Contents

I. Prikry-type Forcings

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One of the central topics of set theory since Cantor has been the study of the power function $\kappa \to 2^\kappa$. The basic problem is to determine all the possible values of 2^κ for a cardinal κ. Paul Cohen [7] proved the independence of CH and invented the method of forcing. Easton [11] building on Cohen's results showed that the function $\kappa \to 2^\kappa$ for regular κ can behave in any prescribed way consistent with the Zermelo-König inequality, which entails $\mathrm{cf}(2^\kappa) > \kappa$. This reduces the study to singular cardinals.

It turned out that the situation with powers of singular cardinals is much more involved. Thus, for example, a remarkable theorem of Silver states that a singular cardinal of uncountable cofinality cannot be the first to violate GCH. The Singular Cardinal Problem is the problem of finding a complete set of rules describing the behavior of the function $\kappa \to 2^\kappa$ for singular κ's.

There are three main tools for dealing with the problem: *pcf* theory, inner model theory and forcing involving large cardinals. The purpose of this chapter is to present the main forcing tools for dealing with powers of singular cardinals. We refer to [19] or to [24] for detailed discussion on the Singular Cardinal Problem.

The chapter should be accessible to a reader with knowledge of forcing (say, chapters VII, VIII of Kunen's book [31]) and familiarity with ultrapowers and elementary embeddings. Thus §§5,26 of Kanamori's book [26] will be more than enough. Only Section 6 requires in addition a familiarity with iterated forcing (for example Baumgartner's paper [5], §§0-2 of Section II of Shelah's book [54], or Cummings' chapter [8] in this handbook). The following sections can be read independently: 1 and 2; 1.1, 3 and 4; 1.1 and 5.1, 5.2; 6.

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1. Prikry Forcings

We describe here the classical Prikry forcing and some variations of it. They were all introduced implicitly or explicitly by K. Prikry in [47].

1.1. Basic Prikry Forcing

Let κ be a measurable cardinal and U a normal ultrafilter over κ.

1.1 Definition. Let $\mathcal{P}$ be the set of all pairs $\langle p, A\rangle$ such that

(1) p is a finite subset of κ,

(2) $A \in U$, and

(3) $\min(A) > \max(p)$.

It is convenient sometimes to view p as an increasing finite sequence of ordinals.

We define two partial orderings on $\mathcal{P}$, the first one conspicuously lacking any useful closure property and the second closed enough to compensate the lack of closure of the first.

1.2 Definition. Let $\langle p, A\rangle, \langle q, B\rangle \in \mathcal{P}$. We say that $\langle p, A\rangle$ *is stronger than* $\langle q, B\rangle$ and denote this by $\langle p, A\rangle \geq \langle q, B\rangle$ iff

(1) p is an end extension of q, i.e. $p \cap (\max(q) + 1) = q$,

(2) $A \subseteq B$, and

(3) $p \setminus q \subseteq B$.

We shall use $\leq$ with the corresponding meaning, and proceed analogously for similar definitions without further comment.

1.3 Definition. Let $\langle p, A\rangle, \langle q, B\rangle \in \mathcal{P}$. We say that $\langle p, A\rangle$ is *a direct (or Prikry) extension* of $\langle q, B\rangle$ and denote this by $\langle p, A\rangle \geq^* \langle q, B\rangle$ iff

(1) $p = q$, and

(2) $A \subseteq B$.

We will force with $\langle \mathcal{P}, \leq\rangle$, and $\langle \mathcal{P}, \leq^*\rangle$ will be used to show that no new bounded subsets are added to κ after the forcing with $\langle \mathcal{P}, \leq\rangle$.

Let us prove a few basic lemmas.

1.4 Lemma. *Let $G \subseteq \mathcal{P}$ be generic for $\langle \mathcal{P}, \leq\rangle$. Then $\bigcup\{p \mid \exists A(\langle p, A\rangle \in G)\}$) is an ω-sequence cofinal in κ.*

Proof. Just note that for every $\alpha < \kappa$ and $\langle q, B\rangle \in \mathcal{P}$ the set

$$D_\alpha = \{\langle p, A\rangle \in \mathcal{P} \mid \langle p, A\rangle \geq \langle q, B\rangle \text{ and } \max(p) > \alpha\}$$

is dense in $\langle \mathcal{P}, \leq\rangle$ above $\langle q, B\rangle$. ⊣

1.5 Lemma. $\langle \mathcal{P}, \leq\rangle$ *satisfies the* κ^+*-c.c.*

Proof. Note that any two conditions having the same first coordinate are compatible: If $\langle p, A\rangle$, $\langle p, B\rangle \in \mathcal{P}$, then $\langle p, A \cap B\rangle$ is stronger than both of them. ⊣

Let us now state three lemmas about $\leq^*$ and its relation to $\leq$. The third one contains the crucial idea of Prikry that makes everything work.

1.6 Lemma. $\leq^* \subseteq \leq$.

This is obvious from the definitions 1.2 and 1.3.

1.7 Lemma. $\langle \mathcal{P}, \leq^*\rangle$ *is* κ*-closed.*

Proof. Let $\langle\langle p_\alpha, A_\alpha\rangle \mid \alpha < \lambda\rangle$ be a $\leq^*$-increasing sequence of length λ for some $\lambda < \kappa$. Then all the p_α's are the same. Set $p = p_0$ and $A = \bigcap_{\alpha<\kappa} A_\alpha$. Then $A \in U$ by κ-completeness of U. So $\langle p, A\rangle \in \mathcal{P}$, and it is stronger than each $\langle p_\alpha, A_\alpha\rangle$ according to $\leq^*$. ⊣

1.8 Lemma (The Prikry condition). *Let* $\langle q, B\rangle \in \mathcal{P}$ *and* σ *be a statement of the forcing language of* $\langle \mathcal{P}, \leq\rangle$*. Then there is a* $\langle p, A\rangle \geq^* \langle q, B\rangle$ *such that* $\langle p, A\rangle \parallel \sigma$ *(i.e.* $\langle p, A\rangle \Vdash \sigma$ *or* $\langle p, A\rangle \Vdash \neg\sigma$*), where, again, we force with* $\langle \mathcal{P}, \leq\rangle$ *and not with* $\langle \mathcal{P}, \leq^*\rangle$*.*

Proof. We identify finite subsets of κ and finite increasing sequences of ordinals below κ, i.e. $[\kappa]^{<\omega}$. Define a partition $h : [B]^{<\omega} \to 2$ as follows:

$$h(s) = \begin{cases} 1, & \text{if there is a } C \text{ such that } \langle q \cup s, C\rangle \Vdash \sigma \ , \\ 0, & \text{otherwise.} \end{cases}$$

U is a normal ultrafilter, so by the Rowbottom theorem (see [26], 7.17 or [25], 70) there is an $A \in U$, $A \subseteq B$ homogeneous for h, i.e. for every $n < \omega$ and every $s_1, s_2 \in [A]^n$, $h(s_1) = h(s_2)$. Now $\langle q, A\rangle$ will decide σ. Otherwise, there would be

$$\langle q \cup s_1, B_1\rangle, \langle q \cup s_2, B_2\rangle \geq \langle q, A\rangle$$

such that $\langle q \cup s_1, B_1\rangle \Vdash \sigma$ and $\langle q \cup s_2, B_2\rangle \Vdash \neg\sigma$. By extending one of these conditions if necessary, we can assume that $|s_1| = |s_2|$. But then $s_1, s_2 \in [A]^{|s_1|}$ and $h(s_1) \neq h(s_2)$, which contradicts the homogeneity of A. ⊣

The above lemma allows us to implement the κ-closure of $\langle \mathcal{P}, \leq^* \rangle$ in the actual forcing $\langle \mathcal{P}, \leq \rangle$. Thus we can conclude the following:

1.9 Lemma. *$\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets of κ.*

Proof. Let $t \in \mathcal{P}$, $\underset{\sim}{a}$ is a name, $\lambda < \kappa$ and

$$t \Vdash \underset{\sim}{a} \subseteq \check{\lambda} .$$

For every $\alpha < \lambda$ denote by σ_α the statement "$\check{\alpha} \in \underset{\sim}{a}$". We define by recursion a $\leq^*$-increasing sequence of conditions $\langle t_\alpha \mid \alpha < \lambda \rangle$ such that $t_\alpha \parallel \sigma_\alpha$ for each $\alpha < \lambda$. Let t_0 be a direct extension of t deciding σ_0; one exists by 1.8. Suppose that $\langle t_\beta \mid \beta < \alpha \rangle$ is defined. Define t_α. First, using 1.7 we find a direct extension t'_α of $\langle t_\beta \mid \beta < \alpha \rangle$. Then by 1.8 choose a direct extension t_α of t'_α deciding σ_α. This completes the definition of $\langle t_\alpha \mid \alpha < \lambda \rangle$. Now let t^* be a direct extension of $\langle t_\alpha \mid \alpha < \lambda \rangle$ (again 1.7 is used). Then $t^* \geq t$ (in fact $t^* \geq^* t$) and $t^* \Vdash \underset{\sim}{a} = \check{b}$ where $b = \{\alpha < \lambda \mid t^* \Vdash \alpha \in \underset{\sim}{a}\}$. ⊣

Let us summarize the situation.

1.10 Theorem. *The following holds in $V[G]$:*

(a) *κ has cofinality $\aleph_0$.*

(b) *All the cardinals are preserved.*

(c) *No new bounded subsets are added to κ.*

Proof. (a) is established by 1.4, (c) by 1.9. Finally, (b) follows from (c) and 1.5. ⊣

If $2^\kappa > \kappa^+$ in V, then in $V[G]$ the Singular Cardinal Hypothesis will fail at κ.

Let $C = \bigcup\{p \mid \exists A(\langle p, A \rangle \in G)\}$. By 1.4, C is an ω-sequence cofinal in κ. It is called a *Prikry sequence for U*. The generic set G can be easily reconstructed from C:

$$G = \{\langle p, A \rangle \in \mathcal{P} \mid p \text{ is an initial segment of } C \text{ and } C \setminus (\max(p) + 1) \subseteq A\} .$$

So, $V[G] = V[C]$.

1.11 Lemma. *C is almost contained in every set in U, i.e.*

$(*)$ *for every $A \in U$ the set $C \setminus A$ is finite.*

Proof. Let $A \in U$. Then the set

$$D = \{\langle p, B \rangle \in \mathcal{P} \mid B \subseteq A\}$$

is dense in $\mathcal{P}$. So, there is a $\langle q, S \rangle \in G \cap D$. But then, for every $\langle q', S' \rangle \geq \langle q, S \rangle$, $q' \setminus q \subseteq S \subseteq A$. Hence, also, $C \setminus q \subseteq A$. ⊣

The above implies that C generates U, i.e. $X \in U$ iff $X \in V$ and $C \setminus X$ is finite.

Mathias [38] pointed out that $(*)$ of 1.11 actually characterizes Prikry sequences:

1.12 Theorem. *Suppose that M is an inner model of* ZFC*, U a normal ultrafilter over κ in M. Assume that C is an ω-sequence satisfying $(*)$. Then C is a Prikry sequence for U over M.*

Proof. We need to show that the set

$$G(C) = \{\langle p, A\rangle \in \mathcal{P} \mid \\ p \text{ is an initial segment of } C \text{ and } C \setminus (\max(p)+1) \subseteq A\}$$

is a generic subset of $\mathcal{P}$ over M. The only nontrivial property to check is that $G(C) \cap D \neq \emptyset$ for every dense open subset $D \in M$ of $\mathcal{P}$. Let us first point out that the following holds in M:

1.13 Lemma. *Let $\langle q, B\rangle \in \mathcal{P}$ and $D \subseteq \mathcal{P}$ be dense open. Then there are $\langle q, A\rangle \geq^* \langle q, B\rangle$ and $m < \omega$ such that for every n with $m \leq n < \omega$ and every $s \in [A]^n$, we have $\langle q \cup s, A \setminus (\max(s)+1)\rangle \in D$.*

Proof. We define a partition $h : [B]^{<\omega} \to 2$ as in 1.8 only replacing "$\Vdash \sigma$" by "$\in D$". Let $A' \in U$, $A' \subseteq B$ be homogeneous for h. Then, starting with some m, for every $n \geq m$ and $s \in [A']^n$ we have $h(s) = 1$. Hence there will be a set $A_s \in U$ such that $\langle q \cup s, A_s\rangle \in D$. Set $A = A' \cap \Delta\{A_s \mid s \in [A']^n, m \leq n < \omega\}$, where

$$\begin{aligned}&\Delta\{A_s \mid s \in [A']^n, m \leq n < \omega\} \\ &= \{\alpha < \kappa \mid \forall n \geq m \forall s \in [A']^n (\max(s) < \alpha \to \alpha \in A_s)\} .\end{aligned}$$

Then, clearly $A \in U$. The condition $\langle q, A\rangle$ is as desired, since for each $n \geq m$ and $s \in [A]^n$ we have $A \setminus (\max(s)+1) \subseteq A_s$ and, so $\langle q \cup s, A \setminus (\max(s)+1)\rangle \in D$. ⊣

Now, let $D \in M$ be a dense open subset of $\mathcal{P}$. For every finite $q \subseteq \kappa$, using 1.13, we pick $m(q) < \omega$ and $A(q) \in U$ such that $\langle q, A(q)\rangle \geq^* \langle q, \kappa \setminus (\max(q)+1)\rangle$ and for every $n \geq m(q)$ and $s \in [A(q)]^n$, $\langle q \cup s, A(q) \setminus (\max(s)+1)\rangle \in D$. Set

$$A = \Delta\{A(q) \mid q \in [\kappa]^{<\omega}\} = \{\alpha < \kappa \mid \forall q \in [\kappa]^{<\omega} (\max q < \alpha \to \alpha \in A(q))\} .$$

There is a $\tau < \kappa$ such that $C \setminus \tau \subseteq A$. Consider $\langle C \cap \tau, A \setminus \tau\rangle$. Since $C \cap \tau$ is finite, $\langle C \cap \tau, A \setminus \tau\rangle \in \mathcal{P}$. Then, for every $n \geq \max(C \cap \tau)$ and $s \in [C \setminus \tau]^n$ we have

$$\langle (C \cap \tau) \cup s, A \setminus (\max(s)+1)\rangle \in D ,$$

since $A \setminus \tau \subseteq A \setminus (C \cap \tau)$. But $C \setminus \tau \subseteq A$, so we can pick $s \in [C \setminus \tau]^n$ for some $n \geq \max(C \cap \tau)$. Then $(C \cap \tau) \cup s \subseteq C$ and $C \setminus (\max(s)+1) \subseteq A \setminus (\max(s)+1)$. Hence, $\langle (C \cap \tau) \cup s, A \setminus (\max(s)+1)\rangle \in G(C) \cap D$. ⊣

1.2. Tree Prikry Forcing

We would now like to eliminate the use of the normality of the ultrafilter U in the previous construction. Note that it was used only once in the proof of the Prikry condition 1.8.

Let us now assume only that U is a κ-complete ultrafilter over κ.

1.14 Definition. A set T is called a *U-tree with a trunk t* iff

(1) T consists of finite increasing sequences of ordinals below κ.

(2) $\langle T, \trianglelefteq \rangle$ is a tree, where $\trianglelefteq$ is the order of end extension of finite sequences, i.e. $\eta \trianglelefteq \nu$ iff $\nu \restriction \mathrm{dom}(\eta) = \eta$.

(3) t is a trunk of T, i.e. $t \in T$ and for every $\eta \in T$, $\eta \trianglerighteq t$ or $t \trianglerighteq \eta$.

(4) For every $\eta \trianglerighteq t$ the set $\mathrm{Suc}_T(\eta) = \{\alpha < \kappa \mid \eta^\frown \langle \alpha \rangle \in T\}$ is in U.

Define $\mathrm{Lev}_n(T) = \{\eta \in T \mid \mathrm{length}(\eta) = n\}$ for every $n < \omega$.

We now define the tree Prikry forcing.

1.15 Definition. The set $\mathcal{P}$ consists of all pairs $\langle t, T \rangle$ such that T is a U-tree with trunk t.

1.16 Definition. Let $\langle t, T \rangle, \langle s, S \rangle \in \mathcal{P}$. We say that $\langle t, T \rangle$ *is stronger than* $\langle s, S \rangle$ and denote this by $\langle t, T \rangle \geq \langle s, S \rangle$ iff $S \supseteq T$.

Note that $S \supseteq T$ implies that $t \trianglerighteq s$ and $t \in S$.

1.17 Definition. Let $\langle t, T \rangle, \langle s, S \rangle \in \mathcal{P}$. We say that $\langle t, T \rangle$ is *a direct (or Prikry) extension of* $\langle s, S \rangle$ and denote this by $\langle t, T \rangle \geq^* \langle s, S \rangle$ iff

(1) $S \supseteq T$, and

(2) $s = t$.

As in the previous section we will force with $\langle \mathcal{P}, \leq \rangle$ and the role of $\leq^*$ will be to provide closure.

1.18 Lemma. *Let $\langle T_\alpha \mid \alpha < \lambda \rangle$ be a sequence of U-trees with the same trunk and $\lambda < \kappa$. Then $T = \bigcap_{\alpha<\lambda} T_\alpha$ is a U-tree having that same trunk.*

Proof. Let t be the trunk of T_0 (and so of every T_α). Suppose that $\eta \in T$ and $\eta \trianglerighteq t$. Then

$$\mathrm{Suc}_T(\eta) = \bigcap\nolimits_{\alpha<\lambda} \mathrm{Suc}_{T_\alpha}(\eta) \ .$$

By κ-completeness of U, $\mathrm{Suc}_T(\eta) \in U$. Hence T is a U-tree with trunk t. $\dashv$

Using 1.18 it is easy to prove lemmas analogous to 1.4-1.7.

1.19 Lemma. *Let $G \subseteq \mathcal{P}$ be generic for $\langle \mathcal{P}, \leq \rangle$. Then*

$$\bigcup \{t \mid \exists T(\langle t, T\rangle \in G)\}$$

is an ω-sequence cofinal in κ.

1.20 Lemma. *$\langle \mathcal{P}, \leq \rangle$ satisfies the κ^+-c.c.*

1.21 Lemma. $\leq^* \subseteq \leq$.

1.22 Lemma. *$\langle \mathcal{P}, \leq^* \rangle$ is κ-closed.*

Let us show that $\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition. The proof is based on the following Ramsey property:

If T is a U-tree and $f : T \to \lambda < \kappa$, then there is an U-tree $S \subseteq T$ such that $f \restriction \mathrm{Lev}_n(S)$ is constant for each $n < \omega$.

We prefer here and later to give a direct proof instead of deducing first a relevant Ramsey property and then proving it.

1.23 Lemma (The Prikry condition). *Let $\langle t, T\rangle \in \mathcal{P}$ and σ be a statement of the forcing language. Then there is a $\langle s, S\rangle \geq^* \langle t, T\rangle$ such that $\langle s, S\rangle \parallel \sigma$.*

Proof. Suppose otherwise. Consider the set $\mathrm{Suc}_T(t)$. We split it into three sets as follows:

$$\begin{aligned} X_0 = \{\alpha \in \mathrm{Suc}_T(t) \mid\ & \exists S_\alpha \subseteq T \text{ a } U\text{-tree with trunk } t^\frown\langle\alpha\rangle \text{ such that} \\ & \langle t^\frown\langle\alpha\rangle, S_\alpha\rangle \Vdash \sigma\} \\ X_1 = \{\alpha \in \mathrm{Suc}_T(t) \mid\ & \exists S_\alpha \subseteq T \text{ a } U\text{-tree with trunk } t^\frown\langle\alpha\rangle \text{ such that} \\ & \langle t^\frown\langle\alpha\rangle, S_\alpha\rangle \Vdash \neg\sigma\} \\ X_2 = \mathrm{Suc}_T(t) \setminus (X_0 & \cup X_1) \ . \end{aligned}$$

Clearly, $X_0 \cap X_1 = \emptyset$, since by 1.18 any two conditions with the same trunk are compatible. Now U is an ultrafilter and $\mathrm{Suc}_T(t) \in U$, so for some $i < 3$, $X_i \in U$. We shrink T to a tree T_1 with the same trunk t, having $\mathrm{Suc}_{T_1}(t) = X_i$ and: If $i < 2$, then let T_1 be S_α above $t^\frown\langle\alpha\rangle$ for every $\alpha \in X_i$; if $i = 2$, then let T_1 be the same as T above $t^\frown\langle\alpha\rangle$ for every $\alpha \in X_2$. We continue by recursion to shrink the initial tree T level by level. Thus define a decreasing sequence $\langle T_n \mid n < \omega\rangle$ of U-trees with trunk t so that

(1) $T_0 = T$.

(2) For every $n > 0$ and $m > n$, $T_m \restriction (n + |t|) = T_n \restriction (n + |t|)$, i.e. after stage n the n-th level above the trunk remains unchanged in all T_m's for $m \geq n$.

(3) For every $n > 0$, if $i < 2$, $\eta \in \mathrm{Lev}_{n+|t|}(T_n)$ and for some U-tree S with trunk η we have $\langle \eta, S\rangle \Vdash {}^i\sigma$, then

(3a) $\langle \eta, (T_n)_\eta \rangle \Vdash {}^i\sigma$, and

(3b) For every $\nu \in \mathrm{Lev}_{n+|t|}(T_n)$ having the same immediate predecessor as η,

$$\langle \nu, (T_n)_\nu \rangle \Vdash {}^i\sigma \ .$$

Here, ${}^0\sigma$ denotes σ, ${}^1\sigma$ denotes $\neg\sigma$ and for a tree R with $r \in R$

$$(R)_r = \{r' \in R \mid r' \trianglerighteq r\} \ .$$

Now, we set $T^* = \bigcap_{n<\omega} T_n$. Clearly, T^* is a U-tree with a trunk t by (2) or by 1.18. Consider $\langle t, T^* \rangle \in \mathcal{P}$. By the assumption, $\langle t, T^* \rangle \nVdash \sigma$. Pick a condition $\langle s, S \rangle \geq \langle t, T^* \rangle$ forcing σ with $n = |s - t|$ as small as possible. Then $s \in \mathrm{Lev}_{n+|t|}(T^*) = \mathrm{Lev}_{n+|t|}(T_n)$. By (3) of the recursive construction,

$$\langle s, (T_n)_s \rangle \Vdash \sigma$$

and for every $s' \in \mathrm{Lev}_{n+|t|}(T_n)$ with the same predecessor as s, $\langle s', (T_n)_{s'} \rangle \Vdash \sigma$. But $T^* \subseteq T_n$, so

$$\langle s, (T^*)_s \rangle \Vdash \sigma \text{ and } \langle s', (T^*)_{s'} \rangle \Vdash \sigma$$

for every s' as above.

Let s^* denote the immediate predecessor of s, i.e. s without its last element. Then $\langle s^*, (T^*)_{s^*} \rangle \Vdash \sigma$ since for every $\langle r, R \rangle \geq \langle s^*, (T^*)_{s^*} \rangle$, $r = s'^\frown r'$ for some $s' \in \mathrm{Lev}_{n+|t|}(T^*)$ and $s' \triangleright s^*$. Hence, $\langle r, R \rangle \geq \langle s', (T^*)_{s'} \rangle \Vdash \sigma$.

But we chose s to be of minimal length such that for some S $\langle s, S \rangle \Vdash \sigma$, yet $|s^*| = |s| - 1$. Contradiction. $\dashv$

Now, as in 1.9 the κ-closure of $\langle \mathcal{P}, \leq^* \rangle$ can be used to derive the following:

1.24 Lemma. *$\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets of κ.*

The conclusions are the same as those of the previous section.

1.25 Theorem. *The following holds in $V[G]$:*

(a) *κ has cofinality $\aleph_0$.*

(b) *All the cardinals are preserved.*

(c) *No new bounded subsets are added to κ.*

1.3. One-Element Prikry Forcing and Adding a Prikry Sequence to a Singular Cardinal

Suppose that κ is a limit of an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ of measurable cardinals. We want to add an ω-sequence dominating every sequence in $\prod_{n<\omega} \kappa_n$, i.e. a sequence $\langle \tau_m \mid m < \omega \rangle \in \prod_{n<\omega} \kappa_n$ such that for every $\langle \rho_m \mid m < \omega \rangle \in (\prod_{n<\omega} \kappa_n) \cap V$ and for all but finitely many m's, $\tau_m > \rho_m$.

Fix a κ_n-complete ultrafilter U_n over κ_n for every $n < \omega$. One can assume normality but it is not necessary.

Let $n < \omega$. We describe first a very simple forcing for adding a one-element Prikry sequence.

1.26 Definition. Let $Q_n = U_n \cup \kappa_n$. If $p, q \in Q_n$ we define $p \geq_n q$ iff either

(1) $p, q \in U_n$ and $p \subseteq q$,

(2) $q \in U_n$ and $p \in q$, or

(3) $p = q \in \kappa_n$.

Thus we can pick a set in U_n, and then shrink it still in U_n or pick an element of this set. In particular, above every condition there is an atomic one. So, the forcing $\langle Q_n, \leq_n \rangle$ is trivial.

Nevertheless we also define a direct extension ordering:

1.27 Definition. Let $p, q \in Q_n$. Set $p \geq_n^* q$ iff $p = q$, or $p, q \in U_n$ and $p \subseteq q$.

The forcing $\langle Q_n, \leq_n, \leq_n^* \rangle$ is called the one-element Prikry forcing . The following lemma follows from the κ_n-completeness of U_n.

1.28 Lemma. *$\langle Q_n, \leq_n^* \rangle$ is κ_n-closed.*

1.29 Lemma. *$\langle Q_n, \leq_n, \leq_n^* \rangle$ satisfies the Prikry condition, i.e. for every $p \in Q_n$ and every statement σ of the forcing language there is a $q \geq_n^* p$ such that $q \parallel \sigma$.*

The proof repeats the first stage of the proof of 1.23.

We now combine Q_n's together.

1.30 Definition. Let $\mathcal{P}$ be the set of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) For every $n < \omega$, $p_n \in Q_n$.

(2) There is an $\ell(p) < \omega$ so that for every $n < \ell(p)$, p_n is an ordinal below κ_n and for every $n \geq \ell(p)$, $p_n \in U_n$.

The orderings $\leq$ and $\leq^*$ are defined on $\mathcal{P}$ in obvious fashion:

1.31 Definition. Let $p = \langle p_n \mid n < \omega\rangle$, $q = \langle q_n \mid n < \omega\rangle \in \mathcal{P}$. We say that $p \geq q$ (resp. $p \geq^* q$) iff for every $n < \omega$, $p_n \geq_n q_n$ (resp. $p_n \geq_n^* q_n$).

For $p = \langle p_n \mid n < \omega\rangle \in \mathcal{P}$ we denote $\langle p_m \mid m < n\rangle$ by $p\restriction n$ and $\langle p_m \mid m \geq n\rangle$ by $p \setminus n$. Let $\mathcal{P}\restriction n = \{p\restriction n \mid p \in \mathcal{P}\}$ and $\mathcal{P} \setminus n = \{p \setminus n \mid p \in \mathcal{P}\}$.

The following splitting lemma is obvious:

1.32 Lemma. *$\mathcal{P} \simeq \mathcal{P}\restriction n \times \mathcal{P} \setminus n$ for every $n < \omega$.*

1.33 Lemma. *For every $n < \omega$, $\langle \mathcal{P} \setminus n, \leq^*\rangle$ is κ_n-closed.*

The above follows from the fact that each U_m with $m \geq n$ is κ_n-complete.

1.34 Lemma. *$\langle \mathcal{P}, \leq, \leq^*\rangle$ satisfies the Prikry condition.*

Proof. Let $p = \langle p_n \mid n < \omega\rangle$ be an element of $\mathcal{P}$ and σ be a statement of the forcing language. Suppose for simplicity that $\ell(p) = 0$. Then let $p_n = A_n \in U_n$ for every $n < \omega$. We want to find a direct extension of p deciding σ. Assume that there is no such extension. Define by recursion on $n < \omega$ a $\leq^*$-increasing sequence $\langle q(n) \mid n < \omega\rangle$ of $\leq^*$-extensions of p such that for every $n < \omega$ the following holds:

(1) If $m \geq n$, then $q(m)\restriction n = q(n)\restriction n$.

