EXTENDER-BASED MAGIDOR-RADIN FORCINGS WITHOUT TOP EXTENDERS

MOTI GITIK AND SITTINON JIRATTIKANSAKUL

Abstract. Continuing [1], we develop a version of Extender-based Magidor-Radin forcing where there is no extender on the top ordinal. As an application, we provide another approach to obtain a failure of SCH on a club subset of an inaccessible cardinal, and a model where the cardinal arithmetic behaviors are different on stationary classes whose union is a club.

1. Introduction

In [1], we developed a Prikry-type forcing which shoots a club subset of κ containing all former regular cardinals from the optimal assumption. Unlike [2], the regular cardinals outside the club remain regular. The forcing in [1] can be viewed as the Magidor-Radin forcing with interleaving quotients, and there were no ultrafilters on the top cardinal required in the forcing construction. In this work, we develop a forcing with the same style, but use Extender based Magidor-Radin forcing instead.

In [3], they provided a consistency results where there are models of ZFC such that there are stationary classes in which the cardinal arithmetic behaves differently with the optimal assumptions. As an application, we provide a ZFC model where GCH fails on a club, and a ZFC model where there are stationary classes in which cardinal arithmetic behaves differently, as stated in Theorem 9.1.

The organization of the paper is the following. In Section 2 we introduce all basic ingredients we need to develop the forcing. From Section 3 to Section 8, we develop the forcing in which a club class of cardinals α with $2^\alpha = \alpha^{++}$. The forcing for building a club class of cardinals is built from approximated forcings, which will be built by recursion. The basic cases are constructed in Section 3. In Section 4 we state all the properties we need to be true, and show that the forcings in the first few levels satisfies the properties. Then the construction proceeds in Section 5, Section 6, and Section 7. The main forcing will then be introduced in 8. Lastly, in Section 9, we sketch a generalization of the forcing to get different cardinal behaviors on different stationary classes.

Although a version of Extender-based forcing and the Extender-based Magidor-Radin forcing looks slightly different from [4], I assume that the readers are familiar with the Extender-Based Magidor-Radin forcing.

Conventions: Without mentioning, we assume that every forcing has the weakest element 1. $p \leq q$ means p is stronger than q. When possible, every name in this paper will be in the simplest form. For sets A and B, $A \sqcup B$ just means $A \cup B$ where $A \cap B = \emptyset$. If f is a function and d is a set, define $f \restriction d$ as $f \restriction [d \cap \mathrm{dom}(f)]$.

The authors were partially supported by ISF grant No. 1216/18.

Throughout the paper, the forcing at level ρ, denoted by P_ρ will be defined. We often abbreviate the $\Vdash_{P_\rho}$ by $\Vdash_\rho$. If $\vec{x} = \langle x_{\alpha,\beta}\rangle$ is a sequence indexed by pairs of ordinals, we define

$$\vec{x} \restriction (\alpha,\beta) = \langle x_{\alpha',\beta'} \mid \alpha' < \alpha \text{ or } (\alpha' = \alpha \text{ and } \beta' < \beta)\rangle,$$

and

$$\vec{x} \restriction \alpha = \vec{x} \restriction (\alpha, 0).$$

2. Basic preparation

From now until Section 8, we have the following hypotheses.

Assumption 2.1. *GCH holds. κ is a strongly inaccessible cardinal. There is a function $\circ : \kappa \to \kappa$ and $\vec{E} = \langle E(\alpha,\beta) \mid \alpha < \kappa, \beta < \circ(\alpha)\rangle$ such that*

(1) $E(\alpha,\beta)$ is an (α,α^{++})-extender, which means that if

$$j_{\alpha,\beta} : V \to \mathrm{Ult}(V, E(\alpha,\beta)) =: M_{\alpha,\beta}$$

is the ultrapower map, then $\mathrm{crit}(j_{\alpha,\beta}) = \alpha$, and $M_{\alpha,\beta}$ computes cardinals correctly up to an including α^{++}.

(2) $\vec{E}$ is coherent*, namely*

$$j_{\alpha,\beta}(\vec{E}) \restriction (\alpha+1) = \vec{E} \restriction (\alpha,\beta).$$

(3) for all α, $\circ(\alpha) < \alpha$.

(4) For every $\gamma < \kappa$, the collection

$$\{\alpha < \kappa \mid \circ(\alpha) \geq \gamma\}$$

is stationary.

Definition 2.2. Let $\alpha < \kappa$. We say that d is a α-*domain* if $d \in [\alpha^{++} \setminus \alpha]^{\leq \alpha}$ and $\alpha \in d$. Define $C(\alpha^+,\alpha^{++})$ as the collection of functions f such that $\mathrm{dom}(f)$ is a α-domain d, and $\mathrm{rng}(f) \subseteq \alpha$. Define the ordering in $C(\alpha^+,\alpha^{++})$ by $f \leq g$ iff $f \supseteq g$.

Note that $C(\alpha^+,\alpha^{++})$ is isomorphic to $\mathrm{Add}(\alpha^+,\alpha^{++})$, the forcing adding α^{++} subsets of α^+.

Remark 2.3. If $|P| \leq \alpha$ and $\dot{C}(\alpha^+,\alpha^{++})$ is a P-name of the forcing interpreted in the extension, then

$$\Vdash_P \text{“}\{\dot{f} \in \dot{C}(\alpha^+,\alpha^{++}) \mid \mathrm{dom}(\dot{f}) = \check{d}, d \in V\} \text{ is dense”}.$$

We identify such and $\dot{f}$ by f with $\mathrm{dom}(f) = d$, and for $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a P-name of an element below α.

Until the end of this section, fix α with $\circ(\alpha) > 0$ and $\beta < \circ(\alpha)$. We introduce some definitions and facts which will be used since Section 7. Fix a α-domain d.

- Define $\mathrm{mc}_{\alpha,\beta}(d) = \{(j_{\alpha,\beta}(\xi), \xi) \mid \xi \in d\}$.
- Define $E_{\alpha,\beta}(d)$ by $X \in E_{\alpha,\beta}(d)$ iff $\mathrm{mc}_{\alpha,\beta}(d) \in j_{\alpha,\beta}(X)$. Then $E_{\alpha,\beta}(d)$ concentrates on the collection $\mathrm{OB}_{\alpha,\beta}(d)$ of (α,β)-d-*objects*, which are functions μ such that
  - $\alpha \in \mathrm{dom}(\mu) \subseteq d, \mathrm{rng}(\mu) \subseteq \alpha$ (in fact, we can assume that $\mathrm{rng}(\mu) \subseteq \mu(\alpha)^{++}$).
    (The reason is that $\mathrm{dom}(\mathrm{mc}_{\alpha,\beta}(d)) = j_{\alpha,\beta}[d] \subseteq j_{\alpha,\beta}(d)$, $j_{\alpha,\beta}(\alpha) \in j_{\alpha,\beta}[d]$, $\mathrm{rng}(\mathrm{mc}_{\alpha,\beta}(d)) = d \subseteq \alpha^{++} = \mathrm{mc}_{\alpha,\beta}(d)(j_{\alpha,\beta}(\alpha))^{++}$).

  - $\circ(\mu(\alpha)) = \beta$, in particular, $\mu(\alpha)$ is strongly inaccessible, $|\operatorname{dom}(\mu)| \leq \mu(\alpha)^{++}$, and μ is order-preserving.
    (The reason is that $j_{\alpha,\beta}(\circ)(\alpha)^{M_{\alpha,\beta}} = \beta$, α is inaccessible, $|\operatorname{dom}(\mathrm{mc}_{\alpha,\beta}(d))| = |d| \leq \alpha^{++}$, and $\mathrm{mc}_{\alpha,\beta}$ is order-preserving.)
- Let $X_\nu \in E_{\alpha,\beta}(d)$ for $\nu < \alpha$. Define the *diagonal intersection*
$$\Delta_{\nu<\alpha} X_\nu = \{\mu \in \mathrm{OB}_{\alpha,\beta}(d) \mid \forall \nu < \mu(\alpha)(\mu \in X_\nu)\}.$$
  Then $\Delta_{\nu<\alpha} X_\nu \in E_{\alpha,\beta}(d)$.
- The measure $E_{\alpha,\beta}(\{\alpha\})$ is normal, and is isomorphic to $E_{\alpha,\beta}(\alpha)$, which is defined by $X \in E_{\alpha,\beta}(\alpha)$ iff $\alpha \in j_{\alpha,\beta}(X)$.
- if $d' \supseteq d$ is an α-domain, there is an *associated projection* from $E_{\alpha,\beta}(d')$ to $E_{\alpha,\beta}(d)$ induced by the map $\pi_{d',d} : \mathrm{OB}_{\alpha,\beta}(d') \to \mathrm{OB}_{\alpha,\beta}(d)$ defined by $\pi_{d',d}(\mu) = \mu \restriction d$ (i.e. $\mu \restriction (d \cap \operatorname{dom}(\mu))$. In particular, there is a projection from $E_{\alpha,\beta}(d)$ to $E_{\alpha,\beta}(\{\alpha\})$.
- Similar as in the proof of Lemma 2 [5], there is a measure-one set $B_d \in E_{\alpha,\beta}(d)$ such that for every $\nu < \alpha$, $\{\mu \in \mathrm{OB}_{\alpha,\beta}(d) \mid \mu(\alpha) = \nu\} \leq \nu^{++}$. We will assume that for every $A \in E_{\alpha,\beta}(d)$, $A \subseteq B_d$.

We now no longer fix β, but still fix α and d.

- μ is an *α-d-object* if μ is an (α,β)-d-object for some $\beta < \circ(\alpha)$. Denote the collection of α-d-object by $\mathrm{OB}_\alpha(d)$. For each pair of α-d-objects μ and τ, define $\mu < \tau$ if $\operatorname{dom}(\mu) \subseteq \operatorname{dom}(\tau)$ and $\mu(\alpha) < \tau(\alpha)$. Equivalently, $\mu < \tau$ iff $\operatorname{dom}(\mu) \subseteq \operatorname{dom}(\tau)$ and for $\gamma \in \operatorname{dom}(\mu)$, $\mu(\gamma) < \tau(\gamma)$.
- Define $X \in \vec{E}_\alpha(d)$ iff X can be written as $X = \cup_{\beta<\circ(\alpha)} X_\beta$ where $X_\beta \in E_{\alpha,\beta}(d)$. Note that for each α-d-object μ, $\{\tau \in \mathrm{OB}_\alpha(d) \mid \mu < \tau\} \in \vec{E}_\alpha(d)$.
- Let $X_\nu \in \vec{E}_\alpha(d)$ for $\nu < \alpha$. The *diagonal intersection*
$$\Delta_{\nu<\alpha} X_\nu = \{\mu \in \mathrm{OB}_\alpha(d) \mid \forall \nu < \mu(\alpha)(\mu \in X_\nu)\}$$
  is in $\vec{E}_\alpha(d)$.
- If $\mu < \tau$, we define $\mu \downarrow \tau = \mu \circ \tau^{-1}$, which is the function whose domain is $\tau[\operatorname{dom}(\mu)]$ and for $\gamma \in \operatorname{dom}(\mu)$, $(\mu \downarrow \tau)(\tau(\gamma)) = \mu(\gamma)$. Since τ is order-preserving, we have that $\mu \downarrow \tau$ is well-defined.
- If X is a set of α-d-object and $\tau \in \mathrm{OB}_\alpha(d)$, define $X \downarrow \tau = \{\mu \downarrow \tau \mid \mu < \tau, \circ(\mu(\alpha)) < \circ(\tau(\alpha))\}$. By the coherence of the extenders, we also assume that every $X \in \vec{E}_\alpha(d)$ is *coherent*, i.e. for every $\tau \in X$, $X \downarrow \tau \in \vec{E}_{\tau(\alpha)}(\tau[d \cap \operatorname{dom}(\tau)])$.
- Let $\vec{\mu} = \langle \mu_0, \cdots, \mu_{n-1} \rangle$ be an increasing sequence of α-d-objects, define $\vec{\mu}(\alpha) = \mu_{n-1}(\alpha)$, which is just an inaccessible cardinal below α. Also write $\operatorname{dom}(\vec{\mu}) = \operatorname{dom}(\mu_{n-1})$. Also, if $\mu_{n-1} < \tau$, we define $\vec{\mu} \downarrow \tau = \langle \mu_0 \downarrow \tau, \cdots, \mu_{n-1} \downarrow \tau \rangle$.
- A is an *α-d-tree* if A consists of nonempty finite increasing sequences of α-d-objects, and A has the following descriptions:
  - $\vec{\mu} \leq_A \vec{\tau}$ iff $\vec{\mu} \sqsubseteq \vec{\tau}$ ($\vec{\mu}$ is an initial segment of $\vec{\tau}$).
  - $\mathrm{Lev}_n(A)$ is the collection of $\langle \mu_0, \cdots, \mu_n \rangle$ in A, so they have lengths $n+1$.
  - We require that $\mathrm{Lev}_0(A) \in \vec{E}_\alpha(d)$.
  - For $\vec{\mu} \in A$, define $\mathrm{Succ}_A(\vec{\mu}) = \{\tau \mid \vec{\mu}^\frown\langle\tau\rangle \in A\}$. We require that $\mathrm{Succ}_A(\vec{\mu}) \in \vec{E}_\alpha(d)$.

- If A is an α-d-tree and $\mu \in \mathrm{Lev}_0(A)$, define $A_{\langle\mu\rangle} = \{\vec{\tau} \mid \langle\mu\rangle^\frown\vec{\tau} \in A\}$, and we recursively define $A_{\langle\mu_0,\cdots,\mu_n\rangle} = (A_{\langle\mu_0\cdots,\mu_{n-1}\rangle})_{\langle\mu_n\rangle}$.
- Fix $d' \subseteq d$ an α-domain and $\vec{\mu} = \langle\mu_0,\cdots,\mu_{n-1}\rangle$ is a finite increasing sequence of α-d-objects, define $\vec{\mu} \restriction d' = \langle\mu_0 \restriction d,\cdots,\mu_{n-1} \restriction d'\rangle$. If we assume that A is an α-d-tree, define $A \restriction d' = \{\vec{\mu} \restriction d' \mid \vec{\mu} \in A\}$. Then $A \restriction d'$ is an α-d'-tree.
- If $d' \supseteq d$ is an α-domain, and A is an α-d-tree, the *pullback of A to d'*, is $\{\vec{\mu} \in [\mathrm{OB}_\alpha(d')]^{<\omega} \mid \vec{\mu}$ is increasing and $\vec{\mu} \restriction d \in A\}$. Note that the pullback is an α-d'-tree.
- A tree A is *generated by* $B \in \vec{E}_\alpha(d)$ if $\mathrm{Lev}_0(A) = B$, and for $\vec{\mu} = \langle\mu_0,\cdots,\mu_{n-1}\rangle \in A$, $\mathrm{Succ}_A(\vec{\mu}) = \{\tau \in B \mid \mu_{n-1} < \tau\}$. Such a tree is an α-d-tree. Furthermore, every α-d-tree A has a sub α-d-tree which is generated by some $B \in \vec{E}_\alpha(d)$: for each $\nu < \alpha$, let $X_\nu = \cap_{\vec{\mu}\in T,\vec{\mu}(\alpha)\leq\nu} \mathrm{Succ}_A(\vec{\mu})$, and $B = \Delta_\nu X_\nu$. We assume that every d-tree A is generated by some $B \subseteq B_d$.
- We write $A(\alpha) = \{\vec{\mu}(\alpha) \mid \vec{\mu} \in A\}$. If A is generated by B, then $A(\alpha) = B(\alpha) = \{\mu(\alpha) \mid \mu \in B\}$.
- If A is an α-d-tree and τ is an object, define $A \downarrow \tau = \{\vec{\mu} \downarrow \tau \mid \forall i(\mu_i < \tau$ and $\circ(\mu_i(\alpha)) < \circ(\tau(\alpha)))\}$. By the coherence, assume that for each τ, $A \downarrow \tau$ is an $\tau(\alpha)$-$\tau[d \cap \mathrm{dom}(\tau)]$-tree, with respects to $\vec{E}_{\tau(\alpha)}(\tau[d \cap \mathrm{dom}(\tau)])$.

Remark 2.4. For every d-tree A and ν, we assume that $\{\vec{\mu} \in A \mid \vec{\mu}(\alpha) = \mu_{|\vec{\mu}|-1}(\alpha) = \nu\}$ has size at most ν^{++}.

3. The first few levels

We consider the forcings at the first ω inaccessible cardinals, so, the extenders are not involved. We first analyze just for the first few inaccessible cardinals concretely, which will be served as the first few basic cases for our induction scheme for the forcings in the general levels, which will be listed later in Section 4.1.

3.1. **The first inaccessible cardinal.** Let α_0 be the least inaccessible cardinal. The following describe the scenario at the level α_0.

- The forcing P_{α_0} consists of $\langle f\rangle$ where $f \in C(\alpha_0^+, \alpha_0^{++})$. For $\langle f\rangle, \langle g\rangle \in P_{\alpha_0}$, define $\langle f\rangle \leq_{\alpha_0} \langle g\rangle$ iff $f \leq^*_{\alpha_0} g$ iff $f \supseteq g$.
- Let $\dot{C}_{\alpha_0}$ be a P_{α_0}-name for the set $\{\alpha_0\}$.
- Let $\dot{P}_{\alpha_0/\alpha_0}$ be a P_{α_0}-name of the trivial forcing, with the obvious extension and the obvious direct extension.
- In $V^{P_{\alpha_0}}$, let $\dot{C}_{\alpha_0/\alpha_0}$ be a $\dot{P}_{\alpha_0/\alpha_0}$-name of the empty set.

The forcing at the first inaccessible cardinal has nothing particularly interesting. The name $\dot{C}_{\alpha_0}$ will be served as the initial approximation of the final club where GCH fails at its limit points. The quotient forcing like $\dot{P}_{\alpha_0/\alpha_0}$ will show its importance later. $\dot{C}_{\alpha_0/\alpha_0}$ will also be considered for an approximation of the final club. It will be more meaningful to write $\dot{P}_{\check{\alpha}_0/\alpha_0}$ since in general, the ordinal which appears for the numerator, like $\check{\alpha}_0$, may be a non-trivial name of an ordinal. Since this is a check name, we omit the check symbol. A trivial remark is that forcing $P_{\alpha_0} * \dot{P}_{\alpha_0/\alpha_0}$ is equivalent to P_{α_0}.

3.2. **The second inaccessible cardinal.** Let $\alpha_0 < \alpha_1$ be the first two inaccessible cardinals.

Definition 3.1. The forcing P_{α_1} consists of two kinds of conditions (apart from the weakest condition). Conditions of different kinds are not compatible.

(1) The first kind consists of $\langle f \rangle$ in $C(\alpha_1^+, \alpha_1^{++})$. For $\langle f \rangle$ and $\langle g \rangle$ which are of first kind, define $\langle f \rangle \leq_{\alpha_1} \langle g \rangle$ iff $\langle f \rangle \leq^*_{\alpha_1} \langle g \rangle$ iff $f \supseteq g$.

(2) The second kind consists of $p = (\langle f_0 \rangle, \langle \dot{P}_{\dot{\xi}/\alpha_0}, \dot{q}_0 \rangle)^\frown \langle f_1 \rangle$, where
- $f_0 \in C(\alpha_0^+, \alpha_0^{++})$.
- $\Vdash_{\alpha_0}$ " $\leq \alpha_0 \leq \dot{\xi} < \alpha_1$ is strongly inaccessible" (in this case, we can assume that $\dot{\xi}$ is α_0, or more formally, $\check{\alpha}_0$).
- $\Vdash_{\alpha_0}$ "$\dot{q}_0 \in \dot{P}_{\dot{\xi}/\alpha_0}$".
- $\mathrm{dom}(f_1)$ is an α_1-domain, and for $\gamma \in \mathrm{dom}(f_1)$, $f_1(\gamma)$ is a $P_{\alpha_0} * \dot{P}_{\dot{\xi}/\alpha_0}$-name, $\Vdash_{P_{\alpha_0} * \dot{P}_{\alpha_0/\alpha_0}}$ "$f_1(\gamma) < \alpha_1$".
- For such a condition, define $p \restriction P_{\alpha_0} = \langle f_0 \rangle$.

