{b}[\nu] \supseteq \underset{\sim}{b}'[\pi_{\lambda_n, \max(a), \max(a')}(\nu)].$$

This means just that the empty condition of one element Prikry forcing forces the inclusion.

6. for every $\nu \in A$ we have

$$\pi''_{\kappa_n, \max(\underset{\sim}{b}[\nu]), \max(\underset{\sim}{b}'[\pi_{\lambda_n, \max(a), \max(a')}(\nu)])} \underset{\sim}{B}[\nu] \subseteq \underset{\sim}{B}'[\pi_{\lambda_n, \max(a), \max(a')}(\nu)]$$

Definition 3.3 Q_{n1} consists of pairs $\langle f, g \rangle$ such that

1. f is a partial function from κ^{++} to λ_n of cardinality at most κ
2. g is a partial function from κ^{+3} to κ_n of cardinality at most κ

Q_{n1} is ordered by extension. Denote this order by $\leq_1$.
So, it is basically the Cohen forcing for adding κ^{+3} Cohen subsets to κ^+.

Definition 3.4 Set $Q_n = Q_{n0} \cup Q_{n1}$. Define $\leq_n^* = \leq_{Q_{n0}} \cup \leq_{Q_{n1}}$.

Define now a natural projection to the first coordinate:

Definition 3.5 Let $p \in Q_n$. Set $(p)_0 = p$, if $p \in Q_{n1}$ and let $(p)_0 = \langle a, A, f \rangle$, if $p \in Q_{n0}$ is of the form $\langle \langle a, A, f \rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g \rangle \rangle$.

Let $(Q_n)_0 = \{(p)_0 \mid p \in Q_n\}$.

Definition 3.6 Let $p, q \in Q_n$. Then $p \leq_n q$ iff either

1. $p \leq_n^* q$

   or

2. $p = \langle \langle a, A, f \rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g \rangle \rangle \in Q_{n0}$, $q = \langle e, h \rangle \in Q_{n1}$ and the following hold:

   (a) $e \supseteq f$

   (b) $h \supseteq g$

   (c) $\mathrm{dom}(e) \supseteq \mathrm{dom}(a)$

   (d) $e(\max(\mathrm{dom}(a))) \in A$

   (e) for every $\beta \in \mathrm{dom}(a), e(\beta) = \pi_{\lambda_n, a(\max(\mathrm{dom}(a)), a(\beta)}(e(\max(\mathrm{dom}(a)))$

   (f) $\mathrm{dom}(h) \supseteq \mathrm{dom}(\underset{\sim}{b})$

   (g) $h(\max(\mathrm{dom}(\underset{\sim}{b})) \in \underset{\sim}{B}[e(\max(\mathrm{dom}(a))]$.
   I.e., we use $e(\max(\mathrm{dom}(a))$ in order to interpret $\underset{\sim}{B}$. Note that by 2d above, it is inside A and so the interpretation makes sense.

(h) for every $\beta \in \text{dom}(\underset{\sim}{b})$

$$h(\beta) = \pi_{\kappa_n, \max(\text{rng}(\underset{\sim}{b}[\nu])), \underset{\sim}{b}(\beta)[\nu]}(h(\max(\text{dom}(\underset{\sim}{b})),$$

where $\nu = e(\max(\text{dom}(a)))$. Recall that we code models by ordinals.

Definition 3.7 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

1. for every $n < \omega$, $p_n \in Q_n$
2. there is $\ell(p) < \omega$ such that
   (a) for every $n < \ell(p)$, $p_n \in Q_{n1}$
   (b) for every $n \geq \ell(p)$, $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b_n}, \underset{\sim}{B_n}, g_n \rangle \rangle \in Q_{n0}$
   (c) for every $n, m \geq \ell(p)$, $\max(\text{dom}(a_n)) = \max(\text{dom}(a_m))$ and $\max(\text{dom}(\underset{\sim}{b_n})) = \max(\text{dom}(\underset{\sim}{b_m}))$
   (d) for every $n \geq m \geq \ell(p)$, $\text{dom}(a_m) \subseteq \text{dom}(a_n)$ and $\text{dom}(\underset{\sim}{b_m})) \subseteq \text{dom}(\underset{\sim}{b_n})$
   (e) for every n, $\ell(p) \leq n < \omega$, and $X \in \text{dom}(a_n)$ the following holds: for each $k < \omega$ the set

$$\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$$

   is finite.
   (f) for every n, $\ell(p) \leq n < \omega$, and $X \in \text{dom}(\underset{\sim}{b_n})$ the following holds: for each $k < \omega$ the set

$$\{m < \omega \mid \exists \nu \in A_m(\neg(\underset{\sim}{b_m}(X)[\nu] \cap H(\chi^{+k}) \prec H(\chi^{+k})))\}$$

   is finite.

We define the orders $\leq, \leq^*$ as in [2].

Definition 3.8 Let $p = \langle p_n | n < \omega \rangle, q = \langle q_n | n < \omega \rangle$ be in $\mathcal{P}$. Define

1. $p \geq q$ iff for each $n < \omega, p_n \geq_n q_n$
2. $p \geq^* q$ iff for each $n < \omega, p_n \geq^*_n q_n$

Definition 3.9 Let $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$. Set $(p)_0 = \langle (p_n)_0 \mid n < \omega \rangle$.

Define $(\mathcal{P})_0 = \{(p)_0 \mid p \in \mathcal{P}\}$.

Finally, the equivalence relation $\longleftrightarrow$ and the order $\rightarrow$ are defined on $(\mathcal{P})_0$ exactly as it was done in [1], [2] and [3]. We extend $\rightarrow$ to $\mathcal{P}$ as follows:

Definition 3.10 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$. Set $q \rightarrow p$ iff

1. $(q)_0 \rightarrow (p)_0$
2. $\ell(p) = \ell(q)$
3. for every $n < \ell(p), p_n$ extends q_n
4. for every $n \geq \ell(p)$, let $p_n = \langle\langle a_n, A_n, f_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle$ and $q_n = \langle\langle a'_n, A'_n, f'_n\rangle, \langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n, g'_n\rangle\rangle$. Require the following:
   (a) $g_n \supseteq g'_n$
   (b) there is $\underset{\sim}{b}''_n$ such that for every $\nu \in A_n$ the following holds:
      i. $\underset{\sim}{b}_n[\nu]$ extends $\underset{\sim}{b}''_n[\nu']$
      ii. $\mathrm{dom}(\underset{\sim}{b}'_n) = \mathrm{dom}(\underset{\sim}{b}''_n)$
      iii. $\pi''_{\kappa_n, \max(\underset{\sim}{b}_n[\nu]), \max(\underset{\sim}{b}'_n[\nu'])} \underset{\sim}{B}_n[\nu] \subseteq \underset{\sim}{B}'_n[\nu']$,
      where $\nu' = \pi_{\lambda_n, \max(\mathrm{rng}(a_n)), \xi}(\nu)$ and $\xi = a_n(\max(\mathrm{dom}(a'_n))$
      iv. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'] \longleftrightarrow_{k_n} \mathrm{rng}(\underset{\sim}{b}''_n)[\nu']$, where ν' is as above and k_n is the k_n's member of a nondecreasing sequence converging to the infinity.
      v. $\mathrm{rng}(\underset{\sim}{b}'_n)[\nu'] \upharpoonright \kappa^{+n+1} = \mathrm{rng}(\underset{\sim}{b}'_n)[\nu'] \upharpoonright \kappa^{+n+1}$

   Here is the main difference between $\rightarrow$ here and those of [1] etc. In the present context we deal with assignment functions b_n's which act over κ_n's but are of cardinalities below κ_n's (as well as the models in $\mathrm{rng}(b_n)$ which are images of those of cardinality κ^+). Thus, assume that n is fixed and $X = b_n(\max(\mathrm{dom}(b_n))$, where $b_n = \underset{\sim}{b}_n[\nu]$ is the interpretation according to some $\nu < \lambda_n < \kappa_n$. Then $|X| = (\nu^0)^{+n+1}$ by 3.1(8c(ii)). Now if we like to realize types inside X, as it was done usually in [1] etc., it may be just impossible since X is too small and so does not contains all the types.

   The way suggested here in order to overcome this difficulty, will be to use 3.1(8j) together with the above definition. It turns out that once working with names it is still possible to prove κ^{++}-c.c. of the final forcing $\langle \mathcal{P}, \rightarrow \rangle$. It will be done in 4.6.

4 Basic Lemmas

In this section we study the properties of the forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$ defined in the previous section.

Lemma 4.1 *Let* $p = \langle p_k \mid k < \omega \rangle \in \mathcal{P}$, $p_k = \langle \langle a_k, A_k, f_k \rangle, \langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k \rangle \rangle$ *for* $k \geq \ell(p)$ *and* X *be a model appearing in an element of* $G(\mathcal{P}'(\kappa^{++}))$. *Suppose that*

(a) $X \notin \bigcup_{\ell(p) \leq k < \omega} \mathrm{dom}(a_k) \cup \mathrm{dom}(f_k)$

(b) X *is a successor model or if it is a limit one with* $\mathrm{cof}(otp_{\kappa^+}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega \rangle$, $q_k = \langle \langle a'_k, A'_k, f'_k \rangle, \langle \underset{\sim}{b}'_k, \underset{\sim}{B}'_k, g_k \rangle \rangle$ *for* $k \geq \ell(q)$, *of* p *so that starting with some* $n \geq \ell(q)$ *we have* $X \in \mathrm{dom}(a'_k)$ *for each* $k \geq n$. *In addition the second part of the condition* p, *i.e.* $\langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k \rangle$ *remains basically unchanged (just names should be lifted to new* A_k*'s).*

The proof is the same as those of the corresponding lemma in [6]

Turn now to a parallel lemma needed for adding elements of $G(\mathcal{P})$.

Lemma 4.2 *Let* $p = \langle p_k \mid k < \omega \rangle \in \mathcal{P}$, $p_k = \langle \langle a_k, A_k, f_k \rangle, \langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k \rangle \rangle$ *for* $k \geq \ell(p)$ *and* X *be a model appearing in an element of* $G(\mathcal{P}')$. *Suppose that*

(a) $X \notin \bigcup_{\ell(p) \leq k < \omega} \mathrm{dom}(\underset{\sim}{b}_k) \cup \mathrm{dom}(g_k)$

(b) X *is a successor model or if it is a limit one with* $\mathrm{cof}(otp_{|X|}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega \rangle$, $q_k = \langle \langle a'_k, A'_k, f'_k \rangle, \langle \underset{\sim}{b}'_k, \underset{\sim}{B}'_k, g_k \rangle \rangle$ *for* $k \geq \ell(q)$, *of* p *so that starting with some* $n \geq \ell(q)$ *we have* $X \in \mathrm{dom}(\underset{\sim}{b}'_k)$ *for each* $k \geq n$.

The ordering $\leq^*$ on $\mathcal{P}$ and $\leq_n$ on Q_{n0} is not closed in the present situation. Thus it is possible to find an increasing sequence of $\aleph_0$ conditions $\langle \langle a_{ni}, A_{ni}, f_{ni} \rangle \mid i < \omega \rangle$ in $(Q_{n0})_0$ with no upperbound. The reason is that the union of maximal models of these conditions, i.e. $\bigcup_{i<\omega} \max(\mathrm{dom}\, a_{ni})$ need not be in $A^{1\kappa^+}$ for any $A^{1\kappa^+}$ in $G(\mathcal{P}')$. The next lemma shows that still $\leq_n$ and so also $\leq^*$ share a kind of strategic closure. The proof is similar to those of [4, 3.5].

Lemma 4.3 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_0 \rangle$ does not add new sequences of ordinals of the length $< \lambda_n$, i.e. it is (λ_n, ∞) – distributive.*

Now as in [4] we obtain the following:

Lemma 4.4 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 4.5 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Let us turn now to the main lemma in the present context:

Lemma 4.6 *$\langle \mathcal{P}, \rightarrow \rangle$ satisfies κ^{++}-c.c.*

Proof. Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++} \rangle$ be a name of an antichain of the length κ^{++}. As in [6], using the κ^{++}-strategic closure of $\mathcal{P}(\kappa^{++})$ and $\mathcal{P}'$ ([6, 1.6]) we find an increasing sequence

$$\langle \langle A^{0\tau}_\alpha, A^{1\tau}_\alpha, C^\tau_\alpha \rangle \mid \tau \in \{\kappa^+, \kappa^{++}\}, \alpha < \kappa^{++} \rangle, \langle \langle A^{0\kappa^+}_\alpha(\kappa^{++}), A^{1\kappa^+}_\alpha(\kappa^{++}), C^{\kappa^+}_\alpha(\kappa^{++}) \rangle \mid \alpha < \kappa^{++} \rangle$$

of elements of $\mathcal{P}' \times \mathcal{P}'(\kappa^{++})$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ so that for every $\alpha < \kappa^{++}$ the following holds:

1. $\langle \langle \langle A^{0\tau}_{\alpha+1}, A^{1\tau}_{\alpha+1}, C^\tau_{\alpha+1} \rangle \mid \tau \in \{\kappa^+, \kappa^{++}\} \rangle, \langle \langle A^{0\kappa^+}_{\alpha+1}(\kappa^{++}), A^{1\kappa^+}_{\alpha+1}(\kappa^{++}), C^{\kappa^+}_{\alpha+1}(\kappa^{++}) \rangle \rangle \rangle \Vdash \underset{\sim}{p}_\alpha = \check{p}_\alpha$
2. if α is a limit ordinal, then $\bigcup \{A^{0\tau}_\beta | \beta < \alpha\} = A^{0\tau}_\alpha$, for each $\tau \in \{\kappa^+, \kappa^{++}\}$
3. if α is a limit ordinal, then $\bigcup \{A^{0\kappa^+}_\beta(\kappa^{++}) | \beta < \alpha\} = A^{0\kappa^+}_\alpha(\kappa^{++})$
4. ${}^{\tau >}A^{0\tau}_{\alpha+1} \subseteq A^{0\tau}_{\alpha+1}$, for each $\tau \in \{\kappa^+, \kappa^{++}\}$
5. ${}^{\kappa^+ >}A^{0\kappa^+}_{\alpha+1}(\kappa^{++}) \subseteq A^{0\kappa^+}_{\alpha+1}(\kappa^{++})$
6. $A^{0\tau}_{\alpha+1}$ is a successor model, for each $\tau \in \{\kappa^+, \kappa^{++}\}$
7. $A^{0\kappa^+}_{\alpha+1}(\kappa^{++})$ is a successor model
8. $\langle \langle \cup A^{1\tau}_\beta \mid \tau \in \{\kappa^+, \kappa^{++}\} \rangle \mid \beta < \alpha \rangle \in (A^{0\kappa^+}_{\alpha+1})^-$ (i.e. the immediate predecessor over $C^{\kappa^+}_{\alpha+1}$)

9. for every $\alpha \leq \beta < \kappa^{++}, \tau \in \{\kappa^+, \kappa^{++}\}$ we have

$$A^{0\tau}_\alpha \in C^\beta(A^{0\tau}_\beta)$$

10. $A^{0\tau}_{\alpha+2}$ is not an immediate successor model of $A^{0\tau}_{\alpha+1}$, for every $\alpha < \kappa^{++}, \tau \in \{\kappa^+, \kappa^{++}\}$.