(2) If $q = \langle q_n \mid n < \omega\rangle \geq q(n)$ decides σ and $\ell(q) = n + 1$ then already $\langle q_m \mid m \leq n\rangle^\frown\langle q(n)_m \mid m > n\rangle$ decides σ and in the same way as q; moreover for every $\tau_n \in q(n)_n$ also $\langle q_m \mid m < n\rangle^\frown\langle \tau_n\rangle^\frown\langle q(n)_m \mid m > n\rangle$ makes the same decision.

The recursive construction is straightforward. At stage n, the κ_n-completeness of the U_m's for $m \geq n$ is used in order to take care of the possibilities for initial sequences of length $n - 1$ below κ_n. The number of such possibilities is $|\prod_{i\leq n-1} \kappa_i| = \kappa_{n-1} < \kappa_n$. Now define $s = \langle s_n \mid n < \omega\rangle$ to be $\langle q(n)_n \mid n < \omega\rangle$. Clearly, $s \in \mathcal{P}$ and $s \geq^* p$. The conclusion is now as in 1.23. Thus let $q = \langle q_n \mid n < \omega\rangle$ be an extension of s forcing σ and with $\ell(q)$ as small as possible. By the assumption, $\ell(q) > 0$. Let $n = \ell(q) - 1$. Now, using (2) of the construction, we conclude that

$$\langle q_m \mid m < n\rangle^\frown\langle \tau_n\rangle^\frown\langle s_m \mid m > n\rangle \Vdash \sigma$$

for every $\tau_n \in q(n)_n = s_n$. But then also $\langle q_m \mid m < n\rangle^\frown\langle s_m \mid m \geq n\rangle \Vdash \sigma$, contradicting the minimality of $\ell(q)$. ⊣

Combining 1.32, 1.33 and 1.34 we obtain the following:

1.35 Lemma. *$\langle \mathcal{P}, \leq\rangle$ does not add new bounded subsets to κ.*

Note that for each $n < \omega$, $\mathcal{P}\restriction n$ is just a trivial forcing "adding" a sequence of length n of ordinals in $\prod_{m\leq n-1} \kappa_m$.

1.36 Lemma. *$\langle \mathcal{P}, \le \rangle$ satisfies the κ^+-c.c.*

Proof. Note that any two conditions $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle$ are compatible provided $\ell(p) = \ell(q)$ and $\langle p_n \mid n < \ell(p) \rangle = \langle q_n \mid n < \ell(q) \rangle$. ⊣

Now let $G \subseteq \mathcal{P}$ be generic for $\langle \mathcal{P}, \le \rangle$. Define an ω-sequence $\langle t_n \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_n$ as follows: $t_n = \tau$ if for some $p = \langle p_m \mid m < \omega \rangle \in G$ with $\ell(p) > n$ $p_n = \tau$.

Using density arguments it is easy to show the following:

1.37 Lemma. *For every $\langle s_n \mid n < \omega \rangle \in (\prod_{n<\omega} \kappa_n) \cap V$ there is an $n_0 < \omega$ such that for every $n \ge n_0$, $t_n > s_n$.*

Combining lemmas together we now obtain the following:

1.38 Theorem. *The following holds in $V[G]$:*

(a) *All cardinals and cofinalities are preserved.*

(b) *No new bounded subsets are added to κ.*

(c) *There is a sequence in $\prod_{n<\omega} \kappa_n$ dominating every sequence in $(\prod_{n<\omega} \kappa_n) \cap V$.*

1.4. Supercompact and Strongly Compact Prikry Forcings

In this section, we present Prikry forcings for supercompact and strongly compact cardinals. The main feature of these forcings is that not only κ changes its cofinality to ω, but also every regular cardinal in the interval $[\kappa, \lambda]$ does so, if we use a λ-supercompact (or strongly compact) cardinal κ. The presentation will follow that of M. Magidor who was the first to use these forcings in his celebrated papers [35, 36].

Fix cardinals $\kappa \le \lambda$. Let $\mathcal{P}_\kappa(\lambda) = \{P \subseteq \lambda \mid |P| < \kappa\}$. Let us recall few basic definitions.

1.39 Definition. An ultrafilter U over $\mathcal{P}_\kappa(\lambda)$ is called *normal* iff

(1) U is κ-complete.

(2) U is fine, i.e. for every $\alpha < \lambda$, $\{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in P\} \in U$.

(3) For every $A \in U$ and every $f : A \to \lambda$ satisfying $f(P) \in P$ for $P \in A$ there are $A' \in U$ and $\alpha' < \lambda$ such that for every $P \in A'$ we have $f(P) = \alpha'$.

1.40 Definition. (1) κ is called λ-*strongly compact* iff there exists a κ-complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

(2) κ is called λ-*supercompact* iff there exists a normal ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

(3) If $P, Q \in \mathcal{P}_\kappa(\lambda)$, then P is *strongly included* in Q iff $P \subseteq Q$ and $\mathrm{otp}(P) < \mathrm{otp}(Q \cap \kappa)$. We denote this by $P \precneqq Q$.

Suppose now that κ is λ-supercompact cardinal, and U is a normal ultrafilter over $\mathcal{P}_\kappa(\lambda)$. The normality of U easily implies the following:

(a) If F is function from a set in U into $\mathcal{P}_\kappa(\lambda)$ such that for all $P \neq \emptyset$ $F(P) \precneqq P$, then F is constant on a set in U.

(b) If for every $Q \in \mathcal{P}_\kappa(\lambda)$, $A_Q \in U$, then $\{P \mid \forall Q \precneqq P \ (P \in A_Q)\} \in U$. (This last set is called the *diagonal intersection* of the system $\{A_Q \mid Q \in \mathcal{P}_\kappa(\lambda)\}$).

For $B \subseteq \mathcal{P}_\kappa(\lambda)$, denote by $[B]^{[n]}$ the set of all n element subsets of B totally ordered by $\precneqq$; denote $\bigcup_{n<\omega}[B]^{[n]}$ by $[B]^{[<\omega]}$. The following is a straightforward analog of the Rowbottom theorem:

If $F : [\mathcal{P}_\kappa(\lambda)]^{[<\omega]} \to 2$, then there is an $A \in U$ such that for every $n < \omega$, F is constant on $[A]^{[n]}$.

We are now ready to define the supercompact Prikry forcing with a normal ultrafilter U over $\mathcal{P}_\kappa(\lambda)$.

The definitions will be the same as in 1.1 with only κ replaced by $\mathcal{P}_\kappa(\lambda)$ and the order on ordinals replaced by $\precneqq$.

1.41 Definition. Let $\mathcal{P}$ be the set of all pairs $\langle\langle P_1, \dots, P_n\rangle, A\rangle$ such that

(1) $\langle P_1, \dots, P_n\rangle$ is a finite $\precneqq$-increasing sequence of elements of $\mathcal{P}_\kappa(\lambda)$,

(2) $A \in U$, and

(3) for every $Q \in A$, $P_n \precneqq Q$.

1.42 Definition. Let $\langle\langle P_1, \dots, P_n\rangle, A\rangle$, $\langle\langle Q_1, \dots, Q_m\rangle, B\rangle \in \mathcal{P}$. Then define $\langle\langle P_1, \dots, P_n\rangle, A\rangle \geq \langle\langle Q_1, \dots, Q_m\rangle, B\rangle$ iff

(1) $n \geq m$,

(2) for every $k \leq m$ $\ P_k = Q_k$,

(3) $A \subseteq B$, and

(4) $\{P_{m+1}, \dots, P_n\} \subseteq B$.

1.43 Definition. Let $\langle\langle P_1, \dots, P_n\rangle, A\rangle, \langle\langle Q_1, \dots, Q_m\rangle, B\rangle \in \mathcal{P}$. Then $\langle\langle P_1, \dots, P_n\rangle, A\rangle \geq^* \langle\langle Q, \dots, Q_m\rangle, B\rangle$ iff

(1) $\langle P_1, \dots, P_n \rangle = \langle Q_1, \dots, Q_m \rangle$, and

(2) $A \subseteq B$.

The next lemmas are proved as in 1.1 with obvious changes from κ to $\mathcal{P}_\kappa(\lambda)$.

1.44 Lemma. $\leq^* \subseteq \leq$.

1.45 Lemma. *$\langle \mathcal{P}, \leq^* \rangle$ is κ-closed.*

1.46 Lemma (The Prikry condition). *Let $\langle q, B \rangle \in \mathcal{P}$ and σ be a statement of the forcing language (i.e. of $\langle \mathcal{P}, \leq \rangle$). Then there is a $\langle p, A \rangle \geq^* \langle q, B \rangle$ such that $\langle p, A \rangle \parallel \sigma$.*

1.47 Lemma. *$\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets to κ.*

1.48 Lemma. *$\langle \mathcal{P}, \leq \rangle$ satisfies the $(\lambda^{<\kappa})^+$-c.c.*

Proof. As in 1.5, any two conditions with the same finite sequence, i.e. of the form $\langle p, A \rangle$ and $\langle p, B \rangle$ are compatible. The number of possibilities for p's now is $\lambda^{<\kappa}$. So we are done. $\dashv$

By the theorem of Solovay (see [55] or [27]), $\lambda^{<\kappa} = \lambda$ if λ is regular or of cofinality $\geq \kappa$ and $\lambda^{<\kappa} = \lambda^+$ in case of $\mathrm{cf}(\lambda) < \kappa$. Note that λ-supercompactness of κ implies actually its $\lambda^{<\kappa}$–supercompactness. We can restate 1.48 using Solovay's theorem as follows:

1.49 Lemma. *$\langle \mathcal{P}, \leq \rangle$ satisfies the μ^+–c.c., where*

$$\mu = \begin{cases} \lambda, & \textit{if } \mathrm{cf}(\lambda) \geq \kappa \\ \lambda^+, & \textit{if } \mathrm{cf}(\lambda) < \kappa \ . \end{cases}$$

Our next lemma presents the main property of the supercompact Prikry forcing. Also, it shows that 1.49 is sharp.

Let G be a generic subset of $\langle \mathcal{P}, \leq \rangle$ and let $\langle P_n \mid 1 \leq n < \omega \rangle$ be the Prikry sequence produced by G, i.e. the sequence such that for every $n < \omega$, there is an $A \in U$ with $\langle \langle P_1, \dots, P_n \rangle, A \rangle \in G$.

1.50 Lemma. *Every $\delta \in [\kappa, \mu]$ of cofinality $\geq \kappa$ (in V) changes its cofinality to ω in $V[G]$, where*

$$\mu = \begin{cases} \lambda, & \textit{if } \mathrm{cf}(\lambda) \geq \kappa \\ \lambda^+, & \textit{if } \mathrm{cf}(\lambda) < \kappa \ . \end{cases}$$

Moreover, for each $\delta \leq \lambda$, $\delta = \bigcup_{n<\omega} (P_n \cap \delta)$, i.e. it is a countable union of old sets each of cardinality less than κ.

Proof. Let $\alpha < \lambda$. The fineness of U implies that $\{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in P\} \in U$. Then, by a density argument, $\alpha \in P_n$ for all but finitely many n's. Hence, for each $\delta \leq \lambda$

$$\delta = \bigcup_{n<\omega}(P_n \cap \delta) \ .$$

This implies that each $\delta \leq \lambda$ of cofinality $\geq \kappa$ in V changes cofinality to ω in $V[G]$, as witnessed by $\langle \sup(P_n \cap \delta) \mid n < \omega\rangle$. In order to finish the proof, we need to deal with λ of cofinality below κ and to show that in this case λ^+ also changes its cofinality to ω. Fix in V a sequence cofinal in λ of regular cardinals $\langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$, a sequence of functions $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ in $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ and an ultrafilter D over $\mathrm{cf}(\lambda)$ including all cobounded subsets of $\mathrm{cf}(\lambda)$, so that

(a) $\alpha < \beta < \lambda^+ \Longrightarrow f_\alpha < f_\beta \pmod D$, and

(b) for every $g \in \prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ there is an $i < \lambda^+$ such that $f_i > g(\mathrm{mod}\ D)$.

Using $\lambda^{<\kappa} = \lambda^+$, it is not hard directly by induction to construct such sequence of f_i's. One can also appeal to general *pcf* considerations; see [1]. Now, by fineness and density again, for every $\alpha < \lambda^+$ and for all but finitely many $n < \omega$ we will have $P_n \supseteq \mathrm{ran}(f_\alpha)$. Hence, for such n's, $\langle \bigcup(P_n \cap \lambda_i) \mid i < \mathrm{cf}(\lambda)\rangle > f_\alpha$. So, $\{\langle \bigcup(P_n \cap \lambda_i) \mid i < \mathrm{cf}(\lambda)\rangle \mid n < \omega\}$ will be an ω-sequence of functions from $(\prod_{i<\mathrm{cf}(\lambda)} \lambda_i) \cap V$ unbounded in $(\prod_{i<\mathrm{cf}(\lambda)} \lambda_i) \cap V$. This implies that λ^+ should have cofinality ω in $V[G]$. $\dashv$

Let us now turn to the strongly compact Prikry forcing. So, we give up normality and assume only that U is a κ-complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$. The construction here is completely parallel to the construction of the tree Prikry forcing in 1.2.

1.51 Definition. A set T is called a *U-tree with trunk t* iff

(1) T consists of finite sequences $\langle P_1, \ldots, P_n\rangle$ of elements of $\mathcal{P}_\kappa(\lambda)$ so that $P_1 \precsim P_2 \precsim \cdots \precsim P_n$.

(2) $\langle T, \trianglelefteq\rangle$ is a tree, where $\trianglelefteq$ is the order of the end extension of finite sequences.

(3) t is a trunk of T, i.e. $t \in T$ and for every $\eta \in T$, $\eta \trianglerighteq t$ or $t \trianglerighteq \eta$.

(4) For every $\eta \trianglerighteq t$,

$$\mathrm{Suc}_T(\eta) = \{Q \in \mathcal{P}_\kappa(\lambda) \mid \eta^\frown\langle Q\rangle \in T\} \in U \ .$$

The definitions of the forcing notion $\mathcal{P}$ and the orders $\leq$ and $\leq^*$ are now exactly the same as those in 1.15, 1.16 and 1.17. $\langle \mathcal{P}, \leq, \leq^*\rangle$ here shares all the properties of the tree Prikry forcing of §1.2 except the κ^+-c.c. Thus the

Lemmas 1.18, 1.21–1.24 are valid in the present context with basically the same proofs. Instead of the κ^+–c.c. we will have here the $(\lambda^{<\kappa})^+$–c.c. Also 1.48–1.50 holds with the same proofs.

Let us summarize the properties of both supercompact and strongly compact Prikry forcings.

1.52 Theorem. *Let G be a generic set for $\langle \mathcal{P}, \leq, \leq^* \rangle$, where $\langle \mathcal{P}, \leq, \leq^* \rangle$ is either supercompact or strongly compact Prikry forcing over $\mathcal{P}_\kappa(\lambda)$. The following holds in $V[G]$:*

(a) *No new bounded subsets are added to κ.*

(b) *Every cardinal in the interval $[\kappa, \mu]$ of cofinality $\geq \kappa$ (as computed in V) changes its cofinality to ω.*

(c) *All the cardinals above μ are preserved, where*

$$\mu = \begin{cases} \lambda, & \textit{if } \mathrm{cf}(\lambda) \geq \kappa \\ \lambda^+, & \textit{if } \mathrm{cf}(\lambda) < \kappa \ . \end{cases}$$

2. Adding Many Prikry Sequences to a Singular Cardinal

In this section we present the extender-based Prikry forcing over a singular cardinal. It is probably the simplest direct way for violating the Singular Cardinal Hypothesis using minimal large cardinal hypotheses. This type of forcing first appeared in [20] in a more complicated form. The presentation here follows [17], Sec.3.

Let, as in 1.3, $\kappa = \bigcup_{n<\omega} \kappa_n$ with $\langle \kappa_n \mid n < \omega \rangle$ increasing and each κ_n measurable. The Prikry forcing described in 1.3 produces basically one Prikry sequence. More precisely, if GCH holds in the ground model, then κ^+-many new ω-sequences are introduced but all of them are coded by the generic Prikry sequence. Here we present a way for adding any number of Prikry sequences into $\prod_{n<\omega} \kappa_n$. In particular this will increase the power of κ as large as one likes without adding new bounded subsets and preserving all the cofinalities.

The basic idea is to use many ultrafilters over each of κ_n's instead of a single one used in 1.3. This leads naturally to extenders over the κ_n's. For the basics about extenders and corresponding large cardinal hypotheses, which are significantly weaker than λ-supercompactness of 1.4, see the fine structure and inner model chapters of this handbook.

Assume GCH and let $\lambda \geq \kappa^+$ be a regular cardinal. Suppose that we want to add to κ or into $\prod_{n<\omega} \kappa_n$ at least λ many Prikry sequences. Our basic assumption will now be that each κ_n is a $(\lambda+1)$-strong cardinal. This means that for every $n < \omega$ there is a $(\kappa_n, \lambda+1)$-extender E_n over κ_n whose ultrapower contains $V_{\lambda+1}$ and which moves κ_n above λ. We fix such E_n and let $j_n : V \to M_n \simeq \mathrm{Ult}(V, E_n)$. For every $\alpha < \lambda$ we define a κ_n–complete ultrafilter $U_{n\alpha}$ over κ_n by setting $X \in U_{n\alpha}$ iff $\alpha \in j_n(X)$. Actually only $U_{n\alpha}$'s with $\alpha \geq \kappa_n$ will be important. Note that a lot of $U_{n\alpha}$'s are comparable in the Rudin-Keisler order $\leq_{RK}$, recalling that

$$U \leq_{RK} W \text{ iff } \exists f : \bigcup W \to \bigcup U \, \forall X \subseteq \bigcup U \, (X \in U \leftrightarrow f^{-1}(X) \in W) \, .$$

Thus for example, if α is a cardinal and $\beta \leq \alpha$, then $U_{n(\alpha+\beta)} \geq_{RK} U_{n,\alpha}$ and $U_{n(\alpha+\beta)} \geq_{RK} \ U_{n,\beta}$.

We will need a strengthening of the Rudin-Keisler order. For $\alpha, \beta < \lambda$ define

$$\alpha \leq_{E_n} \beta \text{ iff } \alpha \leq \beta \text{ and for some } f \in {}^{\kappa_n}\kappa_n, \ j_n(f)(\beta) = \alpha \ .$$

Clearly, then $\alpha \leq_{E_n} \beta$ implies $U_{n\alpha} \leq_{RK} U_{n\beta}$, as witnessed by any $f \in {}^{\kappa_n}\kappa_n$ with $j_n(f)(\beta) = \alpha$: If $A \in U_{n\beta}$, then $\beta \in j_n(A)$. So $\alpha = j_n(f)(\beta) \in j_n(f)``j_n(A) = j_n(f``A)$. Hence $f``A \in U_{n,\alpha}$. Note that, in general, $\alpha < \beta < \lambda$ and $U_{n\alpha} <_{RK} U_{n\beta}$ does not imply $\alpha <_{E_n} \beta$.

The partial order $\langle \lambda, \leq_{E_n} \rangle$ is κ_n-directed, as we see in Lemma 2.1 below. Actually, it is κ_n^{++}-directed, but for our purposes κ_n-directness will suffice. Thus, using GCH, find some enumeration $\langle a_\alpha \mid \alpha < \kappa_n \rangle$ of $[\kappa_n]^{<\kappa_n}$ so that for every regular cardinal $\delta < \kappa_n$ $\langle a_\alpha \mid \alpha < \delta \rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears δ many times in the enumeration. Let $j_n(\langle a_\alpha \mid \alpha < \kappa_n \rangle) = \langle a_\alpha \mid \alpha < j_n(\kappa_n) \rangle$. Then, $\langle a_\alpha \mid \alpha < \lambda \rangle$ will enumerate $[\lambda]^{<\lambda} \supseteq [\lambda]^{<\kappa_n}$ in both M_n and V; this coding will be applied below.

The next lemma is a basic application of commutativity of diagrams corresponding to extenders and their ultrafilters.

2.1 Lemma. *Let $n < \omega$ and $\tau < \kappa_n$. Suppose that $\langle \alpha_\nu \mid \nu < \tau \rangle$ is a sequence of ordinals below λ and $\alpha \in \lambda \setminus (\bigcup_{\nu<\tau} \alpha_\nu + 1)$ codes this sequence, i.e. $a_\alpha = \{\alpha_\nu \mid \nu < \tau\}$. Then $\alpha >_{E_n} \alpha_\nu$ for every $\nu < \tau$.*

Proof. Fix $\nu < \tau$. Consider the following diagram

$$\begin{tikzcd}
 & M_n \\
V \arrow[ur, "j_n"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"'] \\
 & N_{\alpha_\nu} \arrow[uu, bend right, "k_{\alpha_\nu}"']
\end{tikzcd}$$

where $N_\alpha \simeq \mathrm{Ult}(V, U_{n,\alpha})$, $k_\alpha([f]_{U_{n,\alpha}}) = j_n(f)(\alpha)$ and the same with α_ν replacing α. Then $j_n(\langle a_\beta \mid \beta < \kappa_n \rangle) = k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa_n \rangle))$ and $k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa_n \rangle)([id])_{U_{n,\alpha}})) = j_n(\langle a_\beta \mid \beta < \kappa_n \rangle)(\alpha) = a_\alpha = \{\alpha_\mu \mid \mu < \tau\}$. But $\tau < \kappa_n$, so it is fixed by k_α, since $\mathrm{crit}(k_\alpha) \geq \kappa_n$. Hence $i_\alpha(\langle a_\beta \mid \beta < \kappa_n \rangle)([id]_{U_{n,\alpha}})$ is a sequence of ordinals of length τ. Let α_ν^* denote its ν-th element. Then, by elementarity, $k_\alpha(\alpha_\nu^*) = \alpha_\nu$. We can hence define $k_{\alpha_\nu \alpha} : N_{\alpha_\nu} \longrightarrow N_\alpha$ by setting $k_{\alpha_\nu \alpha}([f]_{U_{\alpha_\nu}}) = i_\alpha(f)(\alpha_\nu^*)$. It is easy to see that $k_{\alpha_\nu \alpha}$ is elementary embedding and the following diagram is commutative

$$\begin{tikzcd}
 & M_n & \\
V \arrow[ur, "j_n"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"'] & \\
 & N_{\alpha_\nu} \arrow[u, "k_{\alpha_\nu\alpha}"'] \arrow[uu, bend right=60, "k_{\alpha_\nu}"'] &
\end{tikzcd}$$

Finally, we can define the desired projection $\pi_{\alpha\alpha_\nu}$ of $U_{n,\alpha}$ onto U_{n,α_ν}. Thus let $\pi_{\alpha\alpha_\nu} : \kappa_n \to \kappa_n$ be a function such that $[\pi_{\alpha\alpha_\nu}]_{U_{n,\alpha}} = \alpha^*_\nu$. Then, $j_n(\pi_{\alpha\alpha_\nu})(\alpha) = k_\alpha\ ([\pi_{\alpha,\alpha_\nu}]_{U_{n,\alpha}}) = k_\alpha(\alpha^*_\nu) = \alpha_\nu$. So, $\alpha >_{E_n} \alpha_\nu$. ⊣

Hence we obtain the following:

2.2 Lemma. *For every set $a \subseteq \lambda$ of cardinality less than κ_n, there are λ many α's below λ so that $\alpha >_{E_n} \beta$ for every $\beta \in a$.*

For every $\alpha, \beta < \lambda$ such that $\alpha >_{E_n} \beta$ we fix the projection $\pi_{\alpha\beta} : \kappa_n \to \kappa_n$ defined as in 2.1 witnessing this. Let $\pi_{\alpha\alpha} = id$, the identity map: $\kappa_n \to \kappa_n$.

The following two lemmas are standard.

2.3 Lemma. *Let $\gamma < \beta \leq \alpha < \lambda$. If $\alpha \geq_{E_n} \beta$ and $\alpha \geq_{E_n} \gamma$, then $\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in U_{n\alpha}$.*

Proof. We consider the following commutative diagram

$$\begin{tikzcd}
 & M_n \\
 & N_\alpha \arrow[u, "k_\alpha"'] \\
V \arrow[uur, "j_n"] \arrow[ur, "i_\alpha"] \arrow[r, "i_\beta"] \arrow[dr, "i_\gamma"'] & N_\beta \arrow[u, "k_{\beta\alpha}"'] \arrow[uu, bend right=50, "k_\beta"'] \\
 & N_\gamma \arrow[uu, bend right=50, "k_{\gamma\alpha}"'] \arrow[uuu, bend right=70, "k_\gamma"']
\end{tikzcd}$$

where for $\delta', \delta \in \{\alpha, \beta, \gamma\}$

$$i_\delta : V \longrightarrow N_\delta \simeq \mathrm{Ult}(V, U_{n\delta})$$

$$k_\delta([f]_{U_{n\delta}}) = j_n(f)(\delta)$$

and $k_{\delta'\delta}([f]_{U_{n\delta'}}) = i_\delta(f)([\pi_{\delta\delta'}]_{U_{n\delta}})$.

Then $k_\alpha([\pi_{\alpha\beta}]_{U_{n\alpha}}) = k_\alpha(k_{\beta\alpha}([id]_{U_{n\beta}})) = k_\beta([id]_{U_{n\beta}}) = j_n(id)(\beta) = \beta$. The same is true for γ, i.e.

$$k_\alpha([\pi_{\alpha\gamma}]_{U_{n\alpha}}) = \gamma \ .$$

But $M_n \vDash \gamma < \beta$ and k_α is elementary, so $N_\alpha \vDash [\pi_{\alpha\gamma}]_{U_{n\alpha}} < [\pi_{\alpha\beta}]_{U_{n\alpha}}$. Hence

$$\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in U_{n\alpha} \ .$$

$\dashv$

2.4 Lemma. *Let $\{\alpha_i \mid i < \tau\} \subseteq \alpha < \lambda$ for some $\tau < \kappa_n$. Assume that $\alpha \geq_{E_n} \alpha_i$ for every $i < \tau$. Then there is a set $A \in U_{n\alpha}$ so that for every $i, j < \tau$: $\alpha_i \geq_{E_n} \alpha_j$ implies $\pi_{\alpha\alpha j}(\nu) = \pi_{\alpha_i\alpha_j}(\pi_{\alpha\alpha_i}(\nu))$ for every $\nu \in A$.*

Proof. It is enough to prove the lemma for $\tau = 2$ and then to use the κ_n-completeness of $U_{n\alpha}$. So, let $\beta, \gamma < \alpha$ and assume that $\gamma \leq_{E_n} \beta \leq_{E_n} \alpha$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & M_n \\
 & N_\alpha \arrow[u, "k_\alpha"'] \\
V \arrow[uur, "j_n"] \arrow[ur, "i_\alpha"] \arrow[r, "i_\beta"] \arrow[dr, "i_\gamma"] & N_\beta \arrow[u, "k_{\beta\alpha}"'] \arrow[uu, bend right=60, "k_\beta"'] \\
 & N_\gamma \arrow[u, "k_{\gamma\beta}"'] \arrow[uu, bend right=60, "k_{\gamma\alpha}"'] \arrow[uuu, bend right=80, "k_\gamma"']
\end{tikzcd}$$

where k's and i's are defined as in 2.3.

We need to show that

$$[\pi_{\alpha\gamma}]_{U_{n\alpha}} = [\pi_{\beta\gamma} \circ \pi_{\alpha\beta}]_{U_{n\alpha}} \ .$$

As in 2.3, $k_\alpha([\pi_{\alpha\gamma}]_{U_{n\alpha}}) = \gamma$. On the other hand, again as 2.3,

$$\begin{aligned} & k_\alpha([\pi_{\beta\gamma} \circ \pi_{\alpha\beta}]_{U_{n\alpha}}) = j_n(\pi_{\beta\gamma} \circ \pi_{\alpha\beta})(\alpha) \\ & = j_n(\pi_{\beta\gamma})(j_n(\pi_{\alpha\beta})(\alpha)) = j_n(\pi_{\beta\gamma})(\beta) = \gamma \ . \end{aligned}$$

Since k_α is elementary, we have in N_α the desired equality. $\dashv$

We are now ready to define our first forcing notion. It will resemble the one-element Prikry forcing considered in 1.3 and will be built from two pieces. Fix $n < \omega$.

