A Simpler Short Extenders Forcing- gap 3

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Assume GCH. Let $\kappa = \bigcup_{n<\omega} \kappa_n$, $\langle \kappa_n \mid n < \omega \rangle$ increasing, each κ_n is κ_n^{+n+3} – strong witnessed by $(\kappa_n, \kappa_n^{+n+3})$ – extender E_n. We would like to force $2^\kappa = \kappa^{+3}$ preserving all the cardinals and without adding new bounded subsets to κ. It was done first in [2, Sec.3]. Here we present a different method of doing this. The advantage of the present construction is that the preparation forcing is split completely from the main one. This makes the presentation much simpler and likely to allow a possibility of extensions to arbitrary gaps preserving large cardinals (which was not the case in [2]).

1 The Preparation Forcing

Maymoto and Sharon pointed out that the forcing below reminds a simplified morass. Indeed it implies Velleman's simplified morass with linear limits [4] in a generic extension. We do not know if the objects are equivalent and think that it is not the case due to the intersection properties below. Also it is unclear if such structure exists in L and bigger inner models.

Definition 1.1 The set $\mathcal{P}'$ consists of elements of the form

$$\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle$$

so that the following hold:

(1) $A^{1\kappa^{++}}$ is a closed subset of κ^{+3} of cardinality at most κ^{++}.

(2) $A^{0\kappa^+} \prec H(\kappa^{+3})$ of cardinality κ^+

(3) $A^{1\kappa^+}$ is a set of elementary submodels of $A^{0\kappa^+}$ of cardinality κ^+ including κ^+ so that

   (a) $A^{0\kappa^+} \in A^{1\kappa^+}$

   (b) each element of $A^{1\kappa^+} \backslash \{A^{0\kappa^+}\}$ belongs to $A^{0\kappa^+}$

   (c) (well foundness of inclusion) if B, $C \in A^{1\kappa^+}$ and $B \subsetneqq C$, then $B \in C$.

   In particular $\langle A^{1\kappa^+}, \subset \rangle$ is well-founded. We set

   $otp_{\kappa^+}(A) = \sup\{otp(\langle C, \subset \rangle) \mid C \subseteq \mathcal{P}(A) \cap A^{1\kappa^+}$ is a chain under the inclusion$\}$.

(4) $C^{\kappa^+} : A^{1\kappa^+} \longrightarrow \mathcal{P}(A^{1\kappa^+})$ is so that

   (a) for each $A \in A^{1\kappa^+}$ we require that $C^{\kappa^+}(A)$ is a closed chain (under inclusion) of elements of $\mathcal{P} \cap A^{1\kappa^+}$ of the length $otp_{\kappa^+}(A)$ and there is no chain in $\mathcal{P}(A) \cap A^{1\tau}$ that properly includes $C^{\tau}(A)$.

   In particular this means that there are chains of the maximal length (i.e. $otp_{\kappa^+}(A)$ which was defined as supremum is really a maximum) and $C^{\kappa^+}(A)$ is one of them. Also note that A is always the largest element of $C^{\kappa^+}(A)$. So $otp_{\kappa^+}(A)$ is always a successor ordinal.

   (b) (Coherence)
   if $B \in C^{\kappa^+}(A)$ then $C^{\kappa^+}(B)$ is the initial segment of $C^{\kappa^+}(A)$ starting with B

(c) if $otp_{\kappa^+}(A) - 1$ is a limit ordinal (in such cases we shall refer to A as a limit model and otherwise like to a successor one) then each element of $A \cap A^{1\kappa^+} \backslash \{A\}$ is included (and hence also belongs) to one of the members of $C^{\kappa^+}(A)$.

(5) if $\alpha \in A$ then $A^{1\kappa^{++}} \cap \alpha \in A$

(6) if $\delta \in A^{1\kappa^{++}}$ and $\delta < \sup A$ then $\min(A \backslash \delta) \in A^{1\kappa^{++}}$

(7) if $A, B \in A^{1\kappa^+}$ then $otp(A \cap \kappa^{+3}) = otp(B \cap \kappa^{+3})$ iff $otp_{\kappa^+}(A) = otp_{\kappa^+}(B)$

Further we shall confuse A's with $A \cap \kappa^{+3}$.

(8) (isomorphism condition)
Let $A, B \in A^{1\kappa^+}$ and $otp(A) = otp(B)$ then the structures

$$\langle A, \in, \subseteq, \kappa, C^{\kappa^+}(A), A^{1\kappa^+} \cap A\ ,\ C^{\kappa^+} \restriction (A^{1\kappa^+} \cap A),\ A^{1\kappa^{++}} \cap A, f_A \rangle$$

and

$$\langle B, \in, \subseteq, \kappa, C^{\kappa^+}(B), A^{1\kappa^+} \cap B,\ C^{\kappa^+} \restriction (A^{1\kappa^+} \cap B),\ A^{1\kappa^{++}} \cap B,\ f_B \rangle$$

are isomorphic over $A \cap B$, i.e. the isomorphism π_{AB} between them is the identity on $A \cap B$, where $f_A : \kappa^+ \longleftrightarrow A$, $f_B : \kappa^+ \longleftrightarrow B$ are some fixed bijections.

Note that in particular we will have that $A \cap \kappa^{++} = B \cap \kappa^{++}$. Also, together with the next condition, we will have the opposite implication as well, i.e. $A \cap \kappa^{++} = B \cap \kappa^{++}$ implies $otp(A) = otp(B)$.

(9) (first intersection condition) if $A, B \in A^{1\kappa^+}$ $A \neq B$ and $otp(A) = otp(B)$ then there is $\alpha \in A \cap A^{1\kappa^{++}}$ s.t. $A \cap B = A \cap \alpha$

In particular this condition imply that $A \cap B = A \cap \sup(A \cap B)$ and $\alpha = \min(A \backslash A \cap B)$.

(10) (second intersection condition) if $A, B \in A^{1\kappa^+}$, $otp(A) \geq otp(B)$ and $B \notin A$, then there is $B' \in (A \cup \{A\}) \cap A^{1\kappa^+}$ s.t.

(a) $otp(B') = otp(B)$

Note that $otp(A) = otp(B)$ implies then that B' is A itself.

(b) $A \cap B = B' \cap B$.

(c) either

(i) $\min(B' \setminus \sup(B \cap B') + 1) > \sup B$

or

(ii) $\min(B \setminus \sup(B \cap B') + 1) > \sup B'$ and then also $\min(B \setminus \sup(B \cap B') + 1) > \sup A$.

(d) if $\alpha \in A \cap \sup B$, then $\alpha < \min(B \setminus \sup(B \cap B') + 1)$.

The meaning of (c) is that without the common part B and B' are basically one above another. If B is above B' then it is also above A. The condition (d) claims that A (that does not include B) cannot have ordinals in the interval $(\min(B \setminus \sup(B \cap B') + 1), \sup B)$. Note that by (9) above applied to B and B' there will be $\alpha \in B \cap A^{1\kappa++}$ such that $B \cap B' = B \cap \alpha$. Then this $\alpha \geq \sup(B \cap B')$ and hence it is not in A by (d). Now, if $\sup(A) > \sup(B)$ then $\min(A \setminus \alpha) \in A^{1\kappa^{++}}$, by the condition (6) above.

(11) (immediate predecessors condition)
Let $B \in A^{1\kappa^+}$ be a successor model. Then

(a) B has at most two immediate predecessors in $A^{1\kappa^+}$ (under the inclusion relation). They are required to have the same *otp*. In addition, if $Z \in B \cap A^{1\kappa^+}$ then either $Z = B_i$ or $Z \in B_i$ for $i = 0, 1$, where B_0, B_1 are the immediate predecessors of B in $A^{1\kappa^+}$.

(b) if B' is an immediate predecessor of B and it is limit, then B' is the unique immediate predecessor of B.

So its impossible to split over a limit model. This technical condition will be useful further in 3.5.

(12) (closure of models)

Let $B \in A^{1\kappa^+}$ be a successor model. Then ${}^{\kappa}B \subseteq B$.

(13) If α is a successor element of $A^{1\kappa^{++}}$, then it has cofinality κ^{++}.

(14) $\max A^{1\kappa^{++}} \geq \sup(A^{0\kappa^+} \cap \kappa^{+3})$.

$\square$

Now let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Define $swt(p, B)$ to be

$$\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, D^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \ ,$$

where D^{κ^+} is obtained from C^{κ^+} as follows:
$D^{\kappa^+} = C^{\kappa^+}$ unless B has exactly two immediate predecessors in $A^{1\kappa^+}$. If $B_0 \neq B_1$ are such predecessors of B and, say $B_0 \in C^{\kappa^+}(B)$, then we set $D^{\kappa^+}(B) = C^{\kappa^+}(B_1)^\frown B$. Extend D^{κ^+} on the rest in the obvious fashion just replacing $C^{\kappa^+}(B_0)$ by $C^{\kappa^+}(B_1)$ for models including B and then moving over isomorphic models. Note that $swt(swt(p, B), B) = p$.

Define $q = swt(p, B_1, \ldots, B_n)$ by applying the operation swt n-times:
$p_{i+1} = swt(p_i, B_i)$, for each $1 \leq i \leq n$, where $p_1 = p$ and $q = p_{n+1}$.
The following simple observation will be useful further (3.7).

Lemma 1.2 *Let $p = \langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), C^{\kappa^+}(p)\rangle, A^{1\kappa^{++}}(p)\rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}(p)$. Then there are $B_1, B_2, ..., B_n \in A^{1\kappa^+}$ such that $B \in C^{\kappa^+}(q)(A^{0\kappa^+}(p))$, where*

$$q = \langle\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p), C^{\kappa^+}(q)\rangle, A^{1\kappa^{++}}(p)\rangle = swt(p, B_1, B_2, ..., B_n).$$

Proof. If $B \in C^{\kappa^+}(p)(A^{0\kappa^+}(p)$, then let $q = p$. Otherwise, pick B_1 to be the smallest element of $C^{\kappa^+}(p)(A^{0\kappa^+})$ including B. Let B_{11} be the immediate predecessor of B_1 not in $C^{\kappa^+}(p)(B_1)$. If $B \in C^{\kappa^+}(p)(B_{11})$, then set $q = swt(p, B_1)$. Otherwise pick B_2 to be the the smallest element of $C^{\kappa^+}(p)(B_{11})$ including B. Note that $B_2 \in B_1$. So we go down on ranks. Hence after

finitely many steps a model B_n with $B \in C^{\kappa^+}(p)(B_n)$ will be reached. Then $q = swt(p, B_1, B_2, ..., B_n)$ will be as desired.
$\square$

Definition 1.3 Let $r, q \in \mathcal{P}'$. Then $r \geq q$ (r is stronger than q) iff there is $p = swt(r, B_1, \ldots, B_n)$ for some $B_1, \ldots, B_n$ appearing in r so that the following hold, where

$$p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle$$
$$q = \langle\langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle$$

$$A^{1\kappa^{++}} \cap (\max B^{1\kappa^{++}} + 1) = B^{1\kappa^{++}} \tag{1}$$

$$A^{1\kappa^+} \supseteq B^{1\kappa^+} \tag{2}$$

$$C^{\kappa^+} \restriction B^{1\kappa^+} = D^{\kappa^+} \tag{3}$$

$$B^{0\kappa^+} \in C^{\kappa^+}(A^{0\kappa^+}) \tag{4}$$

(5) for each $A \in A^{1\kappa^+}$, $A \cap B^{0\kappa^+} \in B^{1\kappa^+}$ or there are $B \in B^{1\kappa^+}$ and $\alpha \in B^{1\kappa^{++}}$ such that $A \cap B^{0\kappa^+} = B \cap \alpha$.

Remarks (1) Note that if $t = swt(p, B_0, \ldots, B_n)$ is defined, then $t \geq p$ and $p = swt(swt(p, B_0, \ldots, B_n), B_n, B_{n-1}, \ldots, B_0) = swt(t, B_n, \ldots, B_0) \geq t$. Hence the switching produces equivalent conditions.

(2) We need to allow $swt(p, B)$ for the Δ-system argument. Since in this argument two conditions are combined into one and so C^0 should pick one of them only. Also it is needed for proving a strategical closure of the forcing.

(3) The use of finite sequences $B_0, \ldots, B_n$ is needed in order to insure transitivity of the order $\leq$ on $\mathcal{P}'$.

Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle,\ A^{1\kappa^{++}}\rangle \in \mathcal{P}'$. Set $p\backslash\kappa^{++} = A^{1\kappa^{++}}$. Define $\mathcal{P}'_{\geq\kappa^{++}}$ to be the set of all $p\backslash\kappa^{++}$ for $p \in \mathcal{P}'$.

The next lemma is obvious.

Lemma 1.4 *$\langle \mathcal{P}'_{\geq\kappa^{++}}, \leq \rangle$ is κ^{+3}-closed.*

Set $p \restriction \kappa^{++} = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle$ where $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle,\ A^{1\kappa^{++}}\rangle \in \mathcal{P}'$.

Let $G(\mathcal{P}'_{\geq\kappa^{++}})$ be a generic subset of $\mathcal{P}'_{\geq\kappa^{++}}$. Define $\mathcal{P}'_{<\kappa^{++}}$ to be the set of all $p \restriction \kappa^{++}$ for $p \in \mathcal{P}'$ with $p\backslash\kappa^{++} \in G(\mathcal{P}'_{\geq\kappa^{++}})$.

Let $p \in \mathcal{P}'$ and $q \in \mathcal{P}'_{\geq\kappa^{++}}$. Then $q^\frown p$ denotes the set obtained from p by adding q to the last component of p,i.e. to A^{11}.

The following lemma is trivial.

Lemma 1.5 *Let $p \in \mathcal{P}'$, $q \in \mathcal{P}'_{\geq\kappa^{++}}$ and $q \geq_{\mathcal{P}'_{\geq\kappa^{++}}} p\backslash\kappa^{++}$. Then $q^\frown p \in \mathcal{P}'$ and $q^\frown p \geq p$.*

It follows now that $\mathcal{P}'$ projects to $\mathcal{P}'_{<\kappa^{++}}$.

Let us turn to the chain condition.

Lemma 1.6 *The forcing $\mathcal{P}'_{<\kappa^{++}}$ satisfies κ^{+3}-c.c. in $V^{\mathcal{P}'_{\geq\kappa^{++}}}$.*

Proof. Suppose otherwise. Let us assume that

$$\emptyset \Vdash_{\mathcal{P}'_{\geq\kappa^{++}}} (\langle \underset{\sim}{p}_\alpha = \langle \underset{\sim}{A}^{0\kappa^+}_\alpha,\ \underset{\sim}{A}^{1\kappa^+}_\alpha,\ \underset{\sim}{C}^{\kappa^+}_\alpha\rangle \mid \alpha < \kappa^{+3}\rangle \text{ is an antichain in } \underset{\sim}{\mathcal{P}}'_{<\kappa^{++}})$$

Without loss of generality we can assume that each $A^{0\kappa^+}_\alpha$ is forced to be a successor model, otherwise just extend conditions by adding one additional model on the top. Define by induction, using 1.3, an increasing sequence $\langle q_\alpha \mid \alpha < \kappa^{+3}\rangle$ of elements of $\mathcal{P}'_{\geq\kappa^{++}}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{+3}\rangle$, $p_\alpha = \langle A^{0\kappa^+}_\alpha, A^{1\kappa^+}_\alpha, C^{\kappa^+}_\alpha\rangle$ so that for every $\alpha < \kappa^{+3}$

$$q_\alpha \Vdash_{\mathcal{P}'_{\geq\kappa^3}} \langle \underset{\sim}{A}^{0\kappa^+}_\alpha,\ \underset{\sim}{A}^{1\kappa^+}_\alpha,\ \underset{\sim}{C}^{\kappa^+}_\alpha\rangle = \check{p}_\alpha\ .$$

For a limit $\alpha < \kappa^{+3}$ let

$$\overline{q}_\alpha = \bigcup_{\beta<\alpha} q_\beta \cup \{\sup \bigcup_{\beta<\alpha} q_\beta\}$$

and q_α be its extension deciding $\underset{\sim}{p}^{\alpha}$. Also assume that $\max q_\alpha \geq \sup(A^{00}_\alpha \cap \kappa^{+3})$.