11. $p_\alpha = \langle p_{\alpha n} | n < \omega \rangle$

12. for every $n \geq \ell(p_\alpha)$ the maximal model of $\mathrm{dom}(a_{\alpha n})$ is $A^{0\kappa^+}_{\alpha+1}(\kappa^{++})$ and the maximal model of $\mathrm{dom}(\underset{\sim}{b}_{\alpha n})$ is $A^{0\kappa^+}_{\alpha+1}$, where $p_{\alpha n} = \langle \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n} \rangle, \langle \underset{\sim}{b}_{\alpha n}, \underset{\sim}{B}_{\alpha n}, g_{\alpha n} \rangle \rangle$

Let $p_{\alpha n} = \langle \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n} \rangle, \langle \underset{\sim}{b}_{\alpha n}, \underset{\sim}{B}_{\alpha n}, g_{\alpha n} \rangle \rangle$ for every $\alpha < \kappa^{++}$ and $n \geq \ell(p_\alpha)$. Extending by 4.2 if necessary, let us assume that $A^{0\kappa^+}_\alpha(\kappa^{++}) \in \mathrm{dom}(a_{\alpha n})$ and $A^{0\kappa^+}_\alpha \in \mathrm{dom}(\underset{\sim}{b})$, for every $n \geq \ell(p_\alpha)$. Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^+$ the following holds:

(1) $\ell = \ell(p_\alpha) = \ell(p_\beta)$

(2) for every $n < \ell$ $\quad p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1}

(3) for every n, $\ell \leq n < \omega$ $\quad \langle \mathrm{dom}(a_{\alpha n}), \mathrm{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++} \rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0(\kappa^{++})$

(4) for every n, $\omega > n \geq \ell$ $\quad \mathrm{rng}(a_{\alpha n}) = \mathrm{rng}(a_{\beta n})$.

(5) for every n, $\omega > n \geq \ell$ $\quad A_{\alpha n} = A_{\beta n}$

(6) for every n, $\ell \leq n < \omega$ $\quad \langle \mathrm{dom}(\underset{\sim}{b}_{\alpha n}), \mathrm{dom}(g_{\alpha n}) \mid \alpha < \kappa^{++} \rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$.

Remember that the domain of $\underset{\sim}{b}$ is not a name but rather a set.

(7) for every n, $\omega > n \geq \ell$ $\quad \mathrm{rng}(\underset{\sim}{b}_{\alpha n}) = \mathrm{rng}(\underset{\sim}{b}_{\beta n})$, i.e. it is just the same name in the one element Prikry forcing.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let α be in S. For each $n, \ell \leq n < \omega$, there will be $\beta(\alpha, n) < \alpha$ such that

- $\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_\alpha(\kappa^{++}) \subseteq A^{0\kappa^+}_{\beta(\alpha,n)}(\kappa^{++})$

  and

- $\mathrm{dom}(\underset{\sim}{b}_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} \subseteq A^{0\kappa^+}_{\beta(\alpha,n)}$.

Just recall that $|a_{\alpha n}| < \lambda_n$ and $|\mathrm{dom}(\underset{\sim}{b}_{\alpha n})| < \lambda_n$. Shrink S to a stationary subset S^* so that for some $\alpha^* < \min S^*$ of cofinality κ^+ we will have $\beta(\alpha,n) < \alpha^*$, whenever $\alpha \in S^*, \ell \leq n < \omega$. Now, the cardinality of both $A^{0\kappa^+}_{\alpha^*}$ and $A^{0\kappa^+}_{\alpha^*}(\kappa^{++})$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\alpha, \beta \in S^*, \ell \leq n < \omega$

- $\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_{\alpha}(\kappa^{++}) = \mathrm{dom}(a_{\beta n}) \cap A^{0\kappa^+}_{\beta}(\kappa^{++})$

  and

- $\mathrm{dom}(\underset{\sim}{b}_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} = \mathrm{dom}(\underset{\sim}{b}_{\beta n}) \cap A^{0\kappa^+}_{\beta}$.

Let us add both $A^{0\kappa^+}_{\alpha^*}$ and $A^{0\kappa^+}_{\alpha^*}(\kappa^{++})$ to each $p_\alpha, \alpha \in S^*$. By 4.2, it is possible to do this without adding other additional models except the images of $A^{0\kappa^+}_{\alpha^*}$ under isomorphisms. Thus, $A^{0\kappa^+}_{\alpha^*} \in C^{\kappa^+}(A^{0\kappa^+}_{\alpha})$ and $A^{0\kappa^+}_{\alpha} \in \mathrm{dom}(\underset{\sim}{b}_{\alpha n}) \cap C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1})$. So, 3.1(**??**) was already satisfied after adding $A^{0\kappa^+}_{\alpha}$. The rest of 3.1 does not require adding additional models in the present situation.

Denote the result for simplicity by p_α as well. Note that (again by 4.2 and the argument above) any $A^{0\kappa^+}_{\gamma}$ for $\gamma \in S^* \cap (\alpha^*, \alpha)$ or, actually any other successor or limit model $X \in C^{\kappa^+}(A^{0\kappa}_{\alpha})$ with $\mathrm{cof}(otp_{\kappa^+}(X)) = \kappa^+$, which is between $A^{0\kappa^+}_{\alpha^*}$ and $A^{0\kappa^+}_{\alpha}$ can be added without adding other additional models or ordinals except the images of it under isomorphisms.

Let now $\beta < \alpha$ be ordinals in S^*. We claim that p_β and p_α are compatible in $\langle \mathcal{P}, \rightarrow \rangle$. First extend p_α by adding $A^{0\kappa^+}_{\beta+2}$. As it was remarked above, this will not add other additional models or ordinals except the images of $A^{0\kappa^+}_{\beta+2}$ under isomorphisms to p_α. Let p be the resulting extension. Denote p_β by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$, $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$ and $q_n = \langle \langle a'_n, A_n, f'_n \rangle, \langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n, g'_n \rangle \rangle$. Note that by (5) above the sets of measure one of p_n, q_n are the same. Without loss of generality we may assume that $a_n(A^{0\kappa^+}_{\beta+2}(\kappa^{++}))$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $\mathrm{rng}(a'_n)$ inside $a_n(A^{0\kappa^+}_{\beta+2}(\kappa^{++}))$ over the common parts $\mathrm{dom}(a'_n)$ and $\mathrm{dom}(a_n)$. This will produce $\langle a''_n, A'_n, f'_n \rangle$ which is $k_n - 1$-equivalent to $\langle a'_n, A'_n, f'_n \rangle$ and with $\mathrm{rng}(a''_n) \subseteq a_n(A^{0\kappa^+}_{\beta+2}(\kappa^{++}))$. Doing the above for all $n \geq \ell(p)$ we will obtain $\langle \langle a''_n, A'_n, f'_n \rangle \mid n < \omega \rangle$ equivalent to $\langle \langle a'_n, A'_n, f'_n \rangle \mid n < \omega \rangle$ (i.e. $\langle \langle a''_n, A''_n, f''_n \rangle \mid n < \omega \rangle \longleftrightarrow \langle \langle a'_n, A'_n, f'_n \rangle \mid n < \omega \rangle$).

Let $t = \langle \langle \langle a''_n, A'_n, f'_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle \mid n < \omega \rangle$. Extend t to t' by adding to it

$$\langle A^{0\kappa^+}_{\beta+2}(\kappa^{++}), a_n(A^{0\kappa^+}_{\beta+2}(\kappa^{++}))\rangle$$

as the maximal set for every $n \geq \ell(p)$. Recall that $A^{0\kappa^+}_{\beta+1}(\kappa^{++})$ was its maximal model. So we are adding a top model, also, by the condition (15) above $A^{0\kappa^+}_{\beta+2}(\kappa^{++})$ is not an immediate successor of $A^{0\kappa^+}_{\beta+1}(\kappa^{++})$. Hence no additional models or ordinals are added at all.

Let $t'_n = \langle\langle a'''_n, A'''_n, f'_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle$, for every $n \geq \ell(p)$.

Combine now the first coordinates of p and t' together, i.e. $\langle a_n, A_n, f_n\rangle$'s with those of t'. Thus for each $n \geq \ell(p)$ we add a'''_n to a_n. Add if necessary a new top model to insure 3.1(2(d)). Let $r = \langle r_n | n < \omega\rangle$ be the result, where $r_n = \langle\langle c_n, C_n, h_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle$, for $n \geq \ell(p)$.

Claim 4.6.1 $r \in \mathcal{P}$ and $r \geq p$.

Proof. Fix $n \geq \ell(p)$. The main points here are that a'''_n and a_n agree on the common part and adding of a'''_n to a_n does not require other additions of models except the images of a'''_n under isomorphisms.

The check of the rest of conditions of 3.1 is routine. We refer to [2] or [4] for similar arguments.
$\square$ of the claim.

Now let us turn to the second coordinates of q and r. Recall that for a condition $x \in Q_{n0}$ we denote by $(x)_0$ its first coordinate, i.e. the first triple. If $y = \langle y_n | n < \omega\rangle \in \mathcal{P}$, then $(y)_0$ denotes $\langle (y_n)_0 | n < \omega\rangle$. So, we have $(q)_0 \to (r)_0$. Shrinking if necessary A_n's (the sets of measure one of $(q_n)_0$'s), we can assume that for each $n \geq \ell(p) = \ell(r) = \ell(q)$ the set of measure one for $(r_n)_0$, i.e. C_n projects exactly to A_n by $\pi_{\lambda_n, \max(\mathrm{rng}((r_n)_0)), \max(\mathrm{rng}((q_n)_0)}$. Remember that the interpretations of both $\langle \underset{\sim}{b}_n, \underset{\sim}{B}_n\rangle$ and $\langle \underset{\sim}{b'}_n, \underset{\sim}{B'}_n\rangle$ depend only on a choice of elements of A_n.

Our tusk will be extend r to r^* so that $q \to r^*$. This will show that p and q are compatible. Which provides the desired contradiction.

Fix n, $\omega > n \geq \ell(p)$, large enough. Let η be the maximal coordinate of $(r_n)_0$ (i.e. the ordinal coding $\max(\mathrm{rng}(c_n))$, ζ those of $(p_n)_0$ (which is the same for $(q_n)_0$, since (4) above) and ξ the one corresponding to ζ (of $(q_n)_0$) under $(q_n)_0 \to (r_n)_0$. Denote $\pi''_{\lambda_n,\eta,\xi} C_n$ by D_n. Assuming that $n > 2$, it follows from the definitions of the equivalence relation $\longleftrightarrow$ and of the order $\to$, that $E_{\lambda_n}(\xi)$ (the ξ's measure of the extender) is the same as $E_{\lambda_n}(\zeta)$. Also, $D_n \subseteq A_n$.

Define now a condition

$$r_n^* = \langle\langle c_n, C_n, h_n\rangle, \langle \underset{\sim}{e}_n, \underset{\sim}{E}_n, g_n\rangle\rangle \in Q_{n0}$$

which extends

$$r_n = \langle\langle c_n, C_n, h_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle.$$

The addition will depend only on the coordinate ξ of E_{λ_n}. So we need to deal with each $\nu \in D_n$. Set $\mathrm{dom}(\underset{\sim}{e}_n) = \mathrm{dom}(\underset{\sim}{b}_n) \cup \mathrm{dom}(\underset{\sim}{b}'_n)$. Let $X \in \mathrm{dom}(\underset{\sim}{e}_n)$. If $X \in \mathrm{dom}(\underset{\sim}{b}_n)$, then set

$$\underset{\sim}{e}_n(X)[\rho] = \underset{\sim}{b}_n(X)[\rho],$$

for each $\rho \in C_n$. Now, if X is new, i.e. $X \in \mathrm{dom}(\underset{\sim}{b}'_n) \setminus \mathrm{dom}(\underset{\sim}{b}_n)$, then we consider X_α the model that corresponds to X in p_α under the Δ-system.

Now we use Definition 3.1(8j) to find inside $\underset{\sim}{b}_n(A_\alpha)[\rho]$ some σ realizing over the common part the type of $\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\nu]$. Recall that

$$\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\nu] = \underset{\sim}{b}'_n(A^{0\kappa^+}_{\beta+1})[\nu]$$

and

$$\underset{\sim}{b}_n(X_\alpha)[\nu] = \underset{\sim}{b}'_n(X)[\nu].$$

Set now $\underset{\sim}{e}_n(X)[\rho]$ to be the element of σ corresponding to $\underset{\sim}{b}'_n(X)[\nu]$, for each $\rho \in C_n$ and $\nu = \pi_{\lambda_n,\eta,\xi}(\rho)$.

The following claim suffice in order to complete the argument:

Claim 4.6.2 $r_n^* \in Q_{n0}$, $r_n^* \geq_0 r_n$ and $q_n \to r_n^*$.