2.5 Definition. Let $Q_{n1} = \{f \mid f$ is a partial function from λ to κ_n of cardinality at most $\kappa\}$. We order Q_{n1} by inclusion, which here is denoted by $\leq_1$.

Thus Q_{n1} is basically the usual Cohen forcing for blowing up the power of κ^+ to λ. The only minor change is that the functions take values inside κ_n rather than 2 or κ^+.

2.6 Definition. Let Q_{n0} be the set of triples $\langle a, A, f\rangle$ so that

(1) $f \in Q_{n1}$.

(2) $a \subseteq \lambda$ with

  (2a) $|a| < \kappa_n$,

  (2b) $a \cap \mathrm{dom}(f) = \emptyset$, and

  (2c) a has a $\leq_{E_n}$-maximal element, i.e. an element $\alpha \in a$ such that $\alpha \geq_{E_n} \beta$ for every $\beta \in a$.

(3) $A \in U_{n\,\max(a)}$.

(4) For every $\alpha, \beta, \gamma \in a$, if $\alpha \geq_{E_n} \beta \geq_{E_n} \gamma$, then $\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$ for every $\rho \in \pi_{max(a),\alpha}$"$A$.

(5) For every $\alpha > \beta$ in a and every $\nu \in A$,

$$\pi_{\max(a),\alpha}(\nu) > \pi_{\max(a),\beta}(\nu) \ .$$

The last two conditions can be met by Lemmas 2.3, 2.4.

2.7 Definition. Let $\langle a, A, f\rangle, \langle b, B, g\rangle \in Q_{n0}$. We say that $\langle a, A, f\rangle$ *is stronger than* $\langle b, B, g\rangle$ and denote this by $\langle a, A, f\rangle \geq_0 \langle b, B, g\rangle$ iff

(1) $f \supseteq g$,

(2) $a \supseteq b$, and

(3) $\pi_{\max(a),\max(b)}$"$A \subseteq B$.

We now define a forcing notion Q_n which is an extender analog of the one-element Prikry forcing of 1.3.

2.8 Definition. $Q_n = Q_{n0} \cup Q_{n1}$.

2.9 Definition. The direct extension ordering $\leq^*$ on Q_n is defined to be $\leq_0 \cup \leq_1$.

2.10 Definition. Let $p, q \in Q_n$. Then $p \le q$ iff either

(1) $p \le^* q$, or

(2) $p = \langle a, A, f\rangle \in Q_{n0}, q \in Q_{n1}$ and the following holds:

(2a) $q \supseteq f$,

(2b) $\mathrm{dom}(q) \supseteq a$,

(2c) $q(\max(a)) \in A$, and

(2d) for every $\beta \in a$, $q(\beta) = \pi_{\max(a),\beta}(q(\max(a)))$.

Clearly, the forcing $\langle Q_n, \le\rangle$ is equivalent to $\langle Q_{n1}, \le_1\rangle$, i.e. Cohen forcing. However, the following basic facts relate it to the Prikry-type forcing notion.

2.11 Lemma. *$\langle Q_n, \le^*\rangle$ is κ_n-closed.*

2.12 Lemma. *$\langle Q_n, \le, \le^*\rangle$ satisfies the Prikry condition, i.e. for every $p \in Q_n$ and every statement σ of the forcing language there is a $q \ge^* p$ deciding σ.*

Proof. Let $p = \langle a, A, f\rangle$. Suppose otherwise. By recursion on $\nu \in A$ define an increasing sequence $\langle p_\nu \mid \nu \in A\rangle$ of elements of Q_{n1} with $\mathrm{dom}(p_\nu) \cap a = \emptyset$ as follows. Suppose $\langle p_\rho \mid \rho \in A \cap \nu\rangle$ is defined and $\nu \in A$. Define p_ν as follows: Let $u = \bigcup_{\rho<\nu} p_\rho$. Then $u \in Q_{n1}$. Consider $q = \langle a, A, u\rangle$. Let $q^\frown\langle\nu\rangle = u \cup \{\langle\beta, \pi_{\max(a),\beta}(\nu)\rangle \mid \beta \in a\}$. If there is a $p \ge_1 q^\frown\langle\nu\rangle$ deciding σ, then let p_ν be some such p restricted to $\lambda \setminus a$. Otherwise, set $p_\nu = u$. Note that there will always be a condition deciding σ.

Finally, let $g = \bigcup_{\nu\in A} p_\nu$. Shrink A to a set $B \in U_{n\max(a)}$ so that $p_\nu{}^\frown\langle\nu\rangle = p_\nu \cup \{\langle\beta, \pi_{\max(a),\beta}(\nu)\rangle \mid \beta \in a\}$ decides σ the same way or does not decide σ at all, for every $\nu \in B$. By our assumption $\langle a, B, g\rangle \not\parallel \sigma$. However, pick some $h \ge \langle a, B, g\rangle$, $h \in Q_{n1}$ deciding on σ. Let $h(\max(a)) = \nu$. Then, $p_\nu{}^\frown\langle\nu\rangle$ decides σ. But this holds then for every $\nu \in B$. Hence, already $\langle a, B, g\rangle$ decides σ. Contradiction. $\dashv$

Let us now define the main forcing of this section by putting the blocks of Q_n's together. This forcing is called the extender-based Prikry forcing over a singular cardinal.

2.13 Definition. The set $\mathcal{P}$ consists of sequences $p = \langle p_n \mid n < \omega\rangle$ so that

(1) For every $n < \omega$, $p_n \in Q_n$.

(2) There is an $\ell(p) < \omega$ so that for every $n < \ell(p)$, $p_n \in Q_{n1}$, and for every $n \ge \ell(p)$, $p_n = \langle a_n, A_n, f_n\rangle \in Q_{n0}$ and $a_n \subseteq a_{n+1}$.

2.14 Definition. Let $p = \langle p_n \mid n < \omega\rangle$ and $q = \langle q_n \mid n < \omega\rangle \in \mathcal{P}$. We set $p \ge q$ (resp. $p \ge^* q$) iff for every $n < \omega$, $p_n \ge_{Q_n} q_n$ (resp. $p_n \ge^*_{Q_n} q_n$)..

The forcing $\langle \mathcal{P}, \leq \rangle$ does not satisfy the κ^+-c.c. . However:

2.15 Lemma. *$\langle \mathcal{P}, \leq \rangle$ satisfies the κ^{++}-c.c.*

Proof. Let $\{p(\alpha) \mid \alpha < \kappa^{++}\}$ be a set of elements of $\mathcal{P}$, with $p(\alpha) = \langle p(\alpha)_n \mid n < \omega \rangle$ and $p(\alpha)_n = \langle a(\alpha)_n, A(\alpha)_n, f(\alpha)_n \rangle$ for $n \geq \ell(p(\alpha))$. There is an $S \subseteq \kappa^{++}$ stationary such that for every $\alpha, \beta \in S$ the following holds:

(a) $\ell(p(\alpha)) = \ell(p(\beta)) = \ell$.

(b) For every $n < \ell$, $\{\mathrm{dom}(p(\alpha)_n) \mid \alpha \in S\}$ forms a Δ-system with $p(\alpha)_n$ and $p(\beta)_n$ having the same values on its kernel.

(c) For every $n \geq \ell$, $\{(a(\alpha)_n \cup \mathrm{dom}(f(\alpha)_n) \mid \alpha \in S\}$ forms a Δ-system with $f(\alpha)_n$, $f(\beta)_n$ having the same values on the kernel. Also, if $\alpha, \beta \in S$ then $a(\alpha)_n \cap \mathrm{dom}(f(\beta)_n) = \emptyset$.

Now let $\alpha < \beta$ be in S. We construct a condition $q = \langle q_n \mid n < \omega \rangle$ stronger than both $p(\alpha)$ and $p(\beta)$.

For every $n < \ell$ let $q_n = p(\alpha)_n \cup p(\beta)_n$. Now suppose that $n \geq \ell$. q_n will be of the form $\langle b_n, B_n, g_n \rangle$. Set $g_n = f(\alpha)_n \cup f(\beta)_n$. We would like to define b_n as the union of $a(\alpha)_n$ and $a(\beta)_n$. But 2.6(2(iii)) requires the existence of a maximal element in the $\leq_{E_n}$ order which need not be the case in the simple union of $a(\alpha)_n$ and $a(\beta)_n$. It is easy to fix this. Just pick some $\rho < \lambda$ above $a(\alpha)_n \cup a(\beta)_n$ in the $\leq_{E_n}$ order. Also let $\rho > \sup(\mathrm{dom}(f(\alpha)_n)) + \sup(\mathrm{dom}\ (f(\beta))_n)$. Lemma 2.2 insures that there are such ρ's. Now we set $b_n = a(\alpha)_n \cup a(\beta)_n \cup \{\rho\}$. Let $B'_n = \pi^{-1}_{\rho\alpha^*}(A(\alpha)_n) \cap \pi^{-1}_{\rho\beta^*}(A(\beta)_n)$, where $\alpha^* = \max(a(\alpha)_n)$ and $\beta^* = \max(a(\beta)_n)$. Finally we shrink B'_n to a a set $B_n \in U_{n\rho}$ satisfying 2.6((4), (5)). This is possible by Lemmas 2.3, 2.4. $\dashv$

For $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$ set $p{\restriction}n = \langle p_m \mid m < n \rangle$ and $p \setminus n = \langle p_m \mid m \geq n \rangle$. Let $\mathcal{P}{\restriction}n = \{p{\restriction}n \mid p \in \mathcal{P}\}$ and $\mathcal{P} \setminus n = \{p \setminus n \mid p \in \mathcal{P}\}$. Then the following lemmas are obvious:

2.16 Lemma. *$\mathcal{P} \simeq \mathcal{P}{\restriction}n \times \mathcal{P} \setminus n$ for every $n < \omega$.*

2.17 Lemma. *$\langle \mathcal{P} \setminus n, \leq^* \rangle$ is κ_n-closed. Moreover, if $\langle p^\alpha \mid \alpha < \delta < \kappa \rangle$ is a $\leq^*$ increasing sequence with $\kappa_{\ell(p_0)} > \delta$, then there is a $p \geq^* p^\alpha$ for every $\alpha < \delta$.*

We will now turn to the Prikry condition and establish a more general statement which will allow us to deduce in addition that κ^+ is preserved after forcing with $\langle \mathcal{P}, \leq \rangle$.

Let us introduce first some notation. For $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$ and m with $\ell(p) \leq m < \omega$, let $p_m = \langle a_m, A_m, f_m \rangle$. Denote a_m by $a_m(p)$, A_m by $A_m(p)$ and f_m by $f_m(p)$. Let $\langle \nu_{\ell(p)}, \ldots, \nu_m \rangle \in \prod_{k=\ell(p)}^m A_k(p)$. We denote by

$$p^\frown \langle \nu_{\ell(p)}, \ldots, \nu_m \rangle$$

the condition obtained from p by adding the sequence $\langle \nu_{\ell(p)}, \dots, \nu_m \rangle$, i.e. a condition $q = \langle q_n \mid n < \omega \rangle$ such that $q_n = p_n$ for every n, $n < \ell(p)$ or $n > m$, and if $\ell(p) \le n \le m$ then $q_n = f_n(p) \cup \{\langle \beta, \pi_{\max(a_n(p))\beta}(\nu_n)\rangle \mid \beta \in a_n(p)\}$.

We prove the following analog of 1.13:

2.18 Lemma. *Let $p \in \mathcal{P}$ and D be a dense open subset of $\langle \mathcal{P}, \le \rangle$ above p. Then there are $p^* \ge^* p$ and $n^* < \omega$ such that for every $\langle \nu_0, \dots, \nu_{n^*-1}\rangle \in \prod_{m=\ell(p)}^{\ell(p)+n^*-1} A_m(p^*)$, $p^{*\frown}\langle \nu_0, \dots, \nu_{n^*-1}\rangle \in D$.*

Let us first deduce the Prikry condition from this lemma.

2.19 Lemma. *Let $p \in \mathcal{P}$ and σ be a statement of the forcing language. Then there is a $p^* \ge^* p$ deciding σ.*

Proof of 2.19 from 2.18. Consider $D = \{q \in \mathcal{P} \mid q \ge p \text{ and } q \parallel \sigma\}$. Clearly, D is dense open above p. Apply 2.18 to this D and choose n^* as small as possible and $p^* \ge^* p$ such that for every $q \ge p^*$ with $\ell(q) \ge n^*$, $q \in D$. If $n^* = \ell(p)$, then we are done. Suppose otherwise. Assume for simplicity that $\ell(p) = 0$ and $n^* = 2$. Then let $p^* = \langle p^*_n \mid n < \omega\rangle$ and for every $n < \omega$ let $p^*_n = \langle a^*_n, A^*_n, f^*_n\rangle$. Let $\alpha_0 = \max(a^*_0)$ and $\alpha_1 = \max(a^*_1)$. Then $A^*_0 \in U_{0\alpha_0}$ and $A^*_1 \in U_{1\alpha_1}$. Let $\nu_0 \in A^*_0$ and $\nu_1 \in A^*_1$. Consider $p^{*\frown}\langle \nu_0, \nu_1\rangle$ the condition obtained from p^* by adding ν_0 and ν_1. Clearly, $\ell(p^{*\frown}\langle \nu_0, \nu_1\rangle) = 2$. Hence it decides σ. Now we shrink A^*_1 to $A^*_{1\nu_0}$ so that for every $\nu'_1, \nu''_1 \in A^*_{1\nu_0}$ $p^{*\frown}\langle \nu_0, \nu'_1\rangle$ and $p^{*\frown}\langle \nu_0, \nu''_1\rangle$ decide σ the same way. Let $A^{**}_1 = \bigcap\{A^*_{1\nu_0} \mid \nu_0 \in A^*_0\}$. We shrink now A^*_0 to A^{**}_0 so that for every $\nu'_0, \nu''_0 \in A^{**}_0$ and for every $\nu_1 \in A^{**}_1$ $p^{*\frown}\langle \nu'_0, \nu_1\rangle$ and $p^{*\frown}\langle \nu''_0, \nu_1\rangle$ decide σ in the same way. Let p^{**} be a condition obtained from p^* by replacing in it A^*_0 by A^{**}_0 and A^*_1 by A^{**}_1. Then $p^{**} \ge^* p^*$ and $p^{**} \parallel \sigma$. Contradiction.

Proof of 2.18. The main objective is to reduce the problem to the point where we can use the argument of the corresponding fact in §1.3, as if we were forcing using $\langle U_{n \max(a_n)} \mid n < \omega\rangle$.

We first prove the following crucial claim:

Claim. There is a $p' \ge^* p$, $p' = \langle p'_n \mid n < \omega\rangle$, such that for every $q \ge p'$, $q = \langle q_n \mid n < \omega\rangle$, if $q \in D$, then also

$$\langle p'_n \mid n < \ell(p)\rangle^\frown \langle q_n \restriction a_n(p') \cup f_n(p') \mid \ell(p') \le n < \ell(q)\rangle^\frown \langle p'_n \mid n \ge \ell(q)\rangle \in D\,,$$

where $p'_n = \langle a_n(p'), A_n(p'), f_n(p')\rangle$ for $n \ge \ell(p)$.

Proof of Claim. Choose a function $h : \kappa \leftrightarrow [\kappa]^{<\omega}$, such that for every $n < \omega$, $h \restriction \kappa_n : \kappa_n \leftrightarrow [\kappa_n]^{<\omega}$. Now define by recursion a $\le^*$-increasing sequence $\langle p^\alpha \mid \alpha < \kappa\rangle$ of direct extensions of p, where $p^\alpha = \langle p^\alpha_n \mid n < \omega\rangle$ and, for $n \ge \ell(p)$, $p^\alpha_n = \langle a^\alpha_n, A^\alpha_n, f^\alpha_n\rangle$. Set $p^0 = p$. Suppose that $\alpha < \kappa$ and

$\langle p^\beta \mid \beta < \alpha\rangle$ has been defined. As a recursive assumption we assume the following:

(∗) For every $n < \omega$ and for $\beta, \gamma, \kappa_n \le \beta, \gamma < \kappa$,
$$\text{if } \ell(p) \le m \le n+1, \text{then } a_m^\beta = a_m^\gamma \text{ and } A_m^\beta = A_m^\gamma .$$

Let $\tilde{p}^\alpha$ be $p^{\alpha-1}$ if α is successor ordinal, and a direct extension of $\langle p^\beta \mid \beta < \alpha\rangle$ satisfying (*) if α is a limit ordinal. Note that if $n < \omega$ is the maximal such that $\alpha \ge \kappa_n$ then 2.17 applies, since the parts of p^β's below κ_{n+1} satisfy (*). Now we consider $h(\alpha)$. Let $h(\alpha) = \langle \nu_1, \dots, \nu_k\rangle$. If $\langle \nu_0, \dots, \nu_{k-1}\rangle \notin \prod_{m=\ell(p)}^{\ell(p)+k-1} A_m(\tilde{p}^\alpha)$, then we set $p^\alpha = \tilde{p}^\alpha$, where for $m \ge \ell(p)$, $\tilde{p}_m^\alpha = \langle a_m(\tilde{p}^\alpha), A_m(\tilde{p}^\alpha), f_m(\tilde{p}^\alpha)\rangle$. Otherwise we consider $q = \tilde{p}^{\alpha\frown}\langle \nu_0, \dots, \nu_{k-1}\rangle$. If there is no direct extension of q inside D, then let $p^\alpha = \tilde{p}^\alpha$. Otherwise, let $s = \langle s_n \mid n < \omega\rangle \ge^* q$ be in D. Define $p^\alpha = \langle p_n^\alpha \mid n < \omega\rangle$ then as follows:

(a) For each n with $n \ge \ell(p) + k$ or $n < \ell(p)$, let $p_n^\alpha = s_n$, and

(b) For each n with $\ell(p) \le n \le \ell(p) + k - 1$, $a_n(p^\alpha) = a_n(\tilde{p}^\alpha)$, $A_n(p^\alpha) = A_n(\tilde{p}^\alpha)$, and $f_n(p^\alpha) = f_n(s)\restriction((\mathrm{dom}(f_n(s)) \setminus a_n(\tilde{p}^\alpha))$.

The meaning of this last part of the definition is that we extend for n with $\ell(p) \le n \le \ell(p) + k - 1$ only $f_n(\tilde{p}^\alpha)$ and only outside of $a_n(\tilde{p}^\alpha)$. Clearly such defined p^α satisfies (∗).

Finally, (∗) allows us to put all the $\langle p^\alpha \mid \alpha < \kappa\rangle$ together. Thus we define $p' = \langle p'_n \mid n < \omega\rangle$ as follows:

(i) For $n < \ell(p)$, let $p'_n = \bigcup_{\alpha<\kappa} p_n^\alpha$.

(ii) For $n \ge \ell(p)$, let $f_n(p') = \bigcup_{\alpha<\kappa} f_n(p^\alpha)$, $a_n(p') = a_n(p^{\kappa_n})$, and $A_n(p') = A_n(p^{\kappa_n})$.

Obviously $p' \in \mathcal{P}$ and $p' \ge^* p$. This p' is as desired. Thus, if $q \ge p'$ is in D, then we consider $\alpha = h^{-1}(\langle q_n(\max(a_n(p'))) \mid \ell(p) \le n < \ell(q)\rangle)$. By the construction of $p^\alpha \le^* p'$, $p^{\alpha\frown}\langle q_n(\max(a_n(p'))) \mid \ell(p) \le n < \ell(q)\rangle$ will be in D. Then also $p'^\frown\langle q_n(\max(a_n(p')) \mid \ell(p) \le n < \ell(q)\rangle \in D$, since D is open.

This concludes the proof of the Claim.

Now let $p' \ge^* p$ be given by the claim. Assume for simplicity that $\ell(p) = 0$. We would like to shrink the sets $A_n(p')$ in a certain way. Thus define $p(1) \ge^* p'$ such that:

$(*)_1$ For every $m < \omega$ and $\langle \nu_0, \dots, \nu_{m-1}\rangle \in \prod_{n=0}^{m-1} A_n(p(1))$, if for some $\nu \in A_m(p(1))$, $p(1)^\frown\langle \nu_0, \dots, \nu_{m-1}, \nu\rangle \in D$, then for every $\nu' \in A_m(p(1))$ $p(1)^\frown\langle \nu_0, \dots, \nu_{m-1}, \nu'\rangle \in D$.

Let $m < \omega$ and $\vec{\nu} = \langle \nu_0, \dots, \nu_{m-1} \rangle \in \prod_{n=0}^{m-1} A_n(p')$, where in case of $m = 0$, $\vec{\nu}$ is the empty sequence. Consider the set

$$X_{m,\vec{\nu}} = \{\nu \in A_m(p') \mid p'^\frown \langle \nu_0, \dots, \nu_{m-1}, \nu \rangle \in D\} .$$

Define $A_{m\vec{\nu}}$ to be $X_{m,\vec{\nu}}$, if $X_{m\vec{\nu}} \in U_{m,\max(a_m(p'))}$ and $A_m(p') \setminus X_{m,\vec{\nu}}$, otherwise. Let $A_m = \bigcap \{A_{m,\vec{\nu}} \mid \vec{\nu} \in \prod_{n=0}^{m-1} A_n(p')\}$. Define now $p(1) = \langle p(1)_n \mid n < \omega \rangle$ as follows: for each $n < \omega$ let $p(1)_n = \langle a_n(p'), A_n, f(p') \rangle$. Clearly, such defined $p(1)$ satisfies $(*)_1$.

Then, in a similar fashion we chose $p(2) \geq^* p(1)$ satisfying:

$(*)_2$ For every $m < \omega$ and $\langle \nu_0, \dots, \nu_{m-1} \rangle \in \prod_{n=0}^{m-1} A_n(p(2))$, if for some $\langle \nu_m, \nu_{m+1} \rangle \in A_m(p(2)) \times A_{m+1}(p(2))$,

$$p(2)^\frown \langle \nu_0, \dots, \nu_{m-1} \rangle^\frown \langle \nu_m, \nu_{m+1} \rangle \in D,$$

then for every $\langle \nu'_m, \nu'_{m+1} \rangle \in A_m(p(2)) \times A_{m+1}(p(2))$,

$$p(2)^\frown \langle \nu_0, \dots, \nu_{m-1} \rangle^\frown \langle \nu'_m, \nu'_{m+1} \rangle \in D .$$

Continue and define for every k with $2 \leq k < \omega$ a $p(k) \geq^* p(k-1)$ satisfying $(*)_k$, where $(*)_k$ is defined analogously for k-sequences. Finally, let p^* be a direct extension of $\langle p(k) \mid 1 \leq k < \omega \rangle$. Let $s \geq p^*$ be in D. Set $n^* = \ell(s)$. Consider $\langle s_0(\max(a_0(p^*))), \dots, s_{n^*-1}(\max(a_{n^*-1}(p^*))) \rangle$. Then the choice of p', $p' \leq^* p^*$ and openness of D imply that

$$p^{*\frown} \langle s_0(\max(a_0(p^*)), \dots, s_{n^*-1}(\max(a_{n^*-1}(p^*))) \rangle \in D$$

But $p^* \geq^* p(n^*)$. So, p^* satisfies $(*)_{n^*}$. Hence, for $\langle \nu_0, \dots, \nu_{n^*-1} \rangle \in \prod_{m=0}^{n^*-1} A_m(p^*)$, $p^{*\frown} \langle \nu_0, \dots, \nu_{n^*-1} \rangle \in D$. $\dashv$

Combining these lemmas we obtain the following:

2.20 Proposition. *The forcing $\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets to κ and preserves all the cardinals above κ^+.*

Actually, it is not hard now to show that κ^+ is preserved as well.

2.21 Lemma. *Forcing with $\langle \mathcal{P}, \leq \rangle$ preserves κ^+.*

Proof. Suppose that $(\kappa^+)^V$ is not a cardinal in a generic extension $V[G]$. Recall that $\mathrm{cf}(\kappa) = \aleph_0$ and by 2.20 it is preserved. So, $\mathrm{cf}((\kappa^+)^V) < \kappa$ in $V[G]$. Pick $p \in G$, $\delta < \kappa$ and a name $\underset{\sim}{g}$ so that $\kappa_{\ell(p)} > \delta$ and

$$p \Vdash (\underset{\sim}{g} : \check{\delta} \to (\kappa^+)^V \text{ and } \mathrm{ran}(\underset{\sim}{g}) \text{ is unbounded in } (\kappa^+)^V) .$$

For every $\tau < \delta$ let

$$D_\tau = \{q \in \mathcal{P} \mid q \geq p \text{ and for some } \alpha < \kappa^+,\ q \Vdash \underset{\sim}{g}(\check{\tau}) = \check{\alpha}\} .$$

Define by recursion, using 2.17, a $\leq^*$-increasing sequence $\langle p^\tau \mid \tau < \delta\rangle$ of $\leq^*$-extensions of p so that p^τ satisfies the conclusion 2.18 with $D = D_\tau$. By 2.17, there is a $p^\delta \geq^* p^\tau$ for each $\tau < \delta$.

Now let $\tau < \delta$. By the choice of p^τ there is an $n(\tau) < \omega$ such that for every $\langle \nu_0, \dots, \nu_{n(\tau)-1}\rangle \in \prod_{m=\ell(p)}^{\ell(p)+n(\tau)-1} A_m(p^\delta)$, $p^\delta{}^\frown\langle \nu_0, \dots, \nu_{n(\tau)-1}\rangle \in D_\tau$. This means that for some $\alpha(\nu_0, \dots, \nu_{n(\tau)-1}) < \kappa^+$

$$p^\delta{}^\frown\langle \nu_0, \dots, \nu_{n(\tau)-1}\rangle \Vdash \underset{\sim}{g}(\check{\tau}) = \check{\alpha}(\nu_0, \dots, \nu_{n(\tau)-1}) \ .$$

Set

$$\alpha(\tau) = \sup\{\alpha(\nu_0, \dots, \nu_{n(\tau)-1}) \mid \langle \nu_0, \dots, \nu_{n(\tau)-1}\rangle \in \textstyle\prod_{m=\ell(p)}^{\ell(p)+n(\tau)-1} A_m(p^\delta)\} \ .$$

Then clearly $\alpha(\tau) < \kappa^+$ and

$$p^\delta \Vdash \underset{\sim}{g}(\check{\tau}) < \check{\alpha}(\tau) \ .$$

Now let $\alpha^* = \bigcup_{\tau<\delta} \alpha(\tau)$. Then again $\alpha^* < \kappa^+$ and

$$p^\delta \Vdash \forall \tau < \check{\delta}(\underset{\sim}{g}(\tau) < \check{\alpha}^*) \ .$$

But this is impossible since $p \leq^* p^\delta$ forced that the range of g was unbounded in κ^+. Contradiction. $\dashv$

Finally, let us show that this forcing adds λ ω-sequences to κ. Thus, let $G \subseteq \mathcal{P}$ be generic. For every $n < \omega$ define a function $F_n : \lambda \to \kappa_n$ as follows:

$F_n(\alpha) = \nu$ if for some $p = \langle p_m \mid m < \omega\rangle \in G$ with $\ell(p) > n$, $p_n(\alpha) = \nu$.

Now for every $\alpha < \lambda$ set $t_\alpha = \langle F_n(\alpha) \mid n < \omega\rangle$. Let us show that the set $\{t_\alpha \mid \alpha < \lambda\}$ has cardinality λ. Notice that we cannot claim that all such sequences are new or even distinct due to the Cohen parts of conditions, i.e. the f_n's.