We form a Δ-system. By shrinking if necessary assume that for some stationary $S \subseteq \kappa^{+3}$ and $\delta < \kappa^{+3}$ we have the following for every $\alpha < \beta$ in S:

(a) $A^{0\kappa^+}_\alpha \cap \alpha = A^{0\kappa^+}_\beta \cap \beta \subseteq \delta$

(b) $A^{0\kappa^+}_\alpha \backslash \alpha \neq \emptyset$

(c) $\sup A^{0\kappa^+}_\alpha < \beta$

(d) $\sup \overline{q}_\alpha = \alpha + 1$

(e)
$$\langle A^{0\kappa^+}_\alpha, \in, \leq, \subseteq, \kappa, C^{\kappa^+}_\alpha, f_{A^{0\kappa^+}_\alpha},\ A^{1\kappa^+}_\alpha, q_\alpha \cap A^{0\kappa^+}_\alpha \rangle$$
$$\langle A^{0\kappa^+}_\beta, \in, \leq, \subseteq, \kappa, C^{\kappa^+}_\beta, f_{A^{0\kappa^+}_\beta},\ A^{1\kappa^+}_\beta, q_\beta \cap A^{0\kappa^+}_\beta \rangle$$

are isomorphic over δ, i.e. by isomorphism fixing every ordinal below δ, where

$$f_{A^{0\kappa^+}_\alpha} : \kappa^+ \longleftrightarrow A^{0\kappa^+}_\alpha$$

and

$$f_{A^{0\kappa^+}_\beta} : \kappa^+ \longleftrightarrow A^{0\kappa^+}_\beta$$

are the fixed enumerations. Denote the isomorphism by $\pi_{\alpha\beta}$

We claim that for $\alpha < \beta$ in S we can extend q_β to a condition forcing compatibility of p_α and p_β. Proceed as follows. Pick A to be an elementary submodel of cardinality κ^+ so that

(i) $A^{1\kappa^+}_\alpha, A^{1\kappa^+}_\beta \in A$

(ii) $C^{\kappa^+}_\alpha, C^{\kappa^+}_\beta \in A$

(iii) $q_\beta \in A$.

Extend q_β to $q = q_\beta \cup \sup(A \cap \kappa^{+3})$. Set $p = \langle A, A_\alpha^{1\kappa^+} \cup A_\beta^{1\kappa^+} \cup \{A\}, C_\alpha^{\kappa+} \cup C_\beta^{\kappa^+} \cup \langle A, C_\beta^{\kappa^+}(A_\beta^{0\kappa^+})^\frown A\rangle\rangle$. Let us check that $\langle p, q\rangle \in \mathcal{P}'$. The conditions (1)-(6),(10),(11) of 1.1 hold trivially, (6) holds by (d) above. Let us check (8) and (9).

Suppose $X, Y \in A_\alpha^{1\kappa^+} \cup A_\beta^{1\kappa^+} \cup \{A\}$, $otp(X) = otp(Y)$ and $X \neq Y$. Then $X, Y \in A_\alpha^{1\kappa^+} \cup A_\beta^{1\kappa^+}$. If both X and Y belong to $A_\alpha^{1\kappa^+}$ or $A_\beta^{1\kappa^+}$ then we are done since $\langle p_\alpha, q_\beta\rangle$ and $\langle p_\beta, q_\beta\rangle$ satisfy 1.1(8). So, suppose $X \in A_\alpha^{1\kappa^+}$ and $Y \in A_\beta^{1\kappa^+}$. Let $X' \in A_\alpha^{1\kappa+}$ be the one corresponding to Y under (d), i.e. $\pi_{\alpha\beta}(X') = Y$. Then by 1.1(8) for $\langle p_\alpha, q_\alpha\rangle$ we will have $\xi \in X \cap q_\alpha$ such that $X \cap X' = X \cap \xi$. By (a) and $\pi_{\alpha\beta}(X') = Y$ we have $X' \cap Y = X' \cap \alpha = Y \cap \beta$. Then

$$X \cap Y = X \cap Y \cap \beta = X \cap X' \cap \alpha = X \cap \xi \cap \alpha = X \cap \min(\xi, \alpha).$$

If $\xi \leq \alpha$, then we are done since $\xi \in X$. If $\xi > \alpha$, then $\alpha \in \overline{q}_\alpha \cap \sup X$. Hence by 1.1(11), $\min(X \backslash \alpha)$ is in q_α and, clearly then

$$X \cap \alpha = X \cap \min(X \backslash \alpha)\ .$$

Let us check 1.1(9). Thus suppose that $X, Y \in A_\alpha^{1\kappa^+} \cup A_\beta^{1\kappa^+} \cup \{A\}$ and $otp(X) > otp(Y)$. Again we need only to consider the case when X and Y belong to different $A^{1\kappa+}$'s. Assume for example that $X \in A_\alpha^{1\kappa^+}$ and $Y \in A_\beta^{1\kappa^+}$. As above, we pick $X' \in A_\alpha^{10}$ such that $\pi_{\alpha\beta}(X') = Y$. Now, by 1.1(9) for $\langle p_\alpha, q_\alpha\rangle$, find $X'' \in X \cap A_\alpha^{1\kappa^+}$ such that $otp(X'') = otp(X')$ and $X \cap X' = X'' \cap X'$. Then $X \cap Y = X \cap Y \cap \beta = X \cap X' \cap \alpha = X'' \cap X' \cap \alpha = X'' \cap Y \cap \beta = X'' \cap Y$, since $\sup(X'') < \sup(X) \leq \sup(A_\alpha^{0\kappa^+})$ which is below β by (c) above.

The condition 9(c) holds since the models are part of the Δ-system.

Clearly, $\langle p, q\rangle \geq \langle p_\beta, q_\beta\rangle$. $\langle p, q\rangle \geq \langle p_\alpha, q_\alpha\rangle$ follows using switching of $A_\beta^{0\kappa^+}$ to $A_\alpha^{0\kappa^+}$.

$\square$

Lemma 1.7 *$\mathcal{P}'$ is κ^{++}-strategically closed.*

Proof. We define a winning strategy for the player playing at even stages. Thus suppose $\langle p_j \mid j < i\rangle$, $p_j = \langle\langle A_j^{0\kappa^+}, A_j^{1\kappa^+}, C_j^{\kappa^+}\rangle, A_j^{1\kappa^{++}}\rangle$ is a play according to this strategy upto an even stage $i < \kappa^{++}$. Set first

$$B_i^{0\kappa^+} = \bigcup_{j<i} A_j^{0\kappa^+}, B_i^{1\kappa^+} = \bigcup_{j<i} A_j^{1\kappa^+} \cup \{B_i^{0\kappa^+}\},$$

$$D_i^{\kappa^+} = \bigcup_{j<i} C_j^{\kappa^+} \cup \{\langle B_i^{0\kappa^+}, \{B_i^{0\kappa^+}\} \cup \{C_j^{\kappa^+}(A_j^{0\kappa^+}) \mid \ j \text{is even}\}\rangle\}$$

and

$$B_i^{1\kappa^{++}} = \bigcup_{j<i} B_j^{1\kappa^{++}} \cup \{\sup \bigcup_{j<i} B_j^{1\kappa^{++}}\}.$$

Then pick $A_i^{0\kappa^+}$ to be a model of cardinality κ^+ such that

(a) ${}^{\kappa}A_i^{0\kappa^+} \subseteq A_i^{0\kappa^+}$

(b) $B_i^{0\kappa^+}, B_i^{1\kappa^+}, D_i^{\kappa^+}, B_i^{1\kappa^{++}} \in A_i^{0\kappa^+}$.

Set $A_i^{1\kappa^+} = B_i^{1\kappa^+} \cup \{A_i^{0\kappa^+}\}$, $C_i^{\kappa^+} = D_i^{\kappa^+} \cup \{\langle A_i^{0\kappa^+}, D_i^{\kappa^+}(B_i^{0\kappa^+}) \cup \{A_i^{0\kappa^+}\}\rangle\}$ and $A_i^{1\kappa^{++}} = B_i^{1\kappa^{++}} \cup \{\sup(A_i^{0\kappa^+} \cap \kappa^{+3}\}$. As an inductive assumption we assume that at each even stage $j < i$, p_j was defined in the same fashion. Then $p_i = \langle\langle A_i^{0\kappa^+}, A_i^{1\kappa^+}, C_i^{\kappa^+}\rangle, A_i^{1\kappa^{++}}\rangle$ will be a condition in $\mathcal{P}'$ stronger than each p_j for $j < i$. The switching may be required here once moving from an odd stage to its immediate successor even stage.

□

2 Types of Models

The basic approach here is as in [1] but instead of dealing with types of ordinals we shall consider elementary submodels of $H(\chi^{+k})$ for some χ big enough and $k \leq \omega$ types of such models.

Fix $n < \omega$. Fix using GCH an enumeration $\langle a_\alpha \mid \alpha < \kappa_n\rangle$ of $[\kappa_n]^{<\kappa_n}$ so that for every successor cardinal $\delta < \kappa_n$ the initial segment $\langle a_\alpha \mid \alpha < \delta\rangle$

enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_n(\langle a_\alpha \mid \alpha < \kappa_n\rangle) = \langle a_\alpha \mid \alpha < j_n(\kappa_n)\rangle$ where j_n is the canonical embedding of the $(\kappa_n, \kappa_n^{+n+3})$-extender E_n. Then $\langle a_\alpha \mid \alpha < \kappa_n^{+n+3}\rangle$ will enumerate $[\kappa_n^{+n+3}]^{<\kappa_n^{+n+3}}$ and we fix this enumeration. For each $k \leq \omega$ consider a structure

$$\mathfrak{A}_{n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, E_n, \kappa_n, \chi, \langle a_\alpha \mid \alpha < \kappa_n^{+n+3}\rangle, 0, 1, \ldots, \alpha, \ldots \mid \alpha < \kappa_n^{+k}\rangle$$

in the appropriate language $\mathcal{L}_{n,k}$ with a large enough regular cardinal χ.

Remark.

It is possible to use κ_n^{++} here (as well as in [1]) instead of κ_n^{+k}. The point is that there are only κ_n^{++} many ultrafilters over κ_n and we would like that equivalent conditions use the same ultrafilter. The only parameter that that need to vary is k in $H(\chi^{+k})$.

Let $\mathcal{L}'_{n,k}$ be the expansion of $\mathcal{L}_{n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less than κ_n^{+n+3} let $\mathfrak{A}_{n,k,a}$ be the expansion of $\mathfrak{A}_{n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less than κ_n^{+n+3}. Denote by $tp_{n,k}(b)$ the $\mathcal{L}_{n,k}$-type realized by b in $\mathfrak{A}_{n,k}$. Let $tp_{n,k}(a,b)$ be a the $\mathcal{L}'_{n,k}$-type realized by b in $\mathfrak{A}_{n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [1].

Lemma 2.1 *(a)* $|\{tp_{n,k}(b) \mid b \in H(\chi^{+k})\}| = \kappa_n^{+k+1}$

(b) $|\{tp_{n,\kappa}(a,b) \mid a, b \in H(\chi^{+k})\}| = \kappa_n^{+k+1}$

Proof. (a) The cardinality of the language $\mathcal{L}_{n,k}$ is κ_n^{+k} so the number of formulas is κ_n^{+k}. Now the number of types is $2^{\kappa_n^{+k}} = \kappa_n^{+k+1}$.

(b) The same argument.

□

This lemma implies immediately the following:

Lemma 2.2 *Let* $A \prec \mathfrak{A}_{n,k+1}$ *and* $|A| \geq \kappa_n^{+k+1}$. *Then the following holds:*

(a) for every $a, b \in H(\chi^{+k})$ there $c, d \in A \cap H(\chi^{+k})$ with $tp_{n,k}(a, b) = tp_{n,k}(c, d)$

(b) for every $a \in A$ and $b \in H(\chi^{+k})$ there is $d \in A \cap H(\chi^{+k})$ so that $tp_{n,k}(a \cap H(\chi^{+k}),\ b)$=$tp_{n,k}(a \cap H(\chi^{+k}),\ d)$.

Proof. (a) Note that $tp_{n,k}(a, b) \in A$, by 2.1 and since $|A| \geq \kappa_n^{+k+1}$, so $A \supseteq \kappa_n^{+k+1}$. Now, $H(\chi^{+k+1}) \vDash (\exists x, y \in H(\chi^{+k})\forall\varphi(v, u) \in tp_{n,k}(a, b)\ (H(\chi^{+k}) \vDash \varphi(x, y)))$. But $A \prec H(\chi^{+k+1})$. So

$$A \vDash (\exists x, y \in H(\chi^{+k})\forall\varphi(x, y) \in tp_{n,k}(a, b)(H(\chi^{+k}) \vDash \varphi(x, y)))\ .$$

Pick $c, d \in A$ satisfying this formula. Then $c, d \in H(\chi^{+k})$ and $tp_{n,k}(c, d) = tp_{n,k}(a, b)$.

(b) Similar. $\square$

The next lemma will be crucial further for the chain condition arguments.

Lemma 2.3 *Suppose that $A \prec \mathfrak{A}_{n,k+1}, |A| \geq \kappa_n^{+k+1}$, $B \prec \mathfrak{A}_{n,k}$, and $C \in \mathcal{P}(B) \cap A \cap H(\chi^{+k})$. Then there is D so that*

(a) $D \in A$

(b) $C \subseteq D$

(c) $D \prec A \cap H(\chi^{+k}) \prec H(\chi^{+k})$.

(d) $tp_{n,k}(C, B) = tp_{n,k}(C, D)$.

Proof. As in 2.2., the following formula is true in $H(\chi^{+k+1})$:

$$\exists x \subseteq H(\chi^{+k})((x \prec H(\chi^{+k}))\wedge(x \supseteq C)\wedge(\forall\varphi(y, z) \in tp_{n,k}(C, B)H(\chi^{+k}) \vDash \varphi(C, x))).$$

Then the same holds in A. Let D witness this. Hence $D \in A$, $D \supseteq C$, $D \prec A \cap H(\chi^{+k}) \prec H(\chi^{+k})$ and $tp_{n,k}(C, B) = tp_{n,k}(C, D)$. $\square$

Further we shall add models $B \cap H(\chi^{+k})$ with $B \prec H(\chi^{+k+1})$ or models realizing the same $tp_{n,k}(a,-)$ as those of elementary submodel of $H(\chi^{+k+1})$ intersected with $H(\chi^{+k})$ for any a inside. We will require that for every $k < \omega$, each condition p has an equivalent condition q with every model in it being an elementary submodel of $H(\chi^k)$.

3 The Main Forcing

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.

Fix $n < \omega$. Following [2, Sec.3]. We define first Q_{n0}.

Definition 3.1 Let Q_{n0} be the set of the triples $\langle a, A, f \rangle$ so that:

(1) f is partial function from κ^{+3} to κ_n of cardinality at most κ

(2) a is a partial function of cardinality less than κ_n so that

(a) $\text{dom}(a)$ consists of models and ordinals appearing in elements of $G(\mathcal{P}')$, i.e. if $X \in \text{dom}(a)$, then for some $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}\rangle, A^{1\kappa^{++}}\rangle \in G(\mathcal{P}')$ we have $X = A^{0\kappa^+}$ or $X \in A^{1\kappa^{++}}$.

This means in particular that ordinals in $\text{dom}(a)$ are taken from $A^{1\kappa^{++}}$ only

(b) for each $X \in \text{dom}(a)$ there is $k \leq \omega$ so that $a(X) \subseteq H(\chi^{+k})$.

Also the following holds

(i) $|X| = \kappa^+$ implies $|a(X)| = \kappa_n^{+n+1}$

(ii) $|X| = \kappa^{++}$ implies $|a(X)| = \kappa_n^{+n+2}$ and $a(X) \cap \kappa_n^{+n+3} \in ORD$

Note that in (ii) X is an ordinal but $a(X)$ is not. Actually our main interest is in $a(X) \cap \kappa_n^{+n+3}$ which is required to be an ordinal.

Further passing from Q_{0n} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image of X will be an elementary submodel of $H(\chi^{+k})$. But in general just subsets are allowed here.

(c) if $A, B \in \mathrm{dom}(a)$, $A \in B$ (or $A \subseteq B$) and k is the minimal so that $a(A) \subseteq H(\chi^{+k})$ or $a(B) \subseteq H(\chi^{+k})$, then $a(A) \cap H(\chi^{+k}) \in a(B) \cap H(\chi^{+k})$ (or $a(A) \cap H(\chi^{+k}) \subseteq a(B) \cap H(\chi^{+k})$).

The intuitive meaning is that a is supposed to preserve membership and inclusion. But we cannot literally require this since $a(A)$ and $a(B)$ may be substructures of different structures. So we first go down to the smallest of this structures and then put the requirement on the intersections.

(d) $\mathrm{dom}(a)$ has a maximal model (under inclusion) it is a member of $A^{1\kappa^+}$ for some condition in $G(\mathcal{P}')$. Its image $a(\max a)$ intersected with κ_n^{+n+3} is above all the rest of $\mathrm{rng}(a)$ restricted to κ_n^{+n+3} in the ordering of the extender E_n (via some reasonable coding by ordinals).