Proof. Let us check first that q_n, r_n or basically $\underset{\sim}{b}'_n$ and $\underset{\sim}{c}_n$ agree about the values of models in $\mathrm{dom}(\underset{\sim}{b}'_n) \cap \mathrm{dom}(\underset{\sim}{c}_n)$. Suppose that X is such a model. Then, by the assumptions we made on the Δ-system, $X \in A^{0\kappa^+}_{\alpha^*}$. Also,

$$A^{0\kappa^+}_{\alpha^*} \in \mathrm{dom}(\underset{\sim}{b}'_n) \cap \mathrm{dom}(\underset{\sim}{c}_n),$$

$$otp_{\kappa^+}(A^{0\kappa^+}_{\alpha^*}) = A^{0\kappa^+}_{\alpha^*}(\kappa^{++}) \cap \kappa^{++}$$

and

$$A^{0\kappa^+}_{\alpha^*}(\kappa^{++}) \in \mathrm{dom}(c_n).$$

By 3.1, $\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha^*})$ depends only on the measure indexed by the code of

$$c_n(A^{0\kappa^+}_{\alpha^*}(\kappa^{++})) = a_n(A^{0\kappa^+}_{\alpha^*}(\kappa^{++})) = a'_n(A^{0\kappa^+}_{\alpha^*}(\kappa^{++})).$$

Let δ denotes the index of this measure (or its code). Then for each $\rho \in C_n$ we will have

$$\pi_{\lambda_n,\eta,\delta}(\rho) = \pi_{\lambda_n,\xi,\delta}(\pi_{\lambda_n,\eta,\xi}(\rho)).$$

Hence, restricting $(q_n)_0$ to D_n, i.e. by replacing A_n in $(q_n)_0$ with D_n, we can insure that $\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha^*})$ and $\underset{\sim}{b}'_n(A^{0\kappa^+}_{\alpha^*})$ agree. The same applies to any $X \in A^{0\kappa^+}_{\alpha^*}$ which is in the common domain, since its value too will depend on the δ-th measure of the extender only.

Consider now the maximal model of q_n. By 12, above, it is $A^{0\kappa^+}_{\beta+1}$ and the one of p_n is $A^{0\kappa^+}_{\alpha+1}$. Now, for each $\nu \in A_n$, by the condition (7) on the Δ-system above we have

$$\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\nu] = \underset{\sim}{b}'_n(A^{0\kappa^+}_{\beta+1})[\nu].$$

Pick $\rho \in C_n$. Let $\nu = \pi_{\lambda_n,\eta,\xi}(\rho)$ and $\sigma = \pi_{\lambda_n,\eta,\zeta}(\rho)$. Then

$$\underset{\sim}{e}_n(A^{0\kappa^+}_{\alpha+1})[\rho] = \underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\sigma]$$

and

$$\underset{\sim}{e}_n(A^{0\kappa^+}_{\beta+1})[\rho] = \underset{\sim}{b}'_n(A^{0\kappa^+}_{\beta+1})[\nu].$$

The first equality holds since e_n extends b_n and the second by the same reason as e_n was defined this way above.

The crucial observation is that $\sigma, \nu \in A_n$ (just $D_n \subseteq A_n$) and $\sigma > \nu$, so by Definition 3.1(8j),

$$\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\nu] \subseteq \underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\sigma].$$

Hence, also,

$$\underset{\sim}{b}'_n(A^{0\kappa^+}_{\beta+1})[\nu] \subseteq \underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\sigma],$$

since

$$\underset{\sim}{e}_n(A^{0\kappa^+}_{\beta+1})[\rho] = \underset{\sim}{b}'_n(A^{0\kappa^+}_{\beta+1})[\nu].$$

The same inclusion holds, by Definition 3.1(8j), if we replace $A^{0\kappa^+}_{\alpha+1}$ with any $Y \in \mathrm{dom}(\underset{\sim}{b}_n) \cap C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1})$ such that $\sigma(Y) > \nu$, where $\sigma(Y)$ is the measure corresponding to Y. Thus

$$\underset{\sim}{b}'_n(A^{0\kappa^+}_{\beta+1})[\nu] = \underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\nu] \subseteq \underset{\sim}{b}_n(Y)[\sigma].$$

In the present case we have the least such Y. It is $A^{0\kappa^+}_{\alpha}$. Just below it everything falls into $A^{0\kappa^+}_{\alpha^*}$ the kernel of the Δ-system. Consider now Y's in $\mathrm{dom}(\underset{\sim}{b_n})\backslash C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1})$. If such Y is in $A^{0\kappa^+}_{\alpha}$, it belongs to $A^{0\kappa^+}_{\alpha^*}$ the kernel of the Δ-system. Hence as it was observed in the beginning of the proof of this claim, we have the agreement. Suppose now that $Y \notin A^{0\kappa^+}_{\alpha}$. By the basic properties of $G(\mathcal{P}')$ there will be $Z \in A^{0\kappa^+}_{\alpha}$ such that

$$Y \cap A^{0\kappa^+}_{\alpha} = Z \cap A^{0\kappa^+}_{\alpha}.$$

Then again this Z falls into $A^{0\kappa^+}_{\alpha^*}$ and into the kernel of the Δ-system on which we have the agreement.

This completes the proof of the claim.

$\square$ of the claim.

$\square$

Force with $\langle \mathcal{P}, \to \rangle$. Let $G(\mathcal{P})$ be a generic set. By the lemmas above no cardinals are collapsed. Let $\langle \nu_n \mid n < \omega\rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\lambda_n} \mid n < \omega\rangle$ and $\langle \rho_n \mid n < \omega\rangle$ those for $\langle E_{\kappa_n} \mid n < \omega\rangle$. We can deduce now the following conclusion:

Theorem 4.7 *The following hold in $V[G(\mathcal{P}'(\theta')), G((\mathcal{P}'(\theta)), G(\mathcal{P})]$:*

(1) $\mathrm{cof}(\prod_{n<\omega} \nu_n^{+n+2}/$ *finite* $) = \kappa^{++}$

(2) $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+2}/$ *finite* $) = \kappa^{+3}$

(3) for every unbounded subset a of κ consisting of regular cardinals and disjoint to both $\{\nu_n^{+n+2} \mid n < \omega\}$ and $\{\rho_n^{+n+2} \mid n < \omega\}$, for every ultrafilter D over a which includes all co-bounded subsets of κ we have

$$\mathrm{cof}(\prod a/D) = \kappa^+$$

Proof. Items (1) and (2) follow easily from the construction. Thus, for (1), take the increasing (under the inclusion) enumeration $\langle X_\tau | \tau < \kappa^{++}\rangle$ of the chain of models given by $G(\mathcal{P}'(\kappa^{++}))$. Define a scale of functions $\langle F_\tau \mid \tau < \kappa^{++}\rangle$ in the product $\prod_{n<\omega} \nu_n^{+n+2}$ as follows: let for each $\tau < \kappa^{++}$

$$F'_\tau(n) = f_n(X_\tau),\ \text{if}\ f_n(X_\tau) < \nu_n^{+n+2}$$

and

$$F'_\tau(n) = 0,\ \text{otherwise},$$

where for some $p = \langle p_k | k < \omega \rangle \in G(\mathcal{P})$ with $\ell(p) > n$ we have f_n as the first coordinate of p_n. Now let $\langle F_\tau | \tau < \kappa^{++} \rangle$ be the subsequence of $\langle F'_\tau | \tau < \kappa^{++} \rangle$ consisting of all F'_τ which are not in V.

Similar, for (2), take the increasing (under the inclusion) enumeration $\langle Y_\tau | \tau < \kappa^{+3} \rangle$ of the chain of models of cardinality κ^{++} given by $G(\mathcal{P}')$. Define a scale of functions $\langle H_\tau \mid \tau < \kappa^{++} \rangle$ in the product $\prod_{n<\omega} \rho_n^{+n+2}$ as follows:

$$H'_\tau(n) = g_n(X_\tau), \text{ if } g_n(Y_\tau) < \rho_n^{+n+2}$$

and

$$H'_\tau(n) = 0, \text{ otherwise,}$$

where for some $p = \langle p_k | k < \omega \rangle \in G(\mathcal{P})$ with $\ell(p) > n$ we have g_n as the second coordinate of p_n. Let $\langle H_\tau \mid \tau < \kappa^{++} \rangle$ be the subsequence of $\langle H'_\tau \mid \tau < \kappa^{++} \rangle$ consisting of all H'_τ's which are not in V.

Let us turn to (3) which requires a more delicate analyses of the forcing $\langle \mathcal{P}, \rightarrow \rangle$. We deal with

$$\text{cof}(\prod_{n<\omega} \rho_n^{+n+1} / \text{ finite }).$$

The rest of cases are similar or just standard. The crucial observation here is that given $\langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle \in Q_{n0}$, for some $n < \omega$, it is impossible to change $\text{rng}(b_n)[\nu] \restriction \kappa^{+n+1}$ by passing to an equivalent condition, for any $\nu \in A_n$. Just the definition 3.10(4(b)v) explicitly requires this.

This means, in particular that

$$\text{cof}(\prod_{n<\omega} \rho_n^{+n+1} / \text{ finite }) = \text{cof}(\prod_{n<\omega} \kappa_n^{+n+1} / \text{ finite }),$$

where the connection is provided by $\underset{\sim}{b}_n$'s. But note that the cofinality of the last product is κ^+, since every function their can be bounded by an old function. So we are done.

□

5 The general case.

Let us turn now from $\theta = \kappa^{+3}, \theta' = \kappa^+$ to arbitrary regular θ and θ'. Assume We force with preparation forcings $\mathcal{P}'(\theta')$ followed by $\mathcal{P}'(\theta)$ of [6]. Let $G(\mathcal{P}'(\theta'))$ and $G(\mathcal{P}'(\theta))$ be corresponding generic sets. We work in $V[G(\mathcal{P}'(\theta')), G(\mathcal{P}'(\theta))]$ and define forcing notions Q_{n0}, Q_{n1} and then $\mathcal{P}$ similar to those of Section 2.

For each $n < \omega$ let $\delta_n = \kappa_n^{+\kappa_n^{+n+2}+1}$ and $\delta'_n = \lambda_n^{+\lambda_n^{+n+2}+1}$. Fix some $n < \omega$.

Definition 5.1 Let Q_{n0} be the set consisting of pairs of triples $\langle\langle a, A, f\rangle, \langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle\rangle$ so that:

1. f is partial function from θ' to λ_n of cardinality at most κ
2. a is a partial function of cardinality less than λ_n so that
   (a) There is $\langle\langle A^{0\tau}(\theta'), A^{1\tau}(\theta'), C^\tau(\theta')\rangle | \tau \in s(\theta')\rangle \in G(\mathcal{P}'(\theta'))$ which we call it further **a background condition of** a,

   such that for each $\tau \in s(\theta')$ $\quad A^{0\tau}(\theta')$ is a successor model having unique immediate predecessor $(A^{0\tau}(\theta'))^-$ (i.e. $Pred(A^{0\tau}(\theta')) = \{(A^{0\tau}(\theta'))^-\}$) and $\langle A^{0\tau}(\theta')^- | \tau \in s(\theta')\rangle \in A^{0\kappa^+}(\theta')$. The same holds for $\langle\langle (A^{0\tau}(\theta'))^-, A^{1\tau}(\theta') \setminus \{A^{0\tau}(\theta')\}, C^\tau(\theta') \restriction A^{0\tau}(\theta')\rangle | \tau \in s(\theta')\rangle$, i.e. for each $\tau \in s$ $\quad (A^{0\tau}(\theta'))^-$ is a successor model having unique immediate predecessor $((A^{0\tau}(\theta'))^-)^-$ (i.e. $Pred((A^{0\tau}(\theta'))^-) = \{((A^{0\tau'}(\theta'))^-)^-\}$) and $\langle ((A^{0\tau}(\theta'))^-)^- | \tau \in s(\theta')\rangle \in (A^{0\kappa^+}(\theta'))^-$.
   $\text{dom}(a)$ consists of models appearing in $A^{1\kappa^+}(\theta')$ and in $(A^{1\tau}(\theta'))^-$, $\tau \in s(\theta')$.

   Note that conditions as above are dense in $\mathcal{P}'(\theta')$. Let us refer to them further as **conditions of the right form**.

   (b) for each $X \in \text{dom}(a)$ there is $k \leq \omega$ so that $a(X) \subseteq H(\chi^{+k})$.

   Also the following holds

   (i) $|X| = \kappa^+$ implies $|a(X)| = \lambda_n^{+n+1}$
   (ii) $|X| = \theta'$ implies $|a(X)| = \delta'_n$ and $a(X) \cap (\delta'_n)^+ \in ORD$
   (iii) $A^{0\kappa^+}(\theta'), (A^{0\kappa^+}(\theta'))^-, (A^{0\theta'}(\theta'))^- \in \text{dom}(a)$.

   This way we arranged that λ_n^{+n+1} will correspond to κ^+ and δ'_n to θ'.
   Further let us refer to $A^{0\kappa^+}(\theta')$ as **the maximal model of the domain of** a and to $\langle (A^{0\tau}(\theta'))^- | (A^{0\tau}(\theta'))^- \in \text{dom}(a)\rangle$ as **the maximal sequence of the domain of** a. Denote the first as $\max(\text{dom}(a))$ and the second as $\vec{\max}(\text{dom}(a))$ (or just $\max(a), \vec{\max}(a)$).

Further passing from Q_{0n} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

(c) (Models come from $A^{0\kappa^+}(\theta')$) If $X \in \mathrm{dom}(a)$ and $X \neq A^{0\kappa^+}(\theta')$ then $X \in A^{0\kappa^+}(\theta')$.

The condition puts restriction on models in $\mathrm{dom}(a)$ and allows to control them via the maximal model of cardinality κ^+.

(d) (All the cardinalities are inside $A^{0\kappa^+}(\theta')$) If $(A^{0\tau}(\theta'))^- \in \mathrm{dom}(a)$, then $\tau \in A^{0\kappa^+}(\theta')$.

(e) (No holes) If $X \in A^{1\tau}(\theta') \cap \mathrm{dom}(a)$, for some $\tau \in s(\theta')$, then $(A^{0\tau}(\theta'))^- \in \mathrm{dom}(a)$ as well.

This means that in order to add $X \in A^{1\tau}(\theta')$ to $\mathrm{dom}(a)$ we need first to insure that the maximal model of cardinality as those of X is inside.

(f) If $X, Y \in \mathrm{dom}(a)$, $X \in Y$ (or $X \subseteq Y$) and k is the minimal so that $a(X) \subseteq H(\chi^{+k})$ or $a(Y) \subseteq H(\chi^{+k})$, then $a(X) \cap H(\chi^{+k}) \in a(Y) \cap H(\chi^{+k})$ (or $a(X) \cap H(\chi^{+k}) \subseteq a(Y) \cap H(\chi^{+k})$).