2.22 Lemma. *For every $\beta < \lambda$ there is an α with $\beta < \alpha < \lambda$ such that t_α dominates every t_γ with $\gamma \leq \beta$.*

Proof. Suppose otherwise. Then there is a $p = \langle p_n \mid n < \omega\rangle \in G$ and $\beta < \lambda$ such that

$$p \Vdash \forall \alpha(\beta < \alpha < \lambda \to \exists \gamma \leq \beta\ (\underset{\sim}{t_\alpha} \text{ does not dominate } \underset{\sim}{t_\gamma}))$$

For every $n \geq \ell(p)$ let $p_n = \langle a_n, A_n, f_n\rangle$. Pick some

$$\alpha \in \lambda \setminus (\textstyle\bigcup_{n<\omega} a_n \cup \bigcup \mathrm{dom}(f_n) \cup (\beta + 1)) \ .$$

We extend p to a condition q so that $q \geq^* p$ and for every $n \geq \ell(q) = \ell(p)$, $\alpha \in b_n$, where $q_n = \langle b_n, B_n, g_n \rangle$. Then q will force that t_α dominates every t_γ with $\gamma < \alpha$. This leads to the contradiction. Thus, let $\gamma < \alpha$ and assume that q belongs to the generic subset of $\mathcal{P}$. Then either $t_\gamma \in V$ or it is a new ω-sequence. If $t_\gamma \in V$ then it is dominated by t_α by the usual density arguments. If t_γ is new, then for some $r \geq q$ in the generic set $\gamma \in c_n$ for every $n \geq \ell(r)$, where $r_n = \langle c_n, C_n, h_n \rangle$. But also $\alpha \in c_n$ since $c_n \supseteq b_n$. This implies $F_n(\alpha) > F_n(\gamma)$ (by 2.6(5)) and we are done. $\dashv$

We now have the following conclusion.

2.23 Theorem. *The following holds in $V[G]$:*

(a) *All cardinals and cofinalities are preserved.*

(b) *No new bounded subsets are added to κ; in particular* GCH *holds below κ.*

(c) *There are λ new ω-sequences in $\prod_{n<\omega} \kappa_n$. In particular $2^\kappa \geq \lambda$.*

2.24 Remark. The initial large cardinal assumptions used here are not optimal. We refer to Mitchell's chapter [41] on the Covering Lemma for matters of the consistency strength. In the next section another extender-based Prikry forcing requiring much weaker extenders will be introduced.

It is tempting to extend 2.22 and claim that $\langle t_\alpha \mid \alpha < \lambda \text{ and } t_\alpha \notin V \rangle$ is a scale in $\prod_{n<\omega} \kappa_n$, i.e. for every $t \in \prod_{n<\omega} \kappa_n$ there is an $\alpha < \lambda$ such $t_\alpha \notin V$ and t_α dominates t. Unfortunately this is not true in general. We need to replace $\prod_{n<\omega} \kappa_n$ by the product of a sequence $\langle \lambda_n \mid n < \omega \rangle$ related to λ (basically the Prikry sequence for $U_{n\lambda}$ whenever it is defined). Assaf Sharon [50] made a full analysis of possible cofinalities structure for a similar forcing (the one that will be discussed in the next section). Let us now deal with a special case that cannot be covered by such forcing. Let us assume that for every $n < \omega$, $j_n(\kappa_n) = \lambda$, where $j_n : V \to M_n \simeq \mathrm{Ult}(V, E_n)$ is the canonical embedding. In particular each κ_n is a superstrong cardinal. Then the following holds.

2.25 Lemma. *Let $t \in \prod_{n<\omega} \kappa_n$ in $V[G]$. Then there is an $\alpha < \lambda$ such that $t_\alpha \notin V$ and for all but finitely many $n < \omega$, $t_\alpha(n) > t(n)$.*

Proof. Let $\underset{\sim}{t}$ be a name of t. Pick $p \in G$ forcing "$\underset{\sim}{t} \in \prod_{n<\omega} \check{\kappa}_n$". Define for every $n < \omega$ a set dense open above p:

$$D_n = \{q \in \mathcal{P} \mid q \geq p \text{ and there is a } \nu_n < \kappa_n \text{ such that } q \Vdash \underset{\sim}{t}(n) = \check{\nu}_n\} \; .$$

Apply 2.18 to each of D_n's and construct a $\leq^*$-sequence $\langle p(k) \mid k < \omega \rangle$ of direct extensions of p such that $p(k)$ and D_k satisfy the conclusion of

2.18. Let p^* be a common direct extension of $p(k)$'s. Then for every $k, 1 \leq k < \omega$, there is an $n(k) < \omega$ such that for every $\langle \nu_0, \ldots, \nu_{n(k)-1} \rangle \in \prod_{m=\ell(p)}^{\ell(p)+n(k)-1} A_m(p^*)$,

$$p^{*\frown}\langle \nu_0, \ldots, \nu_{n(k)-1} \rangle \Vdash \underset{\sim}{t}(k-1) = \check{\xi}(\nu_0, \ldots, \nu_{n(k)-1})$$

for some $\xi(\nu_0, \ldots, \nu_{n(k)-1}) < \kappa_{k-1}$. Assume for simplicity of notation that $\ell(p) = 0$. Let $1 \leq k < \omega$. We can assume that $\xi(\nu_0, \ldots, \nu_{n(k)-1})$, defined above, depends really only on $\nu_0, \ldots, \nu_{k-1}$, since its values are below κ_{k-1} and ultrafilters over κ_m's are κ_k-complete for $m \geq k$. Also assume that for every $m > 0$ $A_m(p^*) \cap \kappa_{m-1} = \emptyset$. Now, we replace ξ by a bigger function η depending only on ν_{k-1}. Thus set

$$\eta(\nu_{k-1}) = \bigcup\{\xi(\nu_0, \ldots, \nu_{k-2}, \nu_{k-1}) \mid \langle \nu_0, \ldots, \nu_{k-2} \rangle \in \prod_{m=0}^{k-2} A_m(p^*)\} + \nu_{k-1}.$$

Clearly, $\eta(\nu_{k-1}) < \kappa_{k-1}$. So,

$$p^{*\frown}\langle \nu_0, \ldots, \nu_{k-1} \rangle \Vdash \underset{\sim}{t}(k-1) < \check{\eta}(\nu_{k-1})$$

for every k, $1 \leq k < \omega$ and every $\langle \nu_0, \ldots, \nu_{k-1} \rangle \in \prod_{m=0}^{k-1} A_m(p^*)$. For every $n < \omega$ let $\eta_n : A_n(p^*) \to \kappa_n$ be the restriction of η to κ_n. Let $\alpha_n = \max(a_n(p^*))$. Consider $j_n(\eta_n)(\alpha_n)$ where $j_n : V \to M_n$ is the embedding of the extender E_n. Then $j_n(\eta_n)(\alpha_n) < j_n(\kappa_n) = \lambda$. Choose some α below λ and above $\bigcup_{n<\omega} j_n(\eta_n)(\alpha_n) \cup (\mathrm{dom}(f_n(p^*)))$. Now extend p^* to a condition p^{**} such that $p^{**} \geq^* p^*$ and for every $n < \omega$ $\alpha \in a_n(p^{**})$. Then,

$$p^{**} \Vdash \forall n (\underset{\sim}{t_\alpha}(n) > \eta_n((\underset{\sim}{t_{\alpha_n}}(n)) > \underset{\sim}{t}(n))$$

So we are done. ⊣

The extender-based forcing described in this section can also be used with much stronger extenders than those used here. Thus with minor changes we can deal with E_n's such that $j_n(\kappa_n) < \lambda$ but requiring $j_n(\kappa_{n+1}) > \lambda$. Once $j_n(\kappa_{n+1}) \leq \lambda$ for infinitely many n's then the arguments like one in the proof of the Prikry condition seem to break down completely.

Another probably more exciting direction is to use shorter extenders instead of long ones. Thus it turned out that for λ's below $\kappa^{+\omega_1}$ an extender of length κ_n^{+n} over κ_n for $n < \omega$ suffices. The basic idea is to replace in $p \in \mathcal{P}$ the subset $a_n(p)$ of λ by an order preserving function from λ to κ_n^{+n}. Such defined forcing fails to satisfy κ^{++}-c.c. and actually will collapse λ to κ^+. But using increasing with n similarity of ultrafilters involved in the extenders, it turns out that there is a subforcing satisfying the κ^{++}-c.c. and still producing λ new sequences in $\prod_{n<\omega} \kappa_n$. This approach was implemented in [17] for calculating the consistency strength of various instances of the failure of the Singular Cardinal Hypothesis and, as well, for constructing more complicated cardinal arithmetic configurations.

3. Extender-Based Prikry Forcing with a Single Extender

In this section we present a simplified version of the original extender-based Prikry forcing of [19], Sec 1. Our aim is simultaneously to change the cofinality of a regular cardinal to $\aleph_0$ and blow up its power. Recall that the Prikry forcing of 1.1 and 1.2 does the first part, i.e. change cofinality. As in the previous section, we would like to use an extender instead of a single ultrafilter in order to blow up the power.

Let κ, λ be regular cardinals with $\lambda \geq \kappa^{++}$. Assume that κ is $V_{\kappa+\delta}$-strong for a δ so that $\kappa^{+\delta} = \lambda$. Let E be an extender over κ witnessing this and $j : V \longrightarrow M \simeq \mathrm{Ult}(V, E)$ with $M \supseteq V_{\kappa+\delta}$ be the corresponding elementary embedding. Suppose also that there is a function $f_\lambda : \kappa \to \kappa$ such that $j(f_\lambda)(\kappa) = \lambda$. Notice that such a function always exists for small λ's like $\lambda = \kappa^{++}$, $\lambda = \kappa^{+116}$, $\lambda = \kappa^{+\kappa+1}$ etc., just take mappings $\alpha \to \alpha^{++}$, $\alpha \to \alpha^{+116}$, $\alpha \to \alpha^{+\alpha+1}$. In general, assuming $j(\kappa) > \lambda$, it is not hard to force such f_λ. The idea is to force for every inaccessible $\alpha \leq \kappa$ a generic function from α to α and then to extend the embedding specifying to κ the value λ under the generic function from $j(\kappa)$ to $j(\kappa)$ in M.

If κ is a strong cardinal then for every $\lambda > \kappa$ there is a (κ, λ)-extender E and a function $f : \kappa \to \kappa$ so that $j_E(f)(\kappa) = \lambda$, where $j_E : V \to M \simeq \mathrm{Ult}(V, E)$. The Solovay argument [56], originally used for a supercompact κ, works without change for a strong cardinal κ: Let κ be a strong cardinal and suppose that for some $\lambda > \kappa$ for every (κ, λ)-extender E and every function $f : \kappa \to \kappa$ we have $j_E(f)(\kappa) \neq \lambda$. Let λ be the least such ordinal. Pick a $(\kappa, 2^{2^\lambda})$-extender E^*. Let $j : V \to M \simeq \mathrm{Ult}(V, E^*)$. Then, in M, λ will be the least such that for every (κ, λ)-extender E and every function $f : \kappa \to \kappa$, $j_E(f)(\kappa) \neq \lambda$, since $M \supseteq V_{2^{2^\lambda}}$. Now define a function $g : \kappa \to \kappa$ as follows: $g(\alpha) =$ the least $\beta > \alpha$ such that for every (α, β)-extender E and every function $f : \alpha \to \alpha$, $j_E(f)(\alpha) \neq \beta$, if there is such a β and let $g(\alpha) = 0$ otherwise. Then, clearly, $j(g)(\kappa) = \lambda$. But then $E^* {\restriction} \lambda$ and g provide the contradiction.

Suppose for simplicity that V satisfies GCH. Then we will have ${}^{\kappa^+}V_{\kappa+\delta} \subseteq M$. For every $\alpha < \lambda$ define a κ-complete ultrafilter U_α over κ by setting $X \in U_\alpha$ iff $\alpha \in j(X)$. Notice that U_κ will be normal and each U_α with $\alpha < \kappa$ will be trivial; we shall ignore such U_α and refer to U_κ as the least one. As in Section 2, we define a partial ordering $\leq_E$ on λ:

$$\alpha \leq_E \beta \text{ iff } \alpha \leq \beta \text{ and for some } f \in^\kappa \kappa,\ j(f)(\beta) = \alpha\ .$$

Again, clearly, $\alpha \leq_E \beta$ implies that $U_\alpha \leq_{RK} U_\beta$ as witnessed by any $f \in {}^\kappa\kappa$ with $j(f)(\beta) = \alpha$. In the previous section only the κ directedness (more precisely, κ_n directedness for every $n < \omega$) of the ordering was used. Here we will need more — κ^{++}-directedness. Thus, as in Section 2, fix an enumeration $\langle a_\alpha \mid \alpha < \kappa \rangle$ of $[\kappa]^{<\kappa}$ so that for every regular cardinal $\mu < \kappa$,

$\langle a_\alpha \mid \alpha < \mu\rangle$ enumerates $[\mu]^{<\mu}$ and every element of $[\mu]^{<\mu}$ appears μ many times in the enumeration. Let $j(\langle a_\alpha \mid \alpha < \kappa\rangle) = \langle a_\alpha \mid \alpha < j(\kappa)\rangle$. Then, $\langle a_\alpha \mid \alpha < \lambda\rangle$ will enumerate $[\lambda]^{<\lambda} \supseteq [\lambda]^{<\kappa^{++}}$. For each $\alpha < \lambda$ we consider the following basic commutative diagram:

$$\begin{tikzcd} & M \\ V \arrow[ur, "j"] \arrow[dr, "i_\alpha"'] & \\ & N_\alpha \arrow[uu, "k_\alpha"'] \end{tikzcd}$$

where $i_\alpha : V \longrightarrow N_\alpha \simeq \mathrm{Ult}(V, U_\alpha)$ and $k_\alpha([f]_{U_\alpha}) = j(f)(\alpha)$.

3.1 Lemma. $\mathrm{crit}(k_\alpha) = (\kappa^{++})^{N_\alpha}$.

Proof. It is enough to show that $k_\alpha(\kappa) = \kappa$, since $k_\alpha((\kappa^+)^{N_\alpha}) = \kappa^+$ and $k_\alpha((\kappa^{++})^{N_\alpha}) = \kappa^{++}$ by elementarity. But ${}^\kappa N_\alpha \subseteq N_\alpha$. Hence $(\kappa^+)^{N_\alpha} = \kappa^+$. By $2^\kappa = \kappa^+$, $(\kappa^{++})^{N_\alpha} < \kappa^{++}$. So $(\kappa^{++})^{N_\alpha}$ is the first ordinal moved by k_α.

In order to show that κ is fixed let us use the function $f_\lambda : \kappa \to \kappa$ representing λ in M. Thus by commutativity, $k_\alpha(i_\alpha(f_\lambda)) = j(f_\lambda)$. Clearly, $i_\alpha(f_\lambda) : i_\alpha(\kappa) \to i_\alpha(\kappa)$ and $i_\alpha(f_\lambda){\restriction}\kappa = f_\lambda$. Hence

$$N_\alpha \vDash \forall \tau < \kappa \ (i_\alpha(f_\lambda)(\tau) < \kappa)\ .$$

Using k_α we obtain that

$$M \vDash \forall \tau < k_\alpha(\kappa) \ (j(f_\lambda)(\tau) < k_\alpha(\kappa))\ .$$

But $k_\alpha(\kappa) \le k_\alpha([id]_{U_\alpha}) = \alpha < \lambda$. Hence,

$$M \vDash \forall \tau < k_\alpha(\kappa) \ (j(f_\lambda)(\tau) < \lambda)\ .$$

But $k_\alpha(\kappa) \ge \kappa$ and $j(f_\lambda)(\kappa) = \lambda$. So, $k_\alpha(\kappa)$ must be equal to κ and we are done. ⊣

The following is a consequence of the previous lemma.

3.2 Lemma. *For every α with $\kappa \le \alpha < \lambda$, $\alpha \ge_E \kappa$.*

Proof. By 3.1, $k_\alpha(\kappa) = \kappa$. So, $k_\alpha([g]_{U_\alpha}) = \kappa$ for $g : \kappa \to \kappa$ representing κ in N_α. Then g projects U_α on U_κ and $j(g)(\alpha) = k_\alpha([g]_{U_\alpha}) = \kappa$. ⊣

We can now improve 2.1 to κ^{++}-directedness.

3.3 Lemma. *Let $\langle \alpha_\nu \mid \nu < \kappa^+\rangle$ be a sequence of ordinals below λ. Suppose that $\alpha \in \lambda \setminus (\bigcup_{\nu<\kappa^+} \alpha_\nu + 1)$ codes $\{\alpha_\nu \mid \nu < \kappa^+\}$, i.e. $a_\alpha = \{\alpha_\nu \mid \nu < \kappa^+\}$. Then $\alpha >_E \alpha_\nu$ for every $\nu < \kappa^+$.*

Proof. Let $\nu < \kappa^+$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & M \\
V \arrow[ur, "j"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"'] \\
 & N_{\alpha_\nu} \arrow[uu, bend right, "k_{\alpha_\nu}"']
\end{tikzcd}$$

Then $j(\langle a_\beta \mid \beta < \kappa\rangle) = k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa\rangle))$. Let $\alpha^* = [id]_{U_\alpha}$. Then $k_\alpha(\alpha^*) = \alpha$. So, $a_\alpha = k_\alpha(a^*_{\alpha^*})$, where $a^*_{\alpha^*} = i_\alpha(\langle a_\beta \mid \beta < \kappa\rangle)(\alpha^*)$. But $a_\alpha = \{\alpha_{\nu'} \mid \nu' < \kappa^+\}$ and, by 3.1, $k_\alpha(\kappa^+) = \kappa^+$. So, $a^*_{\alpha^*} = \{\alpha^*_{\nu'} \mid \nu' < \kappa^+\}$, where $k_\alpha(\alpha^*_{\nu'}) = \alpha_{\nu'}$. Now we can define an elementary embedding $k_{\alpha_\nu\alpha} : N_{\alpha_\nu} \longrightarrow N_\alpha$. Set

$$k_{\alpha_\nu\alpha}([f]_{U_{\alpha_\nu}}) = i_\alpha(f)(\alpha^*_\nu)\ .$$

Finally every function representing α^*_ν in N_α will be a projection of U_α onto U_{α_ν} and witness $\alpha_\nu <_E \alpha$. $\dashv$

For $\beta \leq_E \alpha < \lambda$ we fix a projection $\pi_{\alpha\beta} : \kappa \to \kappa$ defined as in 3.3. Let $\pi_{\alpha\alpha} = id$. The following two lemmas were actually proved in the previous section (2.3 and 2.4).

3.4 Lemma. *Let $\gamma < \beta \leq \alpha < \lambda$. If $\alpha \geq_E \beta$ and $\alpha \geq_E \gamma$, then $\{\nu < \kappa \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in U_\alpha$.*

3.5 Lemma. *Let $\alpha, \beta, \gamma < \lambda$ be so that $\alpha \geq_E \beta \geq_E \gamma$. Then there is an $A \in U_\alpha$ so that for every $\nu \in A$*

$$\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu))\ .$$

Consider the following set:

$$\overline{X} = \{\nu < \kappa \mid \exists \nu^* \leq \nu(\nu^* \text{ is inaccessible}, \\ f_\lambda {\restriction} \nu^* : \nu^* \longrightarrow \nu^*, \text{ and } f_\lambda(\nu^*) > \nu)\}\ .$$

Clearly $\overline{X} \in U_\alpha$ for every $\alpha < \lambda$ (ignoring α's below κ). Also the function $g : \overline{X} \to \kappa$ defined by $g(\nu) =$ the maximal inaccessible $\nu^* \leq \nu$ closed under f_λ and with $f_\lambda(\nu^*) > \nu$, projects each U_α onto U_κ. Let us change each $\pi_{\alpha\kappa}$ to g on $\overline{X}$ and for $\nu \in \kappa \setminus \overline{X}$ let $\pi_{\alpha\kappa}(\nu) = 0$. Also change $\pi_{\alpha\beta}$'s a little for $\alpha, \beta > \kappa$. Thus for $\nu \in \kappa \setminus \overline{X}$ let $\pi_{\alpha\beta}(\nu) = 0$. If $\nu \in \overline{X}$ and $\pi_{\alpha\beta}(\nu)$ is below $\pi_{\alpha\kappa}(\nu)$ then change $\pi_{\alpha\beta}(\nu)$ to ν or any ordinal between $\pi_{\alpha\kappa}(\nu)$ and ν. Note that these changes are on a small set since $\kappa \setminus \overline{X} \notin U_\alpha$ for any $\alpha < \lambda$. Hence the changed $\pi_{\alpha\beta}$'s are still projections. The following summarizes the main properties of the U_α's and $\pi_{\alpha\beta}$'s:

(1) $\langle \lambda, \leq_E \rangle$ is a κ^{++}-directed partial ordering.

(2) $\langle U_\alpha \mid \alpha < \lambda \rangle$ is a Rudin-Keisler commutative sequence of κ-complete ultrafilters over κ with projections $\langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \lambda,\ \alpha \geq_E \beta \rangle$.

(3) For every $\alpha < \lambda$ $\pi_{\alpha\alpha}$ is the identity on a fixed set $\overline{X}$ which belongs to every U_β for $\beta < \lambda$.

(4) (Commutativity) For every $\alpha, \beta, \gamma < \lambda$ such that $\alpha \geq_E \beta \geq_E \gamma$, there is a $Y \in U_\alpha$ so that for every $\nu \in Y$

$$\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu)).$$

(5) For every $\alpha < \beta$, $\gamma < \lambda$, if $\gamma \geq_E \alpha, \beta$ then

$$\{\nu < \kappa \mid \pi_{\gamma\alpha}(\nu) < \pi_{\gamma\beta}(\nu)\} \in U_\gamma.$$

(6) U_κ is a normal ultrafilter.

(7) $\kappa \leq_E \alpha$ when $\kappa \leq \alpha < \lambda$.

(8) (Full commutativity at κ) For every $\alpha, \beta < \lambda$ and $\nu < \kappa$, if $\alpha \geq_E \beta$ then $\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\pi_{\alpha\beta}(\nu))$.

(9) (Independence of the choice of projection to κ) For every $\alpha, \beta, \kappa \leq \alpha$, $\beta < \lambda$, and $\nu < \kappa$

$$\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\nu).$$

(10) Each U_α is a P-point ultrafilter, i.e. for every $f \in {}^\kappa\kappa$, if f is not constant $\mod U_\alpha$, then there is a $Y \in U_\alpha$ such that for every $\nu < \kappa$ $|Y \cap f^{-1}\{\nu\}| < \kappa$.

The last property just follows using the set $\overline{X}$ defined above and the normality of U_κ.

A system of ultrafilters and projections satisfying (0)-(9) was called in [19] a *nice system*. Its existence is a bit weaker than the strongness assumption used here. In what follows we will use only such a system in order to define extender based Prikry forcing over κ.

Let us denote $\pi_{\alpha\kappa}(\nu)$ by ν^0, where $\kappa \leq \alpha < \lambda$ and $\nu < \kappa$. By a $\circ$-increasing sequence of ordinals we mean a sequence $\langle \nu_1, \ldots, \nu_n \rangle$ of ordinals below κ so that

$$\nu_1^0 < \nu_2^0 < \cdots < \nu_n^0\ .$$

For every $\alpha < \lambda$ by $X \in U_\alpha$ we shall always mean that $X \subseteq \overline{X}$, in particular it will imply that for $\nu_1, \nu_2 \in X$ if $\nu_1^0 < \nu_2^0$ then $|\{\alpha \in X \mid \alpha^0 = \nu_1^0\}| < \nu_2^0$.

The following weak version of normality holds, since U_α is a P-point: if $X_i \in U_\alpha$ for $i < \kappa$ then also $X = \Delta^*_{i<\kappa} X_i = \{\nu \mid \forall i < \nu^0 \ (\nu \in X_i)\} \in U_\alpha$.

Let $\nu < \kappa$ and $\langle \nu_1, \dots, \nu_n \rangle$ be a finite sequence of ordinals below κ. Then ν is called *permitted* for $\langle \nu_1, \dots, \nu_n \rangle$ if $\nu^0 > \max\{\nu_i^0 \mid 1 \le i \le n\}$.

We shall ignore U_α's with $\alpha < \kappa$ and denote U_κ by U_0.

Let us now define a forcing notion for adding λ ω-sequences to κ.

3.6 Definition. The set of forcing conditions $\mathcal{P}$ consists of all the elements p of the form $\{\langle \gamma, p^\gamma \rangle \mid \gamma \in g \setminus \{\max(g)\} \cup \{\langle \max(g), p^{\max(g)}, T \rangle\}$, where

(1) $g \subseteq \lambda$ of cardinality $\le \kappa$ which has a maximal element in $\le_E$-ordering and $0 \in g$. Further let us denote g by $\mathrm{supp}(p)$, $\max(g)$ by $mc(p)$, T by T^p, and $p^{\max(g)}$ by p^{mc} (mc for the maximal coordinate).

(2) for $\gamma \in g$ p^γ is a finite $\circ$-increasing sequence of ordinals $< \kappa$.

(3) T is a tree with a trunk p^{mc} consisting of $\circ$-increasing sequences. All the splittings in T are required to be on sets in $U_{mc(p)}$, i.e. for every $\eta \in T$, if $\eta \ge_T p^{mc}$ then the set

$$\mathrm{Suc}_T(\eta) = \{\nu < \kappa \mid \eta^\frown \langle \nu \rangle \in T\} \in U_{mc(p)} \ .$$

Also require that for $\eta_1 \ge_T \eta_2 \ge_T p^{mc}$

$$\mathrm{Suc}_T(\eta_1) \subseteq \mathrm{Suc}_T(\eta_2) \ .$$

(4) For every $\gamma \in g$, $\pi_{mc(p),\gamma}(\max(p^{mc}))$ is not permitted for p^γ.

(5) For every $\nu \in \mathrm{Suc}_T(p^{mc})$

$$|\{\gamma \in g \mid \nu \text{ is permitted for } p^\gamma\}| \le \nu^0 \ .$$

(6) $\pi_{mc(p),0}$ projects p^{mc} onto p^0 (so p^{mc} and p^0 are of the same length).

Let us give some intuitive motivation for the definition of forcing conditions. We want to add a Prikry sequence for every $U_\alpha (\alpha < \lambda)$. The finite sequences $p^\gamma (\gamma \in \mathrm{supp}(p))$ are initial segments of such sequences. The support of p has two distinguished coordinates. The first is the 0-coordinate of p and the second is its maximal coordinate. The 0-coordinate or more precisely the Prikry sequence for the normal ultrafilter will be used further in order to push the present construction to $\aleph_\omega$. Also condition (6) will be used for this purpose. The maximal coordinate of p is responsible for extending the Prikry sequences for γ's in the support of p. The tree T^p is a set of possible candidates for extending p^{mc} and by using the projections map $\pi_{mc(p),\gamma}$ for $\gamma \in \mathrm{supp}(p)$, T^p becomes also the set of candidates for extending the p^γ's. Instead of working with a tree, it is possible to replace it by a single set. The proof of the Prikry condition will be then a bit more complicated. Condition (4) means that the information carried by $\max(p^{mc})$ is impossible to project down. The reasons for such a condition are technical. Condition (5) is desired to allow the use of the diagonal intersections.

In contrast to the main forcing of the previous section, we deal with κ coordinates simultaneously (i.e. the support of the condition may have cardinality κ). This causes difficulties since we cannot hope to have full commutativity between κ many ultrafilters.

3.7 Definition. Let $p, q \in \mathcal{P}$. We say that p *extends* q and denote this by $p \geq q$ iff

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.

(2) For every $\gamma \in \mathrm{supp}(q)$, p^γ is an end-extension of q^γ.

(3) $p^{mc(q)} \in T^q$.

(4) For every $\gamma \in \mathrm{supp}(q)$,

$$p^\gamma \setminus q^\gamma = \pi_{mc(q),\gamma}``((p^{mc(q)} \setminus q^{mc(q)}) \restriction (\mathrm{length}(p^{mc}) \setminus (i+1)) \ ,$$

where $i \in \mathrm{dom}(p^{mc(q)})$ is the largest such that $p^{mc(q)}(i)$ is not permitted for q^γ.

(5) $\pi_{mc(p),mc(q)}$ projects $T^p_{p^{mc}}$ into $T^q_{q^{mc}}$.

(6) For every $\gamma \in \mathrm{supp}(q)$ and $\nu \in \mathrm{Suc}_{T^p}(p^{mc})$, if ν is permitted for p^γ, then

$$\pi_{mc(p),\gamma}(\nu) = \pi_{mc(q),\gamma}(\pi_{mc(p),mc(q)}(\nu)) \ .$$

In clause (5) above the following notation is used: let T be a tree and $\eta \in T$, then T_η consists of all finite sequences μ such that $\eta^\frown\mu$ is in T.

Intuitively, we are allowing almost everything to be added on the new coordinates and restrict ourselves to choosing extensions from the sets of measure one on the old coordinates. Actually here we are really extending only the maximal old coordinate and then we are using the projection map. This idea goes back to [13] and further to Mitchell [44].