Recall that the extender E_n acts on κ_n^{+n+3} and our main interest is in Prikry sequences it will produce. So, parts of $\mathrm{rng}(a)$ restricted to κ_n^{+n+3} will play the central role.

(e) if $A, B \in \mathrm{dom}(a)$ and $otp(A) = otp(B)$ then

$$\langle a(A) \cap H(\chi^{+k}), \in \rangle \simeq \langle a(B) \cap H(\chi^{+k}), \in \rangle$$

where k is the minimal so that $a(A) \subseteq H(\chi^{+k})$ or $a(B) \subseteq H(\chi^{+k})$.

Note that it is possible to have for example $a(A) \prec H(\chi^{+6})$ and $a(B) \prec H(\chi^{+18})$. Then we take $k = 6$.

Let π be the isomorphism between

$$\langle a(A) \cap H(\chi^{+k}), \in \rangle, \langle a(B) \cap H(\chi^{+k}), \in \rangle$$

and $\pi_{A,B}$ be the isomorphism between A and B given by 1.1(8). Require that for each $Z \in A \cap \mathrm{dom}(a)$ we have $\pi_{A,B}(Z) \in B \cap \mathrm{dom}(a)$ and

$$\pi(a(Z) \cap H(\chi^{+k})) = a(\pi_{A,B}(Z)) \cap H(\chi^{+k}).$$

(f) if $A, B \in \mathrm{dom}(a)$, $A \not\subseteq B$ and $B \not\subseteq A$ then models and ordinals witnessing 1.1(9,10) are in $\mathrm{dom}(a)$

(g) if $\alpha \in \mathrm{dom}(a)$ (i.e. a member of some $A^{1\kappa^{++}}$ for $A^{1\kappa^{++}}$ in $G(\mathcal{P}'))$ then for each $A \in \mathrm{dom}(a)$ with $\alpha \in A$ the smallest model B of $C^{\kappa^+}(A)$ such that $\alpha \in B$ belongs to $\mathrm{dom}(a)$ as well as all its immediate predecessors. Note that by 1.1(11) there are at most two such models.

(h) if $A, B \in \mathrm{dom}(a)$ and $B \in A$, then the walk via $C^{\kappa^+}(A)$ from A to B is in $\mathrm{dom}(a)$, i.e. the least model $A_0 \in C^{\kappa^+}(A)$ such that $A_0 = B$ or $B \in A_0$ is in $\mathrm{dom}(a)$. If $B \in A_0$, then the immediate predecessor A_1 of A_0 with $B \in A_1$ or $B = A_1$ is in $\mathrm{dom}(a)$. Now, the least model $A_2 \in C^{\kappa^+}(A_1)$ such $A_2 = B$ or $B \in A_2$ is in $\mathrm{dom}(a)$. If $B \in A_2$ then the immediate predecessor A_3 of A_2 with $B \in A_2$ or $B = A_2$ is in $\mathrm{dom}(a)$ and so on.

Note that only finitely many models are involved in such a walk.

(i) if $A \in \mathrm{dom}(a)$ then $C^{\kappa^+}(A) \cap \mathrm{dom}(a)$ is a closed chain. Let $\langle A_i | i < j\rangle$ be its increasing continuous enumeration. For each $l < j$ consider the final segment $\langle A_i | l \leq i < j\rangle$ and its image $\langle a(A_i) | l \leq i < j\rangle$. Find the minimal k so that

$$a(A_i) \subseteq H(\chi^{+k}) \text{ for each } i, l \leq i < j.$$

Then the sequence

$$\langle a(A_i) \cap H(\chi^{+k}) | l \leq i < j\rangle.$$

is increasing and continuous.

Note that k here may depend on l, i.e. on the final segment.

(j) if $A, B \in \mathrm{dom}(a)$, $A \not\subseteq B$, $B \not\subseteq A$ and $\sup A < \sup B$, then $\min(B \setminus \sup A) \in \mathrm{dom}(a)$.

Note in this case necessary $\min(B \backslash \sup A) \in A^{1\kappa++}$. Thus, if $otp(A) = otp(B)$ then by 1.1(9) there is $\alpha \in B \cap A^{1\kappa^{++}}$ s.t. $A \cap B = B \cap \alpha$. By 1.1(10), this α will be as desired.

If $otp(A) > otp(B)$, then 1.1(10(c(ii))) applies and together with 1.1(9), we will have $\alpha = \min(B \backslash \sup(B \cap B') + 1)$ as desired.

If $otp(A) < otp(B)$, then find $A' \in B \cap A^{1\kappa^+}$ witnessing 1.1(10) for B and A. The assumption $\sup A < \sup B$ implies then that (i) of 1.1(10(c)) should hold. Pick $\tau \in A \cap A^{1\kappa^{++}}$ such that $A \cap \tau = A \cap A' = A \cap B$. Then $\tau \in A^{1\kappa^{++}} \backslash B$. By 1.1(6), $\min(B \backslash \tau) \in A^{1\kappa++}$. Hence, $\alpha = \min(B \backslash \tau)$ will be equal to $\min(B \backslash \sup A)$, by 1.1(10(c(ii))) and we are done.

(k) if $A, \alpha \in \mathrm{dom}(a)$ and $\sup A > \alpha$, then $\min(A \backslash \alpha) \in \mathrm{dom}(a)$.

(l) if $A, B \in \mathrm{dom}(a)$ and B is an immediate predecessor of A Then the other immediate predecessor of A is in $\mathrm{dom}(a)$ as well.

(m) if $\langle \alpha_i | i < j \rangle$ is an increasing sequence of ordinals in $\mathrm{dom}(a)$, then $\cup\{\alpha_i | i < j\} \in \mathrm{dom}(a)$.

(n) if $A \in \mathrm{dom}(a)$ is a limit model and $cof(otp_{\kappa^+}(A) - 1) < \kappa_n$ (i.e. the cofinality of the sequence $C^{\kappa^+}(A) \backslash \{A\}$ under the inclusion relation is less than κ_n) then a closed cofinal subsequence of $C^{\kappa^+}(A) \backslash \{A\}$ is in $\mathrm{dom}(a)$. The images of its members under a form a closed cofinal in $a(A)$ sequence.

(o) if $\alpha \in \mathrm{dom}(a)$ is a limit member of $A^{1\kappa^{++}}$ of cofinality less than κ_n, then a closed cofinal in α sequence from $A^{1\kappa^{++}}$ is in $\mathrm{dom}(a)$ as well. The images of its members under a form a closed cofinal in $a(\alpha)$ sequence.

(3) $\{\alpha < \kappa^{+3} \mid \alpha \in \mathrm{dom}(a)\} \cap \mathrm{dom}(f) = \emptyset$.

(4) $A \in E_{n,a(\max(a))}$.

(5) for every ordinals α, β, γ which are elements of $rnga$ or the ordinals coding models of cardinality κ_n^{+n+1} in $\mathrm{rng}(a)$ we have

$$\begin{aligned} &\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \quad \text{implies} \\ &\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho)) \end{aligned}$$

for every $\rho \in \pi``_{\max \mathrm{rng}(a), \alpha}(A)$.

We define now Q_{n1} and $\langle Q_n, \leq_n, \leq_n^* \rangle$ as in [2, Sec.2].

Definition 3.2 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) for every $n < \omega$ $p_n \in Q_n$

(2) there is $\ell(p) < \omega$ such that

  (i) for every $n < \ell(p)$ $p_n \in Q_{n1}$,

  (ii) for every $n \geq \ell(p)$ we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$

  (iii) for every $n, m \geq \ell(p)$ $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$ is a model of cardinality κ^+

(3) for every $n \geq m \geq \ell(p)$ $\mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$

(4) for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.

Lemma 3.3 *Suppose* $p = \langle p_k \mid k < \omega \rangle \in \mathcal{P}$, $p_k = \langle a_k, A_k, f_k \rangle$ *for* $k \geq \ell(p)$, X *is model or an ordinal appearing in an element of* $G(\mathcal{P}')$ *and* $X \notin \bigcup_{\ell(p) \leq k < \omega} \mathrm{dom}(a_k) \cup \mathrm{dom}(f_k)$. *Suppose that*

(a) if X *is model then it is a successor model or if it is a limit one then* $cof(otp_{\kappa^+}(X) - 1) > \kappa$

(b) if X *is an ordinal then it is a successor member of some* $A^{1\kappa^{++}} \in G(\mathcal{P}')$ *or it is a limit of cofinality above* κ.

Then there is a direct extension $q = \langle q_k \mid k < \omega\rangle$, $q_k = \langle b_k, B_k, g_k\rangle$ for $k \geq \ell(q)$, of p so that starting with some $n \geq \ell(q)$ we have $X \in \mathrm{dom}(b_k)$ for each $k \geq n$.

Remark We would like to avoid at this stage adding limit models (or ordinals) of small cofinality since by 3.1(2(n,o) this will require additional adding of sequences of models (or ordinals).

Proof. Note first that it is easy to add to p any A appearing in condition of $G(\mathcal{P}')$ of cardinality κ^+ so that the maximal model of p belongs to $C^{\kappa^+}(A)$. Just at each level $n \geq \ell(p)$ pick an elementary submodel of $H(\chi^{+\omega})$ of cardinality κ_n^{+n+1} including $\mathrm{rng}(a_n)$ as an element. Map A to such a model.

Suppose now we like to add to p some X which does not include the maximal model of p. Denote it by $\max(p)$. Without loss of generality we can assume that $X \in \max(p)$. Just otherwise using genericity of $G(\mathcal{P}')$ find A as above with $X \in A$. Pick a model $A \in \cup\{\mathrm{dom}(a_n) | n < \omega\}$ with $X \in A$ of the smallest possible otp_{κ^+} and among such models one of the least possible $rank$ (the usual one). Suppose for simplicity that $A \in \mathrm{dom}(a_n)$ for each $n < \omega$.

Let $\langle A_i \mid i < otp_{\kappa^+}((A))\rangle$ be increasing continuous enumeration of $C^{\kappa^+}(A))$.

Case A. $X = A_i$ for some $i < otp_{\kappa^+}(A)$.

Let A_{i^*} be the first model of $C^{\kappa^+}(A)$ in $\bigcup_{n \geq \ell(p)} \mathrm{dom}(a_n)$ with $X \in A_{i^*}$.

Let $n < \omega$ be big enough so that $a_n(A_{i^*}) \prec H(\chi^{+n+3})$. Denote by $A_{i^{**}}$ the largest member of $C^{\kappa^+}(A_{i^*}) \backslash \{A_{i^*}\}$ inside $\mathrm{dom}(a_n)$, if $C^{\kappa^+}(A_{i^*}) \cap A_{i^*} \cap \mathrm{dom}(a_n) \neq \emptyset$. Notice that this set may vary once we change n.

Suppose that $A_{i^{**}}$ exists otherwise we may view it as empty and run the same argument.

Now for each $Y \in \mathrm{dom}(a_n)$ with $Y \not\supseteq A_{i^*}$ and $otpY \leq otpA_{i^*}$, we will have that $Y \cap A_{i^*} = Y \cap A_{i^{**}}$, since either

(i) $Y \in A_{i^*}$ and then $Y \in A_{i^{**}} \cup \{A_{i^{**}}\}$ (just use the walk from A_{i^*} to Y. It is supposed to be in the domain of a_n by 3.1(2(h))

or

(ii) $Y \notin A_{i^*}$ and then by 1.1(8) there is $Y' \in A_{i^*}$ with $Y \cap A_{i^*} = Y' \cap Y$. But by 3.1(2(f)), some such Y' must be in $\mathrm{dom}(a_n)$. By the choice of $A_{i^{**}}$ then $Y' \in A_{i^{**}} \cup \{A_{i^{**}}\}$. So, $Y \cap A_{i^*} = Y \cap A_{i^{**}}$.

If $Y \in \mathrm{dom}(a_n)$ with $Y \not\supseteq A_{i^*}$ and $otpY > otpA_{i^*}$, then by 3.1(2(f)), there are $Y' \in Y \cap A^{1\kappa^+} \cap \mathrm{dom}(a_n)$ so that $Y \cap A_{i^*} = Y' \cap A_{i^*}$. Again, by 3.1(2(f)), there is $\alpha \in A_{i*} \cap A^{1\kappa^{++}} \cap \mathrm{dom}(a_n)$ such that $Y' \cap A_{i^*} = \alpha \cap A_{i^*}$. Then, by 3.1(2(g)), we have $\alpha \in A_{i^{**}}$, unless $A_{i^{**}}$ is the immediate predecessor of A_{i^*}. But then X must be equal to $A_{i^{**}}$.

Similar, if $\alpha \in \mathrm{dom}(a_n)$ and $\alpha < \sup A_{i^*}$ then $\min(A_{i^*} \backslash \alpha) \in A_{i^{**}}$.

Now pick X^* to be an element of $a_n(A_{i^*})$ such that $X^* \subseteq a_n(A_{i^*})$, ${}^{\kappa_n>}X^* \subseteq X^*$, $X^* \prec H(\chi^{+n+2})$ and $a_n(A_{i^{**}}) \cap H(\chi^{+n+2}) \in X^*$. By the above, all the relevant information (intersections with models, ordinals etc.) is already inside $a_n(A_{i^{**}})$. Hence we can extend a_n by adding to it the pair $\langle X, X^* \rangle$. Map X^* via all isomorphisms between $a_n(A_{i^*})$ and $a_n(B)$'s for each $B \in \mathrm{dom}(a_n)$ with $otp(B) = otp(A_{i^*})$.

Note that here is the place where we may drop to subsets of $H(\chi)$ and not elementary submodels of it. Just moving $a_n(A_{i^*})$ by isomorphisms may decrease degree of elementarity by one.

Let b_n be the result. Note that if X is an immediate predecessor of a model having an other immediate predecessor X' or X is the immediate successor (in $C^{\kappa^+}(X)$) of a model in $\mathrm{dom}(a_n)$ and some model X' is an other immediate predecessor of X, then adding of X may requires by 3.1(2(l)) adding of X' also. Let us delay the adding of such X' to the next case. Instead we deal with (or allow) a_n's which satisfy all the conditions of 3.1(2) but (l).

Claim A1 b_n satisfies all the conditions of 3.1(2) but (l). Moreover, if a_n satisfies all the conditions of 3.1(2) and X is not an immediate predecessor of a model having an other immediate predecessor then also b_n satisfies all the conditions of 3.1(2).

Proof. Let X' be an images of X which was added to $\mathrm{dom}(a_n)$ and X'' be an other model or ordinal in $\mathrm{dom}(b_n)$, which may be in $\mathrm{dom}(a_n)$ (and it is the case if it is an ordinal) or may be an other image of X which was added to $\mathrm{dom}(a_n)$. We need to show that ordinals and models witnessing the intersection conditions for $X' \cap X''$ are in the domain. Split into few cases.

Case A1.1 $X' = X$ and $X'' \in A_{i^*}$

Then by (i) above $X'' \in A_{i^{**}} \cup A_{i^{**}}$. Hence, $X'' \in X = A_{i-1}$ and we are done.

Case A1.2 $X' = X$ and $X'' \notin A_{i^*}$

Let us split this case into two.

Subcase A1.2.1 $X'' \in \mathrm{dom}(a_n)$

If X'' is an ordinal, then either $X'' > \sup A_{i^*}$, then it is above $\sup X$ as well and we are done. Or $X'' < \sup A_{i^*}$ and then $\alpha = \min A_{i^*} \backslash X''$ is in $\mathrm{dom}(a_n)$. By the choice of $A_{i^{**}}$, then $\alpha \in A_{i^{**}}$. But, clearly, then $\alpha \in X$ and $\alpha = \min X \backslash X''$. So we are done.

Assume now that X'' is a model.

Let us point out that the walk from X'' to X is already in $\mathrm{dom}(a_n)$. We claim that the walk must terminate with A_{i^*}. Suppose otherwise. Thus let Y be the first model of the walk which does not contain A. Compare Y and A_{i^*}. By 1.1(9,10) and 3.1(2(f)), there is $\alpha \in A_{i^*} \cap \mathrm{dom}(a_n)$ such that

$$Y \cap A_{i^*} = A_{i^*} \cap \alpha.$$

By the choice of $A_{i^{**}}$, we must have $\alpha \in A_{i^{**}}$. But on the other hand, $X \subseteq Y \cap A_{i^*} = A_{i^*} \cap \alpha$. In particular, $X \subseteq \alpha$. Which is clearly a contradiction, since $X \supseteq A_{i^{**}}$.