The intuitive meaning is that a is supposed to preserve membership and inclusion. But we cannot literally require this since $a(A)$ and $a(B)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(g) Let $X, Y \in \mathrm{dom}(a)$. Then

  (i) $|X| = |Y|$ implies $|a(X)| = |a(Y)|$

  (ii) $|X| < |Y|$ implies $|a(X)| < |a(Y)|$

(h) The set

$$\{\nu \in s(\theta') | (A^{0\nu}(\theta'))^- \in \mathrm{dom}(a)\}$$

is closed.

(i) The image by a of $A^{0\kappa^+}(\theta')$, i.e. $a(A^{0\kappa^+}(\theta'))$, intersected with $(\delta'_n)^+$ is above all the rest of $\mathrm{rng}(a)$ restricted to $(\delta'_n)^+$ in the ordering of the extender E_n (via some reasonable coding by ordinals).

Recall that the extender E_{λ_n} acts on $(\delta'_n)^+$ and our main interest is in Prikry sequences it will produce. So, parts of $\mathrm{rng}(a)$ restricted to $(\delta'_n)^+$ will play the central role.

(j) If $X \in \mathrm{dom}(a)$ then $C^{|X|}(\theta')(X) \cap \mathrm{dom}(a)$ is a closed chain. Let $\langle X_i | i < j \rangle$ be its increasing continuous enumeration. For each $l < j$ consider the final segment $\langle X_i | l \leq i < j \rangle$ and its image $\langle a(X_i) | l \leq i < j \rangle$. Find the minimal k so that

$$a(X_i) \subseteq H(\chi^{+k}) \text{ for each } i, l \leq i < j.$$

Then the sequence

$$\langle a(X_i) \cap H(\chi^{+k}) | l \leq i < j \rangle$$

is increasing and continuous.

Note that k here may depend on l, i.e. on the final segment.

(k) (The walk is in the domain) If $X \in \mathrm{dom}(a) \cap A^{1\nu}(\theta')$, for some $\nu \in s$, then the general walk from $(A^{0\nu}(\theta'))^-$ to X is in $\mathrm{dom}(a)$.

(l) If $X \in \mathrm{dom}(a) \cap A^{1\nu}(\theta')$, for some $\nu \in s$ is a limit model and $\mathrm{cof}(otp_\nu(X) - 1) < \kappa_n$ (i.e. the cofinality of the sequence $C^\nu(X) \backslash \{X\}$ under the inclusion relation is less than κ_n) then a closed cofinal subsequence of $C^{\kappa^+}(X) \backslash \{X\}$ is in $\mathrm{dom}(a)$. The images of its members under a form a closed cofinal in $a(X)$ sequence.

(m) If $\langle X_i | i < j \rangle$ is an increasing (under the inclusion) sequence of elements of $\mathrm{dom}(a)$ with $X_i \in C^{\tau_i}(\theta')(A^{0\tau_i}(\theta'))$, $i < j$, then $\bigcup_{i<j} X_i \in \mathrm{dom}(a)$ as well.

Note that $\bigcup_{i<j} X_i \in C^{\cup_{i<j}\tau_i}(\theta')(A^{0\cup_{i<j}\tau_i}(\theta'))$. So, in particular, by **??** also $A^{0\cup_{i<j}\tau_i}(\theta') \in \mathrm{dom}(a)$.

(n) (The minimal models condition) Suppose that $X \in \mathrm{dom}(a) \cap C^\xi(\theta')(A^{0\xi}(\theta'))$, for some $\xi \in s(\theta') \backslash \kappa^+ + 1$. Let $\tau \in s(\theta')$ and $X^* \in C^\tau(\theta')(A^{0\tau}(\theta'))$ be such that $\tau < \xi$, $X \in X^*$ and for each $\rho, \tau \leq \rho < \xi$, $Z \in C^\rho(\theta')(A^{0\rho}(\theta'))$ we have $X \in Z$ implies $X^* \in Z$ or $X^* = Z$. Then $X^* \in \mathrm{dom}(a)$ as well as $(X^*)^-$-its immediate predecessor in $C^\tau(\theta')(A^{0\tau}(\theta'))$.

In addition, we require the following:

if $(X^*)^- \notin X$, then for each $H \in a((X^*)^-)$ there is $H' \in a((X^*)^-)$ with $H \in H'$ and $a(X) \subseteq H'$. Moreover, if $|a(X)| \in a((X^*)^-)$, then $|H'| = |a(X)|$. If $|a(X)| \notin a((X^*)^-)$, then $|H'| = \min(a((X^*)^-) \cap ORD \backslash |a(X)|)$.

Note that $X \in A^{0\kappa^+}(\theta') \in \mathrm{dom}(a)$, by **??**. So X^* always exists.

The second part of the condition insures that there will be enough models in $a((X^*)^-)$ to allow extensions which will include $a(X)$.

(o) (Minimal cover condition) Let $Y \in A^{0\xi}(\theta') \cap \mathrm{dom}(a), X \in A^{0\tau}(\theta') \cap \mathrm{dom}(a)$ for some $\xi < \tau$ in s. Suppose that $Y \not\subseteq X$. Then

- $\tau \in Y$ implies that the smallest model of $Y \cap C^{\tau}(\theta')(A^{0\tau}(\theta'))$ including X is in $\mathrm{dom}(a)$
- $\tau \notin Y$ implies that the smallest model of $Y \cap C^{\rho}(\theta')(A^{0\rho}(\theta'))$ including X is in $\mathrm{dom}(a)$, for $\rho = \min(Y \cap s \backslash \tau)$.

(p) (The first models condition) Suppose that $X \in dom(a) \cap C^{\tau}(\theta')(A^{0\tau}(\theta')), Y \in \mathrm{dom}(a) \cap C^{\rho}(\theta')(A^{0\rho}(\theta'))$, $\sup(X) > \sup(Y)$ and $Y \notin X$, for some $\tau < \rho, \tau, \rho \in s(\theta')$. Let $\eta = \min((X \cap s) \backslash \rho)$. Then the first model $E \in X \cap C^{\eta}(\theta')(A^{0\eta}(\theta'))$ which includes Y is in $\mathrm{dom}(a)$.

(q) (Models witnessing Δ-system type are in the domain) If $F_0, F_1, F \in A^{1\mu}(\theta') \cap \mathrm{dom}(a)$ is a triple of a Δ - system type, for some $\mu \in s$, then the corresponding models $G_0, G_0^*, G_1, G_1^*, G^*$, as in the definition of a Δ - system type (see [6](Definition 1.1(????))), are in $\mathrm{dom}(a)$ as well and

$$a(F_0) \cap a(F_1) = a(F_0) \cap a(G_0) = a(F_1) \cap a(G_1).$$

(r) If $F_0, F_1, F \in A^{1\mu}(\theta')$ is a triple of a Δ - system type, for some $\mu \in s$ and $F, F_0 \in \mathrm{dom}(a)$ (or $F, F_1 \in \mathrm{dom}(a)$), then $F_1 \in \mathrm{dom}(a)$ (or $F_0 \in \mathrm{dom}(a)$).

(s) (The isomorphism condition) Let $F_0, F_1, F \in A^{1\mu}(\theta') \cap \mathrm{dom}(a)$ be a triple of a Δ - system type, for some $\mu \in s$. Then

$$\langle a(F_0) \cap H(\chi^{+k}), \in \rangle \simeq \langle a(F_1) \cap H(\chi^{+k}), \in \rangle$$

where k is the minimal so that $a(F_0) \subseteq H(\chi^{+k})$ or $a(F_1) \subseteq H(\chi^{+k})$.
Note that it is possible to have for example $a(F_0) \prec H(\chi^{+6})$ and $a(F_1) \prec H(\chi^{+18})$. Then we take $k = 6$.
Let π be the isomorphism between

$$\langle a(F_0) \cap H(\chi^{+k}), \in \rangle, \langle a(F_1) \cap H(\chi^{+k}), \in \rangle$$

and $\pi_{F_0F_1}$ be the isomorphism between F_0 and F_1. Require that for each $Z \in F_0 \cap \mathrm{dom}(a)$ we have $\pi_{F_0F_1}(Z) \in F_1 \cap \mathrm{dom}(a)$ and

$$\pi(a(Z) \cap H(\chi^{+k})) = a(\pi_{F_0F_1}(Z)) \cap H(\chi^{+k}).$$

3. $\{\alpha < \theta' \mid \alpha \in \mathrm{dom}(a)$ or it is a code of an element of $\mathrm{dom}(a)\} \cap \mathrm{dom}(f) = \emptyset$

4. $A \in E_{\lambda_n, a(\max(a))}$

5. $\min(A) > |dom(a)| + |\operatorname{dom}(\underset{\sim}{b})|$

6. for every ordinals α, β, γ which are elements of $\operatorname{rng}(a)$ or actually the ordinals coding models in $\operatorname{rng}(a)$ we have

$$\alpha \geq_{E_{\lambda_n}} \beta \geq_{E_{\lambda_n}} \gamma \quad \text{implies}$$
$$\pi_{\lambda_n,\alpha,\gamma}(\rho) = \pi_{\lambda_n,\beta,\gamma}(\pi_{\lambda_n,\alpha,\beta}(\rho))$$

for every $\rho \in \pi''_{\lambda_n,\max \operatorname{rng}(a),\alpha}(A)$.

Let us turn now to the second component of a condition, i. e. to $\langle \underset{\sim}{b}, \underset{\sim}{B}, g\rangle$.

7. g is a function from θ to κ_n of cardinality at most κ

8. $\underset{\sim}{b}$ is a name, depending on $\langle a, A\rangle$, of a partial function of cardinality less than λ_n. So, each choice of an element from A gives the actual function which is in V. Note that the relevant forcing is the One Element Prikry Forcing on Extender, which does not change V, i.e. it is trivial.

The following conditions are satisfied:

(a) (Domain)

the domain of $\underset{\sim}{b} \in V$, i.e. it is already decided in the sense that each choice of an element in A will give the same domain.

(b) (Background condition) There is $\langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in G(\mathcal{P}'(\theta))$ which we call it further **a background condition of $\underset{\sim}{b}$**.

(c) (Supports) $s \cap \theta' \subseteq s(\theta')$.

(d) for each $X \in \operatorname{dom}(\underset{\sim}{b})$ and each $\nu \in A$ there is $k \leq \omega$ so that the interpretation according to ν of $\underset{\sim}{b}(X)$ is a subset of $H(\chi^{+k})$.

Moreover,

i. if $|X| = (\theta')^+$, then it is forced that $|\underset{\sim}{b}(X)| = \kappa_n^{+n+1}$, i.e. any choice of an element from A interprets $\underset{\sim}{b}(X)$ in such a way.

ii. if $|X| > (\theta')^+$ then it is forced that $|\underset{\sim}{b}(X)| > \kappa_n^{+n+1}$.

iii. if $|X| < (\theta')^+$, then $A^{0|X|}(\theta') \in \operatorname{dom}(a)$ and for each $\nu \in A$ the interpretation of $\underset{\sim}{b}(X)$ according to ν has cardinality corresponding to those of $|a(A^{0|X|}(\theta')|$, i.e.

$$\pi_{\lambda_n,\max(\operatorname{rng}(a)),|a(A^{0|X|}(\theta')|}(\nu).$$

The above conditions mean that the correspondence splits over θ'. Thus, as in the case $\theta = \kappa^{+3}, \theta' = \kappa^+$ we have models of cardinalities below θ' correspond to those of cardinalities below λ_n and the models of cardinalities $\geq \theta'$ to those of cardinalities κ^{+n+1} and above. In the previous case we had models of cardinalities κ^+ and κ^{++} only. Here we can have plenty of them.

iv. if $|X| = \theta$, then it is forced that $|\underset{\sim}{b}(X)| = \delta_n$ and $\underset{\sim}{b}(X) \cap \delta_n^+ \in ORD$

v. $A^{0\kappa^+}, (A^{0\kappa^+})^-, (A^{0\theta})^-$ are in $\mathrm{dom}(\underset{\sim}{b})$.

Further let us refer to $A^{0\kappa^+}$ as **the maximal model of the domain of** $\underset{\sim}{b}$ and to $\langle (A^{0\tau})^- | (A^{0\tau})^- \in \mathrm{dom}(\underset{\sim}{b})\rangle$ as **the maximal sequence of the domain of** $\underset{\sim}{b}$. Denote it as $\max(\mathrm{dom}(\underset{\sim}{b}))$.

Later passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

(e) (Models come from $A^{0\kappa^+}$) If $X \in \mathrm{dom}(\underset{\sim}{b})$ and $X \neq A^{0\kappa^+}$, then $X \in A^{0\kappa^+}$.

(f) Let $E, F \in \mathrm{dom}(\underset{\sim}{b})$, $E \in F$ (or $E \subseteq F$) and $\nu \in A$. If k is the minimal so that the interpretation of $\underset{\sim}{b}(E)$ according to ν is a subset of $H(\chi^{+k})$ or $\underset{\sim}{b}(F)$ according to ν is a subset of $H(\chi^{+k})$, then

$$\underset{\sim}{b}(E)[\nu] \cap H(\chi^{+k}) \in \underset{\sim}{b}(F)[\nu] \cap H(\chi^{+k})$$

$$(\text{or } \underset{\sim}{b}(E)[\nu] \cap H(\chi^{+k}) \subseteq \underset{\sim}{b}(F)[\nu] \cap H(\chi^{+k})),$$

where in the last two lines we mean the interpretations according to ν. Let us further deal with such interpretations without mentioning this explicitly.

The intuitive meaning is that b is supposed to preserve membership and inclusion. But we cannot literally require this since $b(E)$ and $b(F)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(g) The image by b of $A^{0\kappa^+}$, i.e. $b(A^{0\kappa^+})$, intersected with δ_n^+ is above all (i.e. is forced by each $\nu \in A$ to be such) the rest of $\mathrm{rng}(b)$ restricted to δ_n^+ in the ordering of the extender E_{κ_n} (via some reasonable coding by ordinals).

Recall that the extender E_{κ_n} acts on δ_n^+ and our main interest is in Prikry sequences it will produce. So, parts of $\mathrm{rng}(b)$ restricted to δ_n^+ will play the central role.