3.8 Definition. Let $p, q \in \mathcal{P}$. We say that p is a *direct (or Prikry) extension of* q and denote this by $p \geq^* q$ iff

(1) $p \geq q$, and

(2) for every $\gamma \in \mathrm{supp}(q)$, $p^\gamma = q^\gamma$.

Our strategy is to show that $\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition, that $\langle \mathcal{P}, \leq^* \rangle$ is κ-closed, and that $\langle \mathcal{P}, \leq \rangle$ satisfies the κ^{++}-c.c.

The Prikry condition together with κ closedness of $\langle \mathcal{P}, \leq^* \rangle$ insure that no new bounded subsets of κ are added. The κ^{++}-c.c. takes care of cardinals $\geq \kappa^{++}$. Since κ will change its cofinality to $\aleph_0$, an argument similar to 2.21 will be used to show that κ^+ is preserved. Clause 3.7(4) of the system of ultrafilters and projections insures that at least λ-many ω-sequences will be added to κ.

3.9 Lemma. *The relation $\leq$ is a partial order.*

Proof. Let us check the transitivity of $\leq$. Suppose that $r \leq q$ and $q \leq p$. Let us show that $r \leq p$. Conditions (1) and (2) of Definition 3.7 are obviously satisfied. Let us check (3), i.e. let us show that $p^{mc(r)} \in T^r$. Since $p \geq q \geq r$, $mc(r) \in \operatorname{supp}(q), q^{mc(r)} \in T^r$ and

$$p^{mc(r)} \setminus q^{mc(r)} = \pi_{mc(q),mc(r)}``(p^{mc(q)} \setminus q^{mc(q)}) \ .$$

Also $p^{mc(q)} \in T^q$. By (5) of 3.7 (for q and r) $\pi_{mc(q),mc(r)}$ projects $T^q_{q^{mc}}$ into a subtree of $T^r_{q^{mc(r)}}$. Hence $p^{mc(r)} \in T^r$ and, so condition (3) is satisfied.

Let us check condition (4). Suppose that $\gamma \in \operatorname{supp}(r)$. We need to show that $p^\gamma \setminus r^\gamma = \pi_{mc(r),\gamma}``(p^{mc(r)} \setminus r^{mc(r)})$. In order to simplify the notation, we are assuming here that every element of $p^{mc(r)} \setminus r^{mc(r)}$ is permitted for $r^\gamma$. Since $q \geq r$, $q^\gamma \setminus r^\gamma = \pi_{mc(r),\gamma}``(q^{mc(r)} \setminus r^{mc(r)})$. So, we need to show only that $p^\gamma \setminus q^\gamma = \pi_{mc(r),\gamma}``(p^{mc(r)} \setminus q^{mc(r)})$. Since $p \geq q$, $p^{mc(q)} \in T^q$ and $p^\gamma \setminus q^\gamma = \pi_{mc(q),\gamma}``(p^{mc(q)} \setminus q^{mc(q)})$. Using condition (6) of 3.7 for $q \geq r$ and the elements of $p^{mc(q)} \setminus q^{mc(q)}$, we obtain the following

$$\begin{aligned} p^\gamma \setminus q^\gamma &= \pi_{mc(q),\gamma}``(p^{mc(q)} \setminus q^{mc(q)}) \\ &= \pi_{mc(r),\gamma}``(\pi_{mc(q),mc(r)}``(p^{mc(q)} \setminus q^{mc(q)})) \\ &= \pi_{mc(r),\gamma}``(p^{mc(r)} \setminus q^{mc(r)}) \ . \end{aligned}$$

The last equality holds by condition (4) of 3.7 used for p and q.

Let us check condition (5), i.e. $\pi_{mc(p),mc(r)}$ projects $T^p_{p^{mc}}$ into $T^r_{p^{mc(r)}}$. Since $p \geq q$, $T^p_{p^{mc}}$ is projected by $\pi_{mc(p),mc(q)}$ into $T^q_{q^{mc}}$. Since $q \geq r$, $\pi_{mc(q),mc(r)}$ projects $T^q_{q^{mc}}$ into $T^r_{q^{mc(r)}}$. Now, using condition (6) for p and q with $\gamma = mc(r)$, we obtain condition (5) for p and r.

Finally, let us check condition (6). Let $\gamma \in \operatorname{supp}(r)$, $\nu \in \operatorname{Suc}_{T^p}(p^{mc})$ and suppose that ν is permitted for p^γ. Using condition (5) for p and q, we obtain that $\pi_{mc(p),mc(q)}(\nu) \in \operatorname{Suc}_{T^q}(q^{mc})$. Recall that it was required in Clause 3.6(3) that each splitting has a splitting below it in the tree. Denote $\pi_{mc(p),mc(q)}(\nu)$ by δ. By condition (6) for q and r, $\pi_{mc(q),\gamma}(\delta) =$

$\pi_{mc(r),\gamma}(\pi_{mc(q),mc(r)}(\delta))$. Using (6) for p and q, we obtain

$$\begin{aligned}\pi_{mc(p),\gamma}(\nu) &= \pi_{mc(q),\gamma}(\pi_{mc(p),mc(q)}(\nu))\\ &= \pi_{mc(q),\gamma}(\delta)\\ &= \pi_{mc(r),\gamma}(\pi_{mc(q),mc(r)}(\delta))\ .\end{aligned}$$

Once more using (6) for p and q,

$$\pi_{mc(q),mc(r)}(\pi_{mc(p),mc(q)}(\nu)) = \pi_{mc(p),mc(r)}(\nu)\ .$$

This completes the checking of (6) and also the proof of the lemma. ⊣

The main point of the proof appears in the next lemma.

3.10 Lemma. *Let $q \in \mathcal{P}$ and $\alpha < \lambda$. Then there is a $p \geq^* q$ so that $\alpha \in \mathrm{supp}(p)$.*

Proof. If $\alpha \leq_E mc(q)$, then it is obvious. Thus, if $\alpha \in \mathrm{supp}(q)$, then we can take $p = q$. Otherwise add to q a pair $\langle \alpha, t\rangle$ where t is any $^\circ$-increasing sequence so that $\max(q^{mc})$ is not permitted for t.

Now suppose that $\alpha \not\leq_E mc(q)$. Pick some $\beta < \lambda$ so that $\beta \geq_E \alpha$ and $\beta \geq_E mc(q)$. Without loss of generality we can assume that $\beta = \alpha$. We shall define p to be of the form

$$q' \cup \{\langle \alpha, t, T\rangle\}$$

where q' is constructed from q by removing T^q from the triple $\langle mc(q), q^{mc}, T^q\rangle$, t is an $^\circ$-increasing sequence which projects onto q^0 by $\pi_{\alpha 0}$ and the tree T will be defined below.

First consider the tree T_0 which is the inverse image of $T^q_{q^{mc}}$ by $\pi_{\alpha,mc(q)}$, with t added as the trunk. Then $p_0 = q' \cup \{\langle \alpha, t, T_0\rangle\}$ is a condition in $\mathcal{P}$ which is "almost" an extension and even a direct extension of q. The only concern is that condition (6) of Definition 3.7 may not be satisfied by p_0 and q. In order to repair this, let us shrink the tree T_0 a little.

Denote $\mathrm{Suc}_{T_0}(t)$ by A. For $\nu \in A$ set

$$B_\nu = \{\gamma \in \mathrm{supp}(q) \mid \gamma \neq mc(q) \text{ and } \nu \text{ is permitted for } q^\gamma\}\ .$$

Then $|B_\nu| \leq \nu^0$, since $\pi_{\alpha,mc(q)}(\nu) \in \mathrm{Suc}_{T^q}(q^{mc}), \nu^0 = \pi_{\alpha 0}(\nu) = \pi_{mc(q),0}(\pi_{\alpha mc(q)}(\nu))$, and q being in $\mathcal{P}$ satisfies condition (5) of Definition 3.6. Clearly, for $\nu, \delta \in A$, if $\nu^0 = \delta^0$ then $B_\nu = B_\delta$, and if $\nu^0 > \delta^0$ then $B_\nu \supseteq B_\delta$. Also, if $\nu \in A$ and ν^0 is a limit point of $\{\delta^0 \mid \delta \in A\}$, then $B_\nu = \bigcup\{B_\delta \mid \delta \in A \text{ and } \delta^0 < \nu^0\}$. So the sequence $\langle B_\nu \mid \nu \in A\rangle$ is increasing and continuous (according to the ν^0's). Obviously, $\bigcup\{B_\nu \mid \nu \in A\} = \mathrm{supp}(q) \setminus \{mc(q)\}$. Let $\langle \xi_i \mid i < \kappa\rangle$ be an enumeration of $\mathrm{supp}(q) \setminus \{mc(q)\}$ such that for every $\nu \in A$

$$B_\nu \subseteq \{\xi_i \mid i < \nu^0\}\ .$$

Now pick for every $i \in A$ a set $C_i \subseteq A$, with $C_i \in U_\alpha$ so that for every $\nu \in C_i$ $\pi_{\alpha\xi_i}(\nu) = \pi_{mc(q),\xi_i}(\pi_{\alpha,mc(q)}(\nu))$. Let $C = A^\frown\Delta^*_{i<\kappa} C_i = \{\nu \in A \mid \forall i < \nu^0 (\nu \in C_i)\}$. Then $C \in U_\alpha$.

Now define T to be the tree obtained from T_0 by intersecting every level of T_0 with C. Let us show that condition (6) of Definition 3.7 is now satisfied. Suppose $\gamma \in \operatorname{supp}(q)$. If $\gamma = mc(q)$, then everything is trivial. Assume that $\gamma \in \operatorname{supp}(q) \setminus \{mc(q)\}$. Then for some $i_0 < \kappa$ $\gamma = \xi_{i_0}$. Suppose that some $\nu \in C$ is permitted for q^γ. Then $\xi_{i_0} = \gamma \in B_\nu$. Since $B_\nu \subseteq \{\xi_i \mid i < \nu^0\}$, $i_0 < \nu^0$. Then $\nu \in C_{i_0}$. Hence

$$\pi_{\alpha\xi_{i_0}}(\nu) = \pi_{mc(q),\xi_{i_0}}(\pi_{\alpha,mc(q)}(\nu)) \ .$$

So condition (6) is satisfied by p. Hence, $p \geq^* q$. ⊣

3.11 Lemma. (a) $\langle \mathcal{P}, \geq \rangle$ *satisfies the* κ^{++}*-c.c.*

(b) $\langle \mathcal{P}, \geq^* \rangle$ *is* κ*-closed.*

Proof of (a). Let $\{p_\alpha \mid \alpha < \kappa^{++}\} \subseteq \mathcal{P}$. Without loss of generality, we can assume their supports form a Δ-system and are contained in κ^{++}. Also, we can assume that there are s and $\langle t, T\rangle$ so that for every $\alpha < \kappa^{++}$, $p_\alpha \restriction \alpha = s$ and $\langle p^{mc}_\alpha, T^{p_\alpha}\rangle = \langle t, T\rangle$. Let us then show that p_α and p_β are compatible for every $\alpha, \beta < \kappa^{++}$.

Let $\alpha, \beta < \kappa^{++}$ be fixed. We would like simply to take the union $p_\alpha \cup p_\beta$ and to show that this is a condition stronger than both p_α and p_β. The first problem is that $p_\alpha \cup p_\beta$ may not be in $\mathcal{P}$, since $\operatorname{supp}(p_\alpha \cup p_\beta) = \operatorname{supp}(p_\alpha) \cup \operatorname{supp}(p_\beta)$ may not have a maximal element. In order to fix this, let us add say to p_α some new coordinate δ so that $\delta \geq_E mc(p_\alpha), mc(p_\beta)$. Let p^*_α be the extension of p_α defined in the previous lemma by adding δ as a new coordinate to p_α. Then $p^*_\alpha \cup p_\beta \in \mathcal{P}$. But we do need a condition stronger than both p_α and p_β. The condition $p^*_\alpha \cup p_\beta$ is a good candidate for it. The only concerns here are (5) and (6) of Definition 3.6. Actually, (5) can be easily satisfied by intersecting $T^{p^*_\alpha}_{(p^*_\alpha)^{mc}}$ with $\pi^{-1}_{\delta,mc(p_\beta)}$ “$(T^{p_\beta}_{p^{mc}_\beta})$. In order to satisfy (6), we need to shrink $T^{p^*_\alpha}$ more. The argument of the previous lemma can be used for this.

Proof of (b). Let $\delta < \kappa$ and let $\langle p_i \mid i < \delta\rangle$ be an $\leq^*$-increasing sequence of elements of $\mathcal{P}$. Pick $\alpha < \lambda$ above $\{mc(p_i) \mid i < \delta\}$. Let p be the union of p_i's with T^{p_i} removed. Set $T = \bigcap_{i<\delta} \pi^{-1}_{\alpha,mc(p_i)}$ “T^{p_i}. Also remove all τ's with $\tau^0 \leq \delta$ from this tree. Let t be a $^\circ$-increasing sequence so that $\pi_{\alpha 0}$“$t = p^0_0$. Consider $p \cup \{\langle \alpha, t, T\rangle\}$. Clearly, it belongs to $\mathcal{P}$. Now, as in Lemma 3.9, shrink T to a tree T^i so that $p \cup \{\langle \alpha, t, T^i\rangle\}^* \geq p_i$, where $i < \delta$. Let $T^* = \bigcap_{i<\delta} T^i$ and consider $r = p \cup \{\langle \alpha, t, T^*\rangle\}$. Then $r \geq^* p_i$ for every $i < \delta$. ⊣

3.12 Lemma. *$\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition, i.e. for every statement σ of the forcing language, for every $q \in \mathcal{P}$ there exists $p \geq^* q$ deciding σ.*

Proof. Let σ be a statement and $q \in \mathcal{P}$. In order to simplify the notation we assume that $q = \emptyset$. The object in this proof is to reduce the problem to that of Prikry forcing on some U_α, so the arguments of §1.1 can be applied. In order to find a suitable ordinal α, which will be p^{mc}, pick an elementary submodel N of V_μ for μ sufficiently large to contain all the relevant information of cardinality κ^+ and closed under κ-sequences of its elements. Pick $\alpha < \lambda$ above (in $\leq_E$-ordering) all the elements of $N \cap \lambda$. Let T be a tree so that $\{\langle \alpha, \emptyset, T \rangle\} \in \mathcal{P}$. More precisely, we should write $\{\langle 0, \emptyset \rangle\} \cup \{\langle \alpha, \emptyset, T \rangle\}$. But let us omit the least coordinate when the meaning is clear. If there is a $p \in N$ so that $p \cup \{\langle \alpha, \emptyset, T' \rangle\} \in \mathcal{P}$ and decides σ, for some $T' \subseteq T$, then we are done. Suppose otherwise. Denote $\mathrm{Suc}_T(\langle\rangle)$ by A. We shall define by recursion sequences $\langle p_\nu \mid \nu \in A \rangle$ and $\langle T^\nu \mid \nu \in A \rangle$.

Let $\nu = \min(A)$. Consider $\{\langle \alpha, \langle \nu \rangle, T_{\langle \nu \rangle} \rangle\}$. If there is no $p \in N$ and $T' \subseteq T_{\langle \nu \rangle}$ such that $p \cup \{\langle \alpha, \langle \nu \rangle, T' \rangle\}$ is in $\mathcal{P}$ and decides σ, then set $p_\nu = \emptyset$ and $T^\nu = T_{\langle \nu \rangle}$. Otherwise, pick some p and $T' \subseteq T_{\langle \nu \rangle}$ so that $p \cup \{\langle \alpha, \langle \nu \rangle, T' \rangle\}$ is in $\mathcal{P}$ and decides σ. Set $p_\nu = p$ and $T^\nu = T'$.

Suppose now that p_ξ and T^ξ are defined for every $\xi < \nu$ in A. We shall define p_ν and T^ν. But let us first define p'_ν and p''_ν. Define p''_ν to be the union of all p_ξ's with $\xi \in A \cap \nu$. Let $p'_\nu = \{\langle \gamma, p'^{\gamma}_\nu \rangle \mid \gamma \in \mathrm{supp}(p''_\nu)\}$, where for $\gamma \in \mathrm{supp}(p''_\nu)$, $p'^{\gamma}_\nu = p''^{\gamma}_\nu$ unless ν is permitted for p''^{γ}_ν and then $p'^{\gamma}_\nu = p''^{\gamma}_\nu{}^\frown\langle \pi_{\alpha\gamma}(\nu) \rangle$. If there is no $p \in N$ and T' so that $q = p \cup \{\langle \alpha, \langle \nu \rangle, T' \rangle\} \in \mathcal{P}$, $q \geq^* p'_\nu \cup \{\langle \alpha, \langle \nu \rangle, T_{\langle \nu \rangle} \rangle\}$ and $q \parallel \sigma$, then set $p_\nu = p''_\nu$ and $T^\nu = T_{\langle \nu \rangle}$. Suppose otherwise. Let p, T' be witnessing this. Then set $T^\nu = T'$ and $p_\nu = p''_\nu \cup (p \setminus p'_\nu)$.

This completes the recursive definition. Set $p = \bigcup_{\nu \in A} p_\nu$. For $i < \kappa$ let

$$C_i = \begin{cases} A, & \text{if there is no } \delta \in A \text{ such that } \delta^0 = i, \\ \bigcap\{\mathrm{Suc}_{T^\delta}(\langle \delta \rangle) \mid \delta \in A \text{ and } \delta^0 = i\}, & \text{otherwise.} \end{cases}$$

Note that $C_i \in U_\alpha$ since $A \in U_\alpha$ and this means by our agreement that for $\nu_1, \nu_2 \in A$, if $\nu_1^0 < \nu_2^0$ then $|\{\gamma \in A \mid \gamma^0 = \nu_1^0\}| < \nu_2^0$. Set $A^* = A \cap \Delta^*_{i<\kappa} C_i$. Then for every $\nu \in A^*$ for every $\delta \in A$ if $\delta^0 < \nu^0$ then $\nu \in \mathrm{Suc}_{T^\delta}(\langle \delta \rangle)$. Let S be the tree obtained from T by first replacing $T_{\langle \nu \rangle}$ by T^ν for every $\nu \in A^*$ and then intersecting all levels of it with A^*. ⊣

Claim 3.12.1. *$p \cup \{\langle \alpha, \emptyset, S \rangle\}$ belongs to $\mathcal{P}$.*

Proof. The only nontrivial point here is to show that $p \cup \{\langle \alpha, \emptyset, S \rangle\}$ satisfies condition (5) of the definition of $\mathcal{P}$. So let $\nu \in \mathrm{Suc}_S(\langle\rangle)$. By the definition of $S, \mathrm{Suc}_S(\langle\rangle) = A^*$. Consider the set

$$B_\nu = \{\gamma \in \mathrm{supp}(p) \mid \nu \text{ is permitted for } \underset{\sim}{p}^{\gamma}\}\ .$$

For every $\delta \in A$ let $B_{\nu,\delta} = \{\gamma \in \mathrm{supp}(p_\delta) \mid \nu$ is permitted for $p^\gamma\}$. Then $B_\nu = \bigcup_{\delta \in A} B_{\nu,\delta}$. But, actually the definition of the sequence $\langle p_\delta \mid \delta \in A\rangle$ implies that $B_\nu = \bigcup\{B_{\nu,\delta} \mid \delta \in A$ and $\delta^0 < \nu^0\}$. The number of δ's in A with $\delta^0 < \nu^0$ is $\leq \nu^0$, since for $\delta, \nu \in A$ $\delta^0 < \nu^0$ implies $\delta < \nu^0$, and it means in particular, that for every $\xi < \nu^0$, $|\{\delta \in A \mid \delta^0 = \xi\}| < \nu^0$. So it is enough to show that for every $\delta \in A$, $\delta^0 < \nu^0$ implies $|B_{\nu,\delta}| \leq \nu^0$. Fix some $\delta \in A$ such that $\delta^0 < \nu^0$. Since $\nu \in A^*$ and $\delta^0 < \nu^0$ $\nu \in \mathrm{Suc}_{T^\delta}(\langle\delta\rangle)$. But $p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\} \in \mathcal{P}$. So, by the definition of $\mathcal{P}$, $|B_{\nu,\delta}| \leq \nu^0$. $\dashv$

Then, clearly $p \cup \{\langle \alpha, \emptyset, S\rangle\} \geq^* \langle \alpha, \emptyset, T\rangle\}$.

For $\delta \in \mathrm{Suc}_S(\langle\rangle) = A^*$ let us denote by $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta$ the sequence $\{\langle \gamma, (p^\gamma)_{\pi_{\alpha\gamma}(\delta)}\rangle \mid \gamma \in \mathrm{supp}(p)\} \cup \{\langle \alpha, \langle\delta\rangle, S_{\langle\delta\rangle}\rangle\}$, where

$$(p^\gamma)_{\pi_{\alpha\gamma}(\delta)} = \begin{cases} p^{\gamma\frown}\pi_{\alpha\gamma}(\delta), & \text{if } \delta \text{ is permitted for } p^\gamma, \\ p^\gamma, & \text{otherwise.} \end{cases}$$

Note that $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta$ is a condition and $\pi_{\alpha\gamma}(\delta)$ is added only for γ's which appear in the support of some p_ξ with $\xi^0 < \delta^0$ and hence, with $\xi < \delta$. Also $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta{}^* \geq p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$.

Claim 3.12.2. *For every $\delta \in \mathrm{Suc}_S(\langle\rangle)$ if for some $q, R \in N$,*

$$(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta \leq^* q \cup \{\langle \alpha, \langle\delta\rangle, R\rangle\} \text{ and } q \cup \{\langle \alpha, \langle\delta\rangle, R\rangle\} \Vdash \sigma (\text{resp. } \neg\sigma),$$

then $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta \Vdash \sigma$ (resp. $\neg\sigma$).

Proof. Note that such a $q \cup \{\langle \alpha, \langle\delta\rangle, R\rangle\}$ is a direct extension of $p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$. By the choice of p_δ and T^δ, $p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$ decides σ. But $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta{}^* \geq p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$. $\dashv$

Let us now shrink the first level of S in order to insure that for every δ_1 and δ_2 in the new first level

$$(p \cup \{\langle \alpha, \emptyset, S\rangle\})_{\delta_1} \Vdash \sigma \text{ (resp. } \neg\sigma) \text{ iff } (p \cup \{\langle \alpha, \emptyset, S\rangle\})_{\delta_2} \Vdash \sigma \text{ (resp. } \neg\sigma)$$

Let us denote the shrunken tree again by S.

Claim 3.12.3. *For every $\delta \in \mathrm{Suc}_S(\langle\rangle)$, $(p \cup \{\langle \alpha, \emptyset\rangle\})_\delta \not\parallel \sigma$.*

Proof. Suppose otherwise. Then every δ in $\mathrm{Suc}_S(\langle\rangle)$ will force the same truth value of σ. Suppose, for example, that σ is forced. Then $p \cup \{\langle \alpha, \emptyset, S\rangle\}$ will force σ. Since every $q \geq p \cup \{\langle \alpha, \emptyset, S\rangle\}$ is compatible with one of $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta$ for $\delta \in \mathrm{Suc}_S(\langle\rangle)$. This contradicts the initial assumption. $\dashv$

Now, climbing up level by level in the fashion described above for the first level, construct a direct extension $p^* \cup \{\langle \alpha, \emptyset, S^*\rangle\}$ of $p \cup \{\langle \alpha, \emptyset, S\rangle\}$ so that:

(a) For every $\eta \in S^*$, if for some $q, R \in N$,
$(p^* \cup \{\alpha, \emptyset, S^*\rangle\})_\eta \leq^* q \cup \{\langle\alpha, \langle\eta\rangle, R\rangle\}$ and
$q \cup \{\langle\alpha, \langle\eta\rangle, R\rangle\} \Vdash \sigma$ (resp. $\neg\sigma$), then
$(p^* \cup \{\langle\alpha, \emptyset, S^*\rangle\})_\eta \Vdash \sigma$ (resp. $\neg\sigma$)

(b) If $\eta_1, \eta_2 \in S^*$ are of the same length, then

$$(p^* \cup \{\langle\alpha, \emptyset, S^*\rangle\})_{\eta_1} \Vdash \sigma \text{ (resp. } \neg\sigma) \text{ iff } (p^* \cup \{\langle\alpha, \emptyset, S^*\rangle\})_{\eta_2} \Vdash \sigma \text{ (resp. } \neg\sigma)$$

As in Claim 3.12.3, it is impossible to have any $\eta \in S^*$ so that $(p^* \cup \{\langle\alpha, \emptyset, S^*\rangle\})_\eta$ decides σ. Combining this with (a) we obtain the following.

Claim 3.12.4. *For every $q, R, t \in N$, if $q \cup \{\langle\alpha, t, R\rangle\} \geq p^* \cup \{\langle\alpha, \emptyset, S^*\rangle\}$ then $q \cup \{\langle\alpha, t, R\rangle\}$ does not decide σ.*

Proof. Just note that $q \cup \{\langle\alpha, t, R\rangle\} \geq^* (p^* \cup \{\langle\alpha, \emptyset, S\rangle\})_t$ and use (a). $\dashv$

Pick some $\beta \in N \cap \lambda$ which is $\leq_E$ above every element of $\text{supp}(p^*)$. This is possible since $\text{supp}(p^*) \in N$. Shrink S^* to a tree S^{**}, as in Lemma 3.10 in order to insure the following:

For every $\nu \in \text{Suc}_{S^{**}}(\langle\rangle)$ and $\gamma \in \text{supp}(p^*)$,
if ν is permitted for $(p^*)^\gamma$, then $\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu))$.

Let S^{***} be the projection of S^{**} to β via $\pi_{\alpha,\beta}$. Denote $p^* \cup \{\langle\beta, \emptyset, S^{***}\rangle\}$ by p^{**}. Then $p^{**} \in N$ and $p^{**} \cup \{\langle\alpha, \emptyset, S^{**}\rangle\} \geq^* p^* \cup \{\langle\alpha, \emptyset, S^*\rangle\}$. Since N is an elementary submodel there is some $q \in N$ with $q \geq p^{**}$ deciding σ. Let, for example, $q \Vdash \sigma$. Pick some $t \in S^{**}$ so that $\pi_{\alpha\beta}$"$t = q^\beta$. Such t exists, since by Definition 3.7 q^β belongs to S^{***} which is the image of S^{**} under $\pi_{\alpha\beta}$. Note also that $mc(q) <_E \alpha$ by the choice of N. Let R be a from S^{**}_t by intersecting S^{**}_t with $\pi^{-1}_{\alpha,mc(q)}(T^q)$ and shrinking, if necessary, as in Lemma 3.10 in order to insure the equality of projections $\pi_{\alpha\gamma}$ and $\pi_{mc(q),\gamma} \circ \pi_{\alpha,mc(q)}$ for permitted γ's in $\text{supp}(q)$. Then $q \cup \{\langle\alpha, t, R\rangle\}$ will be a condition stronger than q. Hence, it forces σ. But this contradicts Claim 3.12.4, since $q \cup \{\langle\alpha, t, R\rangle\} \geq p^* \cup \{\langle\alpha, \emptyset, S^*\rangle\}$. This contradiction finishes the proof of Lemma 3.12. $\dashv$

Let G be a generic subset of $\mathcal{P}$. By Lemma 3.10, for every $\alpha < \lambda$ there is a $p \in G$ with $\alpha \in \text{supp}(p)$. Let us denote $\bigcup\{p^\alpha \mid p \in G\}$ by G^α.