If $X' \supseteq A_{i^*}$ then the intersection properties are clear. Suppose that it is not the case. Compare X' with A_{i^*}. By 1.1(9,10) and 3.1(2(f)), there is $\alpha \in A_{i^*} \cap \mathrm{dom}(a_n)$ such that

$$X' \cap A_{i^*} = A_{i^*} \cap \alpha.$$

By the choice of $A_{i^{**}}$, we must have $\alpha \in A_{i^{**}}$. But

$$X' \cap X = X' \cap X \cap A_{i^*} = X \cap A_{i^*} \cap \alpha = X \cap \alpha.$$

and, clearly $\alpha \in X$. In order to deal with the intersection on the other side compare $otpX'$ and $otpA_{i^*}$. If $otpX' < otpA_{i^*}$, then, by 3.1(2(f)) there is $Y \in A_{i^*} \cap \operatorname{dom}(a_n)$ such that $otpY = otpX'$ and

$$X' \cap A_{i^*} = X' \cap Y.$$

By the choice of $A_{i^{**}}$, we must have $Y \in A_{i^{**}} \cup \{A_{i^{**}}\}$. But then $Y \subseteq X$. Hence

$$X' \cap X = X' \cap A_{i^*} \cap X = X' \cap Y \cap X = X' \cap Y.$$

So we are done since both X' and Y are old.

If $otpX' \geq otpA_{i^*}$, then, by 3.1(2(f)) there is $Y \in X' \cup \{X'\} \cap \operatorname{dom}(a_n)$ such that $otpY = otpA_{i^*}$ and

$$X' \cap A_{i^*} = Y \cap A_{i^*}.$$

By 1.1(9,10), there are $\alpha \in A_{i^*} \cap \operatorname{dom}(a_n)$ such that

$$Y \cap A_{i^*} = A_{i^*} \cap \alpha.$$

and $\beta \in Y \cap \operatorname{dom}(a_n)$ such that

$$Y \cap A_{i^*} = Y \cap \beta.$$

In this situation $Z = \pi_{A_{i^*}Y}[X]$ will be added as an isomorphic image of X. Also, $\pi_{A_{i^*}Y}[\alpha] = \beta$. Then

$$X' \cap X = X' \cap X \cap A_{i^*} = Y \cap A_{i^*} \cap X = Y \cap \beta \cap X = Z \cap \beta.$$

But as above, we must have $\alpha \in X$. Hence $\beta \in Z$ and we are done.

Subcase A1.2.2 $X'' \notin \operatorname{dom}(a_n)$

Then X'' is an isomorphic image of X. There is $B \in \operatorname{dom}(a_n)$ isomorphic to A_{i^*} such that $\pi_{A_{i^*}B}[X] = X''$. We need to take care only of the intersections properties of X and X'' one with an other. By 1.1(9,10), there are $\alpha \in A_{i^*} \cap \operatorname{dom}(a_n)$ such that

$$B \cap A_{i^*} = A_{i^*} \cap \alpha.$$

and $\beta \in B \cap \mathrm{dom}(a_n)$ such that

$$Y \cap A_{i^*} = Y \cap \beta.$$

Also, $\pi_{A_{i^*}B}[\alpha] = \beta$. Then, as above, $\alpha \in X$ and so $\beta \in X''$. We have the following:

$$X'' \cap X = X'' \cap B \cap X \cap A_{i^*} = X \cap \alpha = X'' \cap \beta.$$

Case A1.3 $X' \neq X$.

Then X' is an isomorphic image of X. There is $B \in \mathrm{dom}(a_n)$ isomorphic to A_{i^*} such that $\pi_{A_{i^*}B}[X] = X'$. The arguments of the previous cases work here completely the same after we replace A_{i^*} with B and $A_{i^{**}}$ with $\pi_{A_{i^*}B}[A_{i^{**}}]$.
$\square$ of the claim.

Case B $X \notin C^{\kappa^{++}}(A)$.

By the previous case it is possible to add each of A_i's, for $i < otp_{\kappa^+}(A)$. Let us proof by induction on i that it is possible to add $X \in A_i$. Thus, if $i = 0$, then add first A_0. The only possibility for $X \in A_0$ is to be an ordinal. Also, A_0 has no predecessors. Let $X = \alpha$.

Subcase B1 Each $\beta \in A_0 \cap (\cup\{\mathrm{dom}(a_n) | n < \omega\})$ (if any) is less than α.

Assume that n is big enouph such that $a_n(A_0) \prec H(\chi^{+k})$ for some $k \gg 2$. Pick now some $M \in a_n(A_0)$ such that

(1) $|M| = \kappa_n^{+n+2}$

(2) $M \prec H(\chi^{+k-1})$

(3) $M \supseteq \kappa_n^{+n+2}$

(4) $\mathrm{cof}(M \cap \kappa_n^{+n+3}) = \kappa_n^{+n+2}$

(5) for each $N \in \mathrm{rng}(a_n) \cap a_n(A_0)$ we have $N \cap H(\chi^{+k-1}) \in M$.

In particular, $M \cap \kappa_n^{+n+3}$ (the main part of M) is above each $N \cap \kappa_n^{+n+3}$, for every $N \in \mathrm{rng}(a_n) \cap a_n(A_0)$.

Define the image of α to be M. Move this setting to all the elements in $\mathrm{dom}(a_n)$ isomorphic to A_0 (if any). Denote the result by b_n.

Claim B1.1 b_n satisfies all the conditions of 3.1(2) but (l). Moreover, if a_n satisfies all the conditions of 3.1(2) then also b_n satisfies all the conditions of 3.1(2).

Proof. Let us check first 3.1(2(k)). Let Y be a model in $\mathrm{dom}(a_n)$ and α' be an image of α under isomorphism which was added to $\mathrm{dom}(a_n)$. We need to deal with the case when $\alpha' < \sup Y$ and show that $\min Y \backslash \alpha'$ is in $\mathrm{dom}(a_n)$. Thus let A_0' be a model in $\mathrm{dom}(a_n)$ isomorphic to A_0 such that $\alpha' = \pi_{A_0 A_0'}(\alpha)$. Compare Y with A_0'.

Case B1.1.1 $otp(Y) = otp(A_0')$.

We split into two subcases according to 1.1(10(c))(i) or (ii).

Subcase B1.1.1.1 $\min(A_0' \backslash \sup(A_0' \cap Y) + 1) > \sup Y$.

Then $\beta' = \min(A_0' \backslash \sup(A_0' \cap Y) + 1)$ is in $\mathrm{dom}(a_n)$ by 3.1(2(f)). Recall that α was above all the ordinals of $A_0 \cap \mathrm{dom}(a_n)$, hence α' will be such in $A_0' \cap \mathrm{dom}(a_n)$, by 3.1(2(e)). In particular, $\alpha' > \beta'$. But then $\alpha' > \sup Y$, which contradicts our assumption on α' and Y.

Subcase B1.1.1.2 $\min(Y \backslash \sup(A_0' \cap Y) + 1) > \sup A_0'$.

Again, $\beta' = \min(A_0' \backslash \sup(A_0' \cap Y) + 1)$ is in $\mathrm{dom}(a_n)$ by 3.1(2(f)). Also, $\alpha' > \beta'$. But $\alpha' < \sup A_0'$. Hence, $\alpha' < \min Y \backslash \alpha' = \min(Y \backslash \sup(A_0' \cap Y) + 1)$. But $\min(Y \backslash \sup(A_0' \cap Y) + 1) \in \mathrm{dom}(a_n)$, by 3.1(2(k)). So we are done.

Case B1.1.2 $otp(Y) < otp(A_0')$.

Then B' as in 1.1(10) must exists. But this is impossible since $otp_{\kappa^+}(A_0') = otp_{\kappa^+}(A_0) = 1$.

Case B1.1.3 $otp(Y) > otp(A_0')$.

Then we have a set B' as in 1.1(10) for Y and A_0 inside $\mathrm{dom}(a_n)$. Again we split into two cases according to (i) and (ii) of 1.1(10(c)).

Subcase B1.1.3.1 $\min(B' \setminus \sup(A'_0 \cap B') + 1) > \sup A'_0$.

As before, α' should be above $\beta' = \min(A'_0 \setminus \sup(A'_0 \cap Y) + 1)$. By 1.1(10(d)), Y has no elements inside the interval

$$(\min(A'_0 \setminus \sup(A'_0 \cap B') + 1), \sup A_0).$$

We will need now the following useful claim:

Subclaim B1.1.3.1.1 There is $Z \in \mathrm{dom}(a_n)$ such that $Z \supseteq A'_0$ and $otp(Z) = otp(Y)$.

Proof. Consider the walks from $\max(a_n)$ to A'_0 and to Y. Let B_0 be the first point where the walks split. Then B_0 must be a successor point with two immediate predecessors B_{00} and B_{01}. By 3.1(2(h)), then all this models B_0, B_{00}, B_{01} are in $\mathrm{dom}(a_n)$. Assume without loss of generality that $A'_0 \subseteq B_{00}$ and $Y \subseteq B_{01}$. If $Y = B_{01}$, then B_{00} will be as desired. Suppose that $Y \subset B_{01}$. Then just copy it to the B_{00} side by taking $Y_1 = \pi_{B_{01}B_{00}}[Y]$. Then $Y_1 \in \mathrm{dom}(a_n)$, by 3.1(2(e)). If $Y_1 \supseteq A'_0$, then we are done. Otherwise consider the walks from B_{00} to A'_0 and Y_1. After finitely many steps a model as desired will be reached.
$\square$ of the subclaim.

Compare now Y and Z. There is $\xi \in Y \cap \mathrm{dom}(a_n)$ such that $Y \cap Z = Y \cap \xi$. Actually, $\xi = \min(Y \setminus \sup(Y \cap Z))$, by 1.1(9). Remember that $\alpha' \in Z$ and $\alpha' < \sup Y$. Then, by 1.1(10), $\xi = \min(Y \setminus \sup(Y \cap Z)) > \sup Z > \alpha'$. But now clearly, $\xi = \min(Y \setminus \alpha')$ and we are done.

Subcase B1.1.3.2 $\min(A'_0 \setminus \sup(A'_0 \cap B') + 1) > \sup B'$.

Then, by 1.1(10(c(ii)), we have also

$$\min(A'_0 \setminus \sup(A'_0 \cap B') + 1) > \sup Y.$$

Which implies that $\alpha' > \sup Y$ and contradicts our assumption.

This completes the check of 3.1(2(k)).

Let us turn to 3.1(2(h)). Suppose that Y is a model in $\mathrm{dom}(a_n)$ and α' is an image of α added by the isomorphism between A_0 and some $A \in \mathrm{dom}(a_n)$. Assume that $\alpha' \in Y$. We would like to show that the walk from Y to α' is already in $\mathrm{dom}(a_n)$. We claim that the walk must terminate with A. Suppose otherwise. Thus let Z be the first model of the walk which does not contain A. Compare Z and A. By 1.1(9,10) and 3.1(2(f)), there is $\mu \in A \cap \mathrm{dom}(a_n)$ such that

$$Z \cap A = A \cap \mu.$$

Then $\alpha' < \mu$, but recall that each $\beta \in A_0 \cap (\cup\{\mathrm{dom}(a_n) | n < \omega\})$ (if any) is less than α. So, by isomorphism between A_0 and A, each $\beta \in A \cap (\cup\{\mathrm{dom}(a_n) | n < \omega\})$ (if any) is less than α'. In particular, $\mu < \alpha'$. Contradiction.

Let us check now 3.1(2(m). Thus let $\langle \alpha_i | i < j \rangle$ be a strictly increasing sequence of isomorphic images of α. For each $i < j$ there is a model $Y_i \in \mathrm{dom}(a_n)$ isomorphic to A_0 such that $\alpha_i = \pi_{A_0 Y_i}(\alpha) = \alpha_i$. Note if $i, k < j$ are different then $\alpha_k \notin Y_i$. Just, by 1.1(8) the isomorphisms between models are identity on common parts of the models. Now, we pick for each $i < j$ the least ordinal $\tau_i \in Y_{i+1} \backslash Y_i$. There is such, since $\alpha_{i+1} \in Y_{i+1} \backslash Y_i$, $\alpha_{i+1} > \alpha_i \in Y_i$ and so,by 1.1(10(c)) we must have then $\min(Y_{i+1} \backslash \sup(Y_{i+1} \cap Y_i) + 1) > \sup Y_i$. Also, we have $\alpha_i < \sup Y_i < \tau_i \leq \alpha_{i+1}$. By 3.1(2(f)), $\tau_i \in \mathrm{dom}(a_n)$ for each $i < j$. Hence,

$$\bigcup_{i<j} \alpha_i = \bigcup_{i<j} \tau_i \in \mathrm{dom}(a_n).$$

The rest of the conditions hold trivialy.

$\square$ of the claim.

Subcase B2 $\kappa^{+3} \cap A_0 \cap (\cup\{\mathrm{dom}(a_n) | n < \omega\}) \backslash \alpha + 1$ is not empty.

Let $\delta = \min(\kappa^{+3} \cap A_0 \cap (\cup\{\mathrm{dom}(a_n) | n < \omega\}) \backslash \alpha + 1$. Pick n^* large enough so that for each $m \geq n^*$ we have $\delta, A_0 \in \mathrm{dom}(a_m)$ and $a_m(\delta) \prec H(\chi^{+k})$ and $a_n(A_0) \prec H(\chi^{+\ell})$ for $k, \ell \gg 2$. Fix $n \geq n^*$. Assume for simplicity that $a_n(\delta) \prec a_n(A_0) \prec H(\chi^{+k})$. Otherwise we just cut one of $a_n(\delta), a_n(A_0)$, i.e.

we choose k to be the minimal so that $a_n(\delta) \subseteq H(\chi^{+k})$ or $a_n(A_0) \subseteq H(\chi^{+k})$ and intersect the one that not contained with $H(\chi^{+k})$.

Now we proceed as in Case B1 with $a_n(\delta)$ replacing $a_n(A_0)$. Thus pick some $M \in a_n(\delta) \cap a_n(A_0)$ such that

(1) $|M| = \kappa_n^{+n+2}$

(2) $M \prec H(\chi^{+k-1})$

(3) $M \supseteq \kappa_n^{+n+2}$

(4) $\mathrm{cof}(M \cap \kappa_n^{+n+3}) = \kappa_n^{+n+2}$

(5) for each $N \in \mathrm{rng}(a_n) \cap a_n(\delta)$ we have $N \cap H(\chi^{+k-1}) \in M$.

In particular, $M \cap \kappa_n^{+n+3}$ (the main part of M) is above each $N \cap \kappa_n^{+n+3}$, for every $N \in \mathrm{rng}(a_n) \cap a_n(\delta)$.

Define the image of α to be M. Move this setting to all the elements in $\mathrm{dom}(a_n)$ isomorphic to A_0 (if any). Denote the result by b_n.

Claim B2.1 b_n satisfies all the conditions of 3.1(2) but (l). Moreover, if a_n satisfies all the conditions of 3.1(2) then also b_n satisfies all the conditions of 3.1(2).

Proof. Let us check first 3.1(2(k)). Let Y be a model in $\mathrm{dom}(a_n)$ and α' be an image of α under isomorphism which was added to $\mathrm{dom}(a_n)$. We need to deal with the case when $\alpha' < \sup Y$ and show that $\min Y \backslash \alpha'$ is in $\mathrm{dom}(a_n)$. Thus let A_0' be a model in $\mathrm{dom}(a_n)$ isomorphic to A_0 such that $\alpha' = \pi_{A_0 A_0'}(\alpha)$. Denote $\pi_{A_0 A_0'}(\delta)$ by δ'. Compare Y with A_0'.

Case B2.1.1 $otp(Y) = otp(A_0')$.

We split into two subcases according to 1.1(10(c))(i) or (ii).

Subcase B2.1.1.1 $\min(A_0' \backslash \sup(A_0' \cap Y) + 1) > \sup Y$.

Then $\beta' = \min(A_0' \backslash \sup(A_0' \cap Y) + 1)$ is in $\mathrm{dom}(a_n)$ by 3.1(2(f)). Recall that α was above all the ordinals of $\delta \cap \mathrm{dom}(a_n)$, hence α' will be such in

$\delta' \cap \mathrm{dom}(a_n)$, by 3.1(2(e)). In particular, if $\beta' < \delta'$ then $\alpha' > \sup Y$, which contradicts our assumption on α' and Y. Hence, $\beta' \geq \delta'$. By 1.1(9,10), we have $A'_0 \cap Y = A'_0 \cap \beta'$. So, $\alpha' \in Y$ and we are done.

Subcase B2.1.1.2 $\min(Y \backslash \sup(A'_0 \cap Y) + 1) > \sup A'_0$.