Let us, as in [6], denote by $otp_\tau(X) \quad \tau \in s$ the order type of the maximal under inclusion chain of elements in $\mathcal{P}(X) \cap A^{1\tau}$ which is just the order type of $C^\tau(X)$, for $X \in A^{1\tau}$. If $X \in C^\tau(A^{0\tau})$, then $C^\tau(X) = C^\tau(A^{0\tau}) \cap (X \cup \{X\}) = C^\tau(A^{0\tau}) \restriction X+1$. Hence, in this case, $otp_\tau(X) = otp(C^\tau(A^{0\tau}) \restriction X)+1$. Note that $otp_\tau(X)$ is always a successor ordinal below τ^+. Recall that by [6] we have for each $X \in A^{1\tau}$ an element $Y \in C^\tau(A^{0\tau})$ such that $otp_\tau(X) = otp_\tau(Y)$.

Next conditions deal with the connection between the structure over λ_n and those over κ_n. Note that there were no similar structures in the previous papers [4], [6].

(h) (Order types) If $X \in \mathrm{dom}(\underset{\sim}{b}) \cap A^{1\tau}$, then $otp_\tau(A^{0\tau}(\theta')) = otp(C^\tau(\theta')(A^{0\tau}(\theta')) \geq otp_\tau(X)$. Note that by 8(d)iii we have $A^{0\tau}(\theta') \in \mathrm{dom}(a)$.

Denote by $X(\lambda_n)$ the least element Z of $C^\tau(\theta')(A^{1\tau}(\theta'))$ with $otp_\tau(Z) \geq otp_\tau(X)$. It will be the one corresponding to X at the level λ_n.

(i) $X(\lambda_n) \in \mathrm{dom}(a)$.

The next condition insures that the function $otp_\tau(X) \to X(\lambda_n)$ is order preserving.

(j) (Order preservation) If $X, X' \in \mathrm{dom}(\underset{\sim}{b})$, then

- $otp_\tau(X) = otp_\tau(X')$ iff $X(\lambda_n) = X'(\lambda_n)$
- $otp_\tau(X) < otp_\tau(X')$ iff $X(\lambda_n) \subset X'(\lambda_n)$

(k) (Dependence) Let $X \in \mathrm{dom}(\underset{\sim}{b}) \cap C^\tau(A^{0\tau})$. Then $\underset{\sim}{b}(X)$ depends on the value of the one element Prikry forcing with the measure $a(X(\lambda_n))$ over λ_n. More precisely: let $A(X) = \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}{}''A$, then each choice of an element from $A(X)$ already decides $\underset{\sim}{b}(X)$, i.e. whenever $\nu_1, \nu_2 \in A$ and

$$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu_1) = \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu_2)$$

we have

$$\underset{\sim}{b}(X)[\nu_1] = \underset{\sim}{b}(X)[\nu_2].$$

Further let us denote, for $\nu \in A$, the projection of ν to $A(X)$, i.e. $\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(X(\lambda_n))}(\nu)$, by $\nu(X)$.

So $\underset{\sim}{b}(X)$ depends only on members of $A(X)$ rather than those of A.

The next condition is crucial for the κ^{++}-c.c. of the forcing.

(l) (Inclusion condition)

Let $\nu, \nu' \in A, \nu < \nu'$. Then

- $\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu) \in \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu')$ implies

$$\underset{\sim}{b}(A^{0\kappa^+})[\nu] \in \underset{\sim}{b}(A^{0\kappa^+})[\nu'].$$

- If $Y \in \mathrm{dom}(\underset{\sim}{b}) \cap C^{\kappa^+}(A^{0\kappa^+})$ and

$$\pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(A^{0\kappa^+}(\lambda_n))}(\nu) \in \pi^{E_{\lambda_n}}_{\max \mathrm{rng}(a), a(Y(\lambda_n))}(\nu'),$$

  then either

$$\underset{\sim}{b}(A^{0\kappa^+})[\nu] \in \underset{\sim}{b}(Y)[\nu']$$

  or

  the k-type realized by $\underset{\sim}{b}(A^{0\kappa^+})[\nu] \cap H(\chi^{+k})$ is in $\underset{\sim}{b}(Y)[\nu']$, where $k < \omega$ is the least such that $\underset{\sim}{b}(Y)[\nu'] \subseteq H(\chi^{+k+1})$.

  The same holds over any element of $\underset{\sim}{b}(Y)[\nu']$, i.e. $tp_k(z, \underset{\sim}{b}(A^{0\kappa^+})[\nu] \cap H(\chi^{+k})) \in \underset{\sim}{b}(Y)[\nu']$, for any $z \in \underset{\sim}{b}(Y)[\nu']$.

  We require in addition that this $k > 2$.

  Let us allow the above also if $\underset{\sim}{b}(Y)[\nu'] \subseteq H(\chi^{+\omega})$. In this case we take k to be any natural number above 2 and require that once we go up to the higher levels then corresponding k's increase (with n).

We cannot in general require only that

$$\underset{\sim}{b}(A^{0\kappa^+})[\nu] \in \underset{\sim}{b}(Y)[\nu']$$

since extending conditions the sequence C^{κ^+} of the maximal model of a new background condition may go not through the old maximal model. But still having the type inside Y will be enough for our purposes.

It is possible to have $Y \subset X$, but $\nu(X)$ smaller than $\nu'(Y)$ (note that $\nu(Y) < \nu(X)$ in this case by 8j). In such situation the interpretation will reverse the order.

Note that given $\nu' \in A$ the number of possibilities for $\nu \in \nu' \cap A$ is bounded now by $(\nu'^0)^{+(\nu'^0)^{+n+2}+1}$ (i.e. the the ordinal corresponding to δ'_n), as $\nu' < (\nu'^0)^{+(\nu'^0)^{+n+2}+1}$.

(m) If $X \in \mathrm{dom}(\underset{\sim}{b})$ then $C^{|X|}(X) \cap \mathrm{dom}(\underset{\sim}{b})$ is a closed chain. Let $\langle X_i | i < j \rangle$ be its increasing continuous enumeration. For each $l < j$ consider the final segment

$\langle X_i | l \leq i < j \rangle$ and its image $\langle \underset{\sim}{b}(X_i) | l \leq i < j \rangle$. Find the minimal k so that

$$\underset{\sim}{b}(X_i) \subseteq H(\chi^{+k}) \text{ for each } i, l \leq i < j.$$

Then the sequence

$$\langle \underset{\sim}{b}(X_i) \cap H(\chi^{+k}) | l \leq i < j \rangle$$

is increasing and continuous. More precisely, each $\nu \in A$ forces this.

Note that k here may depend on l, i.e. on the final segment.

(n) (The walk is in the domain) If $X \in \mathrm{dom}(\underset{\sim}{b}) \cap A^{1\xi}$, for some $\xi \in s$, then the general walk from $(A^{0\xi})^-$ to X is forced by each $\nu \in A$ to be in $\mathrm{dom}(\underset{\sim}{b})$.

(o) If $X \in \mathrm{dom}(\underset{\sim}{b}) \cap A^{1\xi}$, for some $\xi \in s$ is a limit model and $\mathrm{cof}(otp_\xi(X) - 1) < \kappa_n$ (i.e. the cofinality of the sequence $C^\xi(X) \backslash \{X\}$ under the inclusion relation is less than κ_n) then a closed cofinal subsequence of $C^\xi(X) \backslash \{X\}$ is in $\mathrm{dom}(\underset{\sim}{b})$. The images of its members under b form a closed cofinal in $b(X)$ sequence.

(p) If $\langle X_i | i < j \rangle$ is an increasing (under the inclusion) sequence of elements of $\mathrm{dom}(\underset{\sim}{b})$ with $X_i \in C^{\tau_i}(A^{0\tau_i})$, $i < j$, then $\bigcup_{i<j} X_i \in \mathrm{dom}(\underset{\sim}{b})$ as well.
Note that $\bigcup_{i<j} X_i \in C^{\cup_{i<j}\tau_i}(A^{0 \cup_{i<j}\tau_i})$. So, in particular, by [6](Definition 1.1noholes) also $A^{0\cup_{i<j}\tau_i} \in \mathrm{dom}(\underset{\sim}{b})$.

(q) (The minimal models condition) Suppose that $X \in \mathrm{dom}(\underset{\sim}{b}) \cap C^\xi(A^{0\xi})$, for some $\xi \in s \backslash \kappa^+ + 1$. Let $\tau \in s$ and $X^* \in C^\tau(A^{0\tau})$ be such that $\tau < \xi$, $X \in X^*$ and for each $\rho, \tau \leq \rho < \xi$, $Z \in C^\rho(A^{0\rho})$ we have $X \in Z$ implies $X^* \in Z$ or $X^* = Z$. Then $X^* \in \mathrm{dom}(\underset{\sim}{b})$ as well as $(X^*)^-$-its immediate predecessor in $C^\tau(A^{0\tau})$.
In addition, we require the following:
if $(X^*)^- \notin X$, then for each $H \in \underset{\sim}{b}((X^*)^-)$ there is $H' \in \underset{\sim}{b}((X^*)^-)$ with $H \in H'$ and $\underset{\sim}{b}(X) \subseteq H'$. Moreover, if $|\underset{\sim}{b}(X)| \in \underset{\sim}{b}((X^*)^-)$, then $|H'| = |a(X)|$. If $|\underset{\sim}{b}(X)| \notin \underset{\sim}{b}((X^*)^-)$, then $|H'| = \min(\underset{\sim}{b}((X^*)^-) \cap ORD \backslash |\underset{\sim}{b}(X)|)$.
Note that $X \in A^{0\kappa^+} \in \mathrm{dom}(\underset{\sim}{b})$, by [6] Definition 1.1(noholes). So X^* always exists.
The second part of the condition insures that there will be enough models in $\underset{\sim}{b}((X^*)^-)$ to allow extensions which will include $\underset{\sim}{b}(X)$.

(r) (Minimal cover condition) Let $E \in A^{0\xi} \cap \mathrm{dom}(\underset{\sim}{b}), X \in A^{0\tau} \cap \mathrm{dom}(\underset{\sim}{b})$, for some $\xi < \tau$ in s. Suppose that $E \not\subseteq X$. Then

- $\tau \in E$ implies that the smallest model of $E \cap C^{\tau}(A^{0\tau})$ including X is in $\operatorname{dom}(\underset{\sim}{b})$
- $\tau \notin E$ implies that the smallest model of $E \cap C^{\rho}(A^{0\rho})$ including X is in $\operatorname{dom}(\underset{\sim}{b})$, for $\rho = \min(A \cap s \backslash \tau)$.

(s) (The first models condition) Suppose that $E \in dom(\underset{\sim}{b}) \cap C^{\tau}(A^{0\tau}), F \in \operatorname{dom}(\underset{\sim}{b}) \cap C^{\rho}(A^{0\rho})$, $\sup(E) > \sup(F)$ and $F \notin E$. for some $\tau < \rho, \tau, \rho \in s$. Let $\eta = \min((E \cap s)\backslash\rho)$. Then the first model $H \in A \cap C^{\eta}(A^{0\eta})$ which includes F is in $\operatorname{dom}(\underset{\sim}{b})$.

(t) (Models witnessing Δ-system type are in the domain) If $F_0, F_1, F \in A^{1\kappa^+} \cap \operatorname{dom}(\underset{\sim}{b})$ is a triple of a Δ - system type, then the corresponding models $G_0, G_0^*, G_1, G_1^*, G^*$, as in the definition of a Δ - system type (see [6]), are in $\operatorname{dom}(\underset{\sim}{b})$ as well and

$$\underset{\sim}{b}(F_0) \cap \underset{\sim}{b}(F_1) = \underset{\sim}{b}(F_0) \cap \underset{\sim}{b}(G_0) = \underset{\sim}{b}(F_1) \cap \underset{\sim}{b}(G_1).$$

(u) If $F_0, F_1, F \in A^{1\mu}$ is a triple of a Δ - system type, for some $\mu \in s$ and $F, F_0 \in \operatorname{dom}(\underset{\sim}{b})$ (or $F, F_1 \in \operatorname{dom}(\underset{\sim}{b})$), then $F_1 \in \operatorname{dom}(\underset{\sim}{b})$ (or $F_0 \in \operatorname{dom}(\underset{\sim}{b})$).

(v) (The isomorphism condition) Let $F_0, F_1, F \in A^{1\kappa^+} \cap \operatorname{dom}(\underset{\sim}{b})$ be a triple of a Δ - system type. Then

$$\langle \underset{\sim}{b}(F_0) \cap H(\chi^{+k}), \in \rangle \simeq \langle \underset{\sim}{b}(F_1) \cap H(\chi^{+k}), \in \rangle$$

where k is the minimal so that $\underset{\sim}{b}(F_0) \subseteq H(\chi^{+k})$ or $\underset{\sim}{b}(F_1) \subseteq H(\chi^{+k})$.
Note that it is possible to have for example $\underset{\sim}{b}(F_0) \prec H(\chi^{+6})$ and $\underset{\sim}{b}(F_1) \prec H(\chi^{+18})$. Then we take $k = 6$.
Let π be the isomorphism between

$$\langle \underset{\sim}{b}(F_0) \cap H(\chi^{+k}), \in \rangle, \langle \underset{\sim}{b}(F_1) \cap H(\chi^{+k}), \in \rangle$$

and $\pi_{F_0F_1}$ be the isomorphism between F_0 and F_1. Require that for each $Z \in F_0 \cap \operatorname{dom}(\underset{\sim}{b})$ we have $\pi_{F_0F_1}(Z) \in F_1 \cap \operatorname{dom}(\underset{\sim}{b})$ and

$$\pi(\underset{\sim}{b}(Z) \cap H(\chi^{+k})) = \underset{\sim}{b}(\pi_{F_0F_1}(Z)) \cap H(\chi^{+k}).$$

(w) $\{\alpha < \kappa^{+3} \mid \alpha \in \operatorname{dom}(\underset{\sim}{b})\} \cap \operatorname{dom}(g) = \emptyset$.

(x) For each $\nu \in A$ we have $\underset{\sim}{B}[\nu] \in E_{\kappa_n, \underset{\sim}{b}[\nu](\max(\underset{\sim}{b}))}$.