3.13 Lemma.

(a) *For every $\alpha < \lambda$, G^α is a Prikry sequence for U_α, i.e. an ω-sequence such that for every $X \in U_\alpha$ it is almost contained in X.*

(b) *G^0 is an ω-sequence unbounded in κ.*

(c) *If* $\alpha \neq \beta$ *then* $G^\alpha \neq G^\beta$*, moreover* $\alpha < \beta$ *implies that* G^β *dominates* G^α*.*

Proof. (a) follows from the definition of $\mathcal{P}$. (b) is a trivial consequence of (a). For (c) note that there is a $\gamma < \lambda$ such that $\gamma \geq_E \alpha, \beta$. By Lemma 3.4 then $\{\nu < \kappa \mid \pi_{\gamma\alpha}(\nu) < \pi_{\gamma\beta}(\nu)\} \in U_\gamma$. This together with the definition of $\mathcal{P}$ implies that G^α is dominated by G^β. ⊣

3.14 Lemma. κ^+ *remains a cardinal in* $V[G]$*.*

Proof. Suppose otherwise. Then it changes its cofinality to some $\mu < \kappa$. Let $g : \mu \to (\kappa^+)^V$ be unbounded in $(\kappa^+)^V$. Pick $p \in G$ forcing this. Suppose for simplicity that $\emptyset \Vdash \underset{\sim}{g} : \check{\mu} \to \check{\kappa}^+$ unbounded. Pick an elementary submodel N as in Lemma 3.12. Let $\alpha < \lambda$ be above every element of $N \setminus \lambda$. Pick a tree T so that $\{\langle \alpha, \emptyset, T\rangle\} \in \mathcal{P}$. As in Lemma 3.12, define by induction an $\leq^*$ increasing sequence of direct extensions of $\{\langle \alpha, \emptyset, T\rangle\}$ $\langle q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\} \mid i < \mu\rangle$ so that

(a) $q_i \in N$.

(b) If for some $q, R, t \in N$ and $j < \kappa^+$, $q \cup \{\langle \alpha, t, R\rangle\} \geq q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\}$ and $q \cup \{\langle \alpha, t, R\rangle\} \Vdash \underset{\sim}{g}(\check{i}) = \check{j}$, then

$$(q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\})_t \Vdash \underset{\sim}{g}(\check{i}) = \check{j}\ .$$

Using Lemma 3.11, find S so that $\bigcup_{i<\mu} q_i \cup \{\langle \alpha, \emptyset, S\rangle\} \geq^* q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\}$ for every $i < \mu$. Denote $\bigcup_{i<\mu} q_i$ by p. As in Lemma 3.12, choose $\beta \in N \setminus \lambda$ above $\mathrm{supp}(p)$ and project S into β using $\pi_{\alpha\beta}$. Denote the projection by S^*. Let $p^* = p \cup \{\langle \beta, \emptyset, S^*\rangle\}$. Then $p^* \in N$ and $p^* \cup \{\langle \alpha, \emptyset, S\rangle\} \geq^* p \cup \{\langle \alpha, \emptyset, S\rangle\}$. Since N is an elementary submodel, for every $i < \mu$ there will be $q \in N$, $q \geq p^*$ forcing a value for $\underset{\sim}{g}(i)$. Then, using (b), as in Lemma 3.12 for some $t \in S$ $(p \cup \{\langle \beta, \emptyset, S\rangle\})_t$ will force the same value for $\underset{\sim}{g}(i)$. But $|S| = \kappa$. So, all such values are bounded in κ^+ by some ordinal $\underset{\sim}{\delta}$ which is impossible, since $N \supseteq \kappa^+$ and $N \models (\phi \Vdash (\underset{\sim}{g} : \check{\mu} \to \check{\kappa}^+$ unbounded)). Contradiction. ⊣

Now combining the lemmas together, we obtain the following.

3.15 Theorem. *The following holds in* $V[G]$*:*

(a) κ *has cofinality* $\aleph_0$ *and* $\kappa^{\aleph_0} \geq \lambda$*.*

(b) *All the cardinals are preserved.*

(c) *No new bounded subsets are added to* κ*.*

If κ is a strong cardinal and $\lambda > \kappa$, then by the Solovay argument, described in the beginning of the section, there is a function $f : \kappa \to \kappa$ and a λ-strong embedding $j : V \to M$ so that $j(f)(\kappa) = \lambda$. Now, having f and j we can use the extender-based Prikry forcing over κ, as it was defined above. So, the following holds.

3.16 Theorem. *Let V be a model of* GCH *and let κ be a strong cardinal. Then for every λ there exists a cardinal preserving set generic extension $V[G]$ of V so that*

(a) *No new bounded subsets are added to κ.*

(b) *κ changes its cofinality to $\aleph_0$.*

(c) $2^\kappa \geq \lambda$.

4. Down to $\aleph_\omega$

The forcings of Sections 2 and 3 produce models with κ of cofinality $\aleph_0$, GCH below κ, and 2^κ arbitrarily large. But such κ are quite large. Thus, in Section 2, it is a limit of measurables. In Section 3, it is a former measurable and no cardinals below it were collapsed. We should now like to collapse cardinals below κ and to move it to $\aleph_\omega$. Note that it is impossible to keep 2^κ arbitrarily large once κ is $\aleph_\omega$, since by the celebrated results of S. Shelah [53] $2^{\aleph_\omega} < \min(\aleph_{(2^{\aleph_0})^+}, \aleph_{\omega_4})$ provided that $\aleph_\omega$ is a strong limit. Our goal will be only to produce a finite gap between $\kappa = \aleph_\omega$ and 2^κ. It is possible to generalize this to countable gaps and we refer for this matter to [19], Sec. 3. The possibility of getting uncountable gaps between $\aleph_\omega$ and $2^{\aleph_\omega}$ is a major open problem of cardinal arithmetic.

Let $2 \leq m < \omega$. We construct a model satisfying "$2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$" based on the forcing of the previous section.

The basic ideas for moving down to a small cardinal like $\aleph_\omega$ are due to M. Magidor [35, 36]. H. Woodin, see [9] or [15] was able to replace the use of supercompacts and huge cardinals by Magidor in [15] by strong cardinals. We present here a simplified version of [20, Sec. 2]. The main simplification is an elimination of M-generic sets used there. Another simplification, suggested by Assaf Sharon, allows the removal of bounds $b(p, \gamma)$ of [20], Sec.2.

Fix a $(\kappa, \kappa + m)$-extender E over κ. Let $j : V \to M \simeq \mathrm{Ult}(V, E)$, $\mathrm{crit}(j) = \kappa$, $M \supseteq V_{\kappa+m}$, be the canonical embedding. Assume GCH. Let $\langle U_\alpha \mid \alpha < \lambda\rangle$, $\langle \pi_{\alpha\beta} \mid \alpha, \beta < \lambda,\ \beta \leq_E \alpha\rangle$ be as in the previous section with $\lambda = \kappa^{+m}$ and $f_\lambda : \kappa \to \kappa$ defined by $f_\lambda(\nu) = \nu^{+m}$.

We now define the set of forcing conditions.

4.1 Definition. The set of forcing conditions $\mathcal{P}$ consists of all elements p of the form

$$\{\langle 0, \langle \tau_1, \ldots, \tau_n\rangle, \langle f_0, \ldots, f_n\rangle, F\rangle\} \cup \{\langle \gamma, p^\gamma\rangle \mid \gamma \in g \setminus \{\max(g), 0\}\} \cup \{\langle \max(g), p^{\max(g)}, T\rangle\}\ ,$$

where

(1) $\{\langle 0, \langle \tau_1, \ldots, \tau_n\rangle\rangle\} \cup \{\langle \gamma, p^\gamma\rangle \mid \gamma \in g \setminus \{\max(g), 0\}\} \cup \{\langle \max(g), p^{\max(g)}, T\rangle\}$ is as in Definition 3.6. Let us use the notations introduced there. So, we denote g by $\mathrm{supp}(p), \max(g)$ by $mc(p)$, T by T^p and $p^{\max(g)}$ by p^{mc}. Also, let us denote further $\langle \tau_1, \ldots, \tau_n\rangle$ by p^0, $\langle f_0, \ldots, f_n\rangle$ by f^p, for $i < n$ f_i by f_i^p, n by n^p and F by F^p.

(2) $f_0 \in \mathrm{Col}(\omega, \tau_1)$, $f_i \in \mathrm{Col}(\tau_i^{+m+1}, \tau_{i+1})$ for $0 < i < n$, and $f_n \in \mathrm{Col}(\tau_n^{+m+1}, \kappa)$.

(3) F is a function on the projection of $T_{p^{mc}}$ by $\pi_{mc(p),0}$ so that

$$F(\langle \nu_0, \ldots, \nu_{i-1} \rangle) \in \mathrm{Col}(\nu_{i-1}^{+m+1}, \kappa) \ .$$

Let us denote by $T^{p,0}$ the projection of T by π_{mc0}. For every $\eta \in T^{p,0}_{p^0}$ let F_η be defined by $F_\eta(\nu) = F(\eta^\frown \langle \nu \rangle)$.

Intuitively, the forcing $\mathcal{P}$ is intended to turn κ to $\aleph_\omega$ and simultaneously blowing up its power to κ^{+m+1}. The part of $\mathcal{P}$, which is responsible for blowing up the power of κ is the forcing used in Section 3. The function $f_0, \ldots, f_{n-1}$ provides partial information about collapsing already known elements of the Prikry sequence for U_0. F is a set of possible candidates for collapsing between further, still unknown elements of this sequence. Note, that for $i < n$ we are starting the collapse with τ_i^{+m+1}, i.e. we intend to preserve all $\tau_i, \tau_i^+, \ldots, \tau_i^{+m+1}$. The reason for this appears in the proof of the κ^{++}-c.c. and of the Prikry condition. It looks technical but what is hidden behind is that collapsing indiscernibles (i.e. members of Prikry's sequences for U_α's $(\alpha < \lambda)$) causes collapsing generators, i.e. cardinals between κ and λ. Shelah's bounds on the power of $\aleph_\omega$, [53] suggest that there is no freedom in using collapses below κ without effecting the structure of cardinals above κ as well.

4.2 Definition. Let $p, q \in \mathcal{P}$. We say that p *extends* q and denote this by $p \geq q$ iff

(1) $\{\langle 0, p^0 \rangle\} \cup \{\langle \gamma, p^\gamma \rangle \mid \gamma \in \mathrm{supp}(p) \setminus \{mc(p), 0\}\} \cup \{\langle mc(p), p^{mc}, T^p \rangle\}$
extends
$\{\langle 0, q^0 \rangle\} \cup \{\langle \gamma, q^\gamma \rangle \mid \gamma \in \mathrm{supp}(q) \setminus \{mc(q), 0\}\} \cup \{\langle mc(q), q^{mc}, T^q \rangle\}$
in the sense of Definition 3.7.

(2) For every $i < \mathrm{length}(q^0) = n^q$, $f_i^p \geq f_i^q$.

(3) For every $\eta \in T^{p,0}_{p^0}$, $F^p(\eta) \supseteq F^q(\eta)$.

(4) For every i with $n^q \leq i < n^p$,

$$f_i^p \supseteq F^q((p^0 \setminus q^0){\restriction} i+1) \ .$$

(5) $\min(p^0 \setminus q^0) > \sup(\mathrm{ran}(f_{n^q}))$.

4.3 Definition. Let $p, q \in \mathcal{P}$. We say that p is a *direct extension of* q and denote this by $p \geq^* q$ iff

(1) $p \geq q$, and

(2) for every $\gamma \in \mathrm{supp}(q)$, $p^\gamma = q^\gamma$.

The following lemmas are analogous to the corresponding lemmas of the previous section and they have analogous proofs.

4.4 Lemma. *The relation $\leq$ is a partial order.*

4.5 Lemma. *Let $q \in \mathcal{P}$ and $\alpha < \kappa^{+m}$. Then there is a $p \geq^* q$ so that $\alpha \in \mathrm{supp}(p)$.*

4.6 Lemma. *$\langle \mathcal{P}, \leq \rangle$ satisfies the κ^{++}-c.c.*

For the proof of the last lemma, note only that the number of possibilities for the collapsing part $\langle f_0, , \dots, f_n \rangle, F$ of a condition (in the form of 4.1) is κ^+. It is important that F depends only on the normal ultrafilter of the extender. This way F can be viewed as an element of $\mathrm{Col}(\kappa, i_\kappa(\kappa))$ of $N_\kappa \simeq \mathrm{Ult}(V, U_\kappa)$, which (in V) has cardinality κ^+. Once allowing F to depend on the extender itself, say on the maximal coordinate of a condition, we will have the correspondence to $\mathrm{Col}(\kappa, j(\kappa))$ of $M \simeq \mathrm{Ult}(V, E)$. This set is of cardinality $> \kappa^+$ (in V) and using it, it is easy to produce κ^{++} incompatible conditions.

If $p \in \mathcal{P}$ and $\tau \in p^0$, then the set $\mathcal{P}/p$ of all extensions of p in $\mathcal{P}$ can be split in the obvious fashion into two parts: one everything above τ and the second everything below τ. Denote them by $(\mathcal{P}/p)^{\geq \tau}$ and $(\mathcal{P}/p)^{<\tau}$. Then $\mathcal{P}/p$ can be viewed as $(\mathcal{P}/p)^{\geq \tau} \times (\mathcal{P}/p)^{<\tau}$. The part $(\mathcal{P}/p)^{<\tau}$ consists of finitely many Levy collapses and the part $(\mathcal{P}/p)^{\geq \tau}$ is similar to $\mathcal{P}$ but has a slight advantage, namely the Levy collapses used in it are τ^{+m+1}-closed. Using this observation, one can show the following analog of Lemma 3.11(b):

4.7 Lemma. *If $p \in \mathcal{P}$ and $\tau \in p^0$, then $\langle (\mathcal{P}/p)^{\geq \tau}, \leq^* \rangle$ is τ^{+m+1}-closed.*

Let us now turn to the Prikry condition.

4.8 Lemma. *$\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. Let σ be a statement of the forcing language and $q \in \mathcal{P}$. We shall find $p \geq^* q$ deciding σ. In order to simplify notation, assume that $q = \emptyset$.

Pick an elementary submodel N, $\alpha < \kappa^{+m}$ and T as in Lemma 3.12. Consider condition $\{\langle \alpha, \emptyset, T \rangle\}$. More precisely, we should write $\{\langle 0, \emptyset, \emptyset, \emptyset \rangle \cup \{\langle \alpha, \emptyset, T \rangle\}$. But when the meaning is clear we shall omit $\{\langle 0, \emptyset, \emptyset, \emptyset \rangle\}$. If for some $p \in N$ $\{\langle 0, \emptyset, f, F \rangle\} \cup p \cup \{\langle \alpha, \emptyset, T' \rangle\} \in \mathcal{P}$ and decides σ, for some $T' \subseteq T, f$ and F, then we are done. Suppose otherwise.

As in the proof of 3.12 we first show that it is possible to deal with conditions having fixed support. Once the support is fixed the proof will be more or less like that of 1.20, with small complications due to the involvement of collapses.

Claim 4.8.1. *There are p, F and S in N so that*

(a) $\{\langle 0, \emptyset, \emptyset, F \rangle\} \cup p \cup \{\langle \alpha, \emptyset, S \rangle\} \geq^* \{\langle \alpha, \emptyset, T \rangle\}$.

(b) *If for some* $q \in N, q^0, q^\alpha,\ F', T'$ *and* $\vec{f}$,

$$\{\langle 0, q^0, \vec{f}, F'\rangle\} \cup q \cup \{\langle \alpha, q^\alpha, T'\rangle\}$$

is a an extension of $\{\langle 0, \emptyset, \emptyset, F\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T^*\rangle\}$ *and forces* σ *(or* $\neg\sigma$*) then also*

$$\{\langle 0, q^0, (\vec{f}\restriction \mathrm{length}(q^0))^\frown F(q^0), F\rangle\} \cup (p)_{q^\alpha} \cup \{\langle \alpha, q^\alpha, S_{q^\alpha}\rangle\}$$

forces the same, where $(p)_{q^\alpha}$ *is the set* $\{\langle \gamma, p^{\gamma\frown} t^\gamma\rangle \mid \gamma \in \mathrm{supp}(p)\}$ *with* t^γ *the maximal final segment of* $\pi_{\alpha\gamma}$"q^α *permitted for* p^γ.

Proof. Let A denote $\mathrm{Suc}_T(\langle\rangle)$. Assume that $A \subseteq \kappa$ and for $\nu_1, \nu_2 \in A$, $\nu_1 < \nu_2$ implies $\nu_1^0 < \nu_2^0$. Also assume that only the elements of A appear in T, i.e. $T \subseteq [A]^{<\omega}$. Let $\{\langle q_i^0, \vec{f}_i, q_i^\alpha\rangle \mid i < \kappa\}$ be an enumeration of all triples $\langle q^0, \vec{f}, q^\alpha\rangle$ such that

(i) $q^\alpha \in T$.

(ii) $q^0 = \pi_{\alpha 0}$"q^α.

(iii) If $q^0 = \langle \tau_0, \ldots, \tau_{n-1}\rangle$ then $\mathrm{dom}(\vec{f}) = n$ and $\vec{f}(0) \in \mathrm{Col}(\omega, \tau_0)$, $\vec{f}(1) \in \mathrm{Col}(\tau_0^{+m+1}, \tau_1), \ldots, \vec{f}(n-1) \in \mathrm{Col}(\tau_{n-2}^{+m+1}, \tau_{n-1})$. (Note that we do not enumerate the "last" function from $\mathrm{Col}(\tau_{n-1}^{+m+1}, \kappa)$.)

For every $\nu \in A$, $|\{\rho \in A \mid \rho^0 = \nu^0\}| \le (\nu^0)^{+m}$. So, the number of such triples satisfying $q^0(i) \le \nu^0$ for every $i \le$ length(q^0) is at most $(\nu^0)^{+m}$. We can assume that $\{\langle q_i^0, \vec{f}_i, q_i^\alpha\rangle \mid i < (\nu^0)^{+m}\} = \{\langle q^0, \vec{f}, q^\alpha\rangle \mid \langle q^0, \vec{f}, q^\alpha\rangle$ satisfy the conditions (i), (ii), (iii) above and $q^0(i) \le \nu^0$ for every $i \le \mathrm{length}(q^0)\}$.

Define by recursion sequences $\langle p_i \mid i < \kappa\rangle$, $\langle T^i \mid i < \kappa\rangle, \langle f^i \mid i < \kappa\rangle$ and $\langle F^i \mid i < \kappa\rangle$. Set $p_0 = \emptyset$, $T^0 = T, f^0 = \emptyset$ and $F^0 = \emptyset$.

Suppose that p_j, T^j and F^j are defined for every $j < i$. Define p_i, T^i, f^i and F^i.

Set first $p_i'' = \bigcup_{j<i} p_j$. Let $p_i' = \{\langle \gamma, p'^\gamma\rangle \mid \gamma \in \mathrm{supp}(p_i'')\}$, where for $\gamma \in \mathrm{supp}(p_i'')$, $p_i'^\gamma = p_i''^\gamma$ unless there is a $\nu \in q_i^\alpha$ permitted for $p_i''^\gamma$ and then $p_i'^\gamma = p_i''^{\gamma\frown}$ the maximal final segment of $\pi_{\alpha\gamma}$"q_i^α permitted for $p_i''^\gamma$.

We now wish to define a function F' on the set $q_i^{0\frown}(T_{q_i^\alpha})^0 =_{df} \{q_i^{0\frown}\langle\eta\rangle \mid \langle\eta\rangle \in (T_{q_i^\alpha})^0\}$. Let $\langle\eta\rangle \in (T_{q_i^\alpha a})^0$ (it may be just the empty sequence). Consider the set

$$C = \{j < i \mid q_i^{0\frown}\langle\eta\rangle \text{ extends } q_j^0 \text{ and } q_i^{0\frown}\langle\eta\rangle \in q_j^{0\frown}(T^j)^0\}\ .$$

For every $j \in C$ we have

$$q_j^0(\mathrm{length}(q_j^0) - 1) \le q_i^{0\frown}\langle\eta\rangle(\mathrm{length}(q_i^{0\frown}\langle\eta\rangle) - 1)\ .$$

Then, by the properties of the enumeration $\{\langle q^0_\nu, \vec{f}_\nu, q^\alpha_\nu\rangle \mid \nu < \kappa\}$ we have $j < (q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1))^{+m}$. So

$$C \subseteq (q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1))^{+m} .$$

Now define

$$F'(q^0_i{}^\frown\langle\eta\rangle) = \bigcup_{j\in C} F^j(q^0_i{}^\frown\langle\eta\rangle) .$$

Then

$$F'(q^0_i{}^\frown\langle\eta\rangle) \in \mathrm{Col}(q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1)^{+m+1}, \kappa) ,$$

since $|C| \leq q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1)^{+m}$. Define

$$r^i = \{\langle q^0_i, \vec{f}_i{}^\frown F'(q^0_i), F'\rangle\} \cup p'_i \cup \{\langle\alpha, q^\alpha_i, T_{q^\alpha_i}\rangle\} .$$

If $r^i \notin \mathcal{P}$ or it belongs to $\mathcal{P}$ and there is no $p \in N, T', g$ and F so that $\{\langle 0, q^0_i, \vec{f}_i{}^\frown g, F\rangle\} \cup p \cup \{\langle\alpha, q^\alpha_i, T'\rangle\} \in \mathcal{P}$ extends r^i and decides σ, then set $p_i = p''_i, T^i = T_{q^\alpha_i}, f^i = F'(q^0_i)$ and $F^i = F'$. Otherwise, pick some p, T', g and F witnessing this. Then define $T^i = T'$, $F^i = F, f^i = g$, $F^i(q^0_i) = f^i$. Set $p_i = p''_i \cup p^*$, where $p^* = p \setminus p'_i$.

This completes the recursive definition. Set $p = \bigcup_{i<\kappa} p_i$. Now define a subtree S of T by putting together all the T^i's for $i < \kappa$. The definition is level by level. Thus, if S is defined up to level n and t sits in S on this level, then set

$$\begin{aligned}\mathrm{Suc}_S(t) = \{\nu \in A \mid \nu^0 > \max(t),\ &\text{and for every } i < \nu^0,\\ &\nu \in \mathrm{Suc}_{T^i}(\langle\rangle) \text{ and } \nu \in \mathrm{Suc}_{T^i}(t) \text{ when } t \in T^i\} .\end{aligned}$$

So $\mathrm{Suc}_S(t) \in U_\alpha$.

Let us now put together all the F^i's. Define a function F on a tree $(S)^0$.

Thus let $\eta \in S^0$. Consider the set $C = \{j < \kappa \mid q^0_j \subseteq \eta \in q^0_j{}^\frown(T^j)^0\}$. Let $\ell = \mathrm{length}(\eta) - 1$. Then for each $j \in C$ $q^0_j(\mathrm{length}(q^0_j) - 1) \leq \eta(\ell)$. So, by the choice of the enumeration $\{\langle q^0_\nu, \vec{f}_\nu, q^\alpha_\nu\rangle \mid \nu < \kappa\}$ we have $j < \eta(\ell)^{+m}$. Hence $C \subseteq \eta(\ell)^{+m}$. Define $F(\eta) = \bigcup_{j\in C} F^j(\eta)$. Then $F(\eta) \in \mathrm{Col}(\eta(\ell)^{+m+1}, \kappa)$.

Subclaim 4.8.2. *$\{\langle 0, \emptyset, \emptyset, F\rangle\} \cup p \cup \{\langle\alpha, \emptyset, S\rangle\} \in \mathcal{P}$.*

Proof. The only problem is to show that for every $\nu \in \mathrm{Suc}_S(\langle\rangle)$,

$$|\{\gamma \in \mathrm{supp}(p) \mid \nu \text{ is permitted for } p^\gamma\}| \leq \nu^0 .$$

Thus let $\nu \in \mathrm{Suc}_S(\langle\rangle)$ and $i < \kappa$. If $\langle q^0_i, \vec{f}_i, q^\alpha_i\rangle$ satisfies $\max(q^0_i) < \nu^0$, then $i < \max(q^0_i)^{+m} < \nu^0$. Hence for every $i \geq \nu^0$, ν is not permitted for q^0_i. So after the stage ν^0 we did not add any new coordinate γ with ν permitted for $(p_i)^\gamma$. This means that $\{\gamma \in \mathrm{supp}(p) \mid \nu$ is permitted for $p^\gamma\} = \bigcup_{i<\nu^0}\{\gamma \in \mathrm{supp}(p_i) \mid \nu \text{ is permitted for } p^\gamma\}$ and we are done. ⊣

Denote $\{\langle 0,\emptyset,\emptyset,F\rangle\}\cup p\cup\{\langle\alpha,\emptyset,S\rangle\}$ by p^*. We now show that p^* is as desired. Clearly, $p^*\geq^*\{\langle\alpha,\emptyset,T\rangle\}$. Suppose that for some $q\in N$, q^0,q^α,G R and $\vec{f}$

$$\{\langle 0,q^0,\vec{f},G\rangle\}\cup q\cup\{\langle\alpha,q^\alpha,R\rangle\}\geq p^*$$

and

$$\{\langle 0,q^0,\vec{f},G\rangle\}\cup q\cup\{\langle\alpha,q^\alpha,R\rangle\}\Vdash\sigma\quad(\text{or }\neg\sigma)$$

Let $q^0=\langle\tau_1,\ldots,\tau_n\rangle$ and $\vec{f}=\langle f_0,\ldots,f_n\rangle$. Obviously, $n>0$ since otherwise we will have a direct extension of p^* (and hence of $\{\langle\alpha,\emptyset,T\rangle\}$) deciding σ contrary to the initial assumption. Find $i<\tau_n^{+m}$ such that $\langle q^0,\langle f_0,\ldots,f_{n-1}\rangle,q^\alpha\rangle=\langle q^0_i,\vec{f}_i,q^\alpha_i\rangle$. Consider the condition

$$r^i=\{\langle q^0_i,\vec{f}_i{}^\frown F'(q^0_i),F'\rangle\}\cup p'_i\cup\{\langle\alpha,q^\alpha_i,T_{q^\alpha_i}\rangle\}\ ,$$

defined at stage i of the construction. We have

$$\{\langle 0,q^0,\vec{f},G\rangle\}\cup q\cup\{\langle\alpha,q^\alpha,R\rangle\}\geq^* r^i\ ,$$

since $R\subseteq S_{q^\alpha}\subseteq T_{q^\alpha}$, $F'(\eta)\subseteq F(\eta)$ for η's from the common domain, so that in particular $F'(q^0_i)\subseteq F(q^0_i)\subseteq f_n$. But then

$$\{\langle 0,q^0_i,\vec{f}_i{}^\frown f^i,F^i\rangle\}\cup(p_i)_{q^\alpha_i}\cup\{\langle\alpha,q^\alpha_i,T^i_{q^\alpha_i}\rangle\}\Vdash\sigma\ (\text{or }\neg\sigma)$$

by the choice of f^i,F^i,T^i and p_i at the stage i of the construction. Hence also

$$\{\langle 0,q^0,\langle f_0,\ldots,f_{n-1},F(q^0)\rangle,F\rangle\}\cup(p)_{q^\alpha}\cup\{\langle\alpha,q^\alpha,S_{q^\alpha}\rangle\}$$

forces the same. This completes the proof Claim 4.8.1. $\dashv$

Fix $p^*=\{\langle 0,\emptyset,\emptyset,F\rangle\}\cup p\cup\{\langle\alpha,\emptyset,S\rangle\}$ satisfying the conclusion of 4.8.1.

As in Lemma 3.12, it is possible to show that the assumption "$q\in N$" is not really restrictive. Briefly, if there is some q outside of N which is used to decide σ, then there exists one also inside N. So the following claim will provide the desired contradiction.

Claim 4.8.3. *There is a*

$$p^{**}=\{\langle 0,\emptyset,\emptyset,F\restriction T^*\rangle\}\cup p\cup\{\langle\alpha,\emptyset,T^*\rangle\}\geq^* p^*$$

such that the following holds:

$(*)$ *There are no* $q\in N,q^0,q^\alpha,\vec{f},F'$ *and* T' *such that*

$$p^{**}\leq\{\langle 0,q^0,\vec{f},F'\rangle\}\cup q\cup\{\langle\alpha,q^\alpha,T'\rangle\}\parallel\sigma\ .$$

Proof. We shall construct by recursion a $\leq^*$-increasing sequence $\langle p(\ell) \mid \ell \leq \omega\rangle$ of direct extensions of p^* satisfying for every $\ell \leq \omega$ the following condition:

$(*)_\ell$ There are no $q \in N, q^0, q^\alpha, \vec{f}, F'$ and T' such that $\text{length}(q^0) \leq \ell$ and

$$p(\ell) \leq \{\langle 0, q^0, \vec{f}, F'\rangle\} \cup q \cup \{\langle \alpha, q^\alpha, T'\rangle\} \parallel \sigma \ .$$

Clearly, then $p(\omega)$ will be as desired.