Again, $\beta' = \min(A'_0 \backslash \sup(A'_0 \cap Y) + 1)$ is in $\mathrm{dom}(a_n)$ by 3.1(2(f)). By 1.1(9,10), we have $A'_0 \cap Y = A'_0 \cap \beta'$. Hence, if $\delta' \leq \beta'$ then $\alpha' \in Y$ and we are done. Suppose that $\delta' > \beta'$. Then also $\alpha' > \beta'$. It follows that $\alpha' < \min Y \backslash \alpha' = \min(Y \backslash \sup(A'_0 \cap Y) + 1)$. But $\min(Y \backslash \sup(A'_0 \cap Y) + 1) \in \mathrm{dom}(a_n)$, by 3.1(2(f)). So we are done.

Case B2.1.2 $otp(Y) < otp(A'_0)$.

Then B' as in 1.1(10) must exists. But this is impossible since $otp_{\kappa^+} A'_0 = otp_{\kappa^+} A_0 = 1$.

Case B2.1.3 $otp(Y) > otp(A'_0)$.

Then we have a set B' as in 1.1(10) for Y and A'_0 inside $\mathrm{dom}(a_n)$. Again we split into two cases according to (i) and (ii) of 1.1(10(c)).

Subcase B2.1.3.1 $\min(B' \backslash \sup(A'_0 \cap B') + 1) > \sup A'_0$.

As before, α' should be above $\beta' = \min(A'_0 \backslash \sup(A'_0 \cap Y) + 1)$ unless it is already in Y. By 1.1(10(d)), Y has no elements inside the interval

$$(\min(A'_0 \backslash \sup(A'_0 \cap B') + 1), \sup A'_0).$$

Let Z be as in Subclaim B1.1.3.1.1. Compare now Y and Z. There is $\xi \in Y \cap \mathrm{dom}(a_n)$ such that $Y \cap Z = Y \cap \xi$. Actually, $\xi = \min(Y \backslash \sup(Y \cap Z))$, by 1.1(9). Remember that $\alpha' \in Z$ and $\alpha' < \sup Y$. Then, by 1.1(10), $\xi = \min(Y \backslash \sup(Y \cap Z)) > \sup Z > \alpha'$. But now clearly, $\xi = \min(Y \backslash \alpha')$ and we are done.

Subcase B2.1.3.2 $\min(A'_0 \backslash \sup(A'_0 \cap B') + 1) > \sup B'$.

Then, by 1.1(10(c(ii)), we have also

$$\min(A'_0 \backslash \sup(A'_0 \cap B') + 1) > \sup Y.$$

We assumed that $\alpha' < \sup Y$, so $\alpha' \in A_0' \cap B'$. In particular, then $\alpha' \in Y$ and we are done.

This completes the checking of 3.1(2(k)).

Let us turn to 3.1(2(h)). Suppose that Y is a model in $\mathrm{dom}(a_n)$ and α' is an image of α added by the isomorphism between A_0 and some $A \in \mathrm{dom}(a_n)$. Assume that $\alpha' \in Y$. We would like to show that the walk from Y to α' is already in $\mathrm{dom}(a_n)$.

If the walk to α' must terminate with A then we are done. Suppose otherwise. Thus let Z be the first model of the walk which does not contain A. Compare Z and A. By 1.1(9,10) and 3.1(2(f)), there is $\mu \in A \cap \mathrm{dom}(a_n)$ such that

$$Z \cap A = A \cap \mu.$$

Also there is $S \in Z \cup \{Z\} \cap \mathrm{dom}(a_n)$ such that $otpS = otpA$ and $Z \cap A = S \cap A$ (remember that A is isomorphic to A_0 which is minimal). If S is the final model of the walk from Y to α' then we are done. Suppose otherwise. Let $T \neq S$ be such model.

Note that then necessary $otpT = otpA$, since A is minimal the only other possibility is $otpT > otpA$. But if this happens then there will be $T' \in T$ isomorphic to A and with α' inside by 1.1(10). Which is impossible, since then T' must be one of the immediate predecessors of T or a member of them by 1.1(11). So it is possible to continue the walk contradicting to the choice of T as the final model.

We claim that $T \in \mathrm{dom}(a_n)$. Let us argue as follows. Pick Y_0 to be the last member of the common part of the walks from Y to S and to T. Then it should be a successor model. Let Y_{00}, Y_{01} be its immediate predecessors with $Y_{00} \in C^{\kappa^+}(Y_0)$. Then Y_{01} should include S, since otherwise the walk to a common point α' of both models S, T must go into direction of S, which is not the case. Now, Y_0 must be in $\mathrm{dom}(a_n)$. It follows from the definition of the walk and 3.1(2(h)) applied to $Y, S \in \mathrm{dom}(a_n)$. But then both Y_{00}, Y_{01} are in $\mathrm{dom}(a_n)$ by 3.1(2(h)) and (l). Hence also $S_0 = \pi_{Y_{00}, Y_{01}}[S]$

is in $\mathrm{dom}(a_n)$, by 3.1(2(e)). We have still $\alpha' \in S_0$. If $S_0 = T$, then we are done. Otherwise, pick Y_1 to be the last member of the common part of the walks from Y to S_0 and to T. It should be a successor model below Y_{00}. Let Y_{10}, Y_{11} be its immediate predecessors with $Y_{10} \in C^{\kappa^+}(Y_1)$. Then Y_{11} should include S_0, since otherwise the walk to a common point α' of both models S_0, T must go into direction of S_0, which is not the case. Now, Y_1 must be in $\mathrm{dom}(a_n)$. It follows from the definition of the walk and 3.1(2(h)) applied to $Y, S_0 \in \mathrm{dom}(a_n)$. But then both Y_{10}, Y_{11} are in $\mathrm{dom}(a_n)$ by 3.1(2(h)) and (l). Hence also $S_1 = \pi_{Y_{10},Y_{11}}[S_0]$ is in $\mathrm{dom}(a_n)$, by 3.1(2(e)). We have still $\alpha' \in S_1$. If $S_1 = T$, then we are done. Otherwise, pick Y_2 to be the last member of the common part of the walks from Y to S_1 and to T. It should be a successor model below Y_{10}. Continue as above and define $S_2 \in \mathrm{dom}(a_n)$. If $S_2 \neq T$, then we can continue to go down and to define Y_3 etc. After finally many stages T will be reached.

This completes the checking of 3.1(2(h)).

Let us check now 3.1(2(m)). Thus let $\langle \alpha_i | i < j \rangle$ be a strictly increasing sequence of isomorphic images of α. For each $i < j$ there is a model $Y_i \in \mathrm{dom}(a_n)$ isomorphic to A_0 such that $\alpha_i = \pi_{A_0 Y_i}(\alpha) = \alpha_i$. Note if $i, k < j$ are different then $\alpha_k \notin Y_i$. Just, by 1.1(8) the isomorphisms between models are identity on common parts of the models. Now, we pick for each $i < j$ the least ordinal $\tau_i \in Y_{i+1} \backslash Y_i$. There is such, since $\alpha_{i+1} \in Y_{i+1} \backslash Y_i$, $\alpha_{i+1} > \alpha_i \in Y_i$ and so,by 1.1(10(c)) we must have then $\min(Y_{i+1} \backslash \sup(Y_{i+1} \cap Y_i) + 1) > \sup Y_i$. Also, we have $\alpha_i < \sup Y_i < \tau_i \leq \alpha_{i+1}$. By 3.1(2(f)), $\tau_i \in \mathrm{dom}(a_n)$ for each $i < j$. Hence,

$$\bigcup_{i<j} \alpha_i = \bigcup_{i<j} \tau_i \in \mathrm{dom}(a_n).$$

The rest of the conditions hold trivialy.

$\square$ of the claim.

Suppose now that $i > 0$ and for each $j < i$ it is possible to add elements of A_j. Let us show that it is possible to add elements of A_i. If i is limit, then this is clear since then $A_i = \cup\{A_j | j < i\}$. So assume that i is a successor ordinal and let $X \in A_i \setminus \cup\{A_j | j < i\}$. Suppose first that X is not an ordinal. Note that by 3.1(2(g)), in order to add an ordinal we need anyway to add models first.

We would like to run now a new induction on a walk length from A to X. Let us give a precise definition.

Suppose that $K, L \in A^{1\kappa^+}$ and $L \in K$, where as usual $A^{1\kappa^+}$ is taken from $G(\mathcal{P}')$. Set $wl(K, L) = 0$, if $L \in C^{\kappa^+}(K)$. Let $wl(K, L) = 1$, if there is $M \in C^{\kappa^+}(K)$ such that L is the immediate predecessor of M which is not in $C^{\kappa^+}(M)$. In general, set $wl(K, L) = 2n + 2$, if there is M such that $wl(K, M) = 2n + 1$ and $L \in C^{\kappa^+}(M)$; set $wl(K, L) = 2n + 1$, if there is M such that $wl(K, M) = 2n$ and L is the immediate predecessor of M which is not in $C^{\kappa^+}(M)$.

So the induction will be now on wl - the walk length from models in the domain to one that we like to add. Then the zero stage and all even stages are just as Case A. Stage one and all odd stages basically deal with the situation of adding X to A_i ones X is the immediate predecessor of A_i which is not in C^{κ^+}.

So let us concentrate on this case. Then A_i must have two immediate predecessors A_{i-1} and X. Again by Case A, we can assume that also $A_{i-1} \in \mathrm{dom}(a_n)$. Note that it implies, in particular, that a_n fails to satisfy 3.1(2(l)). The thing will fixed below by adding X.

Let $A_{i^{**}}$ be as in Case A but with A_{i-1} replacing A_{i^*} there.

By 1.1(4(d)), $otpA_{i-1} = otpX$. Also there are $\alpha_1 \in A_{i-1} \cap A^{1\kappa^{++}}$, $\alpha_2 \in X \cap A^{1\kappa^{++}}$ such that $A_{i-1} \cap \alpha_1 = A_{i-1} \cap X = X \cap \alpha_2$, for some $A^{1\kappa^{++}}$ in $G(\mathcal{P}')$. We first add α_1 to $\mathrm{dom}(a_n)$. Note that $\alpha_1 \in A_{i-1} \cap A^{1\kappa^{++}}$. So, the induction can be used to add it to the domain. Assume that already $\alpha_1 \in \mathrm{dom}(a_n)$ and $a_n(\alpha_1) = M$ so that $M \in a_n(A_{i-1})$ and it is an elementary submodel of cardinality κ_n^{+n+2} of $H(\chi^{+k})$ for some $k \gg 2$. Find $X^* \in M$ such that $|X^*| = \kappa_n^{+n+1}$, $X^* \prec H(\chi^{+k-1})$ and X^* realizes over $a_n(A_{i-1}) \cap M$ the same $k-1$-type as $a_n(A_{i-1})$ does. It exists by elementarity, since we replace k by

$k-1$.

If $\sup X < \alpha_1$, then we take X^* to be the image of X. Extend the condition by coping from $a_n(A_{i-1})$ to X^* all the elements of $a_n(A_{i-1})$ and $M \cap H(\chi^{k-1})$. If $\sup X > \alpha_1$, then $\beta_1 = \min(X \backslash A_{i-1} \cap X) > \sup A_{i-1}$. Work then inside $a_n(A_i)$ find a model M' above $\sup a_n(A_{i-1})$ of the same type as M and find X^{**} in it above $\sup a_n(A_{i-1})$ as well resembling X^* above. Add it to be the image of X.

Now copy everything from $A_{i-1} \cap \mathrm{dom}(a_n)$ to X and move this setting to all the elements of $\mathrm{dom}(a_n)$ isomorphic to A_i (if any). Denote the result by b_n.

The above is the heart of the argument. The basic idea goes back to [1]

Claim B3 b_n satisfies all the conditions of 3.1(2) but (l). Moreover, if a_n satisfies all the conditions of 3.1(2) but (l) only for A_i and the models isomorphic to it, then also b_n satisfies all the conditions of 3.1(2).

Proof. Let us check first 3.1(2(f)).

Suppose first that X', X'' are images of X and its elements (obtained by moving from $A_{i-1} \cap \mathrm{dom}(a_n)$ to X) which were added to $\mathrm{dom}(a_n)$.

We need to show witnessing the intersection conditions for $X' \cap X''$ are in the domain. Split the proof into few cases.

Case B3.1 $X' = X$ and $X'' \in A_i$.

Assume that $X'' \neq X'$, otherwise every thing is just trivial. If $X'' \in X$ then we are done again. So we can assume that $X'' = A_{i-1}$ or $X'' \in A_{i-1} \cap \mathrm{dom}(a_n)$. If $X'' = A_{i-1}$, then α_1 (from the definition of X) as well as its image- $\pi_{A_{i-1}X}(\alpha_1) = \alpha_X$ are in the domain of q and we are done. Suppose that $X'' \in A_{i-1} \cap \mathrm{dom}(a_n)$.

Subcase B3.1.1 X'' is not an ordinal.

Then

$$X \cap X'' = X \cap A_{i-1} \cap X'' = A_{i-1} \cap \alpha_1 \cap X'' = \alpha' \cap X''.$$

Hence only α' which is already in $\mathrm{dom}(a_n)$, is needed for the intersection of X'' and X. The opposite way- let $Z = \pi_{A_{i-1}X}[X'']$. Then

$$X \cap X'' = X \cap A_{i-1} \cap X'' = Z \cap A_{i-1} = Z \cap \alpha_X.$$

Again, Z and α_X were added so we are done.

Subcase B3.1.2 X'' is an ordinal.

Let us denote X'' by ξ. Suppose that $\xi < \sup X \cap \kappa^{+3}$. If $\xi < \alpha_1$, then ξ is in the common part of A_{i-1} and X. Otherwise, we will have $\alpha_X = \min(X \backslash \xi)$, by 1.1(10).

Case B3.2 $X' \in X$ and $X'' \in A_i$.

We allow the possibility that one of them is an ordinal. In this case we care only about the intersection on the model side.

Let $Z = \pi_{XA_{i-1}}[X']$. Then Z is in the domain of a_n. Also,

$$X' \cap X'' = X' \cap Z \cap X'' = \alpha_1 \cap Z \cap X''.$$

This takes care of the intersection of X' and X'' from the side of X'. Let us deal with the opposite side, i.e. X'. Take $Y = \pi_{XA_{i-1}}[X'']$. Then

$$X' \cap X'' = X' \cap Y \cap X'' = \alpha_X \cap Y \cap X'.$$

But both Y and α_X were added. So we are done.

Case B3.3 $X' \in X \cup \{X\}$ and $X'' \notin A_i$.

Then there is some $A'' \in \mathrm{dom}(a_n)$ isomorphic to A_i with $X'' \in A''$ being the image of an element of $X \cup \{X\}$ that was added under the isomorphism $\pi_{A_iA''}$. Let $Z = \pi_{A''A_i}[X'']$ and $Z' = \pi_{A_iA''}[X']$.

We deal first with the intersection of X' with X'' on the side of X'. Thus

$$X' \cap X'' = X' \cap A_i \cap A'' \cap X'' = X' \cap A_i \cap \alpha_{A_iA''} \cap Z,$$

where $\alpha_{A_iA''} = \min\{\delta | \delta \in A_i \backslash A''\}$. Then we can use Case B3.1 or B3.2, since all the components of the last intersection are in A_i.

Let us turn to the opposite side. Again,

$$X' \cap X'' = X' \cap A_i \cap A'' \cap X'' = X'' \cap A'' \cap \alpha_{A''A_i} \cap Z',$$

where $\alpha_{A''A_i} = \min\{\delta | \delta \in A'' \backslash A_i\}$.

Now we can move all the members of the last equality to A_i using $\pi_{A''A_i}$ take care of the intersections using Case B3.1 or 3.2 and finally move back the result by $\pi_{A_iA''}$.

Case B3.4 $X' \notin A_i$ and $X'' \notin A_i$.

Then there are some $A', A'' \in \mathrm{dom}(a_n)$ isomorphic to A_i with $X' \in A'$, $X'' \in A''$ and both X', X'' being images of elements of $X \cup \{X\}$ that were added under the isomorphisms $\pi_{A_iA'}$ and $\pi_{A_iA''}$. Let $Z = \pi_{A'A''}[X'']$ and $Z' = \pi_{A_iA''}[X']$.

We deal with the intersection of X' with X'' on the side of X'. The opposite side is similar. Thus

$$X' \cap X'' = X' \cap A' \cap A'' \cap X'' = X' \cap A' \cap \alpha_{A'A''} \cap Z,$$

where $\alpha_{A'A''} = \min\{\delta | \delta \in A' \backslash A''\}$. Note that such ordinal exists by 3.1(f,k). Now the members of the last equality are all in A'. Move them to A_i using $\pi_{A''A_i}$ take care of the intersections using Case B3.1 or 3.2 and finally move back the result by $\pi_{A_iA''}$.

Suppose now that only X' is an image of X or of its element and X'' is old, i.e. in $\mathrm{dom}(a_n)$.

Then there is some $A' \in \mathrm{dom}(a_n)$ isomorphic to A_i with $X' \in A'$ being the image of an element of $X \cup \{X\}$ that was added under the isomorphism $\pi_{A_iA'}$. Split into two cases according to $otpX''$.