From now, we replace $\dot{\xi}$ by α_0. We say that

$$(\langle f_0 \rangle, \langle \dot{P}_{\alpha_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \langle f_1 \rangle \leq_{\alpha_1} (\langle g_0 \rangle, \langle \dot{P}_{\alpha_0/\alpha_0}, \dot{r}_0 \rangle)^\frown \langle g_1 \rangle \text{ iff}$$

$$(\langle f_0 \rangle, \langle \dot{P}_{\alpha_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \langle f_1 \rangle \leq^*_{\alpha_1} (\langle g_0 \rangle, \langle \dot{P}_{\alpha_0/\alpha_0}, \dot{r}_0 \rangle)^\frown \langle g_1 \rangle \text{ iff}$$

$f_0 \supseteq g_0, \mathrm{dom}(f_1) \supseteq \mathrm{dom}(g_1)$, and for $\gamma \in \mathrm{dom}(g_1), (\langle f_0 \rangle, \dot{q}_0) \Vdash_{P_{\alpha_0} * \dot{P}_{\alpha_0/\alpha_0}}$ "$f_1(\gamma) = g_1(\gamma)$".

Let $\dot{C}_{\alpha_1}$ be a P_{α_1}-name such that for p of the first kind, $p \Vdash_{\alpha_1} \dot{C}_{\alpha_1} = \{\alpha_1\}$, and for p of the second kind, $p \Vdash_{\alpha_1}$ "$\dot{C}_{\alpha_1} = \{\alpha_0, \alpha_1\}$". We now define different types of quotients.

- $\dot{P}_{\alpha_1/\alpha_1}$ is a P_{α_1}-name of the trivial forcing, with the obvious extension and the obvious direct extension. In $V^{P_{\alpha_1}}$, let $\dot{C}_{\alpha_1/\alpha_1}$ be a $\dot{P}_{\alpha_1/\alpha_1}$-name of the empty set.
- The quotient $\dot{P}_{\alpha_1/\alpha_0}$ is a P_{α_0}-name of the following forcing notion. Let G be P_{α_0}-generic. The forcing $P_{\alpha_1}[G] := \dot{P}_{\alpha_1/\alpha_0}[G]$ consists of $(\langle \emptyset \rangle)^\frown \langle f \rangle$ where $\Vdash_{\dot{P}_{\alpha_0/\alpha_0}[G]}$ "$f \in C(\alpha_1^+, \alpha_1^{++})$" ($C(\alpha_1^+, \alpha_1^{++})$ is considered in $(V[G])^{\dot{P}_{\alpha_0/\alpha_0}[G]}$), and $\mathrm{dom}(f) \in V$. The extension and the direct extension are the natural ones. Back to the ground model, in $V^{P_{\alpha_0}}$, let $\dot{C}_{\alpha_1/\alpha_0}$ be the $\dot{P}_{\alpha_1/\alpha_0}$-name for $\{\alpha_1\}$. The point of having an empty set in the condition because it is more natural to translate a condition in P_{α_1} of the second kind to a condition in $\dot{P}_{\alpha_1/\alpha_0}$, namely, for each $p = (\langle f_0 \rangle, \langle \dot{P}_{\dot{\xi}/\alpha_0}, \dot{q}_0 \rangle)^\frown \langle f_1 \rangle$ in P_{α_1}, we have that $\Vdash_{\alpha_0}$ "$(\langle \dot{q} \rangle)^\frown \langle f_1 \rangle \in \dot{P}_{\alpha_1/\alpha_0}$". This is because $\dot{q}$ is always interpreted as the empty set in $\dot{P}_{\alpha_0/\alpha_0}$, and f_1 is a function whose range contains names of ordinals in with respect to the correct forcing. Note that $\{p \in P_{\alpha_1} \mid p \text{ is of the second kind}\}$ can be densely embedding in $P_{\alpha_0} * \dot{P}_{\alpha_1/\alpha_0}$ in the sense of $\leq$ and $\leq^*$.

The subforcing of P_{α_1} containing conditions of second kinds is nothing but a two-step iteration of the Cohen forcings, except that the domains can always be decided by the weakest element to be in the ground model.

4. The induction scheme

We are now stating the induction scheme, and point out that it holds for the basic case.

Proposition 4.1 (The induction scheme)**.** *Let α be an inaccessible cardinal.*

1. *The basic properties of the forcing $(P_\alpha, \leq, \leq^*)$.*
   - $|P_\alpha| = \alpha^{++}$.
   - $(P_\alpha, \leq)$ *is* α^{++}*-c.c.*
   - $(P_\alpha, \leq, \leq^*)$ *has the Prikry property.*
2. *The P_α-name of the set $\dot{C}_\alpha$. Let $C_\alpha = \dot{C}_\alpha[G]$ where G is generic over P_α.*
   - $C_\alpha \subseteq \alpha + 1$, $\max(C_\alpha) = \alpha$.
   - *If $\circ(\alpha) = 0$, then $C_\alpha \cap \alpha$ is a bounded subset of α.*
   - *If $\circ(\alpha) > 0$, then $C_\alpha \cap \alpha$ is a club subset of α.*
   - *C_α contains only former inaccessible cardinals.*
3. *Cardinals and cofinalities in the extension.*
   - *If $\circ(\alpha) = 0$, then α remains regular in the extension over P_α.*
   - *If $\circ(\alpha) > 0$, then when we force over P_α, α is singularized and* $\mathrm{cf}(\alpha) = \mathrm{cf}(\omega^{\circ(\alpha)})$ *(the ordinal exponentiation).*
   - *In the extension, for every cardinal $\beta \leq \alpha$, $2^\beta = \beta^+$ or $2^\beta = \beta^{++}$, and $2^\beta = \beta^{++}$ iff $\beta \in \lim(C_\alpha)$.*
   - *For each V-regular $\beta \leq \alpha$, β is singularized iff $\beta \in \lim(C_\alpha)$.*
4. *$\dot{P}_{\alpha/\alpha}$ is always a P_α-name of the trivial forcing $(\{\emptyset\}, \leq, \leq^*)$.*
5. *The factor $\dot{P}_{\alpha/\beta}$ for $\beta < \alpha$.*
   - *$\{p \in P_\alpha \mid p \restriction P_\beta \text{ exists}\}$ densely embeds into $P_\beta * \dot{P}_{\alpha/\beta}$.*
   - $\Vdash_\beta$ "$|\dot{P}_{\alpha/\beta}| = \alpha^{++}, (\dot{P}_{\alpha/\beta}, \leq)$ *is* α^{++}*-c.c.*".
   - $\Vdash_\beta$ "$(\dot{P}_{\alpha/\beta}, \leq^*)$ *is* β^**-closed", where* $\beta^* = \min\{\xi > \beta \mid \xi \text{ is strongly inaccessible}\}$.
   - $\Vdash_\beta$ "$(\dot{P}_{\alpha/\beta}, \leq, \leq^*)$ *has the Prikry property*".
6. *The quotient set $C_{\alpha/\beta}$: In V^{P_β}, consider the properties of $\dot{P}_{\alpha/\beta}$-name of the set $\dot{C}_\alpha$. Let G be P_β-generic over V and H be $\dot{P}_{\alpha/\beta}[G]$-generic over $V[G]$. Let $C_{\alpha/\beta} = \dot{C}_{\alpha/\beta}[G][H]$.*
   - *If $\beta = \alpha$, then $C_{\alpha/\beta} = \emptyset$.*
   - *Suppose $\beta < \alpha$. Then $I = G * H$ is P_α-generic, which introduces the set C_α. Also, G introduces the set C_β. Then $C_{\alpha/\beta} \subseteq (\beta, \alpha]$, and $C_\alpha = C_\beta \sqcup C_{\alpha/\beta}$.*
7. *Double quotients: Let $\gamma \leq \beta \leq \alpha$ and G is P_γ-generic. Then $\dot{P}_{\alpha/\beta}[G]$ is defined as*
   $$\Vdash_{P_\beta[G]} \text{“}p \in \dot{P}_{\alpha/\beta}[G] \textit{ iff } p \in P_\alpha[G * \dot{H}]\text{”},$$
   where $\dot{H}$ is the canonical $P_\beta[G]$-generic.

For a non-triviality, we now show that the forcing P_{α_1} as described in Definition 3.1 satisfies the induction scheme.

Proposition 4.2. *Let $\alpha_0 < \alpha_1$ be the first two inaccessible cardinals. Then P_{α_1} satisfies the induction scheme*

Proof. (1) • The set of conditions in P_{α_1} of the first kind is essentially $C(\alpha_1^+, \alpha_1^{++})$, whose size is $(\alpha^{++})^{\leq\alpha} = \alpha^{++}$. Conditions of the second kind are of the form $(\langle f_0 \rangle, \langle \dot{P}_{\dot{\xi}/\alpha_0}, \dot{q}_0 \rangle)^\frown \langle f_1 \rangle$. We assume that the names are in their simplest form in the sense that $\dot{\xi} = \check{\alpha_0}$, $\dot{q}_0 = \check{\emptyset}$. The part $(\langle f_0 \rangle, \langle \dot{P}_{\dot{\xi}/\alpha_0}, \dot{q}_0 \rangle)$ is in V_{α_1}. Then for each $\gamma \in \mathrm{dom}(f_1)$, $f_1(\gamma)$ is a $P_{\alpha_0} * \dot{P}_{\alpha_0/\alpha_0}$-name of an ordinal below α. By replacing $f_1(\gamma)$ with its nice name, assume that $f_1(\gamma) \in V_{\alpha_1}$. Hence, the number of such f_1's is $(\alpha_1^{++})^{\alpha_1} = \alpha_1^{++}$. Hence, $|P_{\alpha_1}| = \alpha_1^{++}$.

- Suppose that $X = \{p^\gamma \mid \gamma < \alpha_1^{++}\}$ is an antichain of conditions in P_{α_1}. By shrinking X, we may assume that X contains conditions of the same kind. If it contains conditions of the first kind, then the standard Δ-system applies. Suppose X contains conditions of the second kind. By shrinking further, assume there is p_0 such that for every γ, $p^\gamma = p_0{}^\frown\langle f_1^\gamma\rangle$. Then we can apply a standard Δ-system argument on $\{f_1^\gamma \mid \gamma < \alpha_1^{++}\}$, and we are done.
- Obvious, since $\leq$ and $\leq^*$ on P_{α_1} are the same.

(2) Note that $\circ(\alpha_1) = 0$. If G contains conditions of the first kind, then $C_{\alpha_1} = \{\alpha_1\}$, and if G contains conditions of the second kind, then $C_{\alpha_1} = \{\alpha_0, \alpha_1\}$. In both cases, it is a subset of $\alpha_1 + 1$ whose maximum is α_1. Also, $C_{\alpha_1} \cap \alpha_1$ is either $\emptyset$ or $\{\alpha_0\}$ which is bounded in α_1, and C_{α_1} contains only former inaccessible cardinals.

(3) $\circ(\alpha_1) = 0$, and the forcing P_{α_1} is equivalent to either a Cohen forcing $\mathrm{Add}(\alpha_1^+, \alpha_1^{++})$, or a two-step iteration of Cohen forcings $\mathrm{Add}(\alpha_0^+, \alpha_0^{++}) * \mathrm{Add}(\alpha_1^+, \alpha_1^{++})$. In both cases, α_1 remains regular, GCH still holds, and $\lim(C_\alpha) = \{\emptyset\}$.

(4) $\dot{P}_{\alpha_1/\alpha_1}$ is a P_{α_1}-name of the trivial forcing.

(5) Consider $\dot{P}_{\alpha_1/\alpha_0}$.

- For each $p = (\langle f_0\rangle, \langle \dot{P}_{\xi/\alpha_0}, \dot{q}_0\rangle)^\frown\langle f_1\rangle$, consider the map $\pi(p) = (\langle f_0\rangle, (\langle \dot{q}\rangle)^\frown f_1\rangle)$. Clearly, this map is a dense embedding from $\{p \in P_{\alpha_1} \mid p \restriction P_{\alpha_0}\}$ to $P_{\alpha_0} * \dot{P}_{\alpha_1/\alpha_0}$.
- Since P_{α_0} forces GCH, a similar argument as in (1) shows that $\Vdash_{\alpha_0}$ "$|\dot{P}_{\alpha_1/\alpha_0}| = \alpha_1^{++}$, $(P_{\alpha_1/\alpha_0}, \leq)$ is α_1^{++}-c.c., "
- Let G be P_{α_0}-generic. Conditions in $P_{\alpha_1}[G]$ are of the form $(\langle\emptyset\rangle)^\frown\langle f_1\rangle$. We ignore the empty set's part. Note that since $P_{\alpha_0}[G] := \dot{P}_{\alpha_0/\alpha_0}[G]$ is trivial, so f_1 is just a Cohen condition in $V[G]$. We now assume that a condition in $P_{\alpha_1}[G]$ is $\langle f_1\rangle$. Let $\langle f_1^\gamma \mid \gamma < \gamma^*\rangle$ be a decreasing sequence of conditions, where $\gamma^* < \alpha_1$. In V, let $d* = \cup_{\gamma<\gamma^*}\{d \mid \exists p \in P_{\alpha_0}$. Then $d^* \in V$, and let f^* be such that $\mathrm{dom}(f^*) = d^*$, and in $V[G]$, $f^* \leq f_1^\gamma$ for all γ. Then f^* is as required.
- $\Vdash_{\alpha_0}$ " $\leq, \leq^*$ are the same in $\dot{P}_{\alpha_1/\alpha_0}$, hence has the Prikry property".

(6) In $V^{P_{\alpha_1} * \dot{P}_{\alpha_1/\alpha_1}}$, C_{α_1/α_1} is the empty set. In $V^{P_{\alpha_0} * \dot{P}_{\alpha_1/\alpha_0}}$, $C_{\alpha_1/\alpha_0} = \{\alpha_1\} \subseteq (\alpha_0, \alpha_1]$, and in this model, $C_{\alpha-0} \sqcup C_{\alpha_1/\alpha_0} = C_{\alpha_1}$, since it is the same model with the extension $V^{P_{\alpha_1}}$ using conditions of the second kind.

(7) Trivial since the definition is given.

□

Remark 4.3. (1) $P_{\alpha_0} * \dot{P}_{\alpha_1/\alpha_0}$ is equivalent to the subforcing P_{α_1} containing conditions of the second kind, and there is a natural translation from one generic to another. Namely, suppose that $G * H$ is such a generic object. Define $I = \{(p_0, \langle \dot{P}_{\alpha_0/\alpha_0}, \dot{q}\rangle)^\frown p_1 \mid p_0 \in G, \Vdash_{\alpha_0}$ "$(\langle \dot{q}\rangle)^\frown p_1 \in \dot{H}$"$\}$. Then $V[I] = V[G * H]$.

(2) If we force with conditions in P_{α_1} of the second kind, we can obtain an equivalent generic object from $P_{\alpha_0} * \dot{P}_{\alpha_1/\alpha_0}$ naturally. Namely, if I is P_{α_1}-generic containing conditions of the second kind, let

$$G = \{\langle f_0 \rangle \mid \exists \dot{q}, f_1(\langle f_0, \langle \dot{P}_{\alpha_0/\alpha_0}, \dot{q}\rangle)^\frown \langle f_1 \rangle \in I\},$$

and

$$H = \{(\langle \emptyset \rangle)^\frown \langle f_1[G] \rangle \mid \exists f_0, \dot{q}(\langle f_0, \langle \dot{P}_{\alpha_0/\alpha_0}, \dot{q}\rangle)^\frown \langle f_1 \rangle \in I\}.$$

5. Below the first measurable cardinal

Let α be a strongly inaccessible cardinal which is below the first α^* with $\circ(\alpha^*) = 1$. We will assume that α is at least the $\omega+1$-th strongly inaccessible cardinal so that the conditions of arbitrarily length will appear at this stage.

Definition 5.1. P_α consists of the conditions of the following kinds:

- The *pure conditions*, which are conditions of the form $\langle f \rangle$, where $f \in C(\alpha^+, \alpha^{++})$.
- The *impure conditions*, which are conditions of the form

$$(\langle f_0 \rangle^\frown \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown(\langle f_{n-1} \rangle^\frown \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown \langle f \rangle,$$

  for some $n > 0$, where
  - $\alpha_0 < \cdots < \alpha_{n-1} < \alpha$ are inaccessible.
  - for all i, $\Vdash_{\alpha_i}$ "$\alpha_i \leq \dot{\beta}_i < \alpha_{i+1}$", where $\alpha_n = \alpha$.
  - $f_0 \in C(\alpha_0^+, \alpha_0^{++})$ and for $i > 0$, $\text{dom}(f_i) = d_i$ is an α_i-domain (in the sense of V), and for $\zeta \in d_i$,
  $$\Vdash_{P_{\alpha_{i-1}} * \dot{P}_{\dot{\beta}_{i-1}/\alpha_{i-1}}} \text{“} f_i(\zeta) < \alpha_i \text{”}.$$
  In particular,
  $$\Vdash_{P_{\alpha_{i-1}} * \dot{P}_{\dot{\beta}_{i-1}/\alpha_{i-1}}} \text{“} f_i \in \dot{C}(\alpha_i^+, \alpha_i^{++}).$$
  - $\text{dom}(f) = d$ is an α-domain, and for $\zeta \in d$,
  $$\Vdash_{P_{\alpha_{n-1}} * \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}} \text{“} f(\zeta) < \alpha \text{”}.$$
  In particular,
  $$\Vdash_{P_{\alpha_{n-1}} * \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}} \text{“} f \in \dot{C}(\alpha^+, \alpha^{++}) \text{”}.$$
  - for all i, $\Vdash_{\alpha_i}$ "$\dot{q}_i \in \dot{P}_{\dot{\beta}_i/\alpha_i}$".

By recursion, we consider

$$(\langle f_0 \rangle^\frown \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown \langle f_i \rangle$$

as a condition in P_{α_i}. Denote $p \restriction P_{\alpha_i}$ as the condition as bove. We also consider

$$(\langle f_0 \rangle^\frown \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown(\langle f_i \rangle, \langle \dot{P}_{\dot{\beta}_i/\alpha_i}, \dot{q}_i\rangle)$$

as a condition in $P_{\alpha_i} * \dot{P}_{\dot{\beta}_i/\alpha_i}$. Denote such a condition by $p \restriction (i+1)$.

The ordering $\leq_\alpha$ and $\leq_\alpha^*$ will be the same. We only define $\leq_\alpha$. We will also write a pure condition in an impure condition's format. When we mention a condition p, we put the superscript p to every component in the condition. If p is the condition as in the definition, we write $n^p = n$, $\text{top}(p) = f$.

Definition 5.2. Let

$$p_0 = (\langle f_0\rangle^\frown\langle \dot{P}_{\dot\beta_0/\alpha_0}, \dot q_0\rangle)^\frown \cdots {}^\frown(\langle f_{n-1}\rangle^\frown\langle \dot{P}_{\dot\beta_{n-1}/\alpha_{n-1}}, \dot q_{n-1}\rangle)^\frown\langle f\rangle,$$

and

$$p_1 = (\langle g_0\rangle^\frown\langle \dot{P}_{\dot\xi_0/\gamma_0}, \dot r_0\rangle)^\frown \cdots {}^\frown(\langle g_{n-1}\rangle^\frown\langle \dot{P}_{\dot\xi_{n-1}/\gamma_{n-1}}, \dot r_{m-1}\rangle)^\frown\langle g\rangle.$$

We say that $p_0 \leq_\alpha p_1$ iff

- $n = m$.
- for $i < n$, $\alpha_i = \gamma_i$.
- $f_0 \supseteq g_0$, $\langle f_0\rangle \Vdash_{\alpha_0}$ "$\dot\beta_0 = \dot\xi_0$ and $\dot q_0 \leq_{\dot\beta_0/\alpha_0} \dot r_0$".
- for $i > 0$, $d_i^{p^0} \supseteq d_i^{p^1}$, and for $\zeta \in d_i^{p^1}$, $p \restriction i \Vdash_{P_{\alpha_i} * \dot P_{\dot\beta_i/\alpha_i}}$ "$f_i(\zeta) = g_i(\zeta)$".
- for $i > 0$, $(p_0 \restriction i)^\frown\langle f_i\rangle \Vdash_{\alpha_i}$ "$\dot\beta_i = \dot\xi_i$ and $\dot q_i \leq_{\dot\beta_i/\alpha_i} \dot r_i$".
- $\mathrm{dom}(f) \supseteq \mathrm{dom}(g)$ and for $\zeta \in \mathrm{dom}(g)$,
$$p_0 \restriction n \Vdash_{P_{\alpha_{n-1}} * \dot P_{\dot\beta_{n-1}/\alpha_{n-1}}} \text{"}f(\zeta) = g(\zeta)\text{"}.$$

We may also assume that $\dot\xi_i = \dot\beta_i$ for all i. The extension relation does not increase the length of a condition. For a generic G containing a condition p, define C_α as the following: If p is pure, then $C_\alpha = \{\alpha\}$. Assume p is impure and $n = n^p$. Then $p \restriction n \in P_{\alpha_n} * \dot P_{\dot\beta_n/\alpha_n}$. Let $\beta_n = \dot\beta_n[G \restriction P_{\alpha_{n-1}}]$. By Proposition 4.1 (2) and (6), $G \restriction (P_{\alpha_n} * \dot P_{\dot\beta_n/\alpha_n})$ introduces the set $C' = C_{\alpha_{n-1}} \sqcup C_{\beta_{n-1}/\alpha_{n-1}} \subseteq \beta_{n-1} + 1$ with $\max(C') = \beta_{n-1}$. Define $C_\alpha = C' \cup \{\alpha\}$. Still, this forcing does not change the cardinal arithmetic.