(y) for every $\nu \in A$ and every ordinals α, β, γ which are elements of $\mathrm{rng}(\underset{\sim}{b})[\nu]$ or actually the ordinals coding models in $\mathrm{rng}(\underset{\sim}{b})[\nu]$ we have

$$\alpha \geq_{E_{\kappa_n}} \beta \geq_{E_{\kappa_n}} \gamma \quad \text{implies}$$
$$\pi_{\kappa_n,\alpha,\gamma}(\rho) = \pi_{\kappa_n,\beta,\gamma}(\pi_{\kappa_n,\alpha,\beta}(\rho))$$

for every $\rho \in \pi''_{\kappa_n,\max \mathrm{rng}(\underset{\sim}{b}[\nu]),\alpha}(\underset{\sim}{B}[\nu])$.

The definition of the order $\leq_{Q_{n0}}$ on Q_{n0} repeats Definition 3.2. Define Q_{n1} as follows:

Definition 5.2 Q_{n1} consists of pairs $\langle f, g\rangle$ such that

1. f is a partial function from θ' to λ_n of cardinality at most κ

2. g is a partial function from θ to κ_n of cardinality at most κ

Q_{n1} is ordered by extension. Denote this order by $\leq_1$.
So, it is basically the Cohen forcing for adding θ Cohen subsets to κ^+.

The ordered sets $\langle Q_n, \leq_n, \leq_n^* \rangle$ and $\langle \mathcal{P}, \leq, \leq^*, \rightarrow \rangle$ are defined exactly as in Section 2.
The properties of $\langle \mathcal{P}, \leq, \leq^*, \rightarrow \rangle$ are similar to those of the forcing of Section 2.

Lemma 5.3 *Let $p = \langle p_k \mid k < \omega\rangle \in \mathcal{P}$, $p_k = \langle\langle a_k, A_k, f_k\rangle, \langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k\rangle\rangle$ for $k \geq \ell(p)$ and X be a model appearing in an element of $G(\mathcal{P}'(\theta'))$. Suppose that*

(a) $X \notin \bigcup_{\ell(p)\leq k<\omega} \mathrm{dom}(a_k) \cup \mathrm{dom}(f_k)$

(b) X is a successor model or if it is a limit one with $\mathrm{cof}(otp_{|X|}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega\rangle$, $q_k = \langle\langle a'_k, A'_k, f'_k\rangle, \langle \underset{\sim}{b}'_k, \underset{\sim}{B}'_k, g_k\rangle\rangle$ for $k \geq \ell(q)$, of p so that starting with some $n \geq \ell(q)$ we have $X \in \mathrm{dom}(a'_k)$ for each $k \geq n$. In addition the second part of the condition p, i.e. $\langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k\rangle$ remains basically unchanged (just names should be lifted to new A_k's).

Lemma 5.4 *Let $p = \langle p_k \mid k < \omega\rangle \in \mathcal{P}$, $p_k = \langle\langle a_k, A_k, f_k\rangle, \langle \underset{\sim}{b}_k, \underset{\sim}{B}_k, g_k\rangle\rangle$ for $k \geq \ell(p)$ and X be a model appearing in an element of $G(\mathcal{P}')$. Suppose that*

(a) $X \notin \bigcup_{\ell(p)\leq k<\omega} \mathrm{dom}(\underset{\sim}{b}_k) \cup \mathrm{dom}(g_k)$

(b) X is a successor model or if it is a limit one with $\mathrm{cof}(otp_{|X|}(X) - 1) > \kappa$

Then there is a direct extension $q = \langle q_k \mid k < \omega\rangle$, $q_k = \langle\langle a'_k, A'_k, f'_k\rangle, \langle \underset{\sim}{b}'_k, \underset{\sim}{B}'_k, g_k\rangle\rangle$ *for* $k \geq \ell(q)$, *of* p *so that starting with some* $n \geq \ell(q)$ *we have* $X \in \mathrm{dom}(\underset{\sim}{b}'_k)$ *for each* $k \geq n$.

Lemma 5.5 *Let* $n < \omega$. *Then* $\langle Q_{n0}, \leq_0\rangle$ *does not add new sequences of ordinals of the length* $< \lambda_n$, *i.e. it is* (λ_n, ∞) – *distributive.*

Lemma 5.6 $\langle \mathcal{P}, \leq^*\rangle$ *does not add new sequences of ordinals of the length* $< \kappa_0$.

Lemma 5.7 $\langle \mathcal{P}, \leq^*\rangle$ *satisfies the Prikry condition.*

Let us turn now to the chain condition lemma. Its proof is similar to those of 4.6, but contains an additional point.

Lemma 5.8 $\langle \mathcal{P}, \rightarrow\rangle$ *satisfies* κ^{++}*-c.c.*

Proof. Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++}\rangle$ be a name of an antichain of the length κ^{++}. As in [6], using the κ^{++}-strategic closure of $\mathcal{P}(\theta')$ and $\mathcal{P}'(\theta)$ ([6, 1.6]) we find an increasing sequence

$$\langle\langle\langle A^{0\tau}_\alpha(\theta), A^{1\tau}_\alpha(\theta), C^\tau_\alpha(\theta)\rangle \mid \tau \in s_\alpha, \alpha < \kappa^{++}\rangle, \langle\langle A^{0\tau}_\alpha(\theta'), A^{1\tau}_\alpha(\theta'), C^\tau_\alpha(\theta')\rangle \mid \tau \in s'_\alpha, \alpha < \kappa^{++}\rangle\rangle$$

of elements of $\mathcal{P}'(\theta) \times \mathcal{P}'(\theta')$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++}\rangle$ so that for every $\alpha < \kappa^{++}$ the following holds:

1. $\langle\langle\langle A^{0\tau}_{\alpha+1}(\theta), A^{1\tau}_{\alpha+1}(\theta), C^\tau_{\alpha+1}(\theta)\rangle \mid \tau \in s_{\alpha+1}\}\rangle, \langle\langle A^{0\tau}_{\alpha+1}(\theta'), A^{1\tau}_{\alpha+1}(\theta'), C^\tau_{\alpha+1}(\theta')\rangle \mid \tau \in s'_{\alpha+1}\rangle\rangle \Vdash \underset{\sim}{p}_\alpha = \check{p}_\alpha$
2. if α is a limit ordinal, then $s'_\alpha = \bigcup_{\beta<\alpha} s'_\beta$
3. if α is a limit ordinal, then $\bigcup\{A^{0\tau}_\beta(\theta') | \beta < \alpha, \tau \in s'_\beta\} = A^{0\tau}_\alpha(\theta')$
4. if α is a limit ordinal, then $s_\alpha = \bigcup_{\beta<\alpha} s_\beta$
5. if α is a limit ordinal, then $\bigcup\{A^{0\tau}_\beta(\theta) | \beta < \alpha, \tau \in s_\beta\} = A^{0\tau}_\alpha(\theta)$
6. ${}^{\tau>}A^{0\tau}_{\alpha+1}(\theta') \subseteq A^{0\tau}_{\alpha+1}(\theta')$, for each $\tau \in s'_{\alpha+1}$
7. ${}^{\tau>}A^{0\tau}_{\alpha+1}(\theta) \subseteq A^{0\tau}_{\alpha+1}(\theta)$, for each $\tau \in s_{\alpha+1}$
8. $A^{0\tau}_{\alpha+1}(\theta')$ is a successor model, for each $\tau \in s'_{\alpha+1}$

9. $A^{0\tau}_{\alpha+1}(\theta)$ is a successor model, for each $\tau \in s_{\alpha+1}$

10. $\langle\langle \cup A^{1\tau}_{\beta}(\theta') \mid \tau \in s'_{\beta}\rangle \mid \beta < \alpha\rangle \in (A^{0\kappa^+}_{\alpha+1}(\theta'))^-$ (i.e. the immediate predecessor over $C^{\kappa^+}_{\alpha+1}(\theta')$)

11. for every $\alpha \leq \beta < \kappa^{++}, \tau \in s'_{\alpha}$ we have

$$A^{0\tau}_{\alpha}(\theta') \in C^{\beta}(\theta')(A^{0\tau}_{\beta}(\theta'))$$

12. $A^{0\tau}_{\alpha+2}(\theta')$ is not an immediate successor model of $A^{0\tau}_{\alpha+1}(\theta')$, for every $\alpha < \kappa^{++}, \tau \in s'_{\alpha+1}$.

13. $\langle\langle \cup A^{1\tau}_{\beta}(\theta) \mid \tau \in s_{\beta}\rangle \mid \beta < \alpha\rangle \in (A^{0\kappa^+}_{\alpha+1}(\theta))^-$ (i.e. the immediate predecessor over $C^{\kappa^+}_{\alpha+1}(\theta)$)

14. for every $\alpha \leq \beta < \kappa^{++}, \tau \in s_{\alpha}$ we have

$$A^{0\tau}_{\alpha}(\theta) \in C^{\beta}(\theta)(A^{0\tau}_{\beta}(\theta))$$

15. $A^{0\tau}_{\alpha+2}(\theta)$ is not an immediate successor model of $A^{0\tau}_{\alpha+1}(\theta)$, for every $\alpha < \kappa^{++}, \tau \in s_{\alpha+1}$.

16. $p_{\alpha} = \langle p_{\alpha n} | n < \omega\rangle$

17. for every $n \geq \ell(p_{\alpha})$ the maximal model of $\mathrm{dom}(a_{\alpha n})$ is $A^{0\kappa^+}_{\alpha+1}(\theta')$ and the maximal model of $\mathrm{dom}(\underset{\sim}{b}_{\alpha n})$ is $A^{0\kappa^+}_{\alpha+1}(\theta)$, where $p_{\alpha n} = \langle\langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle, \langle \underset{\sim}{b}_{\alpha n}, \underset{\sim}{B}_{\alpha n}, g_{\alpha n}\rangle\rangle$

Let $p_{\alpha n} = \langle\langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle, \langle \underset{\sim}{b}_{\alpha n}, \underset{\sim}{B}_{\alpha n}, g_{\alpha n}\rangle\rangle$ for every $\alpha < \kappa^{++}$ and $n \geq \ell(p_{\alpha})$. Extending by 5.4 if necessary, let us assume that $A^{0\kappa^+}_{\alpha} \in \mathrm{dom}(a_{\alpha n})$ and $A^{0\kappa^+}_{\alpha}(\theta) \in \mathrm{dom}(\underset{\sim}{b})$, for every $n \geq \ell(p_{\alpha})$. Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^+$ the following holds:

(1) $\ell = \ell(p_{\alpha}) = \ell(p_{\beta})$

(2) for every $n < \ell \quad p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1}

(3) for every $n,\ \ell \leq n < \omega \quad \langle \mathrm{dom}(a_{\alpha n}), \mathrm{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_{0}(\theta')$

(4) for every $n,\ \omega > n \geq \ell \quad \mathrm{rng}(a_{\alpha n}) = \mathrm{rng}(a_{\beta n})$.

(5) for every $n,\ \omega > n \geq \ell \quad A_{\alpha n} = A_{\beta n}$

(6) for every n, $\ell \leq n < \omega$ $\langle \text{dom}(\underset{\sim}{b}_{\alpha n}), \text{dom}(g_{\alpha n}) \mid \alpha < \kappa^{++} \rangle$ form a Δ-system with the kernel contained in $A_0^{0\kappa^+}(\theta)$.

Remember that the domain of $\underset{\sim}{b}$ is not a name but rather a set.

(7) for every n, $\omega > n \geq \ell$ $\text{rng}(\underset{\sim}{b}_{\alpha n}) = \text{rng}(\underset{\sim}{b}_{\beta n})$, i.e. it is just the same name in the one element Prikry forcing.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let α be in S. For each $n, \ell \leq n < \omega$, there will be $\beta(\alpha, n) < \alpha$ such that

- $\text{dom}(a_{\alpha n}) \cap A_\alpha^{0\kappa^+}(\theta') \subseteq A_{\beta(\alpha,n)}^{0\kappa^+}(\theta')$

  and

- $\text{dom}(\underset{\sim}{b}_{\alpha n}) \cap A_\alpha^{0\kappa^+}(\theta) \subseteq A_{\beta(\alpha,n)}^{0\kappa^+}(\theta)$.

Just recall that $|a_{\alpha n}| < \lambda_n$ and $|\text{dom}(\underset{\sim}{b}_{\alpha n})| < \lambda_n$. Shrink S to a stationary subset S^* so that for some $\alpha^* < \min S^*$ of cofinality κ^+ we will have $\beta(\alpha, n) < \alpha^*$, whenever $\alpha \in S^*, \ell \leq n < \omega$. Now, the cardinality of both $A_{\alpha^*}^{0\kappa^+}(\theta')$ and $A_{\alpha^*}^{0\kappa^+}(\theta)$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\alpha, \beta \in S^*, \ell \leq n < \omega$

- $\text{dom}(a_{\alpha n}) \cap A_\alpha^{0\kappa^+}(\theta') = \text{dom}(a_{\beta n}) \cap A_\beta^{0\kappa^+}(\theta')$

  and

- $\text{dom}(\underset{\sim}{b}_{\alpha n}) \cap A_\alpha^{0\kappa^+}(\theta) = \text{dom}(\underset{\sim}{b}_{\beta n}) \cap A_\beta^{0\kappa^+}(\theta)$.

Let us add both $A_{\alpha^*}^{0\kappa^+}(\theta')$ and $A_{\alpha^*}^{0\kappa^+}(\theta)$ to each $p_\alpha, \alpha \in S^*$. By 5.3,5.4, it is possible to do this without adding other additional models except the images of this models under isomorphisms. Thus, $A_{\alpha^*}^{0\kappa^+}(\theta) \in (C^{\kappa^+}(\theta))(A_\alpha^{0\kappa^+}(\theta))$ and $A_\alpha^{0\kappa^+}(\theta) \in \text{dom}(\underset{\sim}{b}_{\alpha n}) \cap (C^{\kappa^+}(\theta))(A^{0\kappa^+}(\theta))_{\alpha+1})$. So, 5.1(8q) was already satisfied after adding $A_\alpha^{0\kappa^+}(\theta)$. The rest of 5.1 does not require adding additional models in the present situation.