Case B3.5 $otpX'' \leq otpA'$ or X'' is an ordinal.

Then we need only to deal with the intersection of X' and X'' on the side of X'.

Let $\alpha_{A'X''} = \min\{\delta | \delta \in A' \backslash X''\}$. Note that such ordinal is in $\mathrm{dom}(a_n)$ by 3.1(f,k). Then

$$X' \cap X'' = X' \cap A' \cap X'' = X' \cap A' \cap \alpha_{A'X''}.$$

Now all the components of the last equality are in A', so we can deduce the conclusion as in Case B3.4.

Case B3.6 $otpX'' > otpA'$.

Find first $A'' \in X'' \cap \mathrm{dom}(a_n)$ of the order type $\mathrm{otp}\, A'$ such that $A' \cap X'' = A' \cap A''$. It exists by 3.1(f). Also let $\alpha_{A'A''} = \min\{\delta | \delta \in A' \backslash A''\}$ and $\alpha_{A''A'} = \min\{\delta | \delta \in A'' \backslash A'\}$.

Deal first with the intersection of X' and X'' on the side of X'.

$$X' \cap X'' = X' \cap A' \cap X'' = X' \cap A' \cap A'' = X' \cap A' \cap \alpha_{A'A''}.$$

Now all the components of the last equality are in A', so we can deduce the conclusion as in Case B3.3.

Now we deal with the intersection of X' and X'' on the side of X'.

$$X' \cap X'' = X' \cap A' \cap X'' = X' \cap A' \cap A'' = \pi_{A'A''}[X'] \cap A' \cap \alpha_{A''A'}.$$

Now all the components of the last equality are in A'', so we can deduce the conclusion as in Case B3.3.

This completes checking of 3.1(2(f)).

Let us turn to 3.1(2(h)).

Suppose that Y, U are appear in q and $U \in Y$. We claim that the walk from Y to U is also in q. We may assume that at least one of the elements Y, U is new (i.e. not in p). Split into few cases.

Case B3.7 $Y \notin \mathrm{dom}(a_n)$.

Then there is $A \in \mathrm{dom}(a_n)$ isomorphic to A_i such that $Y \in A$ is the isomorphic image of a model X or of its element obtained by moving from $A_{i-1} \cap \mathrm{dom}(a_n)$ to X. We have $U \in Y$, so $U \in A$.

Without loss of generality we can assume that A is A_i, otherwise just move Y, U to A_i via the isomorphism run the the argument inside A_i and then move the result back to A.

Now, we apply the isomorphism $\pi_{XA_{i-1}}$ to Y, U. Let Y^*, U^* be the images. Then they are in $A_{i-1} \cup \{A_{i-1}\} \cap \mathrm{dom}(a_n)$. Hence the walk from Y^* to U^* is in $\mathrm{dom}(a_n)$. Its image under $\pi_{A_{i-1}X}$ will be then the walk from Y to U and we are done.

Case B3.8 $Y \in \mathrm{dom}(a_n)$.

There is $A \in \mathrm{dom}(a_n)$ isomorphic to A_i such that $U \in A$ is the isomorphic image of a model X or of its element obtained by moving from $A_{i-1} \cap \mathrm{dom}(a_n)$ to X.

Subcase B3.8.1 U is an image of X

If the walk from Y to U terminates at A then we are done. Otherwise there must be a model Z on this walk inside $\mathrm{dom}(a_n)$ which does not contain A. We can take for example, $Z = Y$, if Y does not contain A or the point were the walk from Y to A differs from the one from Y to U, if $Y \supseteq A$. Compare Z and A. By 1.1(9,10) and 3.1(2(f)), there is $\mu \in A \cap \mathrm{dom}(a_n)$ such that

$$Z \cap A = A \cap \mu.$$

Also there is $S \in Z \cup \{Z\} \cap \mathrm{dom}(a_n)$ such that $otpS = otpA$ and $Z \cap A = S \cap A$ (remember that A is isomorphic to A_i which is a minimal including X and U is the image of X). If S is the final model of the walk from Y to U then we are done. Suppose otherwise. Let $T \neq S$ be such model.

Note that then necessary $otpT = otpA$, since A is a minimal including U the only other possibility is $otpT > otpA$. But if this happens then there will be $T' \in T$ isomorphic to A and with U inside by 1.1(10). Which is impossible, since then T' must be one of the immediate predecessors of T or a member of them by 1.1(11). So it is possible to continue the walk contradicting to the choice of T as the final model.

Now $T \in \mathrm{dom}(a_n)$. The rest of the argument repeats completely those of Claim B2.1.

Subcase B3.8.2 U is not an image of X

Suppose for simplicity that $A = A_i$. Then U will be in X and it will be obtained by moving from $A_{i-1} \cap \mathrm{dom}(a_n)$ to X.

If the walk from Y to U goes via X, then using the previous case this walk will be in q and we are done. Suppose that this does not happen.

If $otpX > otpY$, then by 3.1(2(f)) there is $X' \in A$ in q with $otpY = otpX'$ and $Y \cap X = Y \cap X'$. Clearly, $U \in X'$. Note that $Y \neq X'$ unless Y is in X. If $Y \in X$, then both $U, Y \in X$. Hence the argument of the previous case applies. Assume so, that $Y \neq X'$.

Set $K = X$, if $otpX \leq otpY$ and $K = X'$, if $otpX > otpY$.

Let Z be the first model of the walk from Y to U which does not contain K. Compare Z and K. By 1.1(9,10) and 3.1(2(f)), there is there is $S \in Z \cup \{Z\}$ in q such that $otpS = otpK$ and $Z \cap K = S \cap K$. If the walk from Y (or Z which is the same) to U goes through S, then use π_{KS} to copy to S the walk from K to U. The walk from K to U is in q, so the one copied (from S to U) must be in q as well by 3.1(2(e)). But the walk from Y to U is the combination of the walks from Y to S with the walk from S to U and both are in q. So we are done. Just note that the walk from Y to S is in q, since we have either

(a) $K = X$ and then $otpS = otpX$.

So S is an image of X and hence, B3.8.1 applies

or

(b) $K = X'$ and then $otpY = otpX' = otpS$.

In this case we must have $Y = S$ and so the walk is trivial.

Suppose now that the walk from Y (or Z which is the same) to U does not go through S.

Pick Y_0 to be the last member of the common part of the walks from Y to S and to U. Then it should be a successor model. Let Y_{00}, Y_{01} be its immediate predecessors with $Y_{00} \in C^{\kappa^+}(Y_0)$. Then Y_{01} should include S, since otherwise the walk to a common point U of both models S, Y_0 must go into direction of S, which is not the case. Now, Y_0 must be in q. It follows from the definition of the walk and 3.1(2(h)) applied to Y, S in q. But then both Y_{00}, Y_{01} are in q by 3.1(2(h)) and (l). Hence also $S_0 = \pi_{Y_{00},Y_{01}}[S]$ is in q, by 3.1(2(e)). We have still $U \in S_0$. If S_0 is on the walk from Y to U, then we are done exactly as above. Just as above use $\pi_{S_0 K}$ to copy the walk from S_0 to U to the one from K to U. Otherwise, pick Y_1 to be the last member of the common part of the walks from Y to S_0 and to U. It should be a successor model below Y_{00}. Let Y_{10}, Y_{11} be its immediate predecessors with $Y_{10} \in C^{\kappa^+}(Y_1)$. Then Y_{11} should include S_0, since otherwise the walk to a common point U of both models S_0, Y_{10} must go into direction of S_0, which is not the case. Now, Y_1 must be in q. It follows from the definition of the walk and 3.1(2(h)) applied to $Y, S_0 \in q$. But then both Y_{10}, Y_{11} are in q by 3.1(2(h)) and (l). Hence also $S_1 = \pi_{Y_{10},Y_{11}}[S_0]$ is in q, by 3.1(2(e)). We have still $U \in S_1$. If S_1 is on the walk from Y to U, then we are done. Otherwise, pick Y_2 to be the last member of the common part of the walks from Y to S_1 and to U. It should be a successor model below Y_{10}. Continue as above and define $S_2 \in q$. If $S_2 \neq T$, then we can continue to go down and to define Y_3 etc. After finally many stages a model S^* which is on the walk from Y to U will be reached.

This completes the checking of 3.1(2(h)).

The checking of 3.1(2(k)) repeats those of Claim B2.1. The rest of the conditions hold trivialy.

$\square$ of the claim.

It remains only to deal with the case when X is an ordinal. Recall that $i > 0$ is the least with $X \in A_i$. We can assume now that all the immediate predecessors of A_i (and there are at most two and at least one) are already in $\mathrm{dom}(a_n)$. Let us denote $X = \alpha$ and let A'_{i-1} denotes the immediate predecessor different from A_{i-1}, if such exists.

Split now into two cases.

Case B4.1 There is no $Z \in A_i \cap \mathrm{dom}(a_n)$ with $\alpha < \sup(\kappa^{+3} \cap Z)$.

We proceed as in Case A. We assume that $a_n(A_i) \prec H(\chi^{+k})$ for $k \gg 2$. Pick X^* to be an element of $a_n(A_i)$ such that

(1) $X^* \prec H(\chi^{+k-1})$

(2) $X^* \cap \kappa^{+n+3}$ is an ordinal of cofinality κ^{+n+2}

(3) for every $Z \in A_i \cap \mathrm{dom}(a_n)$ we have $a_n(Z) \cap H(\chi^{+k-1}) \in X^*$

We take X^* to be the image of $X = \alpha$. Move this setting now to all the elements of $\mathrm{dom}(a_n)$ isomorphic to A_i (if any). The arguments of Cases A,B apply in order to show the result is a condition.

Case B4.1 There is $Z \in A_i \cap \mathrm{dom}(a_n)$ with $\alpha < \sup(\kappa^{+3} \cap Z)$.

Then we pick $Z \in A_i \cap \mathrm{dom}(a_n)$ with $\min(\kappa^{+3} \cap Z \backslash \alpha)$ as small as possible. Let β be $\min(\kappa^{+3} \cap Z \backslash \alpha)$, for such Z. Using induction we first add β to $\mathrm{dom}(a_n)$. Assume without loss of generality that both $a_n(A_i), a_n(\beta) \prec H(\chi^{+k})$ for $k \gg 2$. Pick X^* to be an element of $a_n(A_i) \cap a_n(\beta)$ such that

(1) $X^* \prec H(\chi^{+k-1})$

(2) $X^* \cap \kappa^{+n+3}$ is an ordinal of cofinality κ^{+n+2}

(3) for every $Z' \in A_i \cap \mathrm{dom}(a_n)$ with $\sup(Z') \cap \kappa^{+3} < \beta$ we have $a_n(Z') \cap H(\chi^{+k-1}) \in X^*$

We take X^* to be the image of $X = \alpha$. Move this setting now to all the elements of $\mathrm{dom}(a_n)$ isomorphic to A_i (if any). The arguments of Cases A,B apply in order to show the result is a condition. The only new possibility that was not considered in the checking of 3.1(2(k)) above is the following: Y a model in $\mathrm{dom}(a_n)$ with $\sup Y > \alpha$, $\alpha \notin Y$ and $otp(Y) < otp(A_i)$. Then use Subclaim B1.1.3.1 to find $W \in \mathrm{dom}(a_n)$ such that $W \supseteq Y$ and

$otp(W) = otp(A_i)$. Suppose first that $\alpha \notin W$. Now compare W and A_i. There is $\xi \in W \cap \mathrm{dom}(a_n)$ such that $A_i \cap W = W \cap \xi$. By 1.1(10(c)), then $\xi = \min(W \backslash \sup(W \cap A_i)) > \sup(A_i) > \alpha$. Hence $\min(Y \backslash \alpha) = \min(Y \backslash \xi)$. But both Y and ξ are in $\mathrm{dom}(a_n)$. So we are done.

Suppose now that $\alpha \in W$. Then $\alpha = \pi_{WA_i}[\alpha]$. Set $Y' = \pi_{WA_i}[Y]$. Then $\sup Y' > \alpha$ since π_{WA_i} is order preserving. Induction on wl can be applied now to Y'. So, $\tau' = \min(Y' \backslash \alpha) \in \mathrm{dom}(a_n)$. Set $\tau = \pi_{A_iW}(\tau')$. Then $\tau \in \mathrm{dom}(a_n)$ and $\tau = \min(Y \backslash \alpha)$ by the elementarity of π_{WA_i}.

This completes the proof of the lemma.

$\square$ of 3.3

Remark 3.4 The proof of 3.3 provides a bit more information. Thus, if X is a model that we like to add to $dom(a_n)$, then

(1) if $X \in C^{\kappa^+}(A)$, for some $A \in \mathrm{dom}(a_n)$, and either

   (a) X is a limit model with $cof(otp_{\kappa^+}(X) - 1) > \kappa$

   or

   (b) X is a successor model, the immediate successor of X in $C^{\kappa^+}(A)$ is not in $\mathrm{dom}(a_n)$ and the same for the immediate predecessor (if it exists at all)

   or

   (c) X is a successor model, the immediate successor of X in $C^{\kappa^+}(A)$ is in $\mathrm{dom}(a_n)$, but X is his unique immediate predecessor, the immediate predecessor (if it exists) is not in $\mathrm{dom}(a_n)$ or it is, but then it is the unique immediate predecessor of X,

then X can be added without adding other additional models or ordinals except the images of X under isomorphisms. I.e. if $A' \in \mathrm{dom}(a_n)$ is the least model including X and $B \in \mathrm{dom}(a_n)$ has the same otp as those of A', then $\pi_{A'B}[X]$ is added also. Note that necessary $A' \in C^{\kappa^+}(A)$, otherwise just use 1.1(10),(4(a)).

(2) if X is a successor model, $X \in C^{\prime\kappa^+}(A)$, for some $A \in \mathrm{dom}(a_n)$, and (b), (c) of the previous case fail, then adding X requires adding of an other immediate predecessor of the immediate successor of X in $C^{\prime\kappa^+}(A)$, by 3.1(2(l)) or of an other its immediate predecessor. Which in turn requires further additions which will be specified in (3) below.

(3) If $X \notin C^{\prime\kappa^+}(A)$, for a smallest $A \in \mathrm{dom}(a_n)$ including X (note that it is possible to have many such A's but all of them will have the same *otp* and will agree about X not being on the sequence $C^{\prime\kappa^+}$, just due to isomorphism. Also its possible in general to have them immediate successors of X. If one likes to fix one, then the one of the least $rank$ can be used), then in order to add X we need to add finitely many models of the walk from A to X. Adding a model B_1 which is an immediate predecessor of a model B not in $C^{\prime\kappa^+}(B)$ for B that was in $\mathrm{dom}(a_n)$ or was added during the walk requires adding the immediate predecessor B_0 of B inside $C^{\prime\kappa^+}(B)$, as well as an ordinal $\alpha_0 \in B_0$ such that
$$B_1 \cap B_0 = B_0 \cap \alpha_0.$$
Adding α_0 may require further adding of the immediate predecessors of B_0 or of some model in $C^{\prime\kappa^+}(B_0)$. This in turn requires adding of new ordinal and so on. The rank (or wl) of the models and the ordinals involved is decreasing. Hence after finitely many additions the process will terminate. Again after each addition we need to take all the isomorphic images.

The ordering $\leq^*$ on $\mathcal{P}$ and $\leq_n$ on Q_{n0} is not closed in the present situation. Thus it is possible to find an increasing sequence of $\aleph_0$ conditions $\langle\langle a_{ni}, A_{ni}, f_{ni}\rangle \mid i < \omega\rangle$ in Q_{n0} with no upperbound. The reason is that the union of maximal models of these conditions, i.e. $\bigcup_{i<\omega} \max(\mathrm{dom}\, a_{ni})$ need not be in A_{11} for any A_{11} in $G(\mathcal{P}')$. The next lemma shows that still $\leq_n$ and so also $\leq^*$ share a kind of strategic closure.

Lemma 3.5 *Let $n < \omega$. Then $\langle Q_{n0}, \leq_n \rangle$ does not add new sequences of ordinals of the length $< \kappa_n$, i.e. it is (κ_n, ∞) – distributive.*

Proof. Let $\delta < \kappa_n$ and $\underset{\sim}{f}$ be a Q_{n0}-name of a function from δ to ordinals. Using genericity of $G(\mathcal{P}')$ (or stationarity of the set $\{A^{0\kappa^+} | A^{0\kappa^+}$ appears in an element of $G(\mathcal{P}')\}$) it is not hard to find elementary submodel M of some $H(\nu)$ for ν big enough so that

(a) Q_{n0}, $\underset{\sim}{f}$, $\mathcal{P}' \in M$

(b) $|M| = \kappa^+$

(c) $M^* = M \cap H(\kappa^{+3})$ appears in $A^{1\kappa^{++}}$ of a condition of $G(\mathcal{P}')$

(d) $cf(M^* \cap \kappa^{++}) = \delta$.