We now define $P_{\alpha/\beta}$. An intuition is that we need $\{p \in P_\alpha \mid p \restriction P_\beta \text{ is defined}\}$ to be densely embedded in $P_\beta * \dot P_{\alpha/\beta}$.

Definition 5.3 (The quotient forcing)**.** Let $\dot P_{\alpha/\alpha}$ be the P_α-name of the trivial forcing $(\{\emptyset\}, \leq, \leq^*)$. In V^{P_α}, let $\dot C_{\alpha/\alpha}$ be the $\dot P_{\alpha/\alpha}$-name of the empty set. Now assume that $\beta < \alpha$. Define $\dot P_{\alpha/\beta}$ as the following. Let G be P_β-generic. Define $P_\alpha[G] = \dot P_{\alpha/\beta}[G]$ as the forcing consisting of conditions of the form

$$p = (\langle P_{\beta'}[G], q'\rangle)^\frown(\langle f_0\rangle, \langle \dot P_{\dot\beta_0/\alpha_0}[G], \dot q_0\rangle) \cdots {}^\frown(\langle f_{n-1}\rangle, \langle \dot P_{\dot\beta_{n-1}/\alpha_{n-1}}[G], \dot q_{n-1}\rangle)^\frown\langle f\rangle$$

where $n \geq 0$ and

1. $\beta \leq \beta' < \alpha$, so $P_{\beta'}[G]$ was already defined by recursion, which is just $\dot P_{\dot\beta'[G]/\beta}[G]$ and $\beta' = \dot\beta'[G]$. Furthermore, $q' \in P_{\beta'}[G]$.
2. If $n > 0$, then $\alpha_0 < \cdots < \alpha_{n-1}$, and for $i < n$,
   - let $d_i = \mathrm{dom}(f_i)$, then d_i is an α_i-domain, $d_i \in V$.
   - for $\zeta \in d_0$, $\Vdash_{P_{\beta'}[G]}$ "$f_0(\zeta) < \alpha_0$", and if $i > 0$, then for $\zeta \in d_i$, $\Vdash_{P_{\alpha_{i-1}}[G] * \dot P_{\dot\beta_{i-1}/\alpha_{i-1}}[G]}$ "$f_i(\zeta) < \alpha_i$".
   - $\Vdash_{P_{\alpha_i}[G]}$ "$\alpha_i \leq \dot\beta_i < \alpha_{i+1}$", where $\alpha_n = \alpha$.
   - $\Vdash_{P_{\alpha_i}[G]}$ "$\dot q_i \in \dot P_{\dot\beta_i/\alpha_i}[G]$".
3. $d := \mathrm{dom}(f)$ is an α-domain, and is in V.
4. Fix $\zeta \in d$. If $n = 0$, then $\Vdash_{P_{\beta'}[G]}$ "$f(\zeta) < \alpha$", otherwise, $\Vdash_{P_{\alpha_{n-1}}[G] * P_{\dot\beta_{n-1}/\alpha_{n-1}}[G]}$ "$f(\zeta) < \alpha$".

Back in V. If $\dot p$ is a P_β-name of a condition in $\dot P_{\alpha/\beta}$, then by density, there is $p_0 \in P_\beta$ such that p_0 decides n, $\alpha_0, \cdots, \alpha_{n-1}$, $\mathrm{dom}(f_0), \cdots, \mathrm{dom}(f_{n-1})$, $\mathrm{dom}(f)$. In this case, we say that p_0 *interprets* $\dot p$. All in all, for such p_0 which interprets all the relevant components of $\dot{p}$, let p_1 be such the interpretation. Write p_0 as $r_0{}^\frown\langle g\rangle$ and by the interpretation, we may write

$$p_1 = (\langle \dot{P}_{\beta'/\beta}, \dot{q}'\rangle)^\frown(\langle f_0\rangle, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0\rangle)\cdots{}^\frown(\langle f_{n-1}\rangle, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown\langle f\rangle.$$

There is a natural concatenation p_0 with p_1, written by $p_0{}^\frown p_1$, which is

$$r = r_0{}^\frown(\langle g\rangle, \langle \dot{P}_{\beta'/\beta}, \dot{q}'\rangle)^\frown\cdots{}^\frown(\langle f_{n-1}\rangle, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown\langle f\rangle.$$

Then $r \in P_\alpha$ with $r \restriction P_\beta$ exists. For p_0 and p_1 in $\dot{P}_{\alpha/\beta}$, we say that $p_0 \leq p_1$ if there is $p \in G^{P_\beta}$ such that p interprets p_0 and p_1, and $p^\frown p_0 \leq_\alpha p^\frown p_1$. Also define $p_0 \leq^* p_1$ if there is $p \in G^{P_\beta}$ such that p interprets p_0 and p_1, and $p^\frown p_0 \leq^*_\alpha p^\frown p_1$ (note that at this level $\leq^*$ and $\leq$ are still the same). One can check that the map $\phi : \{p \in P_\alpha \mid p \restriction P_\beta \text{ exists}\} \to P_\beta * \dot{P}_{\alpha/\beta}$ defined by $\phi(p) = (p \restriction P_\beta, p \setminus P_\beta)$ is a dense embedding, where $p \setminus P_\beta$ is the obvious component of p which is in $\dot{P}_{\alpha/\beta}$.

In V^{P_β}, let $\dot{C}_{\alpha/\beta}$ be a $\dot{P}_{\dot{\beta}/\alpha}$-name of the set described as the following. Let G be P_β-generic. Write

$$p = (\langle P_{\beta'}[G], q'\rangle)^\frown(\langle f_0\rangle, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}[G], \dot{q}_0\rangle)\cdots{}^\frown(\langle f_{n-1}\rangle, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}[G], \dot{q}_{n-1}\rangle)^\frown\langle f\rangle$$

as an element in $P_\alpha[G]$. The part which excludes the top part, i.e.

$$(\langle P_{\beta'}[G], q'\rangle)^\frown(\langle f_0\rangle, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}[G], \dot{q}_0\rangle)\cdots{}^\frown(\langle f_{n-1}\rangle, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}[G], \dot{q}_{n-1}\rangle)$$

is in $P_{\alpha_{n-1}}[G] * \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}[G]$. Let H be generic over the forcing. By our induction scheme, H produces $C_0 \sqcup C_1$, where $C_0 \subseteq (\beta, \alpha_{n-1}]$ (can be empty if $n = 0$), and $C_1 \subseteq (\alpha_{n-1}, \beta_{n-1}]$ (can be empty if β_{n-1}, the interpretation of $\dot{\beta}_{n-1}$, is α_{n-1}). If $n > 0$, then $\max(C_0) = \alpha_{n-1}$, and if $\beta_{n-1} > \alpha_{n-1}$, then $\max(C_1) = \beta_{n-1}$. Let $C_{\alpha/\beta} = C_0 \cup C_1 \cup \{\alpha\}$.

Proposition 5.4. *P_α and the relevant quotients at α satisfy Proposition 4.1.*

Proof.
(1) Similar as the proof of the corresponding properties in Propoisition 4.2.
(2) $\circ(\alpha) = 0$. Then the forcing P_α introduces the set $C_\alpha \subseteq \alpha+1$ where $C_\alpha \setminus \{\alpha\}$ is a bounded subset of α. By induction hypothesis, it is easy to see that C_α contains only former inaccessible cardinals.
(3) The forcing P_α under a certain condition can be factored to $P^0 * \dot{C}(\alpha^+, \alpha^{++})$, where $P^0 \in V_\alpha$, and hence, α is still regular. Note that by induction on α, C_α is still finite, and since P^0 is either empty or a two-step iteration where it forces GCH. Hence, P_α still forces GCH.
(4) Obvious.
(5) Let $\beta < \alpha$.
    - The map $p \mapsto (p \restriction P_\beta, p \setminus P_\beta)$ is a dense embedding from $\{p \in P_\alpha \mid p \restriction P_\beta \text{ exists}\}$ to $P_\beta * \dot{P}_{\alpha/\beta}$.
    - Similar to the proof of the corresponding properties in Proposition 4.1, $\Vdash_\beta$ "$|\dot{P}_{\alpha/\beta}| = \alpha^{++}$ and is α^{++}-c.c."
    - Let $\beta' < \beta^*$ and $\Vdash_\beta$ "$\{p^\gamma \mid \gamma < \beta'\}$ be a $\leq^*$-decreasing sequence of conditions in $\dot{P}_{\alpha/\beta}$". We may assume that $p^\gamma = p_0^\gamma{}^\frown\langle f^\gamma\rangle$. Then $\Vdash_\beta$ "$\{p_0^\gamma \mid \gamma < \beta'\}$ is a $\leq^*$-decreasing sequence in a certain forcing $P_{\alpha^*} * \dot{P}_{\dot{\beta}^*/\alpha^*}$". By induction hypothesis, the two-step iteration is β^*-closed under $\leq^*$. Let p_0^* be such that for all γ, $\Vdash_\beta$ "$p_0^* \leq^* p_0^\gamma$". Now a

similar proof as in the corresponding property of Proposition 4.1 can be used to find f_1^* such that for all γ, $\Vdash_\beta$ "$p_0^* {}^\frown \langle f_1^* \rangle \leq^* p_0^\gamma {}^\frown \langle f_1^\gamma \rangle$".
- Since $\leq$ and $\leq^*$ on $\dot{P}_{\alpha/\beta}$ coincide, the Prikry property holds.

(6) By the construction of $\dot{C}_{\alpha/\beta}$ and the factorization, the property holds.
(7) Obvious by the definition of the double quotient stated in the Proposition 4.1.

□

6. At the first α with $\circ(\alpha) = 1$

We exhibit the forcing at the level of the first cardinal with a positive Mitchell order. Let α be the first such that $\circ(\alpha) = 1$. A variation of the Extender-based Prikry forcing will be introduced. Instead of diving into a full definition all at once, we progress through a series of definitions.

Definition 6.1. A *pure condition* of P_α is $p = \langle f_0, \vec{f}, A, F \rangle$ where there is a *common domain* d such that

(1) A is a d-tree.
(2) $\text{dom}(F) = A(\alpha)$.
(3) for $\nu \in \text{dom}(F)$, $F(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q} \rangle$ where $\Vdash_\nu$ "$\nu \leq \dot{\beta}_\nu < \alpha$ and $\dot{q} \in \dot{P}_{\dot{\beta}_\nu/\nu}$".
(4) $\text{dom}(f) = d$ and $f_0 \in C(\alpha^+, \alpha^{++})$.
(5) $\vec{f} = \langle f_\nu \mid \nu \in A(\alpha) \rangle$.
(6) for each $\nu \in A(\alpha)$, $\text{dom}(f_\nu) = d$ and for $\zeta \in d$, $\Vdash_{P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}}$ "$f_\nu(\zeta) < \alpha$". In particular, $f_\nu(\zeta)$ is a $P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}$-name.

The forcing seems like a version of an Extender-Based Prikry forcing with interleaved forcings. The main difference is that now we have a sequence of Cohen-like functions. The role of the sequence of the Cohen-like functions is that we want the quotient forcings at this level (and also in general) to be highly closed with respect to the direct extension relation. If we just use a Cohen function in the ground model, then the corresponding quotient will no longer be highly closed with respects to the direct extension relation. When we perform a one-step extension, we want to somehow change the Cohen function to a name of a Cohen function with respects to the part of the condition below. The explanation will make a bit more sense once we introduce the one-step extension operation.

We now discuss a one-step extension of a pure condition. Suppose that $p = \langle f_0, \vec{f}, A, F \rangle$ with the common domain d. Let $\langle \mu \rangle \in \text{Lev}_0(A)$ with $\mu(\alpha) = \nu$. The *one-step extension of p by μ* is $r {}^\frown \langle g_0, \vec{g}, A', F' \rangle$ such that

- $r = (\langle f_0 \circ \mu^{-1} \rangle, F(\nu))$. Write $F(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q} \rangle$.
- $A' = \{\vec{\tau} \in A_{\langle \mu \rangle} \mid \tau_0(\alpha) > \beta^*\}$ where $\beta^* = \sup\{\gamma \mid \exists r \in P_\nu (r \Vdash_\nu \text{"}\dot{\beta}_\nu = \gamma)\text{"}\}$.
- $F' = F \restriction (A'(\alpha))$.
- $\text{dom}(g_0) = d$.
- $\Vdash_{P_\nu * \dot{P}_{\dot{\beta}/\nu}}$ "$g_0 = f_\nu \oplus \mu$", i.e. for $\zeta \in d$, if $\zeta \in \text{dom}(\mu)$, $g_0(\zeta) = \check{\mu(\alpha)}$, otherwise, $\Vdash_{P_\nu * \dot{P}_{\dot{\beta}/\nu}}$ "$g_0(\zeta) = f_\nu(\zeta)$" (we can assume tat $g_0(\zeta) = f_\nu(\zeta)$ for $\zeta \in d \setminus \text{dom}(\mu)$).
- $\vec{g} = \langle f_{\nu'} \mid \nu' \in A'(\alpha) \rangle$.

Note that particular, $\langle f_0 \circ \mu^{-1}\rangle \in P_\nu$, and so, r can be considered as a condition in $P_\nu * \dot{P}_{\dot\beta_\nu/\nu}$. Like in a lot of Pirkry-type forcings, a d-tree at α gives us objects to create new blocks below α. The part $\langle g_0, \vec{g}, A', F'\rangle$ looks similar to a pure condition except that for each ζ, we now have that each $g_0(\zeta)$ is a name with respects to the forcing corresponding to where r lives.

We now define a condition in a general form.

Definition 6.2. A condition in P_α is either pure or of the form (which we call *impure*) which is of the form

$$p = (\langle f_0\rangle^\frown \langle \dot{P}_{\dot\beta_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown(\langle f_{n-1}\rangle^\frown\langle \dot{P}_{\dot\beta_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown\langle g_0, \vec{g}, A, F\rangle,$$

for some $n > 0$, and a *common domain* d such that

1. $(\langle f_0\rangle^\frown \langle \dot{P}_{\dot\beta_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown\langle f_{n-1}\rangle \in P_{\alpha_{n-1}}$, where $\alpha_{n-1} < \alpha$.
2. $\Vdash_{\alpha_{n-1}}$ "$\alpha_{n-1} \leq \dot\beta_{n-1} < \alpha, \dot{q}_{n-1} \in \dot{P}_{\dot\beta_{n-1}/\alpha_{n-1}}$".
3. d is an α-domain (we emphasize that $d \in V$).
4. A is a d-tree, $\min(A(\alpha)) > \beta^*$, where $\beta^* = \sup\{\gamma \mid \exists r \in P_{\alpha_{n-1}}(r \Vdash \dot\beta_{n-1} = \gamma)\}$.
5. $\mathrm{dom}(F) = A(\alpha)$, and for each $\nu \in A(\alpha)$, $F(\nu) = \langle \dot{P}_{\dot\beta_\nu/\nu}, \dot{q}\rangle$, where $\Vdash_\nu$ "$\nu \leq \dot\beta_\nu < \alpha$ and $\dot{q} \in \dot{P}_{\dot\beta_\nu/\nu}$".
6. $\vec{g} = \{g_{\nu'} \mid \nu' \in A(\alpha)\}$.
7. $\mathrm{dom}(g_0) = d$ and for all ν', $\mathrm{dom}(g_{\dot\beta_{\nu'}}) = d$.
8. For $\zeta \in d$, $\Vdash_{P_{\alpha_{n-1}} * \dot{P}_{\dot\beta_{n-1}/\alpha_{n-1}}}$ "$g_0(\zeta) < \alpha$", and for all ν', $\Vdash_{P_{\nu'} * \dot{P}_{\dot\beta_{\nu'}/\nu'}}$ "$g_{\nu'}(\zeta) < \alpha$".

We write $p \restriction P_{\alpha_i} = (\langle f_0\rangle^\frown\langle \dot{P}_{\dot\beta_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown\langle f_i\rangle$, so $p \restriction P_{\alpha_i} \in P_{\alpha_i}$. Also write $p \restriction i = (\langle f_0\rangle^\frown\langle \dot{P}_{\dot\beta_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown(\langle f_i\rangle^\frown\langle \dot{P}_{\dot\beta_i/\alpha_i}, \dot{q}_i\rangle)$, and we consider $p \restriction i$ as a condition in $P_{\alpha_i} * \dot{P}_{\dot\beta_i/\alpha_i}$. We put the superscript p to every component, including the common domain, i.e. we write d^p for d. We call $\dot{q}_i$'s the *interleaving part of* p. With p as above, we write $\mathrm{top}(p) = \langle g_0, \vec{g}, A, F\rangle$, $\mathrm{stem}(p) = p \setminus \mathrm{top}(p)$ and say that $\mathrm{stem}(p)$ has n blocks. From the definition, it is straightforward to check that $|P_\alpha| = \alpha^{++}$.

Definition 6.3 (The one-step extension)**.** Let

$$p = (\langle f_0\rangle^\frown \langle \dot{P}_{\dot\beta_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown(\langle f_{n-1}\rangle^\frown\langle \dot{P}_{\dot\beta_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown\langle g_0, \vec{g}, A, F\rangle,$$

with its common domain d, and $\langle \mu\rangle \in \mathrm{Lev}_0(A)$. Say $\nu = \mu(\alpha)$. The *one-step extension of* p *by* μ, denoted by $p + \langle \mu\rangle$, is the condition

$$p' = (\langle f_0\rangle^\frown \langle \dot{P}_{\dot\beta_0/\alpha_0}, \dot{q}_0\rangle)^\frown \cdots {}^\frown(\langle f_{n-1}\rangle^\frown\langle \dot{P}_{\dot\beta_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown r_0{}^\frown r_1,$$

where

1. $r_0 = (g_0 \circ \mu^{-1}, F(\nu))$,
   - $g_0 \circ \mu^{-1}$ has domain $\mathrm{rng}(\mu)$.
   - for $\zeta \in \mathrm{dom}(\mu)$, $(g_0 \circ \mu^{-1})(\mu(\zeta)) = g_0(\zeta)$.
   - Write $F(\nu) = \langle \dot{P}_{\dot\beta_\nu/\nu}/\dot{q}\rangle$.
2. $r_1 = \langle h_0', \vec{h}', A', F'\rangle$,
   - $A' = \{\vec\tau \in A_{\langle\mu\rangle} \mid \tau_0(\alpha) > \beta^*\}$, where $\beta^* = \sup\{\gamma \mid \exists r \in P_\nu(r \Vdash_\nu$ "$\dot\beta_\nu = \gamma$")$\}$.

- $F' = F \upharpoonright A'(\alpha)$.
- $\vec{h} = \{g_{\nu'} \mid \nu' \in A'(\alpha)\}$.
- $\mathrm{dom}(h_0) = d$, and for all ν', $\mathrm{dom}(h_{\nu'}) = d$.
- $\Vdash_{P_\nu * \dot{P}_{\dot{\beta}/\nu}}$ "$h_0 = g_\nu \oplus \mu$", i.e. for $\zeta \in d$, if $\zeta \in \mathrm{dom}(\mu)$, $h_0(\zeta) = \mu(\check{\alpha})$, otherwise, $\Vdash_{P_\nu * \dot{P}_{\dot{\beta}/\nu}}$ "$h_0(\zeta) = g_\nu(\zeta)$" (we may assume that for $\zeta \in d \setminus \mathrm{dom}(\mu)$, $h_0(\zeta) = g_\nu(\zeta)$).
- for $\nu' \in A'(\alpha)$, $h_{\nu'} = g_{\nu'}$

We define $p + \langle\rangle$ as p, and by recursion, define $p + \langle \mu_0, \cdots, \mu_n \rangle = (p + \langle \mu_0, \cdots, \mu_{n-1} \rangle) + \langle \mu_n \rangle$.