Denote the result for simplicity by p_α as well. Note that (again by 5.4 and the argument above) any $A_\gamma^{0\kappa^+}(\theta)$ for $\gamma \in S^* \cap (\alpha^*, \alpha)$ or, actually any other successor or limit model $X \in C^{\kappa^+}(\theta)(A_\alpha^{0\kappa}(\theta))$ with $\text{cof}(otp_{\kappa^+}(X)) = \kappa^+$, which is between $A_{\alpha^*}^{0\kappa^+}(\theta)$ and $A_\alpha^{0\kappa^+}(\theta)$ can be added without adding other additional models or ordinals except the images of it under isomorphisms.

The same holds once we replace θ by θ'.

Let now $\beta < \alpha$ be ordinals in S^*. We claim that p_β and p_α are compatible in $\langle \mathcal{P}, \rightarrow \rangle$. First extend p_α by adding to it both $A^{0\kappa^+}_{\beta+2}(\theta')$ and $A^{0\kappa^+}_{\beta+2}(\theta)$. As it was remarked above, this will not add other additional models or ordinals except the images of this models under isomorphisms to p_α. Let p be the resulting extension. Denote p_β by q. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$, $p_n = \langle \langle a_n, A_n, f_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$ and $q_n = \langle \langle a'_n, A_n, f'_n \rangle, \langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n, g'_n \rangle \rangle$. Note that by (5) above the sets of measure one of p_n, q_n are the same. Without loss of generality we may assume that $a_n(A^{0\kappa^+}_{\beta+2}(\theta'))$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $\mathrm{rng}(a'_n)$ inside $a_n(A^{0\kappa^+}_{\beta+2}(\theta'))$ over the common parts $\mathrm{dom}(a'_n)$ and $\mathrm{dom}(a_n)$. This will produce $\langle a''_n, A'_n, f'_n \rangle$ which is k_n-1-equivalent to $\langle a'_n, A'_n, f'_n \rangle$ and with $\mathrm{rng}(a''_n) \subseteq a_n(A^{0\kappa^+}_{\beta+2}(\theta'))$. Doing the above for all $n \geq \ell(p)$ we will obtain $\langle \langle a''_n, A'_n, f'_n \rangle \mid n < \omega \rangle$ equivalent to $\langle \langle a'_n, A'_n, f'_n \rangle \mid n < \omega \rangle$ (i.e. $\langle \langle a''_n, A''_n, f''_n \rangle \mid n < \omega \rangle \longleftrightarrow \langle \langle a'_n, A'_n, f'_n \rangle \mid n < \omega \rangle$).

Let $t = \langle \langle \langle a''_n, A'_n, f'_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle \mid n < \omega \rangle$. Extend t to t' by adding to it

$$\langle A^{0\kappa^+}_{\beta+2}(\theta'), a_n(A^{0\kappa^+}_{\beta+2}(\theta')) \rangle$$

as the maximal set for every $n \geq \ell(p)$. Recall that $A^{0\kappa^+}_{\beta+1}(\theta')$ was its maximal model. So we are adding a top model, also, by the condition (15) above $A^{0\kappa^+}_{\beta+2}(\theta')$ is not an immediate successor of $A^{0\kappa^+}_{\beta+1}(\theta')$. Hence no additional models or ordinals are added at all.

Let $t'_n = \langle \langle a'''_n, A'''_n, f'_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$, for every $n \geq \ell(p)$.

Combine now the first coordinates of p and t' together, i.e. $\langle a_n, A_n, f_n \rangle$'s with those of t'. Thus for each $n \geq \ell(p)$ we add a'''_n to a_n. Add if necessary a new top model to insure 5.1(2(d)). Let $r = \langle r_n | n < \omega \rangle$ be the result, where $r_n = \langle \langle c_n, C_n, h_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle$, for $n \geq \ell(p)$.

Claim 4.6.1 $r \in \mathcal{P}$ and $r \geq p$.

Proof. Fix $n \geq \ell(p)$. The main points here are that a'''_n and a_n agree on the common part and adding of a'''_n to a_n does not require other additions of models except the images of a'''_n under isomorphisms.

The check of the rest of conditions of 5.1 is routine. We refer to [2] or [4] for similar arguments.

□ of the claim.

Now let us turn to the second coordinates of q and r. Recall that for a condition $x \in Q_{n0}$ we denote by $(x)_0$ its first coordinate, i.e. the first triple. If $y = \langle y_n | n < \omega \rangle \in \mathcal{P}$, then

$(y)_0$ denotes $\langle (y_n)_0 | n < \omega \rangle$. So, we have $(q)_0 \to (r)_0$. Shrinking if necessary A_n's (the sets of measure one of $(q_n)_0$'s), we can assume that for each $n \geq \ell(p) = \ell(r) = \ell(q)$ the set of measure one for $(r_n)_0$, i.e. C_n projects exactly to A_n by $\pi_{\lambda_n, \max(\mathrm{rng}((r_n)_0), \max(\mathrm{rng}((q_n)_0)}$. Remember that the interpretations of both $\langle \underset{\sim}{b}_n, \underset{\sim}{B}_n \rangle$ and $\langle \underset{\sim}{b}'_n, \underset{\sim}{B}'_n \rangle$ depend only on a choice of elements of A_n.

Our tusk will be extend r to r^* so that $q \to r^*$. This will show that p and q are compatible. Which provides the desired contradiction.

Fix n, $\omega > n \geq \ell(p)$, large enough. Let η be the maximal coordinate of $(r_n)_0$ (i.e. the ordinal coding $\max(\mathrm{rng}(c_n))$, ζ those of $(p_n)_0$ (which is the same for $(q_n)_0$, since (4) above) and ξ the one corresponding to ζ (of $(q_n)_0$) under $(q_n)_0 \to (r_n)_0$. Denote $\pi''_{\lambda_n, \eta, \xi} C_n$ by D_n. Assuming that $n > 2$, it follows from the definitions of the equivalence relation $\longleftrightarrow$ and of the order $\to$, that $E_{\lambda_n}(\xi)$ (the ξ's measure of the extender) is the same as $E_{\lambda_n}(\zeta)$. Also, $D_n \subseteq A_n$.
Define now a condition

$$r^*_n = \langle \langle c_n, C_n, h_n \rangle, \langle \underset{\sim}{e}_n, \underset{\sim}{E}_n, g_n \rangle \rangle \in Q_{n0}$$

which extends

$$r_n = \langle \langle c_n, C_n, h_n \rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n \rangle \rangle.$$

The addition will depend only on the coordinate ξ of E_{λ_n}. So we need to deal with each $\nu \in D_n$. Set $\mathrm{dom}(\underset{\sim}{e}_n) = \mathrm{dom}(\underset{\sim}{b}_n) \cup \mathrm{dom}(\underset{\sim}{b}'_n)$. Let $X \in \mathrm{dom}(\underset{\sim}{e}_n)$. If $X \in \mathrm{dom}(\underset{\sim}{b}_n)$, then set

$$\underset{\sim}{e}_n(X)[\rho] = \underset{\sim}{b}_n(X)[\rho],$$

for each $\rho \in C_n$. Now, if X is new, i.e. $X \in \mathrm{dom}(\underset{\sim}{b}'_n) \setminus \mathrm{dom}(\underset{\sim}{b}_n)$, then we consider X_α the model that corresponds to X in p_α under the Δ-system.

Now we use Definition 5.1(8l) to find inside $\underset{\sim}{b}_n(A_\alpha)[\rho]$ some σ realizing over the common part the type of $\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\nu]$. Recall that

$$\underset{\sim}{b}_n(A^{0\kappa^+}_{\alpha+1})[\nu] = \underset{\sim}{b}'_n(A^{0\kappa^+}_{\beta+1})[\nu]$$

and

$$\underset{\sim}{b}_n(X_\alpha)[\nu] = \underset{\sim}{b}'_n(X)[\nu].$$

Set now $\underset{\sim}{e_n}(X)[\rho]$ to be the element of σ corresponding to $\underset{\sim}{b'_n}(X)[\nu]$, for each $\rho \in C_n$ and $\nu = \pi_{\lambda_n,\eta,\xi}(\rho)$.

The following claim suffice in order to complete the argument:

Claim 4.6.2 $r_n^* \in Q_{n0}$, $r_n^* \geq_0 r_n$ and $q_n \to r_n^*$.

Proof. Let us check first that q_n, r_n or basically $\underset{\sim}{b'_n}$ and $\underset{\sim}{c_n}$ agree about the values of models in $\mathrm{dom}(\underset{\sim}{b'_n}) \cap \mathrm{dom}(\underset{\sim}{c_n})$. Suppose that X is such a model. Then, by the assumptions we made on the Δ-system, $X \in A^{0\kappa^+}_{\alpha^*}(\theta)$. Also,

$$A^{0\kappa^+}_{\alpha^*}(\theta) \in \mathrm{dom}(\underset{\sim}{b'_n}) \cap \mathrm{dom}(\underset{\sim}{c_n}),$$

$$otp_{\kappa^+}(A^{0\kappa^+}_{\alpha^*}(\theta)) = otp_{\kappa^+} A^{0\kappa^+}_{\alpha^*}(\theta')$$

and

$$A^{0\kappa^+}_{\alpha^*}(\theta') \in \mathrm{dom}(c_n).$$

By 5.1, $\underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha^*}(\theta))$ depends only on the measure indexed by the code of

$$c_n(A^{0\kappa^+}_{\alpha^*}(\theta')) = a_n(A^{0\kappa^+}_{\alpha^*}(\theta')) = a'_n(A^{0\kappa^+}_{\alpha^*}(\theta')).$$

Let δ denotes the index of this measure (or its code). Then for each $\rho \in C_n$ we will have

$$\pi_{\lambda_n,\eta,\delta}(\rho) = \pi_{\lambda_n,\xi,\delta}(\pi_{\lambda_n,\eta,\xi}(\rho)).$$

Hence, restricting $(q_n)_0$ to D_n, i.e. by replacing A_n in $(q_n)_0$ with D_n, we can insure that $\underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha^*})$ and $\underset{\sim}{b'_n}(A^{0\kappa^+}_{\alpha^*})$ agree. The same applies to any $X \in A^{0\kappa^+}_{\alpha^*}$ which is in the common domain, since its value too will depend on the δ-th measure of the extender only.

Consider now the maximal model of q_n. By 17, above, it is $A^{0\kappa^+}_{\beta+1}(\theta)$ and the one of p_n is $A^{0\kappa^+}_{\alpha+1}(\theta)$. Now, for each $\nu \in A_n$, by the condition (7) on the Δ-system above we have

$$\underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha+1}(\theta))[\nu] = \underset{\sim}{b'_n}(A^{0\kappa^+}_{\beta+1}(\theta))[\nu].$$

Pick $\rho \in C_n$. Let $\nu = \pi_{\lambda_n,\eta,\xi}(\rho)$ and $\sigma = \pi_{\lambda_n,\eta,\zeta}(\rho)$. Then

$$\underset{\sim}{e_n}(A^{0\kappa^+}_{\alpha+1}(\theta))[\rho] = \underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha+1}(\theta))[\sigma]$$

and

$$\underset{\sim}{e_n}(A^{0\kappa^+}_{\beta+1}(\theta))[\rho] = \underset{\sim}{b'_n}(A^{0\kappa^+}_{\beta+1}(\theta))[\nu].$$

The first equality holds since e_n extends b_n and the second by the same reason as e_n was defined this way above.

The crucial observation is that $\sigma, \nu \in A_n$ (just $D_n \subseteq A_n$) and $\sigma > \nu$, so by Definition 5.1(8l),

$$\underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha+1}(\theta))[\nu] \subseteq \underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha+1}(\theta))[\sigma].$$

Hence, also,

$$\underset{\sim}{b'_n}(A^{0\kappa^+}_{\beta+1}(\theta))[\nu] \subseteq \underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha+1}(\theta))[\sigma],$$

since

$$\underset{\sim}{e_n}(A^{0\kappa^+}_{\beta+1}(\theta))[\rho] = \underset{\sim}{b'_n}(A^{0\kappa^+}_{\beta+1}(\theta))[\nu].$$

The same inclusion holds, by Definition 5.1(8l), if we replace $A^{0\kappa^+}_{\alpha+1}(\theta)$ with any $Y \in \mathrm{dom}(\underset{\sim}{b_n}) \cap (C^{\kappa^+}(\theta))(A^{0\kappa^+}_{\alpha+1}(\theta))$ such that $\sigma(Y) > \nu$, where $\sigma(Y)$ is the measure corresponding to Y. Thus

$$\underset{\sim}{b'_n}(A^{0\kappa^+}_{\beta+1}(\theta))[\nu] = \underset{\sim}{b_n}(A^{0\kappa^+}_{\alpha+1}(\theta))[\nu] \subseteq \underset{\sim}{b_n}(Y)[\sigma].$$

In the present case we have the least such Y. It is $A^{0\kappa^+}_{\alpha}(\theta)$. Just below it everything falls into $A^{0\kappa^+}_{\alpha^*}(\theta)$ the kernel of the Δ-system. Consider now Y's in $\mathrm{dom}(\underset{\sim}{b_n}) \setminus (C^{\kappa^+}(\theta))(A^{0\kappa^+}_{\alpha+1}(\theta))$. If such Y is in $A^{0\kappa^+}_{\alpha}(\theta)$, it belongs to $A^{0\kappa^+}_{\alpha^*}(\theta)$ the kernel of the Δ-system. Hence as it was observed in the beginning of the proof of this claim, we have the agreement. Suppose now that $Y \notin A^{0\kappa^+}_{\alpha}(\theta)$. By the basic properties of $G(\mathcal{P}')$ there will be $Z \in A^{0\kappa^+}_{\alpha}(\theta)$ such that

$$Y \cap A^{0\kappa^+}_{\alpha}(\theta) = Z \cap A^{0\kappa^+}_{\alpha}(\theta).$$

Then again this Z falls into $A^{0\kappa^+}_{\alpha^*}(\theta)$ and into the kernel of the Δ-system on which we have the agreement.

This completes the proof of the claim.

$\square$ of the claim.