Set $p(0) = p^*$. Define $p(1)$ to be a condition of the form $\{\langle 0, \emptyset, \emptyset, F\restriction T_1\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T_1\rangle\}$ with T_1 defined below. Consider the three sets

$$\begin{aligned} X_i = \{\nu \in \mathrm{Suc}_S(\langle\rangle) \mid \exists f_0^\nu \in \mathrm{Col}(\omega, \nu^0) \\ (\{\langle 0, \langle \nu^0\rangle, f_0^\nu{}^\frown F(\langle \nu^0\rangle), F\rangle\} \cup p_{\langle\nu\rangle} \cup \{\langle \alpha, \langle\nu\rangle, S_{\langle\nu\rangle}\rangle\} \ \Vdash^i \sigma\}) \ , \end{aligned}$$

where $i < 2$, ${}^0\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$, and

$$X_2 = \mathrm{Suc}_S(\langle\rangle) \setminus (X_0 \cup X_1) \ .$$

There is an $i < 3$ such that $X_i \in U_\alpha$. Let T_1' be the tree obtained from S by intersecting all its levels with X_i. Let $r = \{\langle 0, \emptyset, \emptyset, F\restriction T_1'\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T_1'\rangle\}$. If there is no $q \in N, \vec{f}, \nu, F'$ and T' such that

$$r \leq \{\langle 0, \langle \nu^0\rangle, \vec{f}, F'\rangle\} \cup q \cup \{\langle \alpha, \langle\nu\rangle, T'\rangle\} \parallel \sigma \ ,$$

then set $T_1 = T_1'$ and $p(1) = r$. We claim that this is the only possible case. Otherwise, pick $q, \vec{f} = \langle f_0, f_1\rangle, \nu, F'$ and T' witnessing this and, say, forcing σ. By the previous claim, then

$$\{\langle 0, \langle \nu^0\rangle, f_0{}^\frown F(\nu^0), F\rangle\} \cup p_{\langle\nu\rangle} \cup \{\langle \alpha, \langle\nu\rangle, (T_1')_{\langle\nu\rangle}\rangle\} \Vdash \sigma \ .$$

By the choice of T_1', then $X_0 \in U_\alpha$. Hence, for every $\nu \in \mathrm{Suc}_{T_1}(\langle\rangle)$ there is an $f_0^\nu \in \mathrm{Col}(\omega, \nu^0)$ such that

$$\{\langle 0, \langle \nu^0\rangle, f_0^\nu{}^\frown F(\langle \nu^0\rangle), F\rangle \cup p_{\langle\nu\rangle} \cup \{\langle \alpha, \langle\nu\rangle, (T_1')_{\langle\nu\rangle}\rangle\} \Vdash \sigma \ .$$

Note that the function taking ν^0 to f_0^ν is actually a regressive function on $(X_0)^0$. Find $Y \in U_\alpha$ and $f^* \in \mathrm{Col}(\omega, \kappa)$ such that for every $\nu \in Y$, $f_0^\nu = f^*$. Let T_1 be a tree obtained from T_1' by shrinking all its levels to Y. Set $F_1 = F\restriction T_1$. Finally, let

$$p(1) = \{\langle 0, \emptyset, f^*, F_1\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T_1\rangle\} \ .$$

By the construction, $p^* \leq^* p(1) \Vdash \sigma$, which contradicts the assumption that it is impossible to decide σ by direct extensions of p^*.

Let us define $p(2) = \langle 0, \emptyset, \emptyset, F\restriction T_2\rangle \cup p \cup \{\langle \alpha, \emptyset, T_2\rangle\}$ now. Fix $\nu \in \mathrm{Suc}_{T^1}(\langle\rangle)$. Let $\{\langle f_i, \nu_i\rangle \mid 1 \leq i < (\nu^0)^{+m}\}$ be the enumeration of all pairs $\langle f, \rho\rangle$ such that $\rho \in \mathrm{Suc}_{T_1}(\langle\rangle), \rho^0 = \nu^0$ and $f \in \mathrm{Col}(\omega, \nu^0)$. We would first like to define $T_{2\langle\rho\rangle}$ for every $\rho \in \mathrm{Suc}_{T_1}(\langle\rangle)$ with $\rho^0 = \nu^0$. In order to do this define by recursion on $i < (\nu^0)^{+m}$ sets S_i as follows: for $i = 0$ let $S_0 = (T_1)_{\langle\nu\rangle}$. Suppose that S_j is defined for every $j < i$. Set $S = (\bigcap_{j<i} S_j) \cap (T_1)_{\langle\nu_i\rangle}$. Consider a condition

$$r = \{\langle 0, \langle\nu^0\rangle, f_i, F(\langle\nu^0\rangle), F{\restriction}S\rangle\} \cup (p)_{\langle\nu_i\rangle} \cup \{\langle\alpha, \langle\nu_i\rangle, S\rangle\} \ .$$

Clearly, $r \geq p(1)$. By the choice of $p(1)$, neither r or its direct extensions decide σ. Then, the construction of $p(1)$ from $p(0)$ applied to r (instead of $p(0)$) will produce

$$r^i = \{\langle 0, \langle\nu^0\rangle, f_i, F(\langle\nu^0\rangle), F{\restriction}S_i\rangle\} \cup (p)_{\nu_i} \cup \{\langle\alpha, \langle\nu_i\rangle, S_i\rangle\} \geq^* r$$

satisfying the following: There are no $q \in N, \rho, g_1, g_2, F', S'$ such that

$$r^i \leq \{\langle 0, \langle\nu^0, \rho^0\rangle, \langle f_i, g_1, g_2\rangle, F'\rangle\} \cup (p)_{\langle\nu_i,\rho\rangle} \cup \{\langle\alpha, \langle\nu_i, \rho\rangle, S'\rangle\} \parallel \sigma \ .$$

Now let $(T_2)_{\langle\nu^0\rangle} = \bigcap_{j<(\nu_0)^{+m}} S_j$. Define T_2 to be the tree obtained from T_1 by replacing $(T_1)_{\langle\nu\rangle}$ by $(T_2)_{\langle\nu^0\rangle}$ for each $\nu \in \mathrm{Suc}_{T_1}(\langle\rangle)$. Set $p(2) = \{\langle 0, \emptyset, \emptyset, F{\restriction}T_2\rangle\} \cup p \cup \{\langle\alpha, \emptyset, T_2\rangle\}$. It is easy to see that $p(2)$ satisfies $(*)_2$.

We continue in the same fashion and define $p(n) = \{\langle 0, \emptyset, \emptyset, F{\restriction}T_n\rangle\} \cup p \cup \{\langle\alpha, \emptyset, T_n\rangle\}$ satisfying $(*)_n$ for every n, $2 \leq n < \omega$. Finally let $T_\omega = \bigcap_{n<\omega} T_n$. Set $p(\omega) = \{\langle 0, \emptyset, \emptyset, F{\restriction}T_\omega\rangle \cup p \cup \{\langle\alpha, \emptyset, T_\omega\}$. Then $p(\omega)$ will satisfy $(*)_n$ for every $n < \omega$ and hence (*). $\dashv$

This completes the proof of 4.8. $\dashv$

Using 4.8 as a replacement for 3.12, the arguments of 3.14 show the following:

4.9 Lemma. *κ^+ remains a cardinal in $V^{\mathcal{P}}$.*

Lemma 3.13 transfers directly to the present forcing notion. Thus for G a generic subset of $\mathcal{P}, \alpha < \kappa^{+m}$ define as in Section 3, G^α to be $\bigcup\{p^\alpha \mid p \in G\}$. Let $G^0 = \langle\kappa_0, \kappa_1, \ldots, \kappa_n, \ldots\rangle$.

4.10 Lemma. (a) *For every $\alpha < \kappa^{+m}, G^\alpha$ is a Prikry sequence for U_α.*

(b) *G^0 is an ω-sequence unbounded in κ.*

(c) *If $\alpha \neq \beta$ are then $G^\alpha \neq G^\beta$.*

Let $\alpha < \kappa^{+m}$ and $G^\alpha = \langle\nu_0, \nu_1, \ldots, \nu_n, \ldots\rangle$. An easy density argument provides $n(\alpha) < \omega$ such that either

(i) for all but finitely many n's, $\nu^0_{n+n(\alpha)} = \kappa_n$, or

(ii) for all but finitely many n's, $\nu_n^0 = \kappa_{n+n(\alpha)}$.

Transform G^α into a sequence $G'^\alpha = \langle \nu'_0, \nu'_1, \ldots, \nu'_n, \ldots \rangle$ defined as follows:

$$\nu'_n = \begin{cases} \nu_{n+n(\alpha)}, & \text{if (a) holds,} \\ \nu_{n-n(\alpha)}, & \text{if (b) holds and } n \geq n(\alpha) \text{, and} \\ \kappa_n, & \text{if (b) holds and } n < n(\alpha) \ . \end{cases}$$

Then, for every $n < \omega$, $(\nu'_n)^0 = \kappa_n$.

Assaf Sharon [50] has shown that $\langle G'^\alpha \mid \alpha < \kappa^{+m} \rangle$ is a scale in $\prod_{n<\omega} \kappa_n^{+m}$, i.e. every member of $\prod_{n<\omega} \kappa_n^{+m}$ is dominated by one of G'^α's and $\alpha < \beta$ implies that G'^β dominates G'^α.

The next lemma is obvious.

4.11 Lemma. *If $\aleph_0 < \tau < \kappa$ and τ remains a cardinal in $V[G]$, then for some n and for some $m' \leq m$ $\tau = \kappa_n^{+m'+1}$.*

Implementing $\mathrm{Col}(\nu, \nu^+)$'s also, Sharon [50] was able to collapse each κ_n^+ as well. Thus in his model $\kappa_n^{+m'+1}$ for $1 \leq m' \leq m$ are the only uncountable cardinals below κ. Notice that $\langle \kappa_n^+ \mid n < \omega \rangle$ and $\langle \kappa_n^{+m+1} \mid n < \omega \rangle$ are Prikry sequences for U_{κ^+} and $U_{\kappa^{+m+1}}$ and so correspond to κ^+ and κ^{+m+1} of the ultrapower M by $(\kappa, \kappa{+}m)$-extender E. So, in V, $\mathrm{cf}((\kappa^{+m+1})^M) = \kappa^+$. Also $\langle \kappa_{n+1} \mid n < \omega \rangle$ may be viewed as a sequence corresponding to $j(\kappa)$ which again has cofinality κ^+. Hence, the collapses involved collapse between members of the same cofinality.

Now combining all the lemmas, we obtain the following.

4.12 Theorem. *In a generic extension $V[G]$, $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$.*

5. Forcing Uncountable Cofinalities

In the previous sections we dealt with a singular κ of cofinality $\aleph_0$ or changed the cofinality of a regular κ to $\aleph_0$. Here we would like to deal with forcings changing cofinality to an uncountable value. The first such forcing was introduced by M. Magidor [37]. It changed the cofinality of a regular κ to any prescribed regular value δ below κ. The Magidor forcing adds a closed unbounded in κ sequence of order type δ instead of an ω-sequence added by the Prikry forcing in 1.1. The initial assumption used for this was stronger than just measurability. A measurable cardinal κ of the Mitchell order δ, i.e. $o(\kappa) = \delta$, was used. Later W. Mitchell [45] showed that this assumption is optimal. L. Radin [48] defined a forcing of the same flavor which not only could change the cofinality of κ to $\delta < \kappa$ by shooting a closed unbounded δ-sequence, but also adding a closed unbounded κ-sequence preserving regularity and even measurability of κ. It is not a big deal to add a closed unbounded subset to a regular κ preserving its regularity and also measurability. But what is special about the Radin club is that it consists of cardinals which were regular in the ground model and this way combines together a variety of ways of changing cofinalities. This feature allows results of global character in the cardinal arithmetic. Thus, shortly after the discovery of the Radin forcing, M. Foreman and H. Woodin [12] constructed a model satisfying $2^\tau > \tau^+$ for every τ and Woodin produced a model with $2^\tau = \tau^{++}$ for every τ. Later J. Cummings [9] constructed a model with $2^\tau = \tau^+$ for every regular τ and $2^\tau = \tau^{++}$ for every singular cardinal τ. Recently, C. Merimovich [40], [39] obtained additional results of this type introducing extender based Radin forcing.

5.1. Radin Forcing

Here we will give the basics of Radin forcing. A comprehensive account on the matter containing various beautiful results of Woodin using Radin forcing should appear in the book by J. Cummings and H. Woodin [10]. Originally Radin [48] and then Mitchell [42] defined this forcing axiomatically. We will follow a more concrete approach due to Woodin.

Let $j : V \to M$ be an elementary embedding of V into transitive inner model M, with critical point κ. Define a normal ultrafilter $U(0)$ over κ:

$$X \in U(0) \quad \text{iff} \quad \kappa \in j(X) \ .$$

If $U(0) \in M$, then we define a κ-complete ultrafilter $U(1)$, only not over κ but over V_κ:

$$X \in U(1) \quad \text{iff} \quad \langle \kappa,\ U(0) \rangle \in j(X) \ .$$

Such defined $U(1)$ concentrates on pairs $\langle \nu, F \rangle$ so that ν is a measurable cardinal below κ and F is a normal ultrafilter over ν.

If $U(1) \in M$, then we can continue and define a κ-complete ultrafilter $U(2)$ over V_κ:

$$X \in U(2) \quad \text{iff} \quad \langle \kappa, U(0), U(1) \rangle \in j(X) \ .$$

Continue by recursion and define a sequence

$$\vec{U} = \langle \kappa, U(0), U(1), \dots, U(\alpha), \dots \mid \alpha < \operatorname{length}(\vec{U}) \rangle \ ,$$

where each $U(\alpha)$ will be a κ-complete ultrafilter over V_κ:

$$X \in U(\alpha) \quad \text{iff} \quad \vec{U}\restriction\alpha = \langle \kappa, U(0), U(1), \dots, U(\beta) \cdots \mid \beta < \alpha \rangle \in j(X) \ ,$$

and $\operatorname{length}(\vec{U})$ will be the least α with $\vec{U}\restriction\alpha \notin M$. For example, if $M \supseteq \mathcal{P}(\mathcal{P}(\kappa))$, then $\operatorname{length}(\vec{U})$ will be at least $(2^\kappa)^+$, as we will see below. Let us call $\vec{U}$ and $\vec{U}\restriction\alpha$ $(0 < \alpha < \operatorname{length}(\vec{U}))$ *j-sequences of ultrafilters.*

Fix some α^* with $0 < \alpha^* \le \operatorname{length}(\vec{U})$. Let $\vec{V} = \vec{U}\restriction\alpha^*$. We want to define Radin forcing with the ultrafilter sequence $\vec{V}$. Denote it by $R_{\vec{V}}$. As usual, it will have two orders $\le$ and $\le^*$.

Let us deal first with $\alpha^* = 1$ and $\alpha^* = 2$. Thus, for $\alpha^* = 1$, $\vec{V} = \langle \kappa, U(0) \rangle$. Let $\langle R_{\vec{V}}, \le, \le^* \rangle$ be the usual Prikry forcing with $U(0)$ of 1.1, only instead of writing $\langle t, A \rangle$ (where t is an increasing finite sequence and $A \in U(0)$) we shall write $\langle t, \langle \kappa, U(0) \rangle, A \rangle$.

Now let $\alpha^* = 2$. Then $\vec{V} = \langle \kappa, U(0), U(1) \rangle$. We would like to incorporate both $U(0)$ and $U(1)$ in the process generating the generic cofinal sequence. Thus instead of $A \in U(0)$ in the previous case we allow two sets $A_0 \in U(0)$ and $A_1 \in U(1)$, or equivalently, a set in $U(0) \cap U(1)$. Notice, that we can separate $U(0)$ and $U(1)$ since $U(0)$ concentrates on ordinals and $U(1)$ on pairs $\langle \nu, F \rangle$ with F a normal ultrafilter over ν. An initial condition in $R_{\vec{V}}$ will have a form

$$p = \langle \langle \kappa, U(0), U(1) \rangle, A \rangle$$

with $A \in U(0) \cap U(1)$ and require also that each $a \in A$ is either an ordinal or a pair consisting of a measurable cardinal and a normal ultrafilter over it. In order to extend p pick $a \in A$ and $B \subseteq A$, with $B \in U(0) \cap U(1)$ such that the rank of each member of B is above $\operatorname{rank}(a) + 1$. If a is an ordinal then just add it. We will obtain a one-step extension of p

$$\langle a, \langle \langle \kappa, U(0), U(1) \rangle, B \rangle \rangle \ .$$

If $a = \langle \nu, F \rangle$, then consider $A \cap \nu$. a can be added to p only if this set is in F. Notice that the set $X_A = \{ \langle \nu', F' \rangle \mid A \cap \nu' \in F' \} \in U(1)$ since $A \cap \kappa \in U_0$ and so in M, $\langle \kappa, U_0 \rangle \in j(X_A)$. If $A \cap \nu \in F$, then let $B_\nu \in F$ be a subset of $A \cap \nu$. The following will be one-step extension of p:

$$\langle \langle \langle \nu, F \rangle, B_\nu \rangle, \langle \langle \kappa, U(0), U(1) \rangle, B \rangle \rangle$$

Consider a one-step extension $\langle d, \langle\langle \kappa, U(0), U(1)\rangle, B\rangle\rangle$. If d is an ordinal then repeat the recipe of one-step extension described above. Suppose that $d = \langle\langle \nu, F\rangle, B_\nu\rangle$. We now have two alternatives. The first, just as at step one, is to add an ordinal or a pair but between ν and κ. The second is to add an element of B_ν. Thus $\langle \nu, F\rangle$ will be responsible for producing a Prikry sequence for F. This way, generically a sequence of the type ω^2 will be produced.

We now turn to the general case and give a formal definition of $R_{\vec{V}}$ the Radin forcing with the sequence of ultrafilters $\vec{V}$. First let us introduce some notation. Thus, for a sequence $\vec{F} = \langle F(0), \dots, F(\tau), \dots \mid \tau < \text{length}(\vec{F})\rangle$ let $\bigcap \vec{F} = \bigcap\{F(\tau) \mid \tau < \text{length}(\vec{F})\}$. For an ordinal $d = \nu$ or pair $d = \langle \nu, \vec{F}\rangle$ or a triple $d = \langle \nu, \vec{F}, B\rangle$ let us denote ν by $\kappa(d)$. For a triple $d = \langle \nu, \vec{F}, B\rangle$ by $d \in A$ we shall mean that the two first coordinates of d, i.e. $\langle \nu, \vec{F}\rangle$ belong to A.

The main idea behind this forcing is to use members of finite sequences (that it produces) to give rise to separate blocks that are themselves Radin forcings. In order to realize this idea let us first shrink a bit possibilities of choosing these finite sequences. Let $\vec{F}$ be a sequence of ultrafilters over ν. We would like to use only $\vec{F}$'s which are j-sequences of ultrafilters for some $j : V \to M$. Also, we like to have a set $B \in \cap\vec{F}$ such that each member d of it is a j-sequence for some j with critical point $\kappa(d)$.

To achieve this let us define by recursion classes of sequences:

$$A^{(0)} = \{\vec{F} \mid \vec{F} \text{ is a } j\text{-sequence of ultrafilters for some } j : V \to M\}$$

$$A^{(n+1)} = \{\vec{F} \in A^{(n)} \mid \forall\alpha\ 0 < \alpha < \text{length}(\vec{F})\ (A^{(n)} \cap V_{\kappa(\vec{F})} \in F(\alpha))\}$$

$$\overline{A} = \bigcap\nolimits_{n<\omega} A^{(n)}\ .$$

The main feature of $\overline{A}$ is that if $\vec{F} \in \overline{A}$ then, for $0 < \alpha < \text{length}(\vec{F})$, $F(\alpha)$ concentrates on $\overline{A} \cap V_{\kappa(\vec{F})}$, since then $A^{(n)} \cap V_{\kappa(\vec{F})} \in F(\alpha)$ for every n and hence by countable completeness of $F(\alpha)$, also $\overline{A} \cap V_{\kappa(\vec{F})} \in F(\alpha)$.

Note that each measurable cardinal is in $\overline{A}$. But in the presence of stronger large cardinals, $\overline{A}$ turns to be much wider. We will need the following statement proved by Cummings and Woodin [10]:

5.1 Lemma. *Suppose that E be a (κ, λ)-extender and $j : V \to M \simeq$* $\text{Ult}(V, E)$ *the corresponding elementary embedding, so that $M \supseteq V_{\kappa+2}$ and ${}^\kappa M \subseteq M$. Let $\vec{U}$ be the j-sequence of ultrafilters of the maximal length. Then*

(a) $\text{length}(\vec{U}) \geq (2^\kappa)^+$.

(b) *For every $\alpha < (2^\kappa)^+$, $\vec{U}{\restriction}\alpha \in \overline{A}$.*

Proof. Note that ${}^{\alpha}V_{\kappa+2} \subseteq M$ for every $\alpha < (2^\kappa)^+$. Hence $\vec{U}\restriction\alpha \in M$ for every such α.

Let us first show that for every $\alpha < (2^\kappa)^+$, $\vec{U}\restriction\alpha \in A^{(1)}$. Equivalently, for every β, $0 < \beta < (2^\kappa)^+$, we need to show that $A^{(0)} \cap V_\kappa \in U(\beta)$. By the definition $\vec{U}$, this means that in M, $\vec{U}\restriction\beta \in j(A^0)$. So we need to find in M an embedding constructing $\vec{U}\restriction\beta$. Let E' be the extender $E\restriction[\beta]^{<\omega}$. Then $E' \in M$, since ${}^{\beta}V_{\kappa+2} \subseteq M$. Consider the following commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[dr, "j'"'] & M \arrow[r, "i"] & N \simeq \mathrm{Ult}(M, E') \\
& M' \simeq \mathrm{Ult}(V, E') \arrow[u, "k"'] &
\end{tikzcd}$$

Now, it is not hard to see that $i = j'\restriction M$, since ${}^{\kappa}M \subseteq M$ and E' is an extender over κ. In particular, $i(\kappa) = j'(\kappa)$. Since ${}^{\kappa}V_{\kappa+2} \cap M = {}^{\kappa}V_{\kappa+2} \cap V$, we have $V_{j'(\kappa)+2} \cap N = V_{j'(\kappa)+2} \cap M'$. In addition, $\beta \subseteq \mathrm{ran}(k)$, so $\mathrm{crit}(k) \geq \max(\beta, \kappa^+)$. Let $\vec{U^*}$ be the i-sequence of ultrafilters constructed in M. We show by induction that $U^*(\gamma) = U(\gamma)$ for every $\gamma < \beta$. First note that $k(U) = U$ for every ultrafilter U over κ. Thus $\mathrm{crit}(k) > \kappa$ implies that $U = k``U$. Also, clearly, $k``U \subseteq k(U)$. Finally, using $V_{\kappa+1} \cap M' = V_{\kappa+1} \cap M = V_{\kappa+1}$ and maximality of U as a filter we have $U = k(U)$.

Suppose now that $\gamma < \beta$ and we have already shown $\vec{U^*}\restriction\gamma = \vec{U}\restriction\gamma$. Let $X \subseteq V_\kappa$. Then $X \in U^*(\gamma)$ iff $\vec{U^*}\restriction\gamma \in i(X)$ iff $\vec{U}\restriction\gamma \in i(X)$ iff $\vec{U}\restriction\gamma \in j'(X)$ iff $k(\vec{U}\restriction\gamma) \in j(X)$ (by elementarity of k and since $\vec{U}\restriction\gamma \in M'$) iff $\vec{U}\restriction\gamma \in j(X)$ iff $X \in U(\gamma)$ (since $k(\gamma) = \gamma$ and $k(U(\delta)) = U(\delta)$ for every $\delta < \gamma$).

This concludes the proof of $\vec{U}\restriction\alpha \in A^{(1)}$, for $\alpha < (2^\kappa)^+$. Let us show that $\vec{U}\restriction\alpha \in A^{(n)}$ for every n, $2 \leq n < \omega$ and $\alpha < (2^\kappa)^+$. First, for $n = 2$ we have

$$\begin{aligned}
\vec{U}\restriction\alpha \in A^{(2)} &\text{ iff } \forall\beta < \alpha\ A^{(1)} \cap V_\kappa \in \vec{U}\restriction\beta \\
&\text{ iff } \forall\beta < \alpha\ \vec{U}\restriction\beta \in j(A^{(1)}) \\
&\text{ iff } \forall\beta < \alpha \forall\gamma < \beta\ j(A^{(0)}) \cap V_\kappa \in U(\gamma)\ .
\end{aligned}$$

It is enough to show that $j(A^{(0)}) \cap V_\kappa = A^{(0)} \cap V_\kappa$, since we already proved that $A^{(0)} \cap V_\kappa \in U(\gamma)$ for every $\gamma < (2^\kappa)^+$. Let $\vec{F} \in V_\kappa$ be an i-sequence of ultrafilters for an embedding of either V or M with critical point $\nu = \kappa(\vec{F}) < \kappa$. The length of $\vec{F}$ is below κ, and κ is an inaccessible, so it is easy to find an extender inside V_κ such that the elementary embedding i' of it agrees with i long enough and constructs $\vec{F}$. Hence i' will witness both $\vec{F} \in j(A^{(0)})$ and $\vec{F} \in A^{(0)}$. The same argument works for any $n \geq 2$. Thus we will have

$$\vec{U}\restriction\alpha \in A^{(n)} \text{ iff } \forall\gamma(\gamma + n \leq \alpha \rightarrow j^{n-1}(A^{(0)}) \cap V_\kappa \in U(\gamma))\ ,$$

where j^{n-1} is an application of j $n-1$ many times, or equivalently the embedding $j_{0n-1} : V \to M_{n-1}$ of V into the $n-1$ times iterated ultrapower M_{n-1} of V by E. Again, as above $j^{n-1}(A^{(0)}) \cap V_\kappa = A^{(0)} \cap V_\kappa$. ⊣

Note that using stronger j's it is possible to show that longer ultrafilter sequences are in $\overline{A}$.

We are now ready to define Radin forcing. Let $\vec{V} = \langle U(\alpha) \mid \alpha < \text{length}(\vec{V})\rangle$ be a j-sequence of ultrafilters in $\overline{A}$ for some $j : V \to M$ with $\text{crit}(j) = \kappa$.

5.2 Definition. Let $R_{\vec{V}}$ be the set of finite sequences $\langle d_1, \ldots, d_n, \langle\kappa, \vec{V}\rangle, A\rangle$ such that

(1) $A \in \bigcap \vec{V}$ and $A \subseteq \overline{A}$.

(2) $A \cap V_{\kappa(d_n)+1} = \emptyset$.

(3) For every m with $1 \leq m \leq n$, either

  (3a) d_m is an ordinal, or

  (3b) $d_m = \langle \nu, \vec{F}_\nu, A_\nu\rangle$ for some $\vec{F}_\nu \in \overline{A}, A_\nu \subseteq \overline{A}$ and $A_\nu \in \bigcap \vec{F}_\nu$.

(4) For every $1 \leq i < j \leq n$,

  (4a) $\kappa(d_i) < \kappa(d_j)$, and

  (4b) if d_j is of the form $\langle \nu, F_\nu, A_\nu\rangle$ then $A_\nu \cap V_{\kappa(d_i)+1} = \emptyset$.

Each d_m of the form $\langle \nu, \vec{F}_\nu, A_\nu\rangle$ will give rise to Radin forcing $R_{\vec{F}_\nu}$ with $\vec{F}_\nu$ playing the same role as $\vec{V}$ in $R_{\vec{V}}$.

We define two orders $\leq$ and $\leq^*$ on $R_{\vec{V}}$, where, as usual, $\leq$ will be used to force and $\leq^*$ will provide the closure.

5.3 Definition. Let $p = \langle d_1, \ldots, d_n, \langle\kappa, \vec{V}\rangle, A\rangle$, $q = \langle e_1, \ldots, e_m, \langle\kappa, \vec{V}\rangle, B\rangle \in R_{\vec{V}}$. We say that p *is stronger than* q and denote this by $p \geq q$ iff

(1) $A \subseteq B$.

(2) $n \geq m$.