Definition 6.4 (The direct extension relation)**.** Let

$$p = (\langle f_0 \rangle^\frown \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \cdots {}^\frown (\langle f_{n-1} \rangle^\frown \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1} \rangle)^\frown \langle g_0, \vec{g}, A, F \rangle,$$

and

$$p' = (\langle h_0 \rangle^\frown \langle \dot{P}_{\dot{\xi}_0/\gamma_0}, \dot{r}_0 \rangle)^\frown \cdots {}^\frown (\langle h_{m-1} \rangle^\frown \langle \dot{P}_{\dot{\xi}_{m-1}/\gamma_{m-1}}, \dot{r}_{m-1} \rangle)^\frown \langle t_0, \vec{t}, A', F' \rangle.$$

We say that p is a *direct extension of* p', denoted by $p \leq^*_\alpha p'$, if the following hold.

1. $n = m$.
2. for $i < n$, $\alpha_i = \gamma_i$.
3. $p \upharpoonright n \leq^* p' \upharpoonright n$, i.e.
   - $f_0 \supseteq h_0$.
   - for $i \leq n$, $p \upharpoonright P_{\alpha_i} \Vdash_{\alpha_i}$ "$\dot{\beta}_i = \dot{\xi}_i$ and $\dot{q}_i \leq^*_{\dot{P}_{\dot{\beta}_i/\alpha_i}} \dot{r}_i$" (we can take $\dot{\beta}_i = \dot{\xi}_i$).
   - for $i \in (0, n)$, $\mathrm{dom}(f_i) \supseteq \mathrm{dom}(h_i)$, and for $\zeta \in \mathrm{dom}(h_i)$, $p \upharpoonright i \Vdash_{P_{\alpha_i} * \dot{P}_{\dot{\beta}_i/\alpha_i}}$ "$f_i(\zeta) = h_i(\zeta)$".
4. $d^p \supseteq d^{p'}$.
5. $A \upharpoonright d^{p'} \subseteq A'$.
6. for every $\nu \in A(\alpha)$ and $\vec{\mu} \in A$ with $\vec{\mu}(\alpha) = \nu$,
$$p + \vec{\mu} \upharpoonright P_\nu \Vdash_\nu \text{“}F(\nu)_0 = F'(\nu)_0 \text{ and } F(\nu)_1 \leq^*_{F(\nu)_0} F'(\nu)_1\text{”}.$$
7. For $\zeta \in d^{p'}$,
   - $p \upharpoonright n \Vdash_{P_{\alpha_{n-1}} * \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}}$ "$g_0(\zeta) = t_0(\zeta)$".
   - for $\nu \in A(\alpha)$, write $F(\nu) = (\dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q})$, and every $\vec{\mu}$ with $\vec{\mu}(\alpha) = \nu$, we have
$$p + \vec{\mu} \upharpoonright (n + |\vec{\mu}|) \Vdash_{P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}} \text{“}g_\nu(\zeta) = t_\nu(\zeta)\text{”}.$$

Definition 6.5 (The extension relation)**.** Let

$$p = (\langle f_0 \rangle^\frown \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \cdots {}^\frown (\langle f_{n-1} \rangle^\frown \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1} \rangle)^\frown \langle g_0, \vec{g}, A, F \rangle,$$

and $p' \in P_\alpha$. We say that p is an *extension of* p', denoted by $p \leq_\alpha p'$, if there is $\vec{\mu} \in A^{p'}$, or $\vec{\mu} = \langle\rangle$, such that by letting $p^* = p' + \vec{\mu}$ and write

$$p^* = (\langle h_0 \rangle^\frown \langle \dot{P}_{\dot{\xi}_0/\gamma_0}, \dot{r}_0 \rangle)^\frown \cdots {}^\frown (\langle h_{m-1} \rangle^\frown \langle \dot{P}_{\dot{\xi}_{m-1}/\gamma_{m-1}}, \dot{r}_{m-1} \rangle)^\frown \langle t_0, \vec{t}, A', F' \rangle,$$

we then have that

1. $p \upharpoonright n \leq p^* \upharpoonright m$, namely,
   - $\alpha_{n-1} = \gamma_{m-1}$.
   - $p \upharpoonright P_{\alpha_{n-1}} \leq_{\alpha_{n-1}} p^* \upharpoonright P_{\alpha_{n-1}}$.

- $p \restriction P_{\alpha_{n-1}} \Vdash_{\alpha_{n-1}}$ "$\dot{\beta}_{n-1} = \dot{\gamma}_{m-1}$ and $\dot{q} \leq_{\dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}} \dot{r}_{m-1}$" (we can take $\dot{\beta}_{n-1} = \dot{\gamma}_{m-1}$).

(2) $d^p \supseteq d^{p^*}$.
(3) $A \restriction d^{p^*} \subseteq A'$.
(4) for every $\nu \in A(\alpha)$ and $\vec{\mu} \in A$ with $\vec{\mu}(\alpha) = \nu$,

$$p + \vec{\mu} \restriction P_\nu \Vdash_\nu \text{"}F(\nu)_0 = F'(\nu)_0 \text{ and } F(\nu)_1 \leq^*_{F(\nu)_0} F'(\nu)_1\text{"}.$$

(5) For $\zeta \in d^{p^*}$,
- $p \restriction n \Vdash_{P_{\alpha_{n-1}} * \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}}$ "$g_0(\zeta) = t_0(\zeta)$".
- for $\nu \in A(\alpha)$, write $F(\nu) = \langle \dot{P}_{\dot{\beta}/\nu}, \dot{q} \rangle$, then

$$p + \vec{\mu} \restriction (n + |\vec{\mu}|) \Vdash_{P_\nu * \dot{P}_{\dot{\beta}/\nu}} \text{"}g_\nu(\zeta) = t_\nu(\zeta)\text{"}.$$

Note that equivalently, $p \leq p'$ if there is $\vec{\mu}$ such that p is a condition obtained by extending the interleaving part of a direct extension of $p' + \vec{\mu}$. For $p' \leq p$, the *interpolant of p' and p* is p^* such that there exist unique $\vec{\mu}$ such that $p^* = p + \vec{\mu}$ and p' is obtained by extending the interleaving part of the direct extension of p^*.

Proposition 6.6. *$(P_\alpha, \leq)$ has the α^{++}-chain condition.*

Proof. Let $\{p^\gamma \mid \alpha^{++}\}$ be a collection of conditions in P_α. p_γ can be written as $p_0^\gamma {}^\frown \langle f_0^\gamma, \vec{f}^\gamma, A^\gamma, F^\gamma \rangle$, with the corresponding common domain d^γ. By shrinking the collection, we may assume that there are p_0, d, b such that for all γ, $p_0^\gamma = p_0$, $b = A^\gamma(\alpha)$, and d is the root of the Δ-system $\{d^\gamma \mid \gamma < \alpha^{++}\}$. Since for each $\gamma < \alpha^{++}$, $\zeta \in d$, and $\nu \in b$, $f_0^\gamma(\zeta)$, $f_\nu^\gamma(\zeta) \in V_\alpha$, and $F^\gamma(\nu) \in V_\alpha$, we can shrink the collection of conditions further so that there are $x_{\zeta,0}, x_{\zeta,\nu}, y_\nu$, such that for all $\gamma < \alpha^{++}$, $f_0^\gamma(\zeta) = x_{\zeta,0}$, $f_\nu^\gamma(\zeta) = x_{\zeta,\nu}$, and $F^\gamma(\nu) = y_\nu$. Then any two conditions are compatible.

□

Proposition 6.7. *$(\{p \in P_\alpha \mid p \text{ is pure}\}, \leq^*)$ is α-closed.*

Proof. Let $\beta < \alpha$ and $\langle p^{\beta'} \mid \beta' < \beta \rangle$ be a $\leq^*$-decreasing sequence of conditions in P_α. Write $p^{\beta'} = \langle f_0^{\beta'}, \vec{f}^{\beta'}, A^{\beta'}, F^{\beta'} \rangle$ with its common domain $d^{\beta'}$. Let $d^* = \cup_{\beta' < \beta} d^{\beta'}$, $f_0^* = \cup_{\beta' < \beta} f_0^{\beta'}$. Let $(A^{\beta'})^*$ be the d^*-tree obtained by pulling back $A^{\beta'}$, and $A^* = \cap_{\beta' < \beta} (A^{\beta'})^*$. Shrink A^* further so that $\min(A^*(\alpha)) > \beta$. By induction on $\nu \in A^*(\alpha)$, we may find f_ν^* and $F^*(\nu)$ such that

- for $\zeta \in d^*$, $f_\nu^*(\zeta)$ is "forced" to be equal to $f_\nu^{\beta'}(\zeta)$ for some sufficiently large β' that $\zeta \in \mathrm{dom}(f^{\beta'})$.
- $F^*(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q}_\nu^* \rangle$ is such that $\dot{q}_\nu^*$ is "forced" to be a $\leq^*$-lower bound of $\langle \dot{q}_\nu^{\beta'} \mid \beta' < \beta \rangle$, where $F^{\beta'}(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q}_\nu^{\beta'} \rangle$. This is possible because $\Vdash_\nu$ "$(\dot{P}_{\dot{\beta}_\nu/\nu}, \leq^*)$ is ν^*-closed", where ν^* is the least inaccessible above ν, and $\nu > \beta$.

Then $\langle f^*, \vec{f}^*, A^*, F^* \rangle$, where $\vec{f}^* = \{f_\nu^* \mid \nu \in A^*(\alpha) \text{ is inaccessible}\}$, is as required.
□

Theorem 6.8. *$(P_\alpha, \leq, \leq^*)$ has the Prikry property, i.e. for $p \in P_\alpha$ and a forcing statement φ, there is $p^* \leq^* p$ such that $p^* \parallel \varphi$.*

To prove Theorem 6.8, we start with the following lemma.

Lemma 6.9. *Let $p \in P_\alpha$ and φ be a forcing statement. Then there is $p^* \leq^* p$ such that if $r = r_0{}^\frown \operatorname{top}(r)$, $r \leq p^*$, p' is the interpolant of r and p^*, and $r \parallel \varphi$, then*

$$r_0{}^\frown \operatorname{top}(p') \parallel \varphi \text{ the same way.}$$

Proof. Assume for simplicity that p is pure and write $p = \langle f_0, \vec{f}, A, F\rangle$ with its common domain d. A forcing $\mathbb{A}$ consists of conditions of the form $g = \langle g_0\rangle^\frown \vec{g}$, where there is a common domain d_g such that

- $\operatorname{dom}(g_0) = d_g$, $\vec{g} = \langle g_\nu \mid \nu \in A(\alpha)\rangle$, and for all ν, $\operatorname{dom}(g_\nu) = d_g$.
- for $\zeta \in d_g$, $f_0(\zeta) < \alpha$ and for $\beta < \alpha$ inaccessible, $\Vdash_{P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}}$ "$g_\beta(\zeta) < \alpha$".

For $g^0, g^1 \in \mathbb{A}$, define $g^0 \leq_{\mathbb{A}} g^1$ if $g^0_0 \supseteq g^1_0$, and for $\nu \in A$, $\zeta \in d_{g^1}$, and relevant $r \in P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}$, $r \Vdash_{P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}}$ "$g^0_\nu(\zeta) = g^1_\nu(\zeta)$". Clearly, $\mathbb{A}$ is α^+-closed.

Let $N \prec H_\theta$ for some sufficiently large regular θ, ${}^{<\alpha}N \subseteq N$, $|N| = \alpha$, $d, V_\alpha \subseteq N$, $p, \mathbb{P}, \mathbb{A} \in N$. Build an $\mathbb{A}$-decreasing sequence $\langle f^\gamma \mid \gamma < \alpha\rangle$ below $\langle f_0\rangle^\frown \vec{f}$ such that for every dense open set $D \in N \cap \mathcal{P}(\mathbb{A})$, there are unboundedly many $\gamma < \alpha$ such that $f^\gamma \in D$. Let $f^* = \langle f^*_0\rangle^\frown \vec{f}^*$ be the maximal $\leq^*$-lower bound of $\langle f^\gamma \mid \gamma < \alpha\rangle$ and d^* be its common domain, so $d^* = N \cap \alpha^{++}$. Let A^* be the d^*-tree which is the pullback of A. Note that $A^* \subseteq N$.

We are now going to consider an $\mathbb{A}$-decreasing subsequence $\langle f^{\gamma_\nu} \mid \nu \in A^*(\alpha)\rangle$ of $\langle f^\gamma \mid \gamma < \alpha\rangle$, together with $\langle \dot{q}^\nu_{\nu'} \mid \nu, \nu' \in A^*(\alpha)\rangle$ and $\langle A^\nu \mid \nu \in A^*(\alpha)\rangle$ satisfying a certain property, and

- for each ν', $\langle \dot{q}^\nu_{\nu'} \mid \nu \in A^*(\alpha)\rangle$ is forced to be $\leq^*$-decreasing below $\dot{q}_{\nu'}$, where $F(\nu') = \langle \dot{P}_{\dot{\beta}_{\nu'}/\nu'}, \dot{q}_{\nu'}\rangle$.
- for $\nu' < \nu$, $\dot{q}^\nu_{\nu'} = \dot{q}^{\nu'}_{\nu'}$.

All the proper initial subsequences will be in N. Let $\nu \in A^*(\alpha)$ and suppose that $\langle f^{\gamma_{\nu'}} \mid \nu' < \nu, \nu' \in A^*(\alpha)\rangle$, $\langle \dot{q}^{\nu'}_\rho \mid \nu' < \nu,\ \nu', \rho \in A^*(\alpha)\rangle$ have been constructed. For $\nu' < \nu$, let $\dot{q}^\nu_{\nu'} = \dot{q}^{\nu'}_{\nu'}$. Let f' be the maximal lower bound of the sequence $\langle f^{\gamma_{\nu'}} \mid \nu' < \nu, \nu' \in A^*(\alpha)\rangle$. For $\rho \geq \nu$, Let $\dot{q}^*_\rho$ be a P_ρ-name of a condition in $\dot{P}_{\dot{\beta}_\rho/\rho}$ which is forced to be a $\leq^*$-maximal lower bound of $(\dot{q}^{\nu'}_\rho)_{\nu' < \nu}$. This is possible since $\Vdash_\rho$ "$(\dot{P}_{\dot{\beta}_\rho/\rho}, \leq^*)$ is ν^+-closed" and note that $\langle \dot{q}^*_\rho \mid \rho \geq \nu\rangle \in N$. Consider the following set $D_\nu \subseteq \mathbb{A}$. $g = \langle g_0\rangle^\frown \vec{g} \in D_\nu$ with the common domain d_g, if either $\langle g_0\rangle^\frown \vec{g}$ is incompatible with $\langle f_0\rangle^\frown \vec{f}$, or the following holds:

- for every $\vec{\mu} \in A^*$ with $\vec{\mu}(\alpha) = \nu$, $\operatorname{dom}(\vec{\mu}) \subseteq d_g$.
- there are
  - a P_ν-name of a condition $\dot{q}^{**}_\nu$ in $\dot{P}_{\dot{\beta}_\nu/\nu}$ which is forced to be $\leq^*$ below $\dot{q}^*_\nu$
  - a d_g-tree A^ν with $\min(A^\nu(\alpha)) > \xi^* := \{\xi \mid \exists t \in P_\nu (t \Vdash_\nu \text{"}\dot{\beta}_\nu = \xi\text{"})\}$, and
  - a function F^ν with $\operatorname{dom}(F^\nu) = A^\nu(\alpha)$,
  - for $\rho \in A^\nu(\alpha)$ and all relevant $r \in P_\rho$, $r \Vdash_\rho$ "$F^\nu(\rho)_1 \leq^* \dot{q}^{**}_\rho$",

  such that for every $r \in P_\nu$ and $\dot{q}'$, if there are $h_0, \vec{h}$, A', and F' such that

$$r^\frown \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q}'\rangle^\frown \langle h_0, \vec{h}, A', F'\rangle \leq^* r^\frown \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q}^*_\nu\rangle^\frown \langle g_\nu, \langle g_{\nu'} \mid \nu' \in A^\nu(\alpha)\rangle, A^\nu, F^\nu\rangle,$$

  and

$$r^\frown \langle \dot{P}_{\dot{\xi}_\nu/\nu}, \dot{q}'\rangle^\frown \langle h_0, \vec{h}, A', F'\rangle \parallel \varphi,$$

then

$$r^\frown\langle\dot{P}_{\dot\beta_\nu/\nu},\dot{q}'\rangle^\frown\langle g_{\dot\beta_\nu},\langle g_{\dot\beta_{\nu'}}\mid \nu'\in A^\nu(\alpha)\rangle,A^\nu,F^\nu\rangle \parallel \varphi \text{ the same way.}$$

Claim 6.10. *$D_\nu \in N$ is open dense.*

Proof. The parameters we use to define D_ν are: $\mathbb{A}$, p, P_ν, and $\{\vec\mu\in A^*\mid\vec\mu(\alpha)=\nu\}$. By Remark 2.4, the latter set has size at most ν^{++}, and for each $\vec\mu\in A^*$, by the closure of N, $\vec\mu\in N$, hence, there is an enumeration of such a set in N. Thus, $D_\nu\in N$. To check the openness of D_ν, note that if $\vec g^0\leq_{\mathbb{A}}\vec g^1$ and $\vec g^1\in D_\nu$ with the witnesses $\dot q_\nu^{**}$, A^ν. and F^ν, then $\vec g^0$ is also in D_ν with the same witnesses.

It remains to show that D_ν is dense. Let $g_0{}^\frown\vec g\in\mathbb{A}$. If $\langle g_0\rangle^\frown\vec g\not\parallel\langle f_0\rangle^\frown\vec f$, then we are done. Suppose not, we may assume $\langle g_0\rangle^\frown\vec g\leq_{\mathbb{A}}\langle f_0\rangle^\frown\vec f$. By (1) of Proposition 4.1 for ν, let $\langle r_\xi,\dot q_\xi\mid\xi<(\xi^*)^{++}\rangle$ be an enumeration of elements in $P_\nu*\dot P_{\dot\beta_\nu/\nu}$ (with some repetitions if needed). Build sequences $\langle\langle h_0^\xi\rangle^\frown\vec h^\xi\rangle$, $\langle A_\xi,F_\xi\mid\xi\leq(\xi^*)^{++}\rangle$ such that

- $\langle\langle h_0^\xi\rangle^\frown\vec h^\xi\rangle_{\xi\leq\nu^{++}}$ is $\mathbb{A}$-decreasing, and is below $\langle g_0\rangle^\frown\vec g$.
- $\langle A_\xi\mid\xi\leq\nu^{++}\rangle$ is a $\mathrm{dom}(h_0^\xi)$-tree and for $\xi<\xi'$, $A_{\xi'}$ projects down to a subset of A_ξ, $\min(A_\xi(\alpha))>\xi^*$.
- for $\nu'\in A_\xi(\alpha)$, $\langle F_\xi(\nu')_1\rangle_{\xi\leq\nu^{++}}$ is forced to be $\leq^*$-decreasing below $\dot q^*_{\nu'}$.
- for $\xi<(\xi^*)^{++}$, if there are $h_0',\vec h',A'$, and F' such that
$$r_\xi{}^\frown\langle\dot P_{\dot\beta_\nu/\nu},\dot q_\xi\rangle^\frown\langle h_0',\vec h',A',F'\rangle$$
is a direct extension of
$$t^*:=r_\xi{}^\frown\langle\dot P_{\dot\beta_\nu/\nu},\dot q_\xi\rangle^\frown\langle h_\nu^{\xi+1},\langle h_\rho^{\xi+1}\mid\rho\in A_{\xi+1}(\alpha)\rangle,A_{\xi+1},F_{\xi+1}\rangle,$$
and
$$r_\xi{}^\frown\langle\dot P_{\dot\beta_\nu/\nu},\dot q_\xi\rangle^\frown\langle h_0',\vec h',A',F'\rangle\Vdash\varphi,$$
then t^* decides φ the same way.

The construction is straightforward, and for a limit ξ, we can take any witnesses at the stage ξ as long as the requirements are met. Finally, let $\langle g_0\rangle^\frown\vec g=\langle h_0^{\nu^{++}}\rangle^\frown\vec h^{\nu^{++}}$, $A^\nu=A_{\nu^{++}}$, and $F^\nu=F_{\nu^{++}}$. These will be the witnesses for $\langle g_0\rangle^\frown\vec g\in D_\nu$.

□

Let $\gamma_\nu\geq\sup_{\nu'<\nu}\gamma_{\nu'}$ such that $f^{\gamma_\nu}\in D_\nu$. Also, we obtain the witnesses, A^ν and F^ν. Let $\dot q_\nu^\nu=\dot q_\nu^*$. For $\rho>\nu$, let $\dot q_\rho^\nu$ be the second component of $F^\nu(\rho)$ if exists, otherwise, let $\dot q_\rho^\nu=\dot q_\rho^*$. This completes our analysis.