$\square$

Force with $\langle \mathcal{P}, \rightarrow \rangle$. Let $G(\mathcal{P})$ be a generic set. By the lemmas above no cardinals are collapsed. Let $\langle \nu_n \mid n < \omega \rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\lambda_n} \mid n < \omega \rangle$ and $\langle \rho_n \mid n < \omega \rangle$ those for $\langle E_{\kappa_n} \mid n < \omega \rangle$. The following analog of 4.7 holds here:

Theorem 5.9 *The following hold in $V[G(\mathcal{P}'(\theta')), G((\mathcal{P}'(\theta)), G(\mathcal{P})]$:*

(1) $\mathrm{cof}(\prod_{n<\omega} \nu_n^{+n+2}/$ *finite* $) = \kappa^{++}$

(2) $\mathrm{cof}(\prod_{n<\omega} \nu_n^{+\nu_n^{+n+2}+1} / \text{ finite }) = \theta'$

(3) $\mathrm{cof}(\prod_{n<\omega} \nu_n^{+\nu_n^{+n+2}+2} / \text{ finite }) = (\theta')^+$

(4) for every regular cardinal $\mu \in [\kappa^{++}, (\theta')^+]$,
there is a sequence of regular cardinals $\langle \nu_n(\mu) | n < \omega \rangle$ *such that*

(a) for each $n < \omega$, $\nu_n(\mu) \in [\nu_n^{+n+2}, \nu_n^{+\nu_n^{+n+2}+2}]$

(b) $\mathrm{cof}(\prod_{n<\omega} \nu_n(\mu) / \text{ finite }) = \mu$

(5) $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+2} / \text{ finite }) = (\theta')^{++}$

(6) $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+\rho_n^{+n+2}+1} / \text{ finite }) = \theta$

(7) $\mathrm{cof}(\prod_{n<\omega} \rho_n^{+\rho_n^{+n+2}+2} / \text{ finite }) = \theta^+$

(8) for every regular cardinal $\mu \in [(\theta')^{++}, \theta^+]$,
there is a sequence of regular cardinals $\langle \rho_n(\mu) | n < \omega \rangle$ *such that*

(a) for each $n < \omega$, $\rho_n(\mu) \in [\rho_n^{+n+2}, \rho_n^{+\rho_n^{+n+2}+2}]$

(b) $\mathrm{cof}(\prod_{n<\omega} \rho_n(\mu) / \text{ finite }) = \mu$

(9) for every unbounded subset a *of* κ *consisting of regular cardinals and disjoint to both* $\cup_{n<\omega}[\nu_n^{+n+2}, \nu_n^{+\nu_n^{+n+2}+2}]$ *and* $\cup_{n<\omega}[\rho_n^{+n+2}, \rho_n^{+\rho_n^{+n+2}+2}]$ *, for every ultrafilter* D *over* a *which includes all co-bounded subsets of* κ *we have*

$$\mathrm{cof}(\prod a/D) = \kappa^+$$

Proof. Items (1),(2),(3) and (4) follow easily from the construction, as in [6] or the arguments of 4.7 can be used. Thus, for (3), take the increasing (under the inclusion) enumeration $\langle X_\tau | \tau < (\theta')^+ \rangle$ of the chain of models given by $G(\mathcal{P}'(\theta'))$. Define a scale of functions $\langle F_\tau \mid \tau < (\theta')^+ \rangle$ in the product $\prod_{n<\omega} \nu_n^{+\nu_n^{+n+2}+1}$ as follows:
let for each $\tau < (\theta')^+$

$$F'_\tau(n) = f_n(X_\tau), \text{ if } f_n(X_\tau) < \nu_n^{+\nu_n^{+n+2}+1}$$

and

$$F'_\tau(n) = 0, \text{ otherwise,}$$

where for some $p = \langle p_k | k < \omega \rangle \in G(\mathcal{P})$ with $\ell(p) > n$ we have f_n as the first coordinate of p_n. Let $\langle F_\tau \mid \tau < (\theta')^+ \rangle$ be the subsequence of $\langle F'_\tau \mid \tau < (\theta')^+ \rangle$ consisting of all F'_τ's not in V.

Now, (1),(2) and (4) follow from No Hole Theorem of Shelah [8] or just directly as follows. Let us show (4). Fix a regular cardinal μ in the interval $[\kappa^{++}, (\theta')^+]$. Pick a model $M \prec H(\chi)^V$ for χ big enough such that

- $|M| = \mu$
- $M[G(\mathcal{P}'(\theta')), G(\mathcal{P}'(\theta))] \prec H(\chi)^{V[G(\mathcal{P}'(\theta')), G(\mathcal{P}'(\theta))]}$
- $M \cap H((\theta')^+) \in G(\mathcal{P}'(\theta'))$
- for some $p = \langle p_n | n < \omega \rangle \in G(\mathcal{P}'(\theta'))$ we have $M \cap H((\theta')^+) \in \mathrm{dom}(a_n)$, for each n large enough

Then there is an increasing unbounded in M chain of models $\langle X_\tau | \tau < \mu \rangle$ in $G(\mathcal{P}'(\theta'))$ of cardinalities below μ. Fix such a chain. Let $p = \langle p_n | n < \omega \rangle \in G(\mathcal{P}'(\theta'))$ be so that $M \cap H((\theta')^+) \in \mathrm{dom}(a_n)$, for each n large enough. Let n_0 be such that for each $n > n_0$ we have $M \cap H((\theta')^+) \in \mathrm{dom}(a_n)$. For each $n < \omega$ we set

$$M^*_n = f_n(M \cap H((\theta')^+)),$$

where for some $q \geq p$ in $G(\mathcal{P}'(\theta'))$ with $l(q) > n$, $\quad f_n$ is the first coordinate of q_n. Define now

$$\nu_n(\mu) = |M^*_n|,$$

if $n \geq n_0$ and $|M^*_n|$ is a regular cardinal and

$$\nu_n(\mu) = \omega,$$

otherwise.
Now, let for each $\tau < \mu$

$$F'_\tau(n) = f_n(X_\tau), \text{ if } f_n(X_\tau) \subset M^*_n \text{ of cardinality less than } \nu_n(\mu)$$

and

$$F'_\tau(n) = 0, \text{ otherwise},$$

where for some $p = \langle p_k | k < \omega\rangle \in G(\mathcal{P})$ with $\ell(p) > n$ we have f_n as the first coordinate of p_n. Let $\langle F''_\tau \mid \tau < \mu\rangle$ be the subsequence of $\langle F'_\tau \mid \tau < \mu\rangle$ consisting of all F'_τ's not in V. Finally, we set

$$F_\tau(n) = F''_\tau(n) \cap \nu_n(\mu),$$

for each $n < \omega$ and $\tau < \mu$. The sequence $\langle F_\tau | \tau < \mu\rangle$ will witness (4).

The proof of (5)-(8) is similar. The argument for (9) repeats those of 4.7. Thus, dealing with

$$\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+1} / \text{ finite }),$$

we observe that given a condition $\langle\langle a_n, A_n, f_n\rangle, \langle \underset{\sim}{b}_n, \underset{\sim}{B}_n, g_n\rangle\rangle \in Q_{n0}$, for some $n < \omega$, then it is impossible to change $\mathrm{rng}(b_n)[\nu] \restriction \kappa^{+n+1}$ by passing to an equivalent one, for any $\nu \in A_n$. Just the definition 3.10(4(b)v) explicitly requires this.
This means, in particular that

$$\mathrm{cof}(\prod_{n<\omega} \rho_n^{+n+1} / \text{ finite }) = \mathrm{cof}(\prod_{n<\omega} \kappa_n^{+n+1} / \text{ finite }),$$

where the connection is provided by $\underset{\sim}{b}_n$'s. But note that the cofinality of the last product is κ^+, since every function their can be bounded by an old function. So we are done.
$\square$

6 Some Generalizations

It is possible using the same ideas to realize any finite number of droppings instead of just one. Thus let $m < \omega$ and $\langle \theta_k | k < m\rangle$ be an increasing sequence of regular cardinals in the interval $[\kappa^+, \theta)$. We assume that κ is a limit of a sequence

$$\kappa_{00} < \kappa_{01} < ... < \kappa_{0m} < \kappa_{10} < ... < \kappa_{1m} < ... < \kappa_{n0} < ... < \kappa_{nm} < ...,$$

$n < \omega$ such that for each $n < \omega$ and $k \le m$

$$\kappa_{nk} \text{ is } \kappa_{nk}^{+\kappa_{nk}^{+n+2}+2} \text{ - strong as witnessed by an extender } E_{\kappa_{nk}}.$$

Force with $\mathcal{P}'(\theta_0) * ...\mathcal{P}'(\theta_{m-1}) * \mathcal{P}'(\theta)$. Let G be a generic set.

We define $\langle \mathcal{P}, \le, \le^*, \rightarrow\ \rangle$ in $V[G]$ parallel to those of Sections 2, 4. Just replace there two sequences $\langle \lambda_n | n < \omega\rangle$ and $\langle \kappa_n | n < \omega\rangle$ by $m+1$-many sequences

$$\langle \kappa_{nk} | n < \omega, k \le m\rangle.$$

Force with $\langle \mathcal{P}, \rightarrow \rangle$ over $V[G]$. Let $G(\mathcal{P})$ be a generic subset of $\mathcal{P}$. Let $\langle \nu_{nk} | n < \omega \rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\kappa_{nk}} | n < \omega \rangle$, for each $k \leq m+1$. Denote θ by θ_m and assume that $\theta_0 = \kappa^+$.

The following analog of 4.7, 5.9 holds in this context:

Theorem 6.1 *The following hold in $V[G, G(\mathcal{P})]$:*

(1) for each $k \leq m$ we have

$$\operatorname{cof}(\prod_{n<\omega} \nu_{nk+1}^{+\nu_{nk+1}^{+n+2}+1} / \text{ finite }) = \theta_{k+1}$$

(2) for each $k \leq m$ we have

$$\operatorname{cof}(\prod_{n<\omega} \nu_{nk+1}^{+\nu_{nk+1}^{+n+2}+2} / \text{ finite }) = (\theta_{k+1})^+$$

(3) for every $k \leq m$ and a regular cardinal $\mu \in [\theta_k^+, \theta_{k+1}^+]$, there is a sequence of regular cardinals $\langle \nu_{nk+1}(\mu) | n < \omega \rangle$ such that

(a) for each $n < \omega$, $\nu_{nk+1}(\mu) \in [\nu_{nk+1}^{+n+2}, \nu_{nk+1}^{+\nu_{nk+1}^{+n+2}+2}]$

(b) $\operatorname{cof}(\prod_{n<\omega} \nu_{nk+1}(\mu) / \text{ finite }) = \mu$

(4) for every unbounded subset a of κ consisting of regular cardinals and disjoint to $\bigcup_{n<\omega, k\leq m} [\nu_{nk}^{+n+2}, \nu_{nk}^{+\nu_{nk}^{+n+2}+2}]$, for every ultrafilter D over a which includes all co-bounded subsets of κ we have

$$\operatorname{cof}(\prod a/D) = \kappa^+$$

In a similar fasion, ω many drops can be realized. Let $\langle \theta_k | k < \omega \rangle$ be an increasing sequence of regular cardinals in the interval $[\kappa^+, \theta]$, with $\theta = (\bigcup_{k<\omega} \theta_k)^+$. We assume now that we have a sequence of cardinals

$$\langle \kappa_{nk} | n < \omega, k \leq n \rangle$$

such that

- $\kappa_{nk} < \kappa_{ml}$ whenever $n < m$ or $n = m$ and $k < l$
- for each $k < \omega$ we have $\langle \kappa_{nk} | n \geq k \rangle$ is unbounded in κ

- κ_{nk} is $\kappa_{nk}^{+\kappa_{nk}^{+n+2}+1}$ - strong as witnessed by an extender $E_{\kappa_{nk}}$.

Force with $\mathcal{P}'(\theta_0) * ...\mathcal{P}'(\theta_m) * ...$, $m < \omega$. Let G be a generic set.

We define $\langle \mathcal{P}, \leq, \leq^*, \rightarrow \rangle$ in $V[G]$ parallel to those of Sections 2, 4. Just replace there two sequences $\langle \lambda_n | n < \omega \rangle$ and $\langle \kappa_n | n < \omega \rangle$ by ω-many sequences

$$\langle \kappa_{nk} | n < \omega, k \leq n \rangle,$$

but note at each level $n < \omega$ we have here only finitely many (n) possibilities.

Force with $\langle \mathcal{P}, \rightarrow \rangle$ over $V[G]$. Let $G(\mathcal{P})$ be a generic subset of $\mathcal{P}$. Let $\langle \nu_{nk} | k \leq n < \omega \rangle$ denotes the diagonal Prikry sequence added for the normal measures of the extenders $\langle E_{\kappa_{nk}} | n < \omega \rangle$, for each $k < \omega$. Assume that $\theta_0 = \kappa^+$.

Then the following analog of 4.7, 5.9, 6.1 holds:

Theorem 6.2 *The following hold in $V[G, G(\mathcal{P})]$:*

(1) for each $k < \omega$ we have

$$\mathrm{cof}(\prod_{k \leq n < \omega} \nu_{nk}^{+\nu_{nk+1}^{+n+2}+1} / \textit{ finite }) = \theta_{k+1}$$

(2) for each $k < \omega$ we have

$$\mathrm{cof}(\prod_{k \leq n < \omega} \nu_{nk}^{+\nu_{nk+1}^{+n+2}+2} / \textit{ finite }) = (\theta_{k+1})^+$$

(3) for every $k < \omega$ and a regular cardinal $\mu \in [\theta_k^+, \theta_{k+1}^+]$, there is a sequence of regular cardinals $\langle \nu_{nk+1}(\mu) | k \leq n < \omega \rangle$ such that

(a) for each $k \leq n < \omega$, $\nu_{nk+1}(\mu) \in [\nu_{nk+1}^{+n+2}, \nu_{nk+1}^{+\nu_{nk+1}^{+n+2}+1}]$

(b) $\mathrm{cof}(\prod_{n<\omega} \nu_{nk+1}(\mu) / \textit{ finite }) = \mu$

(4)

$$\mathrm{cof}(\prod_{n<\omega} \nu_{nn}^{+\nu_{nn}^{+n+2}+1} / \textit{ finite }) = \theta$$

(5) for every unbounded subset a of κ consisting of regular cardinals and disjoint to $\bigcup_{k<\omega, k \leq n < \omega} [\nu_{nk}^{+n+2}, \nu_{nk}^{+\nu_{nk}^{+n+2}+2}]$, for every ultrafilter D over a which includes all co-bounded subsets of κ we have

$$\mathrm{cof}(\prod a / D) = \kappa^+$$

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