(e) ${}^{\delta>}M \subseteq M$.

Note that for such M, $M^* = M \cap H(\kappa^{+3})$ must be a limit model, since by 1.1(12) successor models are closed under κ sequences, but M^* is not by (d) above.

We have $C^{\kappa^+}(M^*) \backslash \{M^*\} \subseteq M^*$ and, by elementarity of M, $C^{\kappa^+}(B) \in M$ for each $B \in C^{\kappa^+}(M^*) \backslash \{M^*\}$. Also the cofinality of $C^{\kappa^+}(M^*) \backslash \{M^*\}$ under the inclusion must be δ, since it is an $\in$-increasing continuous sequence of elements of M^* with limit M^* and by (d) above $cf(M^* \cap \kappa^{++}) = \delta$. Fix an increasing continuous sequence $\langle A_i \mid i < \delta \rangle$ of elements of $C^{\kappa^+}(M^*) \backslash \{M^*\}$ such that $\bigcup_{i<\delta} A_i = M^*$, A_0 is a successor model and for each limit model A_i in the sequence A_{i+1} is its immediate successor in $C^{\kappa^+}(M^*)$. By (e), each initial segment of it will be in M. Now we decide inside M one by one values of $\underset{\sim}{f}$ and put models from $\langle A_i \mid i < \delta \rangle$ to be maximal models of conditions used. This way we insure that unions of such conditions is a condition.

We define by induction an increasing sequence of conditions

$$\langle \langle a(i), A(i), f(i) \rangle | i < \delta \rangle.$$

and an increasing continuous subsequence

$$\langle A_{k_i} | i < \delta \rangle \text{ of } \langle A_i | i < \delta \rangle$$

such that for each $i < \delta$

(1) $\langle a(i), A(i), f(i) \rangle \in M$,

(2) $\langle a(i+1), A(i+1), f(i+1) \rangle$ decides $\underset{\sim}{f}(i)$

(3) $A_{k_i}, A_{k_{i+1}} \in \mathrm{dom}(a(i))$ and $A_{k_{i+1}}$ is the maximal model of $\mathrm{dom}(a(i))$

There is no problem with $A(i)$'s and $f(i)$'s in this construction. Thus we have enough completeness to take intersections of $A(i)$'s and unions of $f(i)$'s. The only problematic part is $a(i)$. So let us concentrate only on building of $a(i)$'s.

i=0

Then let us pick some $Y_0 \prec Y_1 \prec H(\chi^\omega)$ of cardinality κ_n^{+n+2}, closed under κ_n^{+n+1} - sequences of its elements and $Y_0 \in Y_1$. Set $a(0) = \langle \langle A_0, Y_0 \rangle, \langle A_1, Y_1 \rangle \rangle$.

i+1

Then we first extend $\langle a(i), A(i), f(i) \rangle$ to a condition $\langle a(i)', A(i)', f(i)' \rangle$ deciding $\underset{\sim}{f}(i)$. Then preform *swt* (see 1.3) to turn $\langle a(i)', A(i)', f(i)' \rangle$ into an equivalent condition $\langle a(i)'', A(i)', f(i)' \rangle$ with $A_{k_i} \in C^{\kappa^+}(\max \mathrm{dom}(a(i)'')$. Pick a successor model A_j (from the cofinal sequence $\langle A_i \mid i < \delta \rangle$) including $\max \mathrm{dom}(a(i)'')$. Set $k_{i+1} = j$ and add it to $\mathrm{dom}(a(i)''$, using **??** and *swt* inside A_j if necessary. Finally we add A_{j+1}, using **??**.

i is a limit ordinal

Then we need to turn $a = \bigcup_{j<i} a(j)$ into condition. For this we will need to add to $\mathrm{dom}(a)$ models and ordinals which are limits of elements of $\mathrm{dom}(a)$. First we extend a by adding to it $\langle A_{k_i}, \bigcup_{j<i} a(A_{k_j}) \rangle$, where $k_i = \cup_{j<i} k_j$. Then for each non decreasing sequence $\langle \alpha_j | j < i \rangle$ of ordinals in $\mathrm{dom}(a)$ we add the pair $\langle \cup_{j<i} \alpha_j, \cup_{j<i}(a(\alpha_j) \cap H(\chi^{+\ell})) \rangle$, if it is not already in the $\mathrm{dom}(a)$, where

$\ell \leq \omega$ the maximal such that for unboundedly many j's in i $a(\alpha_j) \prec H(\chi^{+\ell})$, if the maximum exists or $\ell >> n$ otherwise. Finally, for each model $B \in \mathrm{dom}(a)$ if there is a nondecreasing sequence $\langle B_j | j < i \rangle$ of elements of $C^{\kappa^+}(B)$ in $\mathrm{dom}(a)$ and B is the least possible (under inclusion or with least sup) including the sequence, then we add the pair $\langle \cup_{j<i} B_j, \cup_{j<i}(a(B_j) \cap H(\chi^{+\ell})) \rangle$, if it is not already in the $\mathrm{dom}(a)$, where $\ell \leq \omega$ is the minimum between the least k such that $a(B) \subseteq H(\chi^{+k})$ and the maximal ℓ' such that for unboundedly many j's in i $a(B_j) \prec H(\chi^{+\ell'})$, if the maximum exists

or

it is k, if the maximum does not exists and $k < \omega$,

or

$\ell >> n$, if the maximum does not exists and $k = \omega$.

Denote the result by b.

Claim 3.5.1 b satisfies 3.1(2).

Proof. Let start with 3.1(2(e)). Suppose that $A, B \in dom(b)$ are different and $otp(A) = otp(B)$. Pick $A', B' \in \mathrm{dom}(a)$ to be the smallest possible (under inclusion and *rank*, but actually any choice of a smallest under inclusion alone will do) including A, B respectively.

Subclaim 3.5.1.1 $otp(A') = otp(B')$.

Proof. Suppose otherwise. Let, for example $otp(A') < otp(B')$. Then by 3.1(2(f)), there will be $B'' \in B \cap \mathrm{dom}(a)$ with $otp(A') = otp(B'')$. Note that i is limit so for some $j < i$ big enough both A', B' are in $\mathrm{dom}(a_j)$ which is a part of a condition and so satisfies 3.1. Now, $\pi_{A'B''}[A]$ will be in $\mathrm{dom}(b)$ as well, just all the models from the increasing sequence converging to A are moved by $\pi_{A'B''}$ to form such sequence for $\pi_{A'B''}[A]$. We can assume that $A' \subset B'$ just replacing A by $\pi_{A'B''}[A]$ and A' by B'' if necessary . Considering the walk from B' to A' it is not hard to see that either $A' \in C^{\kappa^+}(B')$ or there is $A'' \in C^{\kappa^+}(B) \cap \mathrm{dom}(a)$ of the same otp as A'. Suppose for simplicity that $A' \in C^{\kappa^+}(B')$. Otherwise just use $\pi_{A'A''}$ to move to A''. But $A \in C^{\kappa^+}(A')$.

Hence $A \in C^{\prime\kappa^+}(B')$. Also $B \in C^{\prime\kappa^+}(B')$, by the choice of B'. $otp(A) = otp(B)$ implies then $A = B$. This contradicts the minimality of B', since we have now

$$B = A \subset A' \subset B'.$$

$\square$ of the subclaim.

Once we have $otp(A') = otp(B')$, the isomorphism $\pi_{A'B'}$ between A', B' and those between $a(A') \cap H(\chi^{+k}), a(B') \cap H(\chi^{+k})$, k is the minimal so that $a(A') \subseteq H(\chi^{+k}), a(B') \subseteq H(\chi^{+k})$, can be used to induce isomorphisms between A and B, $b(A) \cap H(\chi^{+m})$ and $b(B) \cap H(\chi^{+m})$, where m is the minimal so that $b(A) \subseteq H(\chi^{+m}), b(B) \subseteq H(\chi^{+m})$. Note that by the definition of b we must have $m \leq k$. Such isomorphisms will respect those of the members of the sequences converging to A and B, since isomorphisms between members of the sequences are induced in the same way.

Let turn now to 3.1(2(f)). Suppose that $A, B \in dom(b)$, $A \not\subseteq B$ and $B \not\subseteq A$. We may assume that at least one of them is new. Again pick the smallest models A', B' with $A \in A', B \in B'$. Now, as in 3.3, Claim B3, we can use induction on wl - the walk length. Thus basically we need only to consider a situation when $A' \in C^{\prime\kappa^+}(B')$. But then A, B just extend one another and we are done.

Let us check that 3.1(2(g)) holds. Suppose that $\alpha, A \in \mathrm{dom}(b)$ and $\alpha \in A$. If A is a new model then α belongs to one of the members of the sequence of models converging to A. So we can assume that A is an old one, i.e. in $\mathrm{dom}(a)$. Let $\langle \alpha_j | j < i \rangle$ be a nondecreasing sequence from $\mathrm{dom}(a)$ converging to α. By 3.1(2(k)), $\gamma_j = \min(A \backslash \alpha_j) \in \mathrm{dom}(a)$. Then $\langle \gamma_j | j < i \rangle$ will be also a converging to α sequence. Apply 3.1(2(g)) to A and its members. Then either there will be the model $B \in C^{\prime\kappa^+}(A) \cap \mathrm{dom}(a)$ which satisfy 3.1(2(g)) for a final segment of γ_j's, or we will have an increasing sequence of such models. In the former case B must be a successor model and so closed under κ sequences with $a(B)$ closed under $< -\kappa_n$ sequences. Hence α will be in B,

$b(\alpha) \in a(B)$ and we are done. In the later case the union of the sequence of models will be in $\mathrm{dom}(b)$ and it will be as desired.

Let us check 3.1(2(f)).

Suppose that $A, B \in \mathrm{dom}(b)$ and $B \in A$. We need to show that the walk from A to B is in $\mathrm{dom}(b)$. If A is new, then it is limit. So B will belong to a member of the sequence converging to A consisting of elements of $C^{\kappa^+}(A) \cap \mathrm{dom}(a)$. So we can assume without loss of generality that already $A \in \mathrm{dom}(a)$. Pick $B' \in \mathrm{dom}(a)$ to be the smallest model with $B \in B'$. Now, as in 3.3, Claim B3, we can use induction on wl - the walk length. Thus basically we need only to consider a situation when $B' \in C^{\kappa^+}(A)$. But then everything is trivial.

Turn now to 3.1(2(k)) Thus let $A, \alpha \in \mathrm{dom}(a)$ and $\sup A > \alpha$. If α is an old then sup of one of the models converging to A will be above α. 3.1(2(k)) applies and we are done. Suppose so that α is a new. If also A is a new, then sup of one of the models converging to A will be above α. We can replace then A by one of such models. So without loss of generality, we can assume that $A \in \mathrm{dom}(a)$. Let $\langle \alpha_j | j < i \rangle$ be a nondecreasing sequence from $\mathrm{dom}(a)$ converging to α. By 3.1(2(k)), $\gamma_j = \min(A \backslash \alpha_j) \in \mathrm{dom}(a)$. If $\langle \gamma_j | j < i \rangle$ is eventually constant, then the constant value will be as desired. Suppose otherwise. Then $\langle \gamma_j | j < i \rangle$ will be also a converging to α sequence. Apply 3.1(2(g)) to A and its members. It follow that $\alpha \in A$. So $\min(A \backslash \alpha) = \alpha \in \mathrm{dom}(b)$ and we are done.

The condition 3.1(2(l)) is satisfied since all new models that were added are limit models. By 1.1(11(b)), such models are unique immediate predecessors of their immediate successor models. Hence even if some model in a_n got the immediate predecessor it must be the unique one.

The rest of the conditions hold trivially.

$\square$ of the claim.

Now it remains only to add A_{k_i+1} as the top model to $\mathrm{dom}(b)$ which can be done easily using 3.3.

$\square$ of the lemma.

Lemma 3.6 $\langle \mathcal{P}, \leq^* \rangle$ *does not add new sequences of ordinals of the length* $< \kappa_0$.

Proof. Repeat the argument of 3.5 with $\mathcal{P}$ replacing Q_{n0}. Then use 2.10 of [1] to insure 3.2(4). $\square$

The argument of 3.5 can be used in a standard fashion to show the Prikry condition (i.e. the standard argument runs inside elementary submodel M with δ replaced by κ^+).

Lemma 3.7 $\langle \mathcal{P}, \leq^* \rangle$ *satisfies the Prikry condition.*

Finally we define $\rightarrow$ on $\mathcal{P}$ similar to those of [1] or [2].

Using 3.4, the arguments of [2, 3.19] can be used to derive the following.

Lemma 3.8 $\langle \mathcal{P}, \rightarrow \rangle$ *satisfies* κ^{++}*-c.c.*

Proof. Suppose otherwise. Work in V. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa^{++} \rangle$ be a name of an antichain of the length κ^{++}. Using 1.7 we find an increasing sequence $\langle \langle \langle A^{0\kappa^+}_\alpha, A^{1\kappa^+}_\alpha, C^{\kappa^+}_\alpha \rangle, A^{1\kappa^{++}}_\alpha \rangle \mid \alpha < \kappa^{++} \rangle$ of elements of $\mathcal{P}'$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ so that for every $\alpha < \kappa^{++}$ the following holds:

(a) $\langle \langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1} \rangle, A^{1\kappa^+}_{\alpha+1} \rangle \Vdash \underset{\sim}{p}_\alpha = \check{p}_\alpha$

(b) $\bigcup_{\beta<\alpha} A^{0\kappa^+}_\beta = A^{0\kappa^+}_\alpha$

(c) ${}^\kappa A^{0\kappa^+}_{\alpha+1} \subseteq A^{0\kappa^+}_{\alpha+1}$

(d) $A^{0\kappa^+}_{\alpha+1}$ is a successor model

(e) $\langle \cup A^{1\kappa^+}_\beta \mid \beta < \alpha \rangle \in A^{0\kappa^+}_{\alpha+1}$.

(f) for every $\alpha \leq \beta < \kappa^{++}$ we have

$$A^{0\kappa^+}_\alpha \in C^\beta(A^{0\kappa^+}_\beta)$$

(g) $A^{0\kappa^+}_{\alpha+2}$ is not an immediate successor model of $A^{0\kappa^+}_{\alpha+1}$, for every $\alpha < \kappa^{++}$.

(h) $p_\alpha = \langle p_{\alpha n} | n < \omega\rangle$

(i) for every $n \geq \ell(p_\alpha)$ $A^{0\kappa^+}_{\alpha+1}$ is the maximal model of $\mathrm{dom}(a_{\alpha n})$ where $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle$

Let $p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n}\rangle$ for every $\alpha < \kappa^{++}$ and $n \geq \ell(p_\alpha)$. Extending by 3.3 if necessary, let us assume that $A^{0\kappa^+}_{\alpha} \in \mathrm{dom}(a_{\alpha n})$, for every $n \geq \ell(p_\alpha)$. Shrinking if necessary, we assume that for all $\alpha, \beta < \kappa^+$ the following holds:

(1) $\ell = \ell(p_\alpha) = \ell(p_\beta)$

(2) for every $n < \ell$ $p_{\alpha n}$ and $p_{\beta n}$ are compatible in Q_{n1} i.e. $p_{\alpha n} \cup p_{\beta n}$ is a function.

(3) for every n, $\ell \leq n < \omega$ $\langle \mathrm{dom}(a_{\alpha n}), \mathrm{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++}\rangle$ form a Δ-system with the kernel contained in $A^{0\kappa^+}_0$

(4) for every n, $\omega > n \geq \ell$ $\mathrm{rng}(a_{\alpha n}) = \mathrm{rng}(a_{\beta n})$.