Assume that the pullback of A^ν to the d^*-tree has a subtree which is generated by $B^\nu\in E(d^*)$. Let A^{**} be a d^*-tree generated by $\Delta_\nu B^\nu$. Let F^{**} be a function with $\mathrm{dom}(F^{**})=A^*(\alpha)$ and for $\nu\in A^{**}(\alpha)$, $F^{**}(\nu)=\langle\dot P_{\dot\beta_\nu/\nu},\dot q_\nu^{**}\rangle$, $\dot q_\nu^{**}$ is the $\leq^*$-maximal lower bound of $(\dot q_\nu^{\nu'})_{\nu'\in A(\alpha)}$. This is possible since $(\dot q_\nu^{\nu'})_{\nu'\in A(\alpha)}$ stabilizes after the stage $\nu'=\nu$ (equivalently, we take $\dot q_\nu^{**}=\dot q_\nu^\nu$). Then $p^*=\langle f_0^*,\vec f^*,A^{**},F^{**}\rangle\leq^* p$ satisfies.

We now show that p^* satisfies Lemma 6.9. Let $p'\leq p^*$ such that p' decides φ, p' is of the form
$$p'=r^\frown\langle\dot P_{\dot\beta_\nu/\nu},\dot q\rangle^\frown\langle h_0',\vec h',A',F'\rangle,$$

Without loss of generality, assume that $p'\Vdash\varphi$. Let $\bar p$ be the interpolant of p^* and p'. We consider the notions of the proof of Claim 6.10. Say that $r=r_\xi$ and

$\dot{q} = \dot{q}_\xi$. By the construction of A^{**}, we have that A^{**} projects down to a subset of A^ν. This makes $p' \leq^* t^*$, and hence, $t^* \Vdash \varphi$. Thus, $r_\xi {}^\frown \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q}_\xi \rangle {}^\frown \operatorname{top}(\bar{p}) \Vdash \varphi$. This completes the proof of Lemma 6.9.

□

Proof of Theorem 6.8. Let p be a condition and φ be a forcing statement. For simplicity, assume p is pure and p satisfies Lemma 6.9. Write $p = \langle f_0, \vec{f}, A, F \rangle$, d is the common domain for p.

We build a $\leq^*$-decreasing sequence $\langle p^\nu \mid \nu \in A(\alpha) \rangle$ below p by induction.

Assume $p^{\nu'}$ is constructed for $\nu' < \nu$. Let p'_ν be a $\leq^*$-lower bound of $\langle p^{\nu'} \mid \nu' < \nu \rangle$. Write $p'_\nu = \langle f'_0, \vec{f}', A', F' \rangle$ with the common domain d'. For every ξ, let $Q_\xi := P_\xi * \dot{P}_{\dot{\beta}_\xi/\xi}$. Let $\xi^* = \sup\{\gamma \mid \exists r \in P_\nu (r \Vdash_\nu \text{“}\dot{\beta}_\nu = \gamma\text{”})\}$. Fix $\rho > \xi^*$, $\rho \in A'(\alpha)$. Let $\dot{G}_\rho$ be the canonical name for Q_ρ. Define

$$\varphi^0_\rho \equiv \text{“}\exists t \in \dot{G}_\rho (t^\frown \operatorname{top}(\bar{p}) \Vdash \varphi)\text{”}.$$
$$\varphi^1_\rho \equiv \text{“}\exists t \in \dot{G}_\rho (t^\frown \operatorname{top}(\bar{p}) \Vdash \neg\varphi)\text{”}.$$
$$\varphi^1_\rho \equiv \text{“}\nexists t \in \dot{G}_\rho (t^\frown \operatorname{top}(\bar{p}) \parallel \varphi)\text{”},$$

where $\bar{p}$ is the appropriate interpolation, as described in Lemma 6.9. Note that for $r \in Q_\rho$, there are at most one i such that $r \Vdash \varphi^i_\rho$. Enumerate $\langle \mu \in \operatorname{Lev}_0(A') \mid \mu(\alpha) = \rho \rangle$ as $\{\mu_\xi\}_{\xi < \rho^{++}}$. By the closure of $(\dot{P}_{\dot{\beta}_\rho/\rho}, \leq^*)$, we can find $\dot{q}^*_\rho$ such that $\Vdash_\rho$ “$\dot{q}^*_\rho \leq^* F'(\rho)_1$” such that for every $r \in Q_\nu$ and $\xi < \rho^{++}$, there is f^{μ_ξ} with $r \Vdash_{Q_\nu}$ “$f^{\mu_\xi} \leq^* f^*_\nu \circ \mu_\xi^{-1}$”, and if there are f, $\dot{q}$ with $r^\frown(\langle f, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q} \rangle\rangle \leq^* r^\frown(\langle f^{\mu_\xi}, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle\rangle$ which forces φ^i_ρ, then so is $r^\frown(f^{\mu_\xi}, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle)$. Now, for each $\mu = \mu_\xi$, we have $f^\mu = f^{\mu_\xi}$. Let $f^*_\nu = j_{E(\alpha,0)}(\mu \mapsto f^\mu)(\operatorname{mc}(d'))$. Then f^*_ν is forced to be an extension of f'_ν. Say $d^* = \operatorname{dom}(f^*_\nu)$. For $\rho \neq \nu$ including 0, let $f^*_\rho = f'_\rho \cup \{(\xi, 0) \mid \xi \in d^* \setminus d'\}$. Let $F^*(\rho) = F(\rho)$ for $\rho \leq \gamma^*$, otherwise, $F^*(\rho) = \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle$. Take $p^*_\rho = \langle f^*_0, \vec{f}^*, A', F^* \rangle$.

Now assume that A' is generated by B'. By shrinking further, assume that for $\mu \in B'$, $f^*_\nu \circ \mu^{-1} = f^{\mu \restriction d'}$. For $r \in Q_\nu$ and $\mu \in B'$, with $\mu(\alpha) = \rho$, by the Prikry property, and the construction as above there is $r^\mu \leq r$ such that $r^{\mu\frown}(\langle f^*_\nu \circ \mu^{-1}, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle\rangle) \Vdash \varphi^i_\rho$ for a unique i. Let B^r_i be the collection of μ such that by writing $\rho = \mu(\alpha)$,

$$r^{\mu\frown}(\langle f^*_\nu \circ \mu^{-1}, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle\rangle) \Vdash \varphi^i_\rho.$$

There exists unique $i = i_r$ such that $B^r_{i_r}$ is of measure-one. By shrinking further, assume that there is r^* such that for every $\mu \in B^r_{i_r}$, $r^\mu = r^*$. Let $B^* = \Delta_\nu \cap_{r \in Q_\nu} B^r_{i_r}$. Let A^* be generated by B^* and $p^\nu = \langle f^*_0, \vec{f}^*, A^*, F^* \rangle$. This completes the construction of p^ν.

We now change a notation by saying that $p^\nu = \langle f^\nu_0, \vec{f}^\nu, A^\nu, F^\nu \rangle$ and B^ν generates A^ν. Let $p^* = \langle f^*_0, \vec{f}^*, A^*, F^* \rangle$, where A^* is generated by $\Delta_\nu B^\nu$, $f^* = \cup_\nu f^\nu_0$, and for $\rho \in A^*(\alpha)$, $f^*_\rho = \cup_\rho f^\rho_\rho$, and $F^*(\rho) = \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle$, where q^*_ρ is forced to be a $\leq^*$-lower bound of $\langle F^\nu(\rho)_1 \rangle_\nu$. This is possible because for every ρ, $\langle F^\nu(\rho) \rangle_\nu$ stabilizes at $\nu = \rho$. Note that $p^* \leq^* p$.

Claim 6.11. *p^* satisfies the Prikry property.*

Proof. Let $p' \leq p^*$ with $p' \parallel \varphi$, Assume $p' \Vdash \varphi$ and the interpolant of p', p^*, say $\bar{p}$, is such that $\bar{p} = p^* + \vec{\mu}$ with the minimal $n^* = |\vec{\mu}|$. If $n^* = 0$. then we might apply p' for the Prikry property instead. Assume $n^* > 0$.

For simplicity, we establish the case $n^* = 2$. Say $\bar{p} = p^* + \langle \mu_0, \mu_1 \rangle$. Let

$$p' = (g_0, \langle \dot{P}_{\dot{\beta}_0/\nu_0}, \dot{q}_0 \rangle)^\frown (g_1, \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}_1 \rangle)^\frown \operatorname{top}(p').$$

Since p satisfies Lemma 6.9, we have that

$$(g_0, \langle \dot{P}_{\dot{\beta}_0/\nu_0}, \dot{q}_0 \rangle)^\frown (g_1, \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}_1 \rangle)^\frown \operatorname{top}(\bar{p}) \Vdash \varphi.$$

Set $r = (g_0, \langle \dot{P}_{\dot{\beta}_0/\nu_0}, \dot{q}_0 \rangle)$. We use the notation for the construction of p_{ν_1}. Note that $r \Vdash g_1 \leq f^{\mu_1}$ and $r^\frown \langle g_1 \rangle \Vdash_{\nu_1}$ "$\dot{q}_1 \leq \dot{q}^*_{\nu_1}$". We claim that $i_r = 0$. Otherwise, we may assume $i_r = 1$ (the case $i_r = 2$ is similar). Let G be Q_{ν_1}-generic containing $\operatorname{stem}(p^*)$. Then there is $t \in G$ such that $t^\frown \operatorname{top}(\bar{p}) \Vdash \neg\varphi$, but if $t \leq \operatorname{stem}(p^*)$, we get a condition having contradictory decisions, which is a contradiction.

Note that $\mu_1(\alpha) = \nu_1$. We claim that $r^{*\frown}(f^{\mu_1 \restriction d'}, \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}^*_{\nu_1} \rangle)^\frown \operatorname{top}(\bar{p}) \Vdash \varphi$. Otherwise, we use the same argument as above and Lemma 6.9 to get a contradiction.

Consider $p^* + \langle \mu_0 \rangle$. Since $i_r = 0$, we have that for every $\langle \mu \rangle \in \operatorname{Lev}_0(A^{p + \langle \mu_0 \rangle})$, $\mu \restriction d' \in B^r_0$. By a similar argument as above, if $p^\mu = p^* + \langle \mu \rangle$, then

$$r^{*\frown} \langle f^{\mu \restriction d'}, F^*(\mu(\alpha)) \rangle^\frown \operatorname{top}(p^\mu) \Vdash \varphi.$$

By a density argument, $r^{*\frown} \operatorname{top}(p + \langle \mu_0 \rangle) \Vdash \varphi$, which contradicts the minimality of $|\vec{\mu}|$.

□

□

By the Prikry property and the fact the direct extension on P_α restricted to the pure conditions are α-closed, it is standard to verify that all cardinals up to and including α are preserved.

The forcing singularizes α to have cofinality ω, and add α^{++} subsets of α: for $\gamma \in [\alpha, \alpha^{++})$, define $t_\gamma : \omega \to \alpha$ as the following. By a density argument, let $p \in G$ be such that the common domain contains γ. Assume that n^p is the number of the blocks in $p \setminus \operatorname{top}(p)$. For $n > n^p$, find any $p^\gamma \in G$ such that the number of blocks in $p^\gamma \setminus \operatorname{top}(p^\gamma) \geq n$. Write

$$p^\gamma = s_0{}^\frown \cdots {}^\frown s_{n-2}{}^\frown (f_{n-1}, s'_{n-1})^\frown \cdots {}^\frown (f_{k-1}, s'_{n-1})^\frown \langle f, \vec{f}, A, F \rangle.$$

By compatibility between p^γ and p, we have that $f(\gamma)$ has to be of the form $\check{\xi}_0$, $\xi_0 \in \operatorname{dom}(f_{n-1})$, $f_{n-1}(\xi_0) = \check{\xi}_1$, and so on. Define $t_\gamma(n) = f_{n-1} \circ \cdots \circ f_{k-1} \circ f(\gamma)$. Clearly t_α gives a cofinal sequence of α of length ω, and hence, α is singularized to have cofinality ω. Again, by a standard argument with the Prikry property, α^+ is preserved. Since the forcing is α^{++}-c.c., all the cardinals are preserved. One can show that for $\gamma < \gamma'$, there is $p \in G$ such that for every relevant object μ appearing in the tree part, $\gamma, \gamma' \in \operatorname{dom}(\mu)$. From here, use a density argument to show that $t_\gamma <^* t_{\gamma'}$. Hence, the forcing violates the SCH at α.

The set C_α is derived from the generic object as the following. If G is P_α-generic, define $C' = \operatorname{rng}(t_\alpha) \cup \{\alpha\}$. Each condition $p \in G$ is of the form

$$s^\frown (f_k, \langle \dot{P}_{\dot{\xi}_k/\nu_k}, \dot{q} \rangle)^\frown \langle f, \vec{f}, A, F \rangle$$

where $\nu_k = t_\alpha(k+1)$. In this case, the forcing $\dot{P}_{\dot{\xi}_k/\nu_k}$ also derives the set $C^k = C_{\xi_k/\nu_k}$, where $t_\alpha(k+1) = \nu_k < \xi_k < \nu_{k+1} = t_\alpha(k+2)$. Let $C_\alpha = C' \cup \cup_{k<\omega} C^k$. Then $C_\alpha \subseteq \alpha+1$, $\max(C_\alpha) = \alpha$, $C_\alpha \setminus \{\alpha\}$ is a cofinal subset of α, containing a subset of order-type ω. So far, we have verified items (1) through (3) of Proposition 4.1.

Definition 6.12 (The quotient forcing)**.** Let $\dot{P}_{\alpha/\alpha}$ be the P_α-name of the trivial forcing $(\{\emptyset\}, \leq, \leq^*)$. In V^{P_α}, let $\dot{C}_{\alpha/\alpha}$ be the $\dot{P}_{\alpha/\alpha}$-name of the empty set. Now assume that $\beta < \alpha$. Define $\dot{P}_{\alpha/\beta}$ as the following. Let G be P_β-generic. Define $\dot{P}_\alpha[G] = P_{\alpha/\beta}[G]$ as the forcing consisting of conditions of the form

$$p = (\langle P_{\beta'}[G], q' \rangle ^\frown (\langle f_0 \rangle, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}[G], \dot{q}_0 \rangle) \cdots {}^\frown (\langle f_{n-1} \rangle, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}[G], \dot{q}_{n-1} \rangle) ^\frown \langle g_0, \vec{g}, A, F \rangle$$

where $n \geq 0$ and

(1) $\beta \leq \beta' < \alpha$, so $P_{\beta'}[G]$ was already defined by recursion, which is just $P_{\beta'/\beta}[G]$, $q_0 \in P_{\beta'}[G]$.
(2) If $n > 0$, then $\alpha_0 < \cdots < \alpha_{n-1}$, and for $i < n$,
   - let $d_i = \mathrm{dom}(f_i)$, then d_i is an α_i-domain, $d_i \in V$.
   - for $\zeta \in d_0$, $\Vdash_{P_{\beta'}[G]}$ "$f_0(\zeta) < \alpha_0$", and if $i > 0$, then for $\zeta \in d_i$, $\Vdash_{P_{\alpha_{i-1}}[G] * \dot{P}_{\dot{\beta}_{i-1}/\alpha_{i-1}}[G]}$ "$f_i(\zeta) < \alpha_i$".
   - $\Vdash_{P_{\alpha_i}[G]}$ "$\alpha_i \leq \dot{\beta}_i < \alpha_{i+1}$", where $\alpha_n = \alpha$.
   - $\Vdash_{P_{\alpha_i}[G]}$ "$\dot{q}_i \in \dot{P}_{\dot{\beta}_i/\alpha_i}[G]$".
(3) A is a $E(d)$-tree.
(4) $d \in [\alpha^{++}]^{\leq\alpha}$ is the *common domain* for p, i.e. $\mathrm{dom}(g_0) = d$, and $\vec{g} = \langle g_\nu \mid \nu \in A(\alpha) \rangle$ and for each ν, $\mathrm{dom}(g_\nu) = d$.
(5) Fix $\zeta \in d$. If $n = 0$, then $\Vdash_{P_{\beta'}[G]}$ "$g_0(\zeta) < \alpha$", otherwise, $\Vdash_{P_{\alpha_{n-1}}[G] * P_{\dot{\beta}_{n-1}/\alpha_{n-1}}[G]}$ "$g_0(\zeta) < \alpha$".
(6) for $\nu \in A(\alpha)$ and $\zeta \in d$, $\Vdash_{P_\nu[G] * \dot{P}_{\dot{\beta}_\nu/\nu}[G]}$ "$g_\nu(\zeta) < \alpha$".
(7) $\mathrm{dom}(F) = A(\alpha)$.
(8) for $\nu \in \mathrm{dom}(F)$, $F(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}[G], \dot{q} \rangle$, where $\Vdash_{P_\nu[G]}$ "$\nu \leq \dot{\xi}_\nu[G] < \alpha, \dot{q} \in \dot{P}_{\dot{\beta}_\nu/\nu}[G]$"

Back in V. If $\dot{p} \in \dot{P}_{\alpha/\beta}$, then by density, the collection of $p_0 \in P_\beta$ such that p_0 decides n, $\alpha_0, \cdots, \alpha_{n-1}$, $\mathrm{dom}(f_0), \cdots, \mathrm{dom}(f_{n-1})$, the common domain, A, q' (as the equivalent $\dot{P}_{\dot{\beta}'/\beta}$-name, and so on), is open dense. In this case, we say that p_0 *interprets* $\dot{p}$. All in all, for such p_0 which interprets all the relevant components of $\dot{p}$, let p_1 be such the interpretation. Write p_0 as $r_0{}^\frown\langle g \rangle$ and by the interpretation, we may write

$$p_1 = (\langle \dot{P}_{\beta'/\beta}, \dot{q}' \rangle ^\frown (\langle f_0 \rangle, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0 \rangle) \cdots {}^\frown (\langle f_{n-1} \rangle, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1} \rangle) ^\frown \langle f \rangle.$$

There is a natural concatenation p_0 with p_1, written by $p_0{}^\frown p_1$, which is

$$r = r_0{}^\frown (\langle g \rangle, \langle \dot{P}_{\beta'/\beta}, \dot{q}' \rangle)^\frown \cdots {}^\frown (\langle f_{n-1} \rangle, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1} \rangle) ^\frown \langle f \rangle.$$

Then $r \in P_\alpha$ with $r \restriction P_\beta$ exists. We denote p_1 by r/P_β. For p_0 and p_1 in $\dot{P}_{\alpha/\beta}$, we say that $p_0 \leq p_1$ if there is $p \in G^{P_\beta}$ such that p interprets p_0 and p_1, and $p^\frown p_0 \leq_\alpha p^\frown p_1$. Also define $p_0 \leq^* p_1$ if there is $p \in G^{P_\beta}$ such that p interprets p_0 and p_1, and $p^\frown p_0 \leq^*_\alpha p^\frown p_1$. One can check that the map $\phi : \{p \in P_\alpha \mid p \restriction P_\beta$ exists$\} \to P_\beta * \dot{P}_{\alpha/\beta}$ defined by $\phi(p) = (p \restriction P_\beta, p/P_\beta)$ is a dense embedding, where

$p \setminus P_\beta$ is the obvious component of p which is in $\dot{P}_{\alpha/\beta}$. Note that if G is P_β-generic and H is $P_\alpha[G]$-generic, there is a generic I for P_α such that $V[G * H] = V[I]$, where I is generated by $\{p \mid p \restriction P_\beta \text{ exists}, p \restriction P_\beta \in G \text{ and } (p/P_\beta)[G] \in H\}$. If I is P_α-generic and for some $p \in I$, $p \restriction P_\beta$ exists, we can get G which is P_β-generic and H which is $P_\alpha[G]$-generic such that $V[G * H] = V[I]$ where G is generated by $\{p \restriction P_\beta \mid p \in I \text{ and } p \restriction P_\beta \text{ exists}\}$ and $H = \{(p/P_\beta)[G] \mid p \in I \text{ and } p \restriction P_\beta \text{ exists}\}$.

In V^{P_β}, let $\dot{C}_{\alpha/\beta}$ be a $\dot{P}_{\alpha/\beta}$-name of the set described as the following. Let G be P_β-generic. and H be generic over $P_\alpha[G] = \dot{P}_{\alpha/\beta}[G]$. Then let $I = G * H$ be P_α-generic. I derives the set $C_\alpha \subseteq \alpha + 1$ and G derives the set $C_\beta \subseteq \beta + 1$. Let $C_{\alpha/\beta} = C_\alpha \setminus C_\beta$.

The following have the same proof as for P_α essentially. The one that we would like to point out is the closure property.

Proposition 6.13.
- $\Vdash_\beta$ "$(\dot{P}_{\alpha/\beta}, \leq^*)$ *is* α^{++}*-c.c.*"
- $\Vdash_\beta$ "$(\dot{P}_{\alpha/\beta}, \leq, \leq^*)$ *has the Prikry property.*
- $\Vdash_\beta$ "$(\dot{P}_{\alpha/\beta}, \leq^*)$ *is* β^**-closed", where* β^* *is the least inaccessible cardinal greater than* β.

Proof. We only proof item (3). For simplicity, let $\beta' < \beta^*$ and in V^{P_β}, let $\langle p_\gamma \mid \gamma < \beta' \rangle$ be a $\leq^*$-decreasing sequence. Write $p_\gamma = \langle P_\xi[G], q^\gamma \rangle^\frown \langle g_0^\gamma, \vec{g}^\gamma, A^\gamma, F^\gamma \rangle$ with the common domain d^γ. Since $(P_\xi[G], \leq^*)$ is β^*-closed, let q^* be a $\leq^*$-lower bound of q^γ. In V, let $d^* = \cup\{d \mid \exists \gamma \exists p \in P_\beta (p \Vdash_\beta \dot{d}_\gamma = d)\}$. For all β (including 0) with g_β^γ exists, let $\mathrm{dom}(g_\beta^*) = d^*$, and for $\zeta \in d$, $g_\beta^*(\zeta)$ is forced to be the same as the interpretation $g_\beta^*(\zeta)$ for some sufficiently large γ, if exists, otherwise, $g_\beta^\gamma(\zeta) = \check{0}$. Let $A^* = \cap_\gamma \cap_p \{A \mid A \text{ is the pullback of } A^{\gamma,p}\}$ where $p \Vdash_\beta$ "$\dot{A}^\gamma = A^{\gamma,p}$". By shrinking, assume $\min(A^*(\alpha)) > \beta$. Finally, for each $\gamma \in A^*(\alpha)$, the forcing which is relevant to $F^\gamma(\alpha)$ (for any γ) is greater than γ-closed in the direct extension, and $\gamma > \beta$, so we can find F^* such that $\langle P_\xi[G], q^* \rangle^\frown \langle g^*, \vec{g}^*, A^*, F^* \rangle$ is a $\leq^*$-lower bound of $\langle p_\gamma \mid \gamma < \beta' \rangle$. $\square$

With all the definitions, one can verify the rest of Proposition 4.1.

7. The general levels

Let $\alpha < \kappa$ be inaccessible. We may assume that α is greater than the first β with $\circ(\beta) = 1$. This forcing will generalize all of the forcings in previous sections.

Definition 7.1. A condition in P_α is of the form

$$p = \mathrm{stem}(p)^\frown \mathrm{top}(p).$$

We have two cases.

1. $\mathrm{stem}(p)$ is empty. In this case, p is said to be *pure.*
2. $\mathrm{stem}(p)$ is non-empty. In this case, p is said to be *impure.* Then $\mathrm{stem}(p)$ is of the form

$$(s_0, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \cdots ^\frown (s_{n-1}, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1} \rangle),$$

   for some $n > 0$. We say that the number of blocks in $\mathrm{stem}(p)$ is n. We have that
   - $\alpha_0 < \cdots < \alpha_{n-1} < \alpha$.

- for all i, $\Vdash_{\alpha_i}$ "$\alpha_i \leq \dot{\beta}_i < \alpha_{i+1}$", where $\alpha_n = \alpha$.
- $(s_0, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \cdots {}^\frown s_{n-1} \in P_{\alpha_{n-1}}$.
- $\Vdash_{\alpha_{n-1}}$ "$\dot{q}_{n-1} \in \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}$".

Equivalently, $\operatorname{stem}(p) \in P_{\alpha_{n-1}} * \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}$.

$\operatorname{top}(p)$ also depends on $\operatorname{stem}(p)$ and α. We have several cases.

(1) The case where p is pure.
  (a) $\circ(\alpha) = 0$. Then $\operatorname{top}(p) = \langle f \rangle$, $f \in C(\alpha^+, \alpha^{++})$.
  (b) $\circ(\alpha) > 0$. In this case, $\operatorname{top}(p) = \langle f_0, \vec{f}, A, F \rangle$, where
    - $\vec{f} = \langle f_\beta \mid \beta < \alpha \text{ is inaccessible} \rangle$.
    - there is a *common domain* d, which is an α-domain, $\operatorname{dom}(f_0) = d$ and for all β, $\operatorname{dom}(f_\beta) = d$.
    - $f \in C(\alpha^+, \alpha^{++})$ and for each inaccessible $\beta < \alpha$, and $\zeta \in d$, $\Vdash_\beta$ "$f_\beta(\zeta) < \alpha$".
    - A is a d-tree, with respect to $\vec{E}_\alpha(d)$.
    - $\operatorname{dom}(F) = A(\alpha)$.
    - for $\nu \in \operatorname{dom}(F)$, $F(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q} \rangle$ where $\Vdash_\nu$ "$\nu \leq \dot{\beta}_\nu < \alpha$ and $\dot{q} \in \dot{P}_{\dot{\beta}_\nu/\nu}$".

(2) The case where p is impure, say $\operatorname{stem}(p) \in P_{\alpha'} * \dot{P}_{\dot{\beta}'/\alpha'} =: Q$.
  (a) $\circ(\alpha) = 0$. Then $\operatorname{top}(p) = \langle f \rangle$, $\operatorname{dom}(f) = d \in V$ is an α-domain and for $\zeta \in d$, $\Vdash_Q$ "$f(\zeta) < \alpha$".
  (b) $\circ(\alpha) > 0$. In this case, $\operatorname{top}(p) = \langle f_0, \vec{f}, A, F \rangle$, where there is a *common domain* $d \in [\alpha^{++}]^{\leq \alpha}$, $d \in V$, d is an α-domain such that
    - A is a d-tree, with respect to $\vec{E}_\alpha(d)$, $\min(A(\alpha)) > \sup\{\gamma \mid \exists r \in P_{\alpha_{n-1}}(r \Vdash \dot{\beta}_{n-1} = \gamma)\}$.
    - $\vec{f} = \langle f_\nu \mid \nu \in A(\alpha) \rangle$.
    - $\operatorname{dom}(F) = A(\alpha)$.
    - for $\nu \in \operatorname{dom}(F)$, $F(\nu) = \langle \dot{P}_{\dot{\xi}_\nu/\nu}, \dot{q} \rangle$ where $\Vdash_\nu$ "$\nu \leq \dot{\xi}_\nu < \alpha$ and $\dot{q} \in \dot{P}_{\dot{\beta}_\nu/\nu}$".
    - $\operatorname{dom}(f_0) = d$ and for all ν, $\operatorname{dom}(f_\nu) = d$.
    - for $\zeta \in d$, $\Vdash_Q$ "$f_0(\zeta) < \alpha$".
    - for $\nu \in A(\alpha)$ and $\zeta \in d$, $\Vdash_{P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}}$ "$f_\beta(\zeta) < \alpha$".

Definition 7.2 (The one-step extension)**.** Assume $\circ(\alpha) > 0$. Let $p = \operatorname{stem}(p)^\frown \langle f_0, \vec{f}, A, F \rangle$ with the common domain d. Let $\langle \mu \rangle \in \operatorname{Lev}_0(A)$ with $\mu(\alpha) = \nu$. The *one-step extension of p by μ*, denoted by $p + \langle \mu \rangle$, is the condition $p' = \operatorname{stem}(p')^\frown \langle g_0, \vec{g}, A', F' \rangle$ such that

(1) if $\circ(\mu(\alpha)) = 0$, then $\operatorname{stem}(p') = \operatorname{stem}(p)^\frown (f_0 \circ \mu^{-1}, F(\mu(\alpha)))$, where $\operatorname{dom}(f_0 \circ \mu^{-1}) = \operatorname{rng}(\mu)$, for $\gamma \in \operatorname{dom}(\mu)$, $f_0 \circ \mu^{-1}(\mu(\gamma)) = f_0(\gamma)$.
(2) if $\circ(\mu(\alpha)) > 0$, then $\operatorname{stem}(p') = \operatorname{stem}(p)^\frown (\langle f_0 \circ \mu^{-1}, \langle f_\beta \circ \mu^{-1} \mid \beta \in (A \downarrow \mu)(\mu(\alpha)), A \downarrow \mu, F' \rangle, F(\mu(\alpha)))$, where $\operatorname{dom}(F') = (A \downarrow \mu)(\mu(\alpha))$, and for ν, $F'(\nu) = F(\nu)$.
(3) Write Q as the forcing in which $\operatorname{stem}(p')$ lives. Say $Q = P_{\mu(\alpha)} * \dot{P}_{\dot{\beta}/\mu(\alpha)}$. Then
  - $\Vdash_Q$ "$g_0 = f_{\mu(\alpha)} \oplus \mu$", namely $\operatorname{dom}(g_0) = d$, for $\zeta \in \operatorname{dom}(\mu)$, $g_0(\zeta) = \mu(\zeta)$, and for the other ζ, $g_0(\zeta) = f_{\mu(\alpha)}(\zeta) =$

- $\vec{g} = \{g_{\beta'} \mid \beta' \in \{\vec{\tau} \in A_{\langle\mu\rangle} \mid \tau_0(\alpha) > \xi^*\}$, where $\xi^* = \sup\{\gamma \mid \exists r \in P_{\mu(\alpha)}(r \Vdash_{\mu(\alpha)} \dot{\beta}_{\mu(\alpha)} = \gamma\}$.
- $A' = A_{\langle\mu\rangle} \mid \tau_0(\alpha) > \xi^*\}$.
- $F' = F \restriction (A'(\alpha))$.

We define $p+\langle\rangle$ as p, and by recursion, define $p+\langle\mu_0,\cdots,\mu_n\rangle = (p+\langle\mu_0,\cdots,\mu_{n-1}\rangle)+\langle\mu_n\rangle$.

Definition 7.3 (The direct extension relation)**.** Let $p = \mathrm{stem}(p)^\frown \mathrm{top}(p)$ and $p' = \mathrm{stem}(p')^\frown \mathrm{top}(p')$. We say that p is a *direct extension of* p', denoted by $p \leq^*_\alpha p'$, if the following hold.

(1) $\mathrm{stem}(p) \leq^* \mathrm{stem}(p')$ (in some $Q := P_{\alpha'} * \dot{P}_{\dot{\beta}'/\alpha'}$).
(2) If $\circ(\alpha) = 0$, write $\mathrm{top}(p) = \langle f\rangle$ and $\mathrm{top}(p') = \langle g\rangle$, then $\mathrm{dom}(f) \supseteq \mathrm{dom}(g)$, and for $\zeta \in \mathrm{dom}(g)$, $\Vdash_Q$ "$f(\zeta) = g(\zeta)$".
(3) Suppose $\circ(\alpha) > 0$. Write $\mathrm{top}(p) = \langle f_0, \vec{f}, A, F\rangle$ and $\mathrm{top}(p') = \langle g_0, \vec{g}, A', F'\rangle$. Let d^p and $d^{p'}$ be the common domains for p and p', respectively. Then
   - $d^p \supseteq d^{p'}$.
   - $A \restriction d^{p'} \subseteq A'$.
   - for $\zeta \in d^{p'}$, $\Vdash_Q$ "$f_0(\zeta) = g_0(\zeta)$".
   - for $\nu \in A(\alpha)$ and $\vec{\mu} \in A$ with $\vec{\mu}(\alpha) = \nu$, say $F(\nu) = \langle \dot{P}_{\dot{\xi}_\nu/\nu}, \dot{q}\rangle$, and for $\zeta \in d^{p'}$, we have
   $$p + \vec{\mu} \restriction (P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}) \Vdash_{P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}} \text{“}f_\nu(\zeta) = g_\nu(\zeta)\text{”}.$$
   - for $\nu \in A(\alpha)$ and $\vec{\mu} \in A$ with $\vec{\mu}(\alpha) = \nu$,
   $$p + \vec{\mu} \restriction P_\nu \Vdash_\nu \text{“}F(\nu)_0 = F'(\nu)_0 \text{ and } F(\nu)_1 \leq^*_{F(\nu)_0} F'(\nu)_1\text{”}$$
   (the last direct extension is intentional).

Definition 7.4 (The extension relation)**.** Let $p = \mathrm{stem}(p)^\frown \mathrm{top}(p)$ and $p' = \mathrm{stem}(p')^\frown \mathrm{top}(p')$. We say that p is a *extension of* p', denoted by $p \leq_\alpha p'$, if the following hold.

(1) The case $\circ(\alpha) = 0$. Then
   - $\mathrm{stem}(p) \leq \mathrm{stem}(p')$ in some $Q = P_{\alpha'} * \dot{P}_{\dot{\beta}'/\alpha'}$.
   - Write $\mathrm{top}(p) = \langle f\rangle$ and $\mathrm{top}(p') = \langle g\rangle$. Then $\mathrm{dom}(f) \supseteq \mathrm{dom}(g)$ and for $\zeta \in \mathrm{dom}(g)$, $\mathrm{stem}(p) \Vdash_Q$ "$f(\zeta) = g(\zeta)$".
(2) The case $\circ(\alpha) > 0$. Then there is $\vec{\mu}$ (possibly empty) such that if $p^* = p'+\vec{\mu}$, and we write $\mathrm{top}(p) = \langle f, \vec{f}, A, F\rangle$ and $\mathrm{top}(p^*) = \langle g, \vec{g}, A^*, F^*\rangle$, d^p and d^* are the common domains for p and p^*, respectively, then
   - $\mathrm{stem}(p) \leq \mathrm{stem}(p^*)$ in some $Q = P_{\alpha'} * \dot{P}_{\dot{\beta}'/\alpha'}$.
   - $d^p \supseteq d^{p^*}$.
   - $A \restriction d^{p^*} \subseteq A^*$.
   - for $\zeta \in d^{p^*}$, $\Vdash_Q$ "$f_0(\zeta) = g_0(\zeta)$".
   - for $\nu \in A(\alpha)$ and $\vec{\mu} \in A$ with $\vec{\mu}(\alpha) = \nu$, say $F(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}, \dot{q}\rangle$, and for $\zeta \in d^{p'}$, we have
   $$p + \vec{\mu} \restriction (P_\nu * \dot{P}_{\dot{\xi}_\nu/\nu}) \Vdash_{P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}} \text{“}f_\nu(\zeta) = g_\nu(\zeta)\text{”}.$$
   - for $\nu \in A(\alpha)$ and $\vec{\mu} \in A$ with $\vec{\mu}(\alpha) = \nu$,
   $$p + \vec{\mu} \restriction P_\nu \Vdash_\nu \text{“}F(\nu)_0 = F^*(\nu)_0 \text{ and } F(\nu)_1 \leq^*_{F(\nu)_0} F^*(\nu)_1\text{”}.$$

Equivalently, $p \leq p'$ if there is $\vec{\mu}$ such that p is a condition obtained by extending the interleaving part of a direct extension of $p' + \vec{\mu}$. We call p^* the *interpolant* of p and p'. To be precise, p^* is the unique condition such that $p^* = p + \vec{\mu}$ for some $\vec{\mu}$, p' is obtained by extending the interleaving part of a direct extension of p'.

Proposition 7.5. *$(P_\alpha, \leq)$ has the α^{++}-chain condition.*

Proof. Similar to the proof of Proposition 6.6. □

Proposition 7.6. *$(\{p \in P_\alpha \mid p \text{ is pure}\}, \leq^*)$ is α-closed.*

Proof. Similar to the proof of Proposition 6.7. □

Let $\beta < \circ(\alpha)$. Let $\langle f_0 \rangle^\frown \langle f_{\dot{\xi}_\nu} \mid \nu \in B \rangle$, $B \in \cap_{\gamma < \circ(\alpha)} E(\alpha,\gamma)(\{\alpha\})$, there is $d \in [\alpha^{++}]^{\leq \alpha}$ such that $\mathrm{dom}(f_0) = d$, for all ν, $\mathrm{dom}(f_{\dot{\xi}_\nu}) = d$, and each $\zeta \in d$, $\Vdash_{P_\nu * \dot{P}_{\dot{\xi}_\nu/\nu}}$ "$f_{\dot{\xi}_\nu} < \alpha$". Let $X \in E(\alpha,\beta)(d)$ and for each $\mu \in X$, $\vec{g}_\mu = \langle g_0 \rangle^\frown \langle g_{\dot{\xi}_{\nu'}} \mid \nu' \in B \downarrow \mu \rangle \leq \langle f_0 \circ \mu^{-1} \rangle^\frown \langle f_{\dot{\xi}_{\nu'}} \circ \mu^{-1} \mid \nu' \in B \downarrow \mu \rangle$, where $B \downarrow \mu = \{\nu' \in B \cap \mu(\alpha) \mid \circ(\nu') < \circ(\mu(\alpha))\}$. Let $\vec{g} = j_{E(\alpha,\beta)}(\mu \mapsto \vec{g}_\mu)(\mathrm{mc}_{\alpha,\beta}(d))$. Then

(1) $\vec{g} = \langle f_0 \rangle^\frown \langle f_{\dot{\xi}_{\nu'}} \mid \nu' \in B, \circ(\nu') < \beta \rangle$.

(2) $\vec{g} \leq \vec{f} \restriction \{\nu' \in B \mid \circ(\nu') < \beta\}$.

The point is $\vec{g} \leq j_{E(\alpha,\beta)}(\mu \mapsto (\vec{f} \restriction B \downarrow \mu) \circ \mu^{-1})(\mathrm{mc}_{\alpha,\beta}(d))$. $j_{E(\alpha,\beta)}(d)(\mu \mapsto B \downarrow \mu^{-1})(\mathrm{mc}_{\alpha,\beta}(d)) = \{\nu' \in B \mid \circ(\nu') < \beta\}$, and for each ν', $j_{E(\alpha,\beta)}(f_{\dot{\xi}_{\nu'}}) \circ \mathrm{mc}_{\alpha,\beta}(d) = f_{\dot{\xi}_{\nu'}}$.

Theorem 7.7. *$(P_\alpha, \leq, \leq^*)$ has the Prikry property, i.e. for $p \in P_\alpha$ and a forcing statement φ, there is $p^* \leq^* p$ such that $p^* \parallel \varphi$.*

If $\circ(\alpha) = 0$, any $p \in P_\alpha$ is a finite iteration of Prikry-type forcings, hence, it has the Pirkry property. The proof for $\circ(\alpha) = 1$ is similar to the proof of Theorem 6.8. We assume $\circ(\alpha) > 1$.

Lemma 7.8. *Let $p \in P_\alpha$ and φ be a forcing statement. Then there is $p^* \leq^* p$ such that if $r = r_0{}^\frown \mathrm{top}(r)$, $r \leq p^*$, p' is the interpolant of r and p^*, and $r \parallel \varphi$, then*

$$r_0{}^\frown \mathrm{top}(p') \parallel \varphi \textit{ the same way.}$$

Proof. The proof is essentially the same as the proof of Lemma 6.9. □

proof of Theorem 7.7. Assume for simplicity that p is pure and write $p = \langle f_0, \vec{f}, A, F \rangle$. Let d be the common domain of p. Build a $\leq^*$-decreasing sequence $\langle p^\nu \mid \nu \in A(\alpha) \rangle$ below p be induction.