Shrink now to the set S consisting of all the ordinals below κ^{++} of cofinality κ^+. Let α be in S. For each $n, \ell \leq n < \omega$, there will be $\beta(\alpha) < \alpha$ such that

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} \subseteq A^{0\kappa^+}_{\beta(\alpha,n)}.$$

Just recall that $|a_{\alpha n}| < \kappa_n$. Shrink S to a stationary subset S^* so that for some $\alpha^* < \min S^*$ of cofinality κ^+ we will have $\beta(\alpha, n) < \alpha^*$, whenever $\alpha \in S^*, \ell \leq n < \omega$. Now, the cardinality of $A^{0\kappa^+}_{\alpha^*}$ is κ^+. Hence, shrinking S^* if necessary, we can assume that for each $\alpha, \beta \in S^*, \ell \leq n < \omega$

$$\mathrm{dom}(a_{\alpha n}) \cap A^{0\kappa^+}_{\alpha} = \mathrm{dom}(a_{\beta n}) \cap A^{0\kappa^+}_{\beta}.$$

Let us add $A^{0\kappa^+}_{\alpha^*}$ to each $p_\alpha, \alpha \in S^*$. By 3.3, 3.4(1(a)) it is possible to do this without adding other additional models or ordinals except the images of $A^{0\kappa^+}_{\alpha^*}$ under isomorphisms. Denote the result for simplicity by p_α as well. Note that (again by 3.3, 3.4(1(a))) any $A^{0\kappa^+}_\gamma$ for $\gamma \in S^* \cap (\alpha^*, \alpha)$ or, actually any other successor or limit model $X \in C^{\kappa^+}(A^{0\kappa}_\alpha)$ with $cof(otp_{\kappa^+}(X)) = \kappa^+$, which is between $A^{0\kappa^+}_{\alpha^*}$ and $A^{0\kappa^+}_\alpha$ can be added without adding other additional models or ordinals except the images of it under isomorphisms.

Let now $\beta < \alpha$ be ordinals in S^*. We claim that p_β and p_α are compatible in $\langle \mathcal{P}, \to \rangle$.

First extend p_α by adding $A^{0\kappa^+}_{\beta+2}$ As it was remarked above this will not add other additional models or ordinals except the images of $A^{0\kappa^+}_{\beta+2}$ under isomorphisms to p_α.

Let p be the resulting extension. Assume that $\ell(q) = \ell(p)$. Otherwise just extend q in an appropriate manner to achieve this. Let $n \geq \ell(p)$ and $p_n = \langle a_n, A_n, f_n \rangle$. Let $q_n = \langle b_n, B_n, g_n \rangle$. Without loss of generality we may assume that $a_n(A^{0\kappa^+}_{\beta+2})$ is an elementary submodel of $\mathfrak{A}_{n,k_n}$ with $k_n \geq 5$. Just increase n if necessary. Now, we can realize the $k_n - 1$-type of $\mathrm{rng}(b_n)$ inside $a_n(A^{0\kappa^+}_{\beta+2})$ over the common parts $\mathrm{dom}(b_n)$ and $\mathrm{dom}(a_n)$. This will produce $q'_n = \langle b'_n, B_n, g_n \rangle$ which is k_n-1-equivalent to q_n and with $\mathrm{rng}(b'_n) \subseteq a_n(A^{0\kappa^+}_{\beta+2})$. Doing the above for all $n \geq \ell(p)$ we will obtain $q' = \langle q'_n \mid n < \omega \rangle$ equivalent to q (i.e. $q' \longleftrightarrow q$).

Extend q' to q'' by adding to it $\langle A^{0\kappa^+}_{\beta+2}, a_n(A^{0\kappa^+}_{\beta+2}) \rangle$ as the maximal set for every $n \geq \ell(p)$. Recall that $A^{0\kappa^+}_{\beta+1}$ was its maximal model. So we are adding a top model, also, by the condition (g) above $A^{0\kappa^+}_{\beta+2}$ is not an immediate successor of $A^{0\kappa^+}_{\beta+1}$. Hence no additional models or ordinals are added at all. Let $q''_n = \langle b''_n, B_n, g_n \rangle$, for every $n \geq \ell(p)$.

Combine now p and q'' together. Thus for each $n \geq \ell(p)$ we add b''_n to a_n as well as all of its isomorphic images by $\pi_{A^{0\kappa^+}_{\beta+2} X}$, for every X in $\mathrm{dom}(a_n)$ which is isomorphic to $A^{0\kappa^+}_{\beta+2}$. The rest of the parts are combined in the obvious fashion (we put together the functions and intersect sets of measure one moving first to the same measure). Add if necessary a new top model to insure 3.1(2(d)). Let $r = \langle r_n | n < \omega \rangle$ be the result, where $r_n = \langle c_n, C_n, h_n \rangle$, for $n \geq \ell(p)$.

Claim 3.8.1 $r \in \mathcal{P}$ and $r \geq p$.

Proof. Fix $n \geq \ell(p)$. The main points here are that b''_n and a_n agree on the common part and adding of b''_n to a_n does not require other additions of models or of ordinals except the images of b''_n under isomorphisms.

Thus let $A \in \mathrm{dom}(b''_n) \setminus \mathrm{dom}(a_n)$ be a model. Let $B \in \mathrm{dom}(a_n) \setminus \mathrm{dom}(b''_n)$. Note that it is the main possibility. Once we know how to handle it, dealing with isomorphic images can be reduced to the present case as it was done in 3.3. Suppose first that $B \not\supseteq A$. This implies that $B \not\supseteq A^{0\kappa^+}_{\alpha}$ and $B \not\subseteq A^{0\kappa^+}_{\alpha}$ (just if $B \subset A^{0\kappa^+}_{\alpha}$, then $B \in A^{0\kappa^+}_{\alpha^*}$ and so it is in $\mathrm{dom}(b_n)$).

Then

$$A \cap B = A \cap A^{0\kappa^+}_{\beta+2} \cap B = A \cap A^{0\kappa^+}_{\alpha} \cap B = A \cap \rho,$$

for some $\rho \in A^{0\kappa^+}_{\alpha} \cap A^{1\kappa^{++}}_{\alpha} \cap \mathrm{dom}(a_n)$, since both $B, A^{0\kappa^+}_{\alpha} \in \mathrm{dom}(a_n)$ and 3.1(2(f)) holds. But now we must to have this ρ in $A^{0\kappa^+}_{\alpha^*}$ and then in $\mathrm{dom}(b_n)$. So, $a_n(\rho) = b_n(\rho) = b''_n(\rho)$. Hence,

$$b''_n(A) \cap a_n(B) = b''_n(A) \cap a_n(A^{0\kappa^+}_{\alpha}) \cap a_n(B) = b''_n(A) \cap a_n(\rho) = b''_n(A) \cap b''_n(\rho),$$

due to the choice of the type of $\mathrm{rng}(b''_n)$.

Consider now the side of B of the intersection. So now B is a model. Compare it with $A^{0\kappa^+}_{\beta+2}$. If $otp(B) < otp(A^{0\kappa^+}_{\beta+2})$ then there is $D \in \mathrm{dom}(a_n) \cap A^{0\kappa^+}_{\beta+2}$ of the order type of B and such that

$$B \cap A^{0\kappa^+}_{\beta+2} = B \cap D.$$

But then, again $D \in A^{0\kappa^+}_{\alpha^*}$ and so in $\mathrm{dom}(b_n)$. Hence $D \cap A$ can be handled on the side of D, i.e. $D \cap A = E \cap \xi$, for some $E \in D \cup \{D\} \cap \mathrm{dom}(b_n)$ and an ordinal $\xi \in D \cap \mathrm{dom}(b_n)$. But $D \in A^{0\kappa^+}_{\alpha^*}$, hence $E, \xi \in A^{0\kappa^+}_{\alpha^*}$. This implies that $E, \xi \in \mathrm{dom}(a_n)$. So $a_n(E) = b_n(E), a_n(\xi) = b_n(\xi)$.

If $otp(B) \geq otp(A^{0\kappa^+}_{\beta+2})$ then there is $B' \in (B \cup \{B\}) \cap \mathrm{dom}(a_n)$ such that $otp(B') = otp(A^{0\kappa^+}_{\beta+2})$ and

$$B \cap A^{0\kappa^+}_{\beta+2} = B' \cap A^{0\kappa^+}_{\beta+2}.$$

Now we move to the B- side using $\pi_{A^{0\kappa^+}_{\beta+2}B'}[A]$.

The same argument works once $B \supseteq A$, since then necessary, $otp(B) \geq otp(A^{0\kappa^+}_{\beta+2})$.

The above shows 3.1(2(f)).

Let us check 3.1(2(h)). Suppose that $A \in \text{dom}(b''_n), B \in \text{dom}(a_n)$. Again, it is the main possibility. Once we know how to handle it, dealing with isomorphic images can be reduced to the present case as it was done in 3.3.

Case 3.8.1.1 $A \supset B$.

Then $B \in A^{0\kappa^+}_{\beta+2} \subseteq A^{0\kappa^+}_{\alpha}$ and hence $B \in A^{0\kappa^+}_{\alpha^*} \cap \text{dom}(a_n)$. Which implies that $B \in \text{dom}(b_n)$. But then the walk from A to B in $\text{dom}(b''_n)$ and we are done.

Case 3.8.1.2 $A \subset B$.

If $otp(B) < otp(A^{0\kappa^+}_{\beta+2})$ then there is $D \in \text{dom}(a_n) \cap A^{0\kappa^+}_{\beta+2}$ of the order type of B and such that

$$B \cap A^{0\kappa^+}_{\beta+2} = B \cap D.$$

So $D \supset A$ and hence $A \in D$. But D must be in $A^{0\kappa^+}_{\alpha^*} \cap \text{dom}(a_n)$. Then A as an element of D must be in this intersection as well. So, both A, B are in $\text{dom}(a_n)$. Hence the walk from B to A is in $dom(a_n)$ and we are done.

If $otp(B) \geq otp(A^{0\kappa^+}_{\beta+2})$ then there is $B' \in (B \cup \{B\}) \cap \text{dom}(a_n)$ such that $otp(B') = otp(A^{0\kappa^+}_{\beta+2})$ and

$$B \cap A^{0\kappa^+}_{\beta+2} = B' \cap A^{0\kappa^+}_{\beta+2}.$$

Now we can use $\pi_{A^{0\kappa^+}_{\beta+2}B'}$. It is identity on the common part of $A^{0\kappa^+}_{\beta+2}, B'$ and so does not move A. The walk from $A^{0\kappa^+}_{\beta+2}$ to A will be copied to those from B' to A. Once on the B-side we can run induction on wl as it was done in 3.3.

Now let us turn to 3.1(2(k)). Suppose that $A, \xi \in \text{dom}(a_n) \cup \text{dom}(b''_n)$ and $\sup A > \xi$.

Case 3.8.1.3 $A \in \mathrm{dom}(b''_n)$

If $\xi \in \mathrm{dom}(b''_n)$ then we are done. Suppose otherwise. Then, $\xi \notin A^{0\kappa^+}_{\beta+2}$. Consider $\rho = \min(A^{0\kappa^+}_{\beta+2} \backslash \xi)$. By 3.1(2(k)), $\rho \in \mathrm{dom}(a_n)$. But then $\rho \in A^{0\kappa^+}_{\alpha^*}$ and, so it is in $\mathrm{dom}(b_n)$. Now, the least $\mu \in A \backslash \rho$ will be in $\mathrm{dom}(b''_n)$ and will be as desired.

Case 3.8.1.4 $A \in \mathrm{dom}(a_n) \backslash \mathrm{dom}(b''_n)$.

Assume that $\xi \in \mathrm{dom}(b''_n) \backslash \mathrm{dom}(a_n)$. Compare A with $A^{0\kappa^+}_{\beta+2}$.

Subcase 3.8.1.4.1 $otp(A) < otp(A^{0\kappa^+}_{\beta+2})$.

There is $C \in \mathrm{dom}(a_n)$ such that $A \subseteq C$ and $otp(C) = otp(A^{0\kappa^+}_{\beta+2})$. Note that $A^{0\kappa^+}_{\beta+2} \in C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1})$ (by (f) above) and $A^{0\kappa^+}_{\beta+2} \not\supseteq A$, so the walk from $A^{0\kappa^+}_{\alpha+1}$ to A splits from $C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1})$ above $A^{0\kappa^+}_{\beta+2}$. Now it is not hard to find such C. Move now to $A^{0\kappa^+}_{\beta+2}$. Set $A' = \pi_{C A^{0\kappa^+}_{\beta+2}}[A]$. Then A' is in $\mathrm{dom}(a_n) \cap A^{0\kappa^+}_{\alpha}$. Hence it is in $\mathrm{dom}(b_n)$. So $\rho = \min(A' \backslash \xi) \in \mathrm{dom}(b''_n)$. If $\xi \in C$, then $\pi_{A^{0\kappa^+}_{\beta+2} C}(\rho)$ will be as desired.

Suppose that $\xi \notin C$. Compare C with $A^{0\kappa^+}_{\beta+2}$. There will be $\mu \in C \cap \mathrm{dom}(a_n)$ such that

$$C \cap A^{0\kappa^+}_{\beta+2} = C \cap \mu.$$

By 1.1(8), $C, A^{0\kappa^+}_{\beta+2}$ are isomorphic over $C \cap A^{0\kappa^+}_{\beta+2}$. So

$$C \cap \mu = C \cap \sup(C \cap A^{0\kappa^+}_{\beta+2}) = A^{0\kappa^+}_{\beta+2} \cap \sup(C \cap A^{0\kappa^+}_{\beta+2}).$$

Then ξ must be above $\sup(C \cap A^{0\kappa^+}_{\beta+2})$ since $\xi \notin C$. But $\xi < \sup(A)$, so by 1.1(10), $\sup(A^{0\kappa^+}_{\beta+2}) < \mu$. Hence $\mu = \min(C \backslash \xi)$. Now, clearly, $\min(A \backslash \xi) = \min(A \backslash \mu)$. But $A, \mu \in \mathrm{dom}(a_n)$, so we are done.

Subcase 3.8.1.4.2 $otp(A) \geq otp(A^{0\kappa^+}_{\beta+2})$.

Pick then $C \in C^{\kappa^+}(A^{0\kappa^+}_{\alpha+1})$ of the otp equal to $otp(A)$. As in 3.3 (induction on wl), $C \in \mathrm{dom}(a_n)$. By (f) above and the coherence of C^{κ^+}, we have $A^{0\kappa^+}_{\beta+2} \in C^{\kappa^+}(C)$. So, $\xi \in C$. There is $\delta \in A \cap \mathrm{dom}(a_n)$ such that

$$A \cap C = A \cap \delta.$$

But $\xi \in C \backslash A$ and $\xi < \sup A$, so by 1.1(10), we must have $\sup C < \delta$. Hence $\delta = \min(A \backslash \xi)$ and we are done.

The rest of the conditions hold trivially.

$\square$ of the claim.

Now we have $r \geq p, q''$. Hence, $p \rightarrow r$ and $q \rightarrow r$. Contradiction.

$\square$

4 *pcf* Structure With a Jump

The construction of the previous section gives the following pcf-structure – $b_{\kappa^{+3}}$ consists of indiscernibles for κ_n^{+n+3}, $b_{\kappa^{++}}$ consists of indiscernibles for κ_n^{+n+2} and b_{κ^+} includes all the rest. In particular $b_{\kappa^{+3}} = \{\rho^+ | \rho \in b_{\kappa^{++}}\}$. Here we would like to sketch another construction in which $b_{\kappa^{++}}$ and $b_{\kappa^{+3}}$ will be far a part. This was done first in [3, Sec.4]. The present approach is much simpler.

We split κ_n's into even and odd. For even κ_n's we do exactly the same forcing Q_{n0} as in the previous section. For odd n's use Q_{n0} but rng(a) will be a Q_{n-1}-name. Thus for $A \in \operatorname{dom}(a)$ let $a(A)$ be of cardinality κ_n^{+n+1} if $A \in A^{10}$ or of κ_n^{+n+2} if $A \in A^{11}$ but $cf(\sup A)$= the indiscernible for κ_{n-1}^{+n} or κ_{n-1}^{+n+1} accordingly.

The need of models of such cardinality and not smaller than κ_n^{+3} is to catch all types inside. The arguments of the previous section work here smoothly. Cofinalities of supremums of image models insure that in the generic extension there will be a scale $\langle f_i \mid i < \kappa^{+3} \rangle$ in $\prod\limits_{\substack{n<\omega \\ n \text{ odd}}} \rho_n^{+n+3}$ such that for every $i < \kappa^+$ of cofinality κ^{++} we will have f_i to be the exact upperbound of $\langle f_j \mid j < i \rangle$ and $cf f_i(n) = \rho_{n-1}^{+n+1}$ for each odd $n < \omega$, where ρ_n are indiscernibles for normal measures of E_n's. Also

$$cf\left(\prod_{\substack{n<\omega \\ n \text{ odd}}} \rho_n^{+n+2} \Big/ _{\text{finite}}\right) = cf\left(\prod_{\substack{n<\omega \\ n \text{ odd}}} \rho_n^{+n+1} \Big/ _{\text{finite}}\right) = \kappa^+ .$$

Acknowledgment.

The results of this paper were presented at the conference Computational Prospects of Infinity, at the Institute of Mathematical Sciences, Singapore. We would like thank the organizers and specially to Qi Feng for their hospitality.

We are grateful to Assaf Sharon for encouraging us to write this paper, for discussions on the subject, for his corrections and eagerness to have all the details. We would like to thank to Assaf Rinot for the help with T_EX.

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