Assume $p^{\nu'}$ is constructed for $\nu' < \nu$. Let p'_ν be a $\leq^*$-lower bound of $\langle p^{\nu'} \mid \nu' < \nu \rangle$. Write $p'_\nu = \langle f'_0, \vec{f}', A', F' \rangle$ with the common domain d'. For every ξ, let $Q_\xi = P_\xi * \dot{P}_{\dot{\beta}_\xi/\xi}$. Let $\gamma^* = \{\gamma \mid \exists r \in P_\nu (r \Vdash_\nu \text{"}\dot{\beta}_\nu = \gamma\text{"})\}$. Fix $\rho > \xi^*$, $\rho \in A'(\alpha)$. Let $\dot{G}_\rho$ be the canonical name for Q_ρ. Define

$$\varphi^0_\rho \equiv \exists t \in \dot{G}_\rho (t^\frown \mathrm{top}(\bar{p}) \Vdash \varphi)\text{"}$$
$$\varphi^1_\rho \equiv \exists t \in \dot{G}_\rho (t^\frown \mathrm{top}(\bar{p}) \Vdash \neg\varphi)\text{"}$$
$$\varphi^2_\rho \equiv \nexists t \in \dot{G}_\rho (t^\frown \mathrm{top}(\bar{p}) \parallel \varphi)\text{"},$$

where $\bar{p}$ is the appropriate interpolation, as described in Lemma 7.8. Enumerate Q_ν as $\{r_\xi\}_{\xi < (\gamma^*)^{++}}$ (repetition is fine here). We are building $\langle p_{\nu,\xi} \mid \xi \leq (\xi^*)^{++} \rangle$

which is $\leq^*$-decreasing below p'_ν. At limit ξ, take any $p_{\nu,\xi}$ which is a $\leq^*$-lower bound of $\langle p_{\nu,\xi'} \mid \xi' < \xi \rangle$. Suppose $p_{\nu,\xi}$ is constructed and let $p_{\nu,\xi} = \langle f_0^\xi, \vec{f}^\xi, A^\xi, F^\xi \rangle$ and d^ξ be the common domain. Let $\rho \in A^\xi(\alpha)$. By the closure of $(\dot{P}_{\dot{\beta}/\rho}, \leq^*)$, there is $\dot{q}^*_\rho$ such that for every $\mu \in A^\xi(\alpha)$ with $\mu(\alpha) = \rho$, by the Prikry property, there are r^μ, f_0^μ, $\vec{f}^\mu$ A^μ, and F^μ with

$$r^\mu {}^\frown (f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle) \leq^*$$
$$r_\xi {}^\frown (f_\nu^\xi \circ \mu^{-1} \vec{f} \restriction (A^\xi \downarrow \mu)(\rho), A^\xi \downarrow \mu, F^\xi \restriction (A^\xi \downarrow \mu)(\rho), F^\xi(\rho)),$$

and there is unique $i = i_{\mu,r}$ such that

$$r^\mu {}^\frown (f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle) \Vdash \varphi_\rho^{i_{\mu,r_\xi}}.$$

For $\beta < \circ(\alpha)$. there is unique $i_{r_\xi,\beta}$ such that the collection of μ with $\circ(\mu(\alpha)) = \beta$ and $i_{\mu,r_\xi} = i_{r_\xi,\beta}$ is of measure-one. Let $B_{\xi,\beta} := B_{r_\xi,\beta}$ be such a set. By shrinking, assume further that there is r^*_ξ such that for every $\mu \in B_{r_\xi,\beta}$, $r^\mu = r^*_\xi$. We now have two cases.

Case 1: For every β, $i_{r_\xi,\beta} = 2$. In this case, let $p_{\nu,\xi} = \langle f_0^\xi, \vec{f}^\xi, A^*, F^\xi \restriction A^*(\alpha) \rangle$, where A^* is generated by $\cup_{\beta < \circ(\alpha)} B_{r_\xi,\beta}$.

Case 2: There is β such that $i_{r_\xi,\beta} < 2$. Let $g_\nu = j_{E(\alpha,\beta)}(\mu \mapsto f^\mu)(\mathrm{mc}_{\alpha,\beta}(d^\xi))$. Then $g_\nu \supseteq j_{E(\alpha,\beta)}(\mu \mapsto f_\nu^\xi \circ \mu^{-1})(\mathrm{mc}_{\alpha,\beta}(d^\xi)) = f_\nu^\xi$. Let $d^* = \mathrm{dom}(f_\nu^{\xi+1})$. For $\rho \neq \nu$ including 0. Assume now that A^μ is generated by B^μ. Let $B^{<\beta} = j_{E(\alpha,\beta)}(\mu \mapsto B^\mu)(\mathrm{mc}_{\alpha,\beta}(d))$. We have $B^{<\beta} = \cap_{\beta' < \beta} E(\alpha, \beta')(d^*)$. Let $\vec{g}^{<\beta} = j_{E(\alpha,\beta)}(\mu \mapsto \vec{f}^\mu)(\mathrm{mc}_{\alpha,\beta}(d^\xi))$. Then $\vec{g} = \langle g_{\nu'} \mid \nu' \in B^{<\beta}(\alpha) \rangle$ and each $\mathrm{dom}(g_{\nu'}) = d^*$. Let $F^{<\beta} = j_{E(\alpha,\beta)}(\mu \mapsto F^\mu)(\mathrm{mc}_{\alpha,\beta}(d^\xi))$. Let B^β be the collection of $\tau \in \mathrm{OB}_{\alpha,\beta}(d^*)$ such that

- $\tau \restriction d^\xi \in B_{r_\xi,\beta}$. Write $\mu = \tau \restriction d^\xi$ and $\rho = \mu(\alpha)$.
- $B^{<\beta} \downarrow \tau := \{\sigma \circ \tau^{-1} \mid \sigma \in B^{<\beta}\}$ is equal to B^μ.
- for $g_\nu \circ \tau^{-1} = f^\mu$ and for $\eta \in (B^{<\beta} \downarrow \tau)(\rho)$, $g_\eta^{<\beta} \circ \tau^{-1} = f_\eta^\mu$.
- $F^{<\beta} \restriction (B^{<\beta} \downarrow \tau)(\rho) = F^\mu$.

We now take $B^{>\beta}$ as the collection of $\tau \in \cup_{\beta' > \beta} \mathrm{OB}_{\alpha,\beta'}(d^*)$ such that $\mu := \tau \restriction d^\xi \in \mathrm{Lev}_0(A^\xi)$, and $(B^{<\beta} \cup B^\beta) \downarrow \tau \in \cap_{\beta' \leq \beta} E(\tau(\alpha), \beta')(\tau[d^* \cap \mathrm{dom}(\tau)])$. Let $B^* = B^{<\beta} \cup B^\beta \cup B^{>\beta}$. Let $g_0 = f^\xi \cup \{(\zeta, 0) \mid \zeta \in d^* \setminus d^\xi\}$. For $\rho \in B^{>\beta}(\alpha)$, let $g_\rho = f_\rho^\xi \cup \{(\zeta, 0) \mid \zeta \in d^* \setminus d^\xi\}$. Let A^* be generated by B^*. Let F^* be such that for $\rho \in A^*(\alpha)$, if $\circ(\rho) < \beta$, $F^*(\rho) = F^{<\beta}(\rho)$. If $\circ(\rho) = \beta$, let $F^*(\rho) = \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}^*_\rho \rangle$. If $\circ(\rho) > \beta$, let $F^*(\rho) = F^\xi(\rho)$. Finally, let $p_{\nu,\xi+1} = \langle g_0, \vec{g}, A^*, F^* \rangle$. This finishes the construction of $p_{\nu,\xi+1}$. Finally, we let $p_\nu = p_{\nu,(\xi^*)^{++}}$. Note that $\min(A^*(\alpha)) > \gamma^* \geq \nu$.

We now change the notations slightly. Let $p^\nu = \langle f_0^\nu, \vec{f}^\nu, A^\nu, F^\nu \rangle$. Assume A^ν is generated by B^ν. Let A^* be generated by $B^* := \Delta_\nu B^\nu$. Let $f_0^* = \cup_\nu f_0^\nu$. For $\rho \in A^*(\alpha)$, let $f_\rho^* = \cup_\nu f_\rho^\nu$, $F^*(\rho) = \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}_\rho \rangle$, where $\dot{q}_\rho$ is forced to be a $\leq^*$-lower bound of $\langle F^\nu(\rho)_1 \rangle_\nu < \rho$. This is possible because the closure of $(\dot{P}_{\dot{\beta}_\rho/\rho}, \leq^*)$ is at least ρ^+.

Claim 7.9. *p^* satisfies the Prikry property.*

Proof. If there is $p' \leq^* p^*$ deciding φ, then we may use p' instead. Suppose $p' \leq p^*$, p' is impure, and $p' \parallel \varphi$. Assume $p' \Vdash \varphi$, assume $\mathrm{stem}(p')$ has the minimum number of blocks n^*. We will demonstrate the case $n^* = 2$. Let $\bar{p}$ be the interpolant of p^* and p', so $\bar{p} = p^* + \langle \mu_0, \mu_1 \rangle$. Let

$$p' = (g_0, \vec{g}_0, A_0, F_0, \langle \dot{P}_{\dot{\beta}_0/\nu_0}, \dot{q}_0 \rangle)^\frown (g_1, \vec{g}_1, A_1, F_1, \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}_1 \rangle)^\frown \operatorname{top}(p').$$

Say that the tree part in $\operatorname{top}(p')$ is T. Since p satisfies Lemma 7.8, we have that

$$(g_0, \vec{g}_0, A_0, F_0, \langle \dot{P}_{\dot{\beta}_0/\nu_0}, \dot{q}_0 \rangle)^\frown (g_1, \vec{g}_1, A_1, F_1, \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}_1 \rangle)^\frown \operatorname{top}(\bar{p}) \Vdash \varphi.$$

Set $r = (g_0, \vec{g}_0, A_0, F_0, \langle \dot{P}_{\dot{\beta}_0/\nu_0}, \dot{q}_0 \rangle)$, $\nu = \nu_0$, $Q_\nu = P_\nu * \dot{P}_{\dot{\beta}_0/\nu}$ (which is $P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}$). Assume $\circ(\mu_1(\alpha)) = \beta'$. Note that $\mu_1(\alpha) = \nu_1$. Let $\mu = \mu \restriction d^\xi$, where d^ξ is described when we construct $p_{\nu,\xi+1}$. We now use the notations for the construction of p^ν. Let $r = r_\xi$.

Claim 7.10. $i_{r_\xi,\beta} = 0$.

Proof. We divide into cases, depending on β'. Suppose for a contradiction that $i_{r_\xi,\beta} = 1$ (the case $i_{r_\xi,\beta} = 2$ is similar).

Case 1: $\beta' = \beta$. Write $\nu_1 = \rho$. Then $\Vdash_\rho$ "$\dot{P}_{\dot{\beta}_1/\nu_1} = \dot{P}_{\dot{\beta}_\rho/\rho}$". Then note that

$$\vec{r}_0 := r_\xi^{*\frown}(g_1, \vec{g}_1, A_1, F_1 \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}_1 \rangle) \leq r_\xi{}^\frown(f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}_\rho^* \rangle) =: \vec{r}_1.$$

Let G be $Q_\rho := P_\rho * \dot{P}_{\dot{\beta}_\rho/\rho}$-generic containing $\vec{r}_0$, hence containing $\vec{r}_1$. Then there is $t \in G$ such that $t^\frown \bar{p} \Vdash \neg\varphi$. We can take $t \in G$ such that $t \leq \vec{r}_1$, but this contradicts with the fact that $\vec{r}_1{}^\frown \operatorname{top}(\bar{p}) \Vdash \varphi$.

Case 2: $\beta' < \beta$. Pick any $\tau \in \operatorname{Lev}_0(A')$, say $\mu = \tau \restriction d'$ and $\rho = \tau(\alpha)$. Note that $B^{<\beta} \downarrow \tau = B^\mu$. We can see that $\mu_1 \circ \tau^{-1} \in B^\mu$, and with other properties of τ. Let p'' be obtained by extending the r part of $p' + \langle \tau \rangle$ to r^μ. We then have that

$$p'' \leq r_\xi^{*\frown}(f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}_\rho^* \rangle)^\frown \operatorname{top}(p^* + \langle \mu_0, \tau \rangle).$$

Let G be Q_ρ-generic containing $\operatorname{stem}(p'')$. Then it contains $r_\xi^{*\frown}(f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}_\rho^* \rangle)$. Find $t \in G$ such that $t^\frown \operatorname{top}(\bar{p}) \Vdash \neg\varphi$ and $t \leq \operatorname{stem}(p'')$, but then ${}^\frown \operatorname{top}(\bar{p})$ gives contradictory decisions on φ, a contradiction.

Case 3: $\beta' > \beta$. Then take any $\tau' \in \operatorname{Lev}_0(A_1)$ with $\circ(\tau'(\mu_1(\alpha)) = \beta$. $\tau' = \tau \circ \mu_1^{-1}$ for some τ with $\circ(\tau(\alpha)) = \beta$. Write $\mu = \tau \restriction d^\xi$, $\rho = \tau(\alpha)$. Let p'' be obtained by extending p' with τ' is a similar fashion as the one-step extension and extend the r part to p^μ. Then $p'' \restriction Q_\rho$ exists and

$$p'' \leq r_\xi^{*\frown}(f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}_\rho^* \rangle)^\frown$$
$$(g_{1,\rho} \oplus \tau', \langle g_{1,\eta} \mid \eta \in (A_1)_{\tau'}(\mu_1(\alpha)) \rangle (A_1)_{\langle \tau' \rangle}, F_1 \restriction (A_1)_{\langle \tau' \rangle}(\mu_1(\alpha)), \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}_1 \rangle)^\frown \operatorname{top}(p').$$

Let G be Q_ρ-generic containing $p'' \restriction Q_\rho$. Then G contains $r_\xi^{*\frown}(f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}_\rho^* \rangle)$. Find $t \in G$ such that $t \leq p'' \restriction Q_\rho$ and $t^\frown \operatorname{top}(\bar{p}) \Vdash \neg\varphi$, but then

$$t^\frown(g_{1,\rho} \oplus \tau', \langle g_{1,\eta} \mid \eta \in (A_1)_{\tau'}(\mu_1(\alpha)) \rangle (A_1)_{\langle \tau' \rangle}, F_1 \restriction (A_1)_{\langle \tau' \rangle}(\mu_1(\alpha)), \langle \dot{P}_{\dot{\beta}_1/\nu_1}, \dot{q}_1 \rangle)^\frown \operatorname{top}(p')$$

is stronger than $t^\frown \operatorname{top}(\bar{p})$ and p', so the condition gives contradictory decisions, a contradiction. □

All in all, we have that $i_{r_\xi,\beta} = 0$. A similar proof as before shows that for every τ with $\rho = \tau(\alpha)$, $\mu = \tau \restriction d^\xi$, and $\circ(\rho) = \beta$, we have that

$$r_\xi^{*\frown}(f^\mu, \vec{f}^\mu, A^\mu, F^\mu, \langle \dot{P}_{\dot{\beta}_\rho/\rho}, \dot{q}_\rho^* \rangle)^\frown \operatorname{top}(\bar{t}) \Vdash \varphi$$

for the appropriate interpolant $\bar{t}$. Note that then for each extension of $r^*_\xi{}^\frown \operatorname{top}(p^* + \langle \mu_0 \rangle)$ can be extended further to a condition t where an object τ^* with $\circ(\tau^*) = \beta$ is used. With the fact that $i_{r_\xi,\beta} = 0$, we have that $t \Vdash \varphi$. By a density argument, we have that $r^*_\xi{}^\frown \operatorname{top}(p^* + \langle \mu_0 \rangle) \Vdash \varphi$, and this contradicts with the minimality of n^*. □

This completes the proof of Theorem 7.7. □

We now consider the club introduced by P_α and the cardinal arithmetic. By the Prikry property, all the forcings below P_α preserve all cardinals, and $(\{p \in P_\alpha, p \text{ is pure}, \leq^*)$ is α-closed, one can show that all cardinals below α are preserved. Since P_α has the α^{++}-chain condition, all cardinals from α^{++} and above are preserved. For generality, we consider the case $\circ(\alpha) > 0$. Let G be P_α-generic. Then for each $\nu < \alpha$ such that by letting $Q_\nu = P_\nu * \dot{P}_{\dot{\beta}_\nu/\nu}$, we have that $G \restriction Q_\nu$ exists. $G \restriction Q_\nu$ is Q_ν-generic, and it introduces a set $C^\nu \cup C^{\beta_\nu/\nu}$ where $\beta_\nu = \dot{\beta}_\nu[G \restriction P_\nu]$, $C^\nu \subseteq \nu+1$ with $\max(C^\nu)$, $C^{\beta_\nu/\nu} \subseteq (\nu, \beta_\nu]$ such that $\max(C^{\beta_\nu/\nu}) = \beta_\nu$ if $\beta_\nu > \nu$, otherwise, $C^{\beta_\nu/\nu} = \emptyset$. Let $C_\alpha = (\cup_{\{\nu \mid G \restriction Q_\nu \text{ exists}\}}(C^\nu \cup C^{\beta_\nu/\nu})) \cup \{\alpha\}$. Since $\circ(\alpha) > 0$, we can perform one-step extension of any condition so that $\{\nu \mid G \restriction Q_\nu \text{ exists}\}$ is unbounded in α. Like in the extender-based Magidor-Radin forcing, $\{\nu \mid Q_\nu \text{ exists}\}$ has a tail of order-type $\omega^{\circ(\alpha)}$. Hence, in $V[G]$, α is singularized to have cofinality $\operatorname{cf}(\omega^{\circ(\alpha)})$. From here and the Prikry property, one can show that α^+ is preserved. Also, note that for $\nu < \nu'$, with the way we constructed the sets, we have that $C^\nu \cup C^{\beta_\nu/\nu}$ is an initial segment of $C^{\nu'}$, so it is an initial segment of C_α. Thus, $\lim(C_\alpha) = (\cup_{\{\nu \mid G \restriction Q_\nu \text{ exists}\}}(\lim(C^\nu) \cup \lim(C^{\beta_\nu/\nu}))) \cup \{\alpha\}$ Fix $\xi \in C_\alpha$ with $\xi < \alpha$. Then $\xi \in C^\nu \cup C^{\beta_\nu/\nu}$ for some ν. Forcing with G can be factored into $G \restriction Q_\nu * G/Q_\nu$. We can also form the quotient P_α/Q_ν where the conditions look similar to the conditions of P_α, except that all the components lie above β_ν. One can verify that $\Vdash_{Q_\nu}$ "$(P_\alpha/Q_\nu, \leq, \leq^*)$ has the Prikry property and $(P_\alpha/Q_\nu, \leq^*)$ is $\dot{\beta}^*_\nu$-closed" where $\dot{\beta}^*_\nu$ is forced to be the first inaccessible above $\dot{\beta}_\nu$. Also, G is isomorphic to $G_0 * G_1$ where G_0 is Q_ν-generic and G_1 is $P_\alpha/Q_\nu[G]$-generic. The forcing P_α/Q_ν does not affect cardinals above β_ν. Now, note that by Proposition 4.1 items (3) and (6), we have that either $2^\xi = \xi^+$ and $2^\xi = \xi^{++}$, and $2^\xi = \xi^{++}$ iff $\xi \in \lim(C^\nu) \cup \lim(C^{\beta_\nu/\nu})$. Hence, the cardinal arithmetic below α satisfies (3) of Proposition 4.1. Since $\alpha \in \lim(C_\alpha)$, it remains to show that $2^\alpha = \alpha^{++}$.

Work with a pure condition $p \in G$. Enumerate $\{\nu \mid G \restriction P_\nu \text{ exists }\}$ increasingly as $\{\nu_i \mid i < \omega^{\operatorname{cf}(\alpha)}$. . Fix $\gamma \in [\alpha, \alpha^{++})$. By a density argument, let $p^\gamma \leq p$, $p^\gamma \in G$ be such that if $\operatorname{top}(p^\gamma) = \langle f^\gamma, \vec{f}^\gamma, A^\gamma, F^\gamma \rangle$, then for every object μ which appears in A^γ, $\gamma \in \operatorname{dom}(\mu)$. Suppose that $\operatorname{stem}(p^\gamma) \in P_{\nu_i^\gamma} * \dot{P}_{\dot{\beta}_{\nu_i^\gamma}/\nu_i^\gamma}$. For $i \leq i_\gamma$, define $t_\gamma(i) = 0$. For $i > \gamma$, there is an extension $p^{\gamma,i} \in G$ such that

1. $p^{\gamma,i} \restriction P_{\nu_i}$ exists.
2. by writing p^{γ_i} as

$$(s_0, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \cdots ^\frown (s_{n-1}, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1} \rangle)^\frown \langle f, \vec{f}, A, F \rangle,$$

   then $(s_0, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0 \rangle)^\frown \cdots ^\frown s_k \in P_{\nu_i}$, and
   - $f(\gamma)$ is a check-name $\check{\gamma}_0$, then $\gamma_0 \in f_{n-1}$, where f_{n-1} is the first coordinate of s_{n-1}.

- by recursion, $\gamma_0, \cdots, \gamma_{l-1}$ is defined for $l < n-k-1$, then $\gamma_{l-1} \in \operatorname{dom}(f_{n-l})$, where f_{n-l} is the first coordinate of s_{n-l}, and $f_{n-l}(\gamma_{l-1})$ is a check-name γ_l.

We define $t_\gamma(i) = f_k(\gamma_{n-k-1})$. For $\gamma < \gamma'$, there is a condition $p^{\gamma,\gamma'} \in G$ such that if $A^{\gamma,\gamma'}$ is the tree appearing in $\operatorname{top}(p^{\gamma,\gamma'}$, we have that for every μ appearing in $A^{\gamma,\gamma'}$, $\gamma, \gamma' \in \operatorname{dom}(\mu)$ and $\mu(\gamma) < \mu(\gamma')$. From this, it can be shown that $t_\gamma <^* t_{\gamma'}$, which means there is i^* such that for $i > i^*$, $t_\gamma(i) < t_{\gamma'}(i)$. This gives α^{++} different functions from $\omega^{\operatorname{cf}(\alpha)}$ to α. It is easy to show that α is a strong limit cardinal, and so in $V[G]$, $2^\alpha = \alpha^{\operatorname{cf}(\alpha)} \geq \alpha^{++}$. Since P_α is α^{++}-c.c., $2^\alpha = \alpha^{++}$ as desired.

Definition 7.11 (The quotient forcing)**.** Let $\dot{P}_{\alpha/\alpha}$ be the P_α-name of the trivial forcing $(\{\emptyset\}, \leq, \leq^*)$. In V^{P_α}, let $\dot{C}_{\alpha/\alpha}$ be the $\dot{P}_{\alpha/\alpha}$-name of the empty set. Now assume that $\beta < \alpha$. Define $\dot{P}_{\alpha/\beta}$ as the following. Let G be P_β-generic. Define $P_\alpha[G] = \dot{P}_{\alpha/\beta}[G]$ as the forcing consisting of conditions of the form $\operatorname{stem}(p)^\frown \operatorname{top}(p)$, where

(1) $\operatorname{stem}(p)$ is of the form

$$(P_{\beta'}[G], q')^\frown(s_0, \langle \dot{P}_{\beta_0/\alpha_0}[G], \dot{q}_0)^\frown(s_{n-1}, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}[G], \dot{q}_{n-1}\rangle),$$

for some n (if $n = 0$, then $\operatorname{stem}(p)$ is only $(P_{\beta'}[G], q')$) such that
  - $P_{\beta'}[G] = \dot{P}_{\dot{\beta}'/\beta}[G]$, and $q' \in P_{\beta'}$.
  - if $n > 0$, then $\alpha_0 < \cdots < \alpha_{n-1}$, and for $i < n$,
    - if $\circ(\alpha_i) = 0$, $s_i = \langle f_i \rangle$, and if $\circ(\alpha_i) > 0$, $s_i = \langle f_i, \vec{f_i}, A_i, F_i \rangle$, where $d_i = \operatorname{dom}(f_i)$ is an α_i-domain, $d_i \in V$.
    - for $\zeta \in d_0$, $\Vdash_{P_{\beta'}[G]}$ "$f_0(\zeta) < \alpha_0$" and if $i > 0$, then for $\zeta \in d_i$, $\Vdash_{P_{\alpha_{i-1}}[G] * \dot{P}_{\dot{\beta}_{i-1}/\alpha_{i-1}}[G]}$ "$f_i(\zeta) < \alpha_i$".
    - $\Vdash_{P_{\alpha_i}[G]}$ "$\alpha_i \leq \dot{\beta}_i < \alpha_{i+1}$", where $\alpha_n = \alpha$.
    - $\Vdash_{P_{\alpha_i}[G]}$ "$\dot{q}_i \in \dot{P}_{\dot{\beta}_i/\alpha_i}[G]$".
    - if $\circ(\alpha_i) > 0$,
      - A_i is a d_i-tree with respects to $\vec{E}_{\alpha_i}(d_i)$ (in the sense of V).
      - $\vec{f_i} = \langle f_{i,\nu} \mid \nu \in A_i(\alpha_i) \rangle$.
      - for each ν, $\operatorname{dom}(f_{i,\nu}) = d_i$.
      - for $\zeta \in d_i$, $\Vdash_{P_\nu[G] * \dot{P}_{\dot{\beta}_\nu/\nu}[G]}$ "$f_{i,\nu}(\zeta) < \alpha_i$".
      - $\operatorname{dom}(F_i) = A_i(\alpha_i)$.
      - for $\nu \in A_i(\alpha_i)$, $F_i(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}[G], \dot{q} \rangle$, $\Vdash_{P_\nu[G]}$ "$\nu \leq \dot{\beta}_\nu < \alpha_i$" and $\Vdash_{P_\nu[G]}$ "$\dot{q} \in \dot{P}_{\dot{\beta}_\nu/\nu}[G]$".

(2) if $\circ(\alpha) = 0$, then $\operatorname{top}(p)$ is $\langle f \rangle$, and if $\circ(\alpha) > 0$, then $\operatorname{top}(p) = \langle f, \vec{f}, A, F \rangle$, where there is a *common domain* d, which is an α-domain (in the sense of V) such that
  - If $\circ(\alpha) = 0$, then $\operatorname{dom}(f) = d$ and for $\zeta \in d$, $\Vdash_{P_{\beta'}[G]}$ "$f(\zeta) < \alpha$".
  - Assume $\circ(\alpha) > 0$. Then,
    - A is a d-tree with respects to $\vec{E}_\alpha(d)$ (in the sense of V).
    - $\operatorname{dom}(F) = d$ and for $\nu \in \operatorname{dom}(F)$, $F(\nu) = \langle \dot{P}_{\dot{\beta}_\nu/\nu}[G], \dot{q} \rangle$ where $\Vdash_{P_\nu[G]}$ "$\nu \leq \dot{\beta}_\nu < \alpha$ and $\dot{q} \in \dot{P}_{\dot{\beta}_\nu/\nu}[G]$".
    - $\operatorname{dom}(f) = d$, $\vec{f} = \langle f_\nu \mid \nu \in A(\alpha) \rangle$, and for all ν, $\operatorname{dom}(f_\nu) = d$.

– for $\zeta \in d$, $\Vdash_{P_{\alpha_{n-1}}[G]*\dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}[G]}$ "$f(\zeta) < \alpha$" and for $\nu \in A(\alpha)$, $\Vdash_{P_\nu[G]*\dot{P}_{\dot{\beta}_\nu/\nu}[G]}$ "$f_\nu(\zeta) < \alpha$".

Back in V. If $\dot{p} \in \dot{P}_{\alpha/\beta}$, then by density, the collection of $p_0 \in P_\beta$ such that p_0 decides n, $\alpha_0, \cdots, \alpha_{n-1}$, $\mathrm{dom}(f_0), \cdots, \mathrm{dom}(f_{n-1})$, the common domain, A_i, A, q' (as the equivalent $\dot{P}_{\dot{\beta}'/\beta}$-name, and so on), is open dense. In this case, we say that p_0 *interprets* $\dot{p}$. All in all, for such p_0 which interprets all the relevant components of $\dot{p}$, let p_1 be such the interpretation. Assume $\circ(\beta) > 0$ and $\circ(\alpha) > 0$ (the other cases are simpler) write p_0 as $r_0{}^\frown\langle g, \vec{g}, B, H\rangle$ and by the interpretation, we may write

$$p_1 = (\langle \dot{P}_{\beta'/\beta}, \dot{q}'\rangle)^\frown(s_0, \langle \dot{P}_{\dot{\beta}_0/\alpha_0}, \dot{q}_0\rangle) \cdots {}^\frown(s_{n-1}, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown\langle f, \vec{f}, A, F\rangle.$$

There is a natural concatenation p_0 with p_1, written by $p_0{}^\frown p_1$, which is

$$r = r_0{}^\frown(\langle g, \vec{g}, B, H\rangle, \langle \dot{P}_{\beta'/\beta}, \dot{q}'\rangle)^\frown \cdots {}^\frown(s_{n-1}, \langle \dot{P}_{\dot{\beta}_{n-1}/\alpha_{n-1}}, \dot{q}_{n-1}\rangle)^\frown\langle f, \vec{f}, A, F\rangle.$$

Then $r \in P_\alpha$ with $r \restriction P_\beta = p_0$ exists. Denote r/P_β the term p_1. For P_β-names p_0 and p_1 in $\dot{P}_{\alpha/\beta}$, we say that $p_0 \leq p_1$ if there is $p \in G^{P_\beta}$ such that p interprets p_0 and p_1, and $p^\frown p_0 \leq_\alpha p^\frown p_1$. Also define $p_0 \leq^* p_1$ if there is $p \in G^{P_\beta}$ such that p interprets p_0 and p_1, and $p^\frown p_0 \leq^*_\alpha p^\frown p_1$. One can check that the map $\phi : \{p \in P_\alpha \mid p \restriction P_\beta \text{ exists}\} \to P_\beta * \dot{P}_{\alpha/\beta}$ defined by $\phi(p) = (p \restriction P_\beta, p/P_\beta)$ is a dense embedding, where $p \setminus P_\beta$ is the obvious component of p which is in $\dot{P}_{\alpha/\beta}$. Note that if G is P_β-generic and H is $P_\alpha[G]$-generic, there is a generic I for P_α such that $V[G * H] = V[I]$, where I is generated by $\{p \mid p \restriction P_\beta \text{ exists}, p \restriction P_\beta \in G$ and $(p/P_\beta)[G] \in H\}$. If I is P_α-generic and for some $p \in I$, $p \restriction P_\beta$ exists, we can get G which is P_β-generic and H which is $P_\alpha[G]$-generic such that $V[G*H] = V[I]$, where G is generated by $\{p \restriction P_\beta \mid p \in I$ and $p \restriction P_\beta$ exists$\}$ and $H = \{(p/P_\beta)[G] \mid p \in I$ and $p \restriction P_\beta$ exists$\}$.

In V^{P_β}, let $\dot{C}_{\alpha/\beta}$ be a $\dot{P}_{\alpha/\beta}$-name of the set described as the following. Let G be P_β-generic. and H be generic over $P_\alpha[G] = \dot{P}_{\alpha/\beta}[G]$. Then let $I = G * H$ be P_α-generic. I derives the set $C_\alpha \subseteq \alpha + 1$ and G derives the set $C_\beta \subseteq \beta + 1$. Let $C_{\alpha/\beta} = C_\alpha \setminus C_\beta$.

Proposition 7.12.
- $\Vdash_\beta$ "$(\dot{P}_{\alpha/\beta}, \leq)$ *is* α^{++}*-c.c.*"
- $\Vdash_\beta$ "$\dot{P}_{\alpha/\beta}, \leq, \leq^*)$ *has the Prikry property.*
- $\Vdash_\beta$ "$(\dot{P}_{\alpha/\beta}, \leq^*)$ *is* β^**-closed", where* β^* *is the least inaccessible cardinal greater than* β.

We conclude that from all the analysis, Proposition 4.1 holds for P_α and all relevant quotients at α.

8. The main forcing

We are now defining our main forcing $\mathbb{P}$. The forcing $\mathbb{P} = \cup_{\{\alpha<\kappa \mid \alpha \text{ is inaccessible}\}} P_\alpha$. For p and p' in $\mathbb{P}$, define $p \leq p'$ if $p \in P_\alpha$, $p' \in P_{\alpha'}$, $\alpha \geq \alpha'$, $p \restriction P_{\alpha'}$ exists, and $p \restriction P_{\alpha'} \leq_{\alpha'} p$. The forcing is κ^+-c.c. Let G be $\mathbb{P}$-generic. Then if $p \in G$ is such that $p \restriction P_\alpha$ exists, then $G \restriction P_\alpha$ is P_α-generic. We briefly describe $\mathbb{P}/P_\alpha$ for $\alpha < \kappa$ inaccessible. Recall that for $\alpha \leq \eta < \kappa$, $\Vdash_\alpha$ "$\{p/P_\alpha \mid p \in P_\eta, p \restriction P_\alpha \text{ exists}\}$ is densely embedded in $\dot{P}_{\eta/\alpha}$". For $\alpha < \kappa$ inaccessible, let $\mathbb{P}/P_\alpha$ as the collection $\{p/P_\alpha \mid p \in \mathbb{P}, p \restriction P_\alpha \text{ exists}\}$. Note that the notation makes sense, since $p \in P_\eta$ for some η. For $p_0, p_1 \in \mathbb{P}/P_\alpha$, define $p_0 \le p_1$ (in V^{P_α}) if there is $p \in P_\alpha$ such that $p^\frown p_0 \le_{\mathbb{P}} p \frown p_1$.

Remark 8.1. V^{P_α}, for every $p \in \mathbb{P}/P_\alpha$, there is η such that $p \in \dot{P}_{\eta/\alpha}$.

This introduces the set C_α. Let $C = \cup_\alpha \{C_\alpha \mid G \restriction P_\alpha \text{ is } P_\alpha\text{-generic}\}$. Then $C \subseteq \kappa$ is a club. The next theorem shows that the cardinal arithmetic should be as expected.

Theorem 8.2. *Let $\dot{f}$ be a $\mathbb{P}$-name of a function from β to ordinals such that $\beta < \kappa$ and G is $\mathbb{P}$-generic. Then $f \in V[G \restriction P_\alpha]$ for some $\alpha < \kappa$.*

Proof. We show by a density argument. Let $p \in \mathbb{P}$ and $\dot{f}$ be a $\mathbb{P}$-name of functions from β to ordinals, where $\beta < \kappa$. For simplicity, assume p is an empty condition. Let $M \prec H_\theta$ for some sufficiently large regular θ, $\beta \subseteq M$, $\dot{f}, p, \mathbb{P} \in M$, $V_{M\cap\kappa} \subseteq M$, and $\circ(M \cap \kappa) \ge \beta$. Say $\alpha = M \cap \kappa$. We are going to build $p^* \in P_\alpha$ of the form $p^* = \langle f, \vec{f}, A, F\rangle$. Let $f, \vec{f}$, and A be any objects. Fix $\gamma < \beta$ and $\nu \in A(\alpha)$ such that $\circ(\nu) = \gamma$. Let Y_ν be a maximal antichain of relevant collections in P_ν. For each $r \in Y_\nu$, let G_r be P_ν-generic containing r. Since $V_\alpha \subseteq M$, $M[G] \cap \kappa = M \cap \kappa$. Find $q \in \mathbb{P}/G$ such that q decides $\dot{f}(\gamma)[G]$. By elementarity, we may find such a q in $M[G]$. Then $q \in P_\xi/G$ for some $\xi < \alpha$. Back in M, let $\dot{\xi}$ and $\dot{q}$ be the names for such ξ and q. Define $F(\nu) = \langle \dot{P}_{\dot{\xi}/\nu}, \dot{q}\rangle$. For ν with $\circ(\nu) \ge \beta$, we assign $F(\nu)$ to be any value. This completes the construction of F. By our design, we have that p^* decides $\dot{f}$, and hence, $p^* \Vdash_{\mathbb{P}} \dot{f} \in V^{P_\alpha}$.

□

Corollary 8.3. *κ is inaccessible in $V^{\mathbb{P}}$.*

Proof. By Theorem 8.2, if κ is collapsed, then the witness function has to be in V^{P_α} for some $\alpha < \kappa$, but κ is preserved in P_α, a contradiction. The same argument shows that κ is regular. Finally, for every $\beta < \kappa$, the value 2^β must be determined in V^{P_α} for some sufficiently large α because the forcing can be factored so that the quotient forcing after the stage β is β^+-closed under the direct extension, □

Corollary 8.4. *Every cardinal is preserved in $V^{\mathbb{P}}$.*

Proof. Similar to the previous corollary. □

Corollary 8.5. *For $\beta < \kappa$ the value 2^β is determined in $V^{\mathbb{P}_\alpha}$ for some $\alpha \in (\beta, \kappa)$.*

Theorem 8.6. *In $V^{\mathbb{P}}$, κ is inaccessible, there is a club $D \subseteq \kappa$ such that for $\beta \in D$, $2^\alpha = \alpha^{++}$ and for $\alpha \notin D$, $2^\beta = \alpha^+$.*

Proof. Let C be the club derived from $\mathbb{P}$ and $D = \lim(C)$. Then D satisfies the theorem. □

9. Getting different cardinal behaviors on stationary classes

Assume GCH. Let κ be a strongly inaccessible cardinal. For each $\gamma < \kappa$, let $f_\gamma : \kappa \to \kappa$. Assume that for each γ, there is a coherent sequence of extenders $\vec{E}_\gamma$, on a set $X_\gamma \subseteq \kappa$ and $\circ^\gamma : X_\gamma \to \kappa$ such that

- $\vec{E}_\gamma = \langle E_\gamma(\alpha, \beta) \mid \beta < \circ^\gamma(\alpha)\rangle$.
- each $E_\gamma(\alpha, \beta)$ is an $(\alpha, \alpha^{+f_\gamma(\alpha)})$ extender witnesses α being $\alpha^{+f_\gamma(\alpha)}$-strong.
- $\circ^\gamma(\alpha) < \alpha$.

- for $\nu < \kappa$, $\{\alpha \mid \circ^\gamma(\alpha) \geq \nu\}$ is stationary.

Then we can proceed a similar forcing construction, except that the corresponding Cohen part at α will be $C(\alpha^+, \alpha^{f_\gamma(\alpha)})$. Let $\mathbb{P}^{\langle \vec{f}_\gamma \mid \gamma < \kappa \rangle}$ be the corresponded forcing.

Theorem 9.1. *In the forcing $\mathbb{P}^{\langle f_\gamma \mid \gamma < \kappa \rangle}$, all the cardinals are preserved, the forcing produces a club $C \subseteq \cup_{\gamma < \kappa} X_\gamma$ such that for each $0 < \xi < \kappa$ regular and $\gamma < \kappa$, the collection of α with $\mathrm{cf}(\alpha) \geq \xi$ and $2^\alpha = \alpha^{+f_\gamma(\alpha)}$ is stationary.*

Proof Sketch. Fix $\xi > 0$ and a $\mathbb{P}$-name of a club subset of κ $\dot{D}$. Let p be a condition, $\dot{D}$ a name of a club subset of κ. Let $M \prec H_\theta$ where θ is sufficiently large, $\dot{D}, p, \mathbb{P}^{\langle f_\beta \mid \beta < \kappa \rangle} \in M$, $V_{M \cap \kappa} \subseteq M$, and $\circ^\gamma(M \cap \kappa) \geq \xi$. Let $\alpha = M \cap \kappa$. We are now extending p to a condition whose top level is α. Let $p = \langle f, \vec{f}, A, F \rangle \in P_\alpha$, where $f, \vec{f}, A$ can be any sensible components. For each $\nu \in A(\alpha)$, let $F(\nu)$ be a condition that decides an element $\dot{\xi}$ which is the minimum of the interpretation of $\dot{D} \setminus (\nu + 1)$. By elementarity, $\dot{\xi}$ is decided to be below α. Then the final condition forces that α is in $\dot{C} \cap \dot{D}$, and forces that $2^\alpha = \alpha^{f_\gamma(\alpha)}$, and $\mathrm{cf}(\alpha) \geq \xi$. □

Example 9.2. Start from GCH, κ carrying a $(\kappa, \kappa^{+\kappa})$-extender. Then it is possible that for $\gamma < \kappa$, there is a sequence coherent sequence of extenders $\vec{E}_\gamma$ on a stationary set $X_\gamma \subseteq \kappa$ where each $E_\gamma(\alpha, \beta)$ witnesses α being $\alpha^{+\gamma}$-strong. Let $f_\gamma : \xi \mapsto \gamma$. Then the forcing $\mathbb{P}^{\langle f_\gamma \mid \gamma < \kappa \rangle}$ forces that κ is inaccessible, and in V_κ and each $\gamma < \kappa$, there is a stationary class $S_\gamma \subseteq \kappa$ such that for $\alpha \in S_\gamma$, $2^\alpha = \alpha^{+\gamma}$.

References

[1] M. Gitik and S. Jirattikansakul, "Another method to add a closed unbounded set of former regulars," *submitted*, 2022. arXiv:2206.05693.

[2] M. Gitik, "On closed unbounded sets consisting of former regulars," *The Journal of Symbolic Logic*, vol. 64, no. 1, pp. 1–12, 1999.

[3] M. Gitik and C. Merimovich, "Power function on stationary classes," *Annals of Pure and Applied Logic*, vol. 140, no. 1, pp. 75–103, 2006. Cardinal Arithmetic at work: the 8th Midrasha Mathematicae Workshop.

[4] C. Merimovich, "Extender-based magidor-radin forcing," *Israel Journal of Mathematics*, vol. 182, pp. 439–480, 2011.

[5] S. Jirattikansakul, "Blowing up the power of a singular cardinal of uncountable cofinality with collapses," *Annals of Pure and Applied Logic*, vol. 174, no. 6, 2023.

School of Mathematical Sciences, Tel Aviv University, Tel Aviv-Yafo, Tel Aviv, Israel, 6997801

Email address: gitik@tauex.tau.ac.il

School of Mathematical Sciences, Tel Aviv University, Tel Aviv-Yafo, Tel Aviv, Israel, 6997801

Email address: jir.sittinon@gmail.com