(3) There are $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that for $1 \leq k \leq m$, either

  (3a) $e_k = d_{i_k}$, or

  (3b) $e_k = \langle \nu, \vec{F}_\nu, B_\nu\rangle$ and then $d_{i_k} = \langle \nu, \vec{F}_\nu, C_\nu\rangle$ with $C_\nu \subseteq B_\nu$.

(4) Let $i_1, \ldots, i_m$ be as in (3). Then the following holds for every $j, 1 \leq j \leq n$:

(4a) if $j > i_m$, then $d_j \in B$ or d_j is of the form $\langle \nu, \vec{F}_\nu, C_\nu \rangle$ with $\langle \nu, \vec{F}_\nu \rangle \in B$ and $C_\nu \subseteq B \cap \nu$.

(4b) if $j < i_m$, then for the least k with $j < i_k$, e_k is of the form $\langle \nu, \vec{F}_\nu, B_\nu \rangle$ so that

(i) if d_j is an ordinal then $d_j \in B_\nu$, and

(ii) if $d_j = \langle \rho, \vec{T}, S \rangle$ then $\langle \rho, \vec{T} \rangle \in B_\nu$ and $S \subseteq B_\nu$.

5.4 Definition. Let $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle$, $q = \langle e_1, \ldots, e_m, \langle \kappa, \vec{V} \rangle, B \rangle \in R_{\vec{V}}$. We say that p *is a direct extension of* q and denote this by $p \geq^* q$ iff

(1) $p \geq q$, and

(2) $n = m$.

Intuitively, $\langle R_{\vec{V}}, \leq, \leq^* \rangle$ is like the Prikry forcing only once some point of the form $\langle \nu, \vec{F}_\nu \rangle$ was produced, it starts to act completely autonomously and eventually adds its own sequence.

As in the case of the Prikry forcing, any two conditions in $R_{\vec{V}}$ having the same finite sequences are compatible. So we obtain the following analogue of 1.5:

5.5 Lemma. *$\langle R_{\vec{V}}, \leq \rangle$ satisfies the κ^+-c.c.*

Suppose that $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$. Let, for some m with $1 \leq m \leq n$, $d_m = \langle \nu_m, \vec{V}_m, A_m \rangle$. Set $p^{\leq m} = \langle d_1, \ldots, d_m \rangle$ and

$$p^{>m} = \langle d_{m+1}, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \ .$$

Then $p^{\leq m} \in R_{\vec{V}_m}$ and $p^{>m} \in R_{\vec{V}}$. Let for $\vec{W} \in \overline{A}$ and $q \in R_{\vec{W}}$

$$R_{\vec{W}}/q = \{ r \in R_{\vec{W}} \mid r \geq q \} \ .$$

5.6 Lemma. *$R_{\vec{V}}/p \simeq R_{\vec{V}_m}/p^{\leq m} \times R_{\vec{V}}/p^{>m}$.*

5.7 Lemma. *$\langle R_{\vec{V}}/p^{>m}, \leq^* \rangle$ is ν_m-closed.*

This together with the Prikry condition (the next lemma) will suffice to prove the preservation of cardinals. Thus let $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$ and ξ be a cardinal. If $\xi > \kappa$, then we use 5.5. Let $\xi \leq \kappa$. Then we pick the last m, $1 \leq m \leq n$ with d_m of the form $\langle \nu_m, \vec{V}_m, A_m \rangle$ such that $\nu_m < \xi$, if it exists. Work with $R_{\vec{V}}/p^{>m}$ in this case. Otherwise we continue to deal with $R_{\vec{V}}$. Suppose for simplicity that such m does not exist, i.e. $\xi \leq \nu_m$ for every m, $1 \leq m \leq n$ with $d_m = \langle \nu_m, \vec{V}_m, A_m \rangle$.

Let

$$\rho = \min(\{ \kappa, \kappa(d_m) \mid 1 \leq m \leq n \text{ and } d_m \text{ is of form } \langle \nu_m, \vec{V}_m, A_m \rangle \} \setminus \xi) \ .$$

Assume for simplicity that $\rho = \kappa$. If length$(\vec{V}) = 1$, then $R_{\vec{V}}$ is just the Prikry forcing and it preserves cardinals. Suppose that length$(\vec{V}) > 1$. Let $\delta < \kappa$. Extend p to p^* by shrinking A to $A \setminus V_{\delta+1}$. Then $\langle R_{\vec{V}}/p^*, \leq^* \rangle$ will be δ-closed. Using the Prikry condition, one can see that $\langle R_{\vec{V}}/p^*, \leq \rangle$ does not add new subsets to δ. But δ was any cardinal below κ. So ξ is not collapsed even if $\xi = \kappa$ and we are done.

Let us now turn to the Prikry condition. The main new point here is that we are allowed to extend a given condition by picking elements from different ultrafilters of the sequence $\vec{V}$. So maybe different choices will decide some statement σ differently. The heart of the matter will be to show that this really does not happen. Actually, we can pass from one choice of an ultrafilter to another, remaining with compatible conditions.

5.8 Lemma. *$\langle R_{\vec{V}}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. Let $p \in R_{\vec{V}}$ and σ be a statement of the forcing language. We need to find $p^* \geq^* p$ that decides σ. Suppose there is no such p^*. Assume for simplicity that $p = \langle \langle \kappa, \vec{V} \rangle, A \rangle$.

For every $\vec{d} = \langle d_1, \ldots, d_n \rangle \in [V_\kappa]^n$ consider

$$\langle d_1, \ldots, d_n \rangle^\frown p =_{df} \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \setminus V_{\kappa(d_n)+1} \rangle \ .$$

Suppose that it is a condition in $R_{\vec{V}}$. Let

$$\begin{aligned} \widetilde{A}(\vec{d}) = \{ d \in A \mid \ & \text{either } d \text{ is an ordinal and then } \vec{d}^\frown d^\frown p \in R_{\vec{V}} \\ & \text{or } d \text{ is of the form } \langle \nu, \vec{F}_\nu \rangle \text{ and then } \vec{d}^\frown \langle \nu, \vec{F}_\nu, A \cap V_\nu \rangle^\frown p \in R_{\vec{V}} \}. \end{aligned}$$

Clearly, $\widetilde{A}(\vec{d}) \in \bigcap \vec{V}$. We split $\widetilde{A}$ into three sets: First, set

$$A_0(\vec{d}) = \{ d \in \widetilde{A}(\vec{d}) \mid \text{either (i) or (ii)} \}$$

where

(i) d is an ordinal and there is a B_d such that

$$\vec{d}^\frown d^\frown p \leq^* \langle \vec{d}^\frown d, \langle \kappa, \vec{V} \rangle, B_d \rangle \Vdash \sigma \ , \text{ or}$$

(ii) d is of the form $\langle \nu, \vec{F}_\nu \rangle$ and there are B_d and b_d such that

$$\vec{d}^\frown \langle \nu, \vec{F}_\nu, A \cap V_\nu \rangle^\frown p \leq^* \langle \vec{d}^\frown \langle \nu, \vec{F}_\nu, b_d \rangle, \langle \kappa, \vec{V} \rangle, B_d \rangle \Vdash \sigma \ .$$

Then, let $A_1(\vec{d})$ be the same as $A_0(\vec{d})$ but with σ replaced by $\neg\sigma$. Finally, set

$$A_2(\vec{d}) = \widetilde{A}(\vec{d}) \setminus (A_0(\vec{d}) \cup A_1(\vec{d})) \ .$$

For every $\alpha < \mathrm{length}(\vec{V})$ choose $i_\alpha \le 2$ such that $A_{i_\alpha}(\vec{d}) \in U(\alpha)$. Set $A(\alpha, \vec{d}) = A_{i_\alpha}(\vec{d})$. If $\vec{d}^\frown p \notin R_{\vec{V}}$ then let $A(\alpha, \vec{d}) = A$. Set

$$A(\alpha) = \{d \in A \mid \forall \vec{d} = \langle d_1, \ldots, d_n\rangle \in [V_\kappa]^n \\ (\text{if } \max\{\kappa(d_k) \mid 1 \le k \le n\} < \kappa(d), \text{ then } d \in A(\alpha, \vec{d}))\} \ .$$

This is the kind of diagonal intersection which is appropriate for our setting. We claim that $A(\alpha) \in U(\alpha)$. Thus, for every $\vec{d} \in [V_\kappa]^n$ we have $A(\alpha, \vec{d}) \in U(\alpha)$. So, in M, $\langle \kappa, V{\restriction}\alpha\rangle \in j(A(\alpha, \vec{d}))$ for every $\vec{d} \in [V_\kappa]^n$. Clearly $\kappa(\langle \kappa, U{\restriction}\alpha\rangle) = \kappa$. Hence, by the definition of $A(\alpha)$, $\langle \kappa, V{\restriction}\alpha\rangle \in j(A(\alpha))$.

Define now $A^* = \bigcup_{\alpha < \mathrm{length}(\vec{V})} A(\alpha)$. Obviously $A^* \in \bigcap_{\alpha < \mathrm{length}(\vec{V})} U(\alpha)$. Consider $p^* = \langle \langle \kappa, \vec{V}\rangle, A^*\rangle$. By our initial assumption there is no direct extension of p^* deciding σ. Pick $\langle \langle d_1, \ldots, d_{n+1}\rangle, \langle \kappa, \vec{V}\rangle, B\rangle$ to be an extension of p^* deciding σ with n as small as possible. Suppose, for example, that it forces σ. Pick $\alpha < \mathrm{length}(\vec{V})$ such that $d_{n+1} \in A(\alpha)$. Let $\vec{d} = \langle d_1, \ldots, d_n\rangle$. Then $\vec{d}^\frown p \in R_{\vec{V}}$. By the definition of $A(\alpha)$, $d_{n+1} \in A(\alpha, \vec{d})$. By the choice of $A(\alpha, \vec{d})$, then $A(\alpha, \vec{d}) = A_0(\vec{d})$. This means that for every $d \in A(\alpha) \setminus V_{\kappa(d_n)+1}$ there are $\widetilde{d}$ and B such that

$$\vec{d}^\frown d^\frown p \le^* \langle \vec{d}^\frown \widetilde{d}, \langle \kappa, \vec{V}\rangle, B\rangle\rangle \Vdash \sigma \ .$$

Obviously we can replace p by p^* here. In what follows we show that for some C

$$p^* \le \langle \langle d_1, \ldots, d_n\rangle, \langle \kappa, \vec{V}\rangle, C\rangle \Vdash \sigma \ .$$

This will contradict the minimality of n and, in turn, our initial assumption.

We shrink first the sets in $U(\beta)$ for every $\beta < \alpha$ (if there are any). Suppose that $\alpha > 0$. The case $\alpha = 0$ is similar and slightly easier. For every $d \in A(\alpha) \setminus V_{\kappa(d_n)+1}$ of the form $\langle \nu, \vec{F}_\nu\rangle$ pick some b_d and B_d so that $\vec{d}^\frown d^\frown p^* \le^* \langle \vec{d}, \langle \langle \nu, \vec{F}_\nu\rangle, b_d\rangle, \langle \kappa, \vec{V}\rangle, B_d\rangle \Vdash \sigma$. We take a diagonal intersection of the B_d's. Thus, let

$$B^* = \{e \in A^* \mid \forall d \in V_{\kappa(e)} \ (\text{if } B_d \text{ is defined then } e \in B_d)\}$$

For every $\beta < \mathrm{length}(\vec{V})$, $B^* \in U(\beta)$, since clearly for every $d \in V_\kappa$ with B_d defined $\langle \kappa, \vec{V}{\restriction}\beta\rangle \in j(B_d)$ due to $B_d \in \bigcap \vec{V}$, so $\langle \kappa, \vec{V}{\restriction}\beta\rangle \in j(B^*)$,

Note that by the choice of $\overline{A}$ and 5.1.(3(b)) $b_d \in \bigcap_{W \in \vec{F}_\nu} W$, where each $W \in \vec{F}_\nu$ is a ν-complete ultrafilter over V_ν. Consider $A^{<\alpha} = j(\langle b_d \mid d \in A(\alpha)\rangle)(\vec{V}{\restriction}\alpha)$ (recall that $A(\alpha) \in U(\alpha)$ implies that $\vec{V}{\restriction}\alpha \in j(A(\alpha))$). Then, by elementarity, $A^{<\alpha} \in U(\beta)$ for every $\beta < \alpha$. Also, note that the set $A'(\alpha) = \{d \in A(\alpha) \mid A^{<\alpha} \cap V_{\kappa(d)} = b_d\} \in U(\alpha)$, since $j(A^{<\alpha}) \cap V_{\kappa(\vec{V}{\restriction}\alpha)} = j(A^{<\alpha}) \cap V_\kappa = A^{<\alpha} = j(\langle b_d \mid d \in A(\alpha)\rangle)(\vec{V}{\restriction}\alpha)$ and hence $\vec{V}{\restriction}\alpha \in j(A'(\alpha))$.

Set $A^{\le\alpha} = (A^{<\alpha} \cup A'(\alpha)) \cap A^*$. Then $A^{\le\alpha} \in U(\beta)$ for every $\beta \le \alpha$.

Now let us shrink the sets in $U(\beta)$ for all $\beta > \alpha$ (if there are any). Actually, we need to care for only β's with $A^{\leq\alpha} \notin U(\beta)$. Consider the set

$$A^{>\alpha} = \{\langle \nu, \vec{F}\rangle \in A^* \mid \exists \xi < \text{length}(\vec{F})(A'(\alpha) \cap V_\nu \in F(\xi))\} \ .$$

Then $A^{>\alpha} \in U(\beta)$ for every β with $\alpha < \beta < \text{length}(\vec{V})$. Set

$$A^{**} = (A^{\leq\alpha} \cup A^{>\alpha}) \cap B^*$$

Clearly, $A^{**} \in \bigcap \vec{V}$. Consider a condition $p^{**} = \langle\langle\kappa, \vec{V}\rangle,\ A^{**}\rangle$ and $q = \langle d_1, \ldots, d_n\rangle^\frown p^{**}$. By the choice of n, neither q nor its direct extensions can decide σ. Pick some $r \geq q$ forcing $\neg\sigma$. Let $r = \langle e_1, \ldots, e_m, \langle\kappa, \vec{V}\rangle, C\rangle$. There is a $k \leq m$ such that $\kappa(d_n) = \kappa(e_k)$, by Definition 5.2(3). Consider three cases.

Case 1. $k = m$.

Then choose some $d \in A(\alpha) \cap C$ such that $C \cap \nu \in \cap \vec{F}_\nu$ where $d = \langle \nu, \vec{F}_\nu\rangle$. By the choice of $A(\alpha)$ and B^* there is a b_d such that

$$\langle d_1, \ldots, d_n, \langle\langle \nu, \vec{F}_\nu\rangle, b_d\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \setminus V_{\kappa(d)+1}\rangle \Vdash \sigma \ .$$

Clearly we can shrink $A^{**} \setminus V_{\kappa(d)+1}$ to C. Then,

$$\langle e_1, \ldots, e_m, \langle\langle \nu, \vec{F}_\nu\rangle, b_d \cap C\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \cap C \setminus V_{\kappa(d)}\rangle$$

will be a common extension of r and $\langle d_1, \ldots, d_n,\ \langle\langle \nu, \vec{F}_\nu\rangle, b_d\rangle\langle\kappa, \vec{V}\rangle, A^{**} \cap C \setminus V_{\kappa(d)}\rangle$, which is clearly impossible since they disagree about σ.

Case 2. $k < m$ and for $k < j \leq m$, $e_j \in A^{<\alpha}$.

Pick $d \in A'(\alpha) \cap C$, $d = \langle \nu, \vec{F}_\nu\rangle$ such that $C \cap A^{<\alpha} \cap V_\nu \in \cap \vec{F}_\nu$. Then, by the choice of $A'(\alpha)$, $b_d = A^{<\alpha} \cap V_\nu$. So,

$$\langle d_1, \ldots, d_n, \langle\langle \nu, \vec{F}_\nu\rangle, A^{<\alpha} \cap V_\nu\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \setminus V_{\kappa(d)+1}\rangle\rangle \Vdash \sigma \ .$$

But

$$\begin{aligned} &\langle\langle e_1, \ldots, e_m\rangle, \langle\langle \nu, \vec{F}_\nu\rangle, C \cap A^{<\alpha} \cap V_\nu\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \cap C \setminus V_{\kappa(d)+1}\rangle \\ &\qquad \geq \langle d_1, \ldots, d_n,\ \langle\langle \nu, \vec{F}_\nu\rangle, A^{<\alpha} \cap \nu\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \cap C \setminus V_{\kappa(d)+1}\rangle \ , \end{aligned}$$

since $e_j \in A^{<\alpha}$ for every $k < j \leq m$, $\kappa(d_n) = \kappa(e_k)$ and $r \geq q$. Also, clearly,

$$\langle\langle e_1, \ldots, e_m\rangle, \langle\langle \nu, \vec{F}_\nu\rangle, C \cap A^{<\alpha} \cap \nu\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \cap C \setminus V_{\kappa(d)+1}\rangle\rangle \geq r \ .$$

But this is impossible, since $r \Vdash \neg\sigma$.

Case 3. $k < m$ and there is a j with $k < j \leq m$ such that $e_j \notin A^{<\alpha}$.

Let j^* be the minimal j with $k < j \leq m$ and $e_j \notin A^{<\alpha}$. Then $e_{j^*} \in A'(\alpha) \cup A^{>\alpha}$. If $e_{j^*} \in A'(\alpha)$, then

$$\begin{aligned}&\langle e_1, \dots, e_{j^*-1}, \langle\langle \nu, \vec{F}_\nu\rangle, E \cap A^{<\alpha}\rangle, \langle \kappa, \vec{V}\rangle, A^{**} \setminus V_{\kappa(e_{j^*})+1}\rangle \\ &\qquad \geq \langle d_1, \dots, d_n, \langle\langle \nu, \vec{F}_\nu\rangle, A^{<\alpha} \cap V_\nu\rangle \ \langle \kappa, \vec{V}\rangle, A^{**} \setminus V_{\nu+1}\rangle \Vdash \sigma \ ,\end{aligned}$$

by minimality of j^*, where $e_{j^*} = \langle\langle \nu, \vec{F}_\nu\rangle, E\rangle$. But, $\langle e_1, \dots, e_{j^*-1}, \langle\langle \nu, \vec{F}_\nu\rangle, E \cap A^{<\alpha}\rangle, \langle \kappa, \vec{V}\rangle, A^{**} \setminus V_{\kappa(e_{j^*})+1}\rangle$ and r are compatible, which is impossible since $r \Vdash \neg\sigma$.

So, assume that $\langle \nu, \vec{F}_\nu\rangle \in A^{>\alpha} \setminus A^{\leq(\alpha)}$, where $e_{j^*} = \langle\langle \nu, \vec{F}_\nu\rangle, E\rangle$. We have $E \in \cap\vec{F}_\nu$. By the choice of $A^{>\alpha}$, for some $\xi <$ length$(\vec{F}_\nu)$ $A'(\alpha) \cap V_\nu \in F_\nu(\xi)$. Hence $A'(\alpha) \cap E \in F_\nu(\xi)$. Pick some $\langle \tau, \vec{G}_\tau\rangle \in (A'(\alpha) \cap E) \setminus V_{\kappa(e_{j^*-1})+1}$ such that $E \cap \tau \in \cap\vec{G}_\tau$. This can be done since $\vec{F}_\nu$ is a j'-sequence for some j' and $E \in \cap\vec{F}_\nu$. Now we can extend r by adding to it $\langle \tau, \vec{G}_\tau\rangle$. This will reduce the situation to the one considered above, i.e. $e_{j^*} \in A'(\alpha)$. This completes the proof of the lemma. ⊣

Now let G be a generic subset of $R_{\vec{V}}$. Combining the previous lemmas together, we obtain the following:

5.9 Theorem. *$V[G]$ is a cardinal preserving extension of V.*

Consider the following crucial set:

$$\begin{aligned}C_G = \{\kappa(d) < \kappa \mid \exists p \in G \\ &(d \text{ is one of the elements of the finite sequence of } p)\} \ .\end{aligned}$$

5.10 Lemma. *C_G is a closed unbounded subset of κ.*

Proof. C_G is unbounded since for every condition $p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$ and every ordinal $\tau < \kappa$ we can find some $\nu \in A \cap (\kappa \setminus \tau)$ and extend p by adding ν to its finite sequence $\langle d_1, \dots, d_n\rangle$.

Let us show that C_G is closed. Thus, let for some $\tau < \kappa$ some

$$p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle \Vdash \check{\tau} \notin \underset{\sim}{C}_G \ .$$

Clearly, $\tau \neq \kappa(d_i)$ for any i, $1 \leq i \leq n$. If $\tau > \kappa(d_n)$, then we shrink A to $A \setminus (\tau + 1)$. By the definition of the forcing ordering $\leq$,

$$\langle d_1, \dots, d_n, \langle \kappa, \vec{V}\rangle, A \setminus (\tau + 1)\rangle \Vdash \sup(\underset{\sim}{C}_G \cap \check{\tau}) = \check{\kappa}(d_n) \ .$$

Suppose now that $\tau < \kappa(d_n)$. Let $i^* < n$ be the least such that $\tau < \kappa(d_{i^*+1})$. If d_{i^*+1} is an ordinal, then again by the definition of the forcing ordering $\leq$, p forces that $\underset{\sim}{C}_G$ will not have elements in the open interval

$(\kappa(d_{i^*}), d_{i^*+1})$, where $d_0 = 0$. So, let $d_{i^*+1} = \langle \nu, \vec{F}_\nu, B_\nu \rangle$. Then $B_\nu \setminus (\tau+1) \in \bigcap \vec{F}_\nu$ and the extension of p

$$\langle d_1, \ldots, d_{i^*}, \langle \nu, \vec{F}_\nu, B_\nu \setminus (\tau+1) \rangle, d_{i^*+2}, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \Vdash \sup(\underset{\sim}{C}_G \cap \check{\tau}) = \check{\kappa}(d_{i^*}) \ .$$

Combining all the cases together we conclude that there is always an extension of p forcing that τ is not a limit of elements of C_G. ⊣

The next question will be crucial for the issue of changing cofinalities:

What is the order type of C_G?

For every τ with $0 < \tau < \kappa$, $U(\tau)$ concentrates on the set $X_\tau = \{\langle \nu, \vec{F}_\nu \rangle \mid \vec{F}_\nu$ is a sequence of ν-complete ultrafilters over V_ν of length$\tau < \nu\}$. Clearly, $\{X_\tau \mid 0 < \tau < \kappa\}$ are disjoint. We can add to them also $X_0 = \kappa$ and $X_\kappa = \{\langle \nu, \vec{F}_\nu \rangle \mid \vec{F}_\nu$ is a sequence of ν-complete ultrafilters over ν of length$\nu\}$. Using this partition and an easy induction it is not hard to see the following.

5.11 Lemma. *Let δ, $0 < \delta < \kappa$, length$\vec{V}) = \delta$, and $G \subseteq R_{\vec{V}}$ be generic. Then, in $V[G]$, a final segment of C_G has order type ω^δ, where ω^δ is the ordinal power. Moreover, $\langle \kappa, \vec{V}$, $\bigcup\{X_\tau \mid 0 < \tau < \delta\}\rangle$ forces the order type of C_G to be ω^δ. In particular,* otp$(C_G) = \delta$ *if δ is an uncountable cardinal.*

Combining this with 5.9 we obtain the following:

5.12 Theorem. *Let* length$(\vec{V}) = \delta < \kappa$ *be a cardinal, and let $G \subseteq R_{\vec{V}}$ be generic. Then $V[G]$ is a cardinal preserving extension of V in which κ changes its cofinality to* $\mathrm{cf}(\delta)^V$.

Notice that if $\delta > 0$ then $R_{\vec{V}}$ changes cofinalities also below κ. Hence new bounded subsets are added to κ. Mitchell [45] showed that once one changes the cofinality of κ to some uncountable $\delta < \kappa$ preserving cardinals, then new bounded subsets of κ must appear, provided the ground model was the core model. On the other hand, it is possible to prepare a ground model and then force in order to change cofinality of κ to an uncountable δ without adding new bounded subsets. This was first done by Mitchell [44], combining iterated ultrapowers and forcing. A pure forcing construction was given in [13].

If we force with $R_{\vec{V}}$ having length$(\vec{V}) = \kappa$, then κ changes its cofinality to ω again.

5.13 Lemma. *Suppose that* length$(\vec{V}) = \kappa$ *and $G \subseteq R_{\vec{V}}$ generic. Then, in $V[G]$,* $\mathrm{cf}(\kappa) = \aleph_0$.

Proof. Let $\langle X_\tau \mid \tau < \kappa\rangle$ be the partition defined before 5.11. Then, $\bigcup_{\tau<\kappa} X_\tau \in \cap\vec{V}$, since for every $\tau < \kappa$ $X_\tau \in U(\tau)$ and $\vec{V} = \langle U(0),\ldots, U(\tau),\ldots \mid \tau<\kappa\rangle$. Let $X = \bigcup_{\tau<\kappa} X_\tau$. Consider

$$Y = \{\langle \nu, \vec{F}_\nu\rangle \in X \mid \bigcup\{X_\tau \mid \tau < \text{length}(\vec{F}_\nu)\} \cap V_\nu \in \cap\vec{F}_\nu\} \cup \kappa\ .$$

Clearly, $Y \in \bigcap\vec{V}$. Now pick some $p = \langle d_1,\ldots,d_n,\ \langle\kappa,\vec{V}\rangle, A\rangle \in G$ with $A \subseteq Y$. Let

$$C = \{\langle \nu, \vec{F}_\nu\rangle \in V_\kappa \mid \exists E \in \bigcap\vec{F}_\nu(\langle \nu, \vec{F}_\nu, E\rangle \text{ appears in a condition in } G)\}\ .$$

Then, $C\setminus(\kappa(d_n)+1) \subseteq A$. A simple density argument shows that for every $\tau<\kappa$, C will contain unboundedly many members of X_τ. Let

$$C' = \{\nu < \kappa \mid \exists \vec{F}(\langle \nu, \vec{F}\rangle \in C)\}\ .$$

Clearly, C' is just the set of all limit points of C_G. Also, for every $\nu \in C'$ there is a unique $\vec{F}_\nu$ with $\langle\nu,\vec{F}_\nu\rangle \in C$. We define an increasing sequence $\langle \nu_n \mid n<\omega\rangle$ of elements of C' as follows: $\nu_0 = \min(C')$, $\nu_{n+1} = \min\{\nu \in C' \mid \exists\vec{F}_\nu\ \langle\nu,\vec{F}_\nu\rangle \in X_{\nu_n}\}\setminus(\nu_n+1))$.

Set $\nu_\omega = \bigcup_{n<\omega}\nu_n$. We claim that $\nu_\omega = \kappa$. Otherwise there is a $\tau<\kappa$ such that $\langle\nu_\omega,\vec{F}\rangle \in C\cap X_\tau$ for some (unique) $\vec{F}$, since C' is closed and $C\subseteq A\subseteq Y\subseteq X = \bigcup_{\tau<\kappa}X_\tau$. Then there is a $q\geq p$ in G with $\langle\langle\nu_\omega,\vec{F}\rangle, B\rangle$ appearing in q for some $B\in\cap\vec{F}$. We require also $B \subseteq \bigcup\{X_{\tau'} \mid \tau' < \tau\}\cap V_{\nu_\omega}$. This is possible since $q\geq p$, $A\subseteq Y$, $\nu_\omega > \kappa(d_n)$, and hence $\bigcup\{X_{\tau'} \mid \tau'<\tau\}\cap V_{\nu_\omega} \in \bigcap\vec{F}$. Now, by the definition of X_τ, we have $\tau<\nu_\omega$. So, there is an $n<\omega$ with $\nu_n > \max(\tau,\min(B))$. But $\nu_n\in C'$, hence $\langle\nu_n,\vec{F}_{\nu_n}\rangle$ should be in B, for some (unique) $\vec{F}_{\nu_n}$. The same holds for each ν_m with $n\leq m<\omega$. In particular, $\langle\nu_{n+1},\vec{F}_{\nu_{n+1}}\rangle \in \bigcup_{\tau'<\tau}X_{\tau'}$. But it was picked to be in X_{ν_n} which is disjoint to each $X_{\tau'}$ for $\tau'<\nu_n$. Contradiction. ⊣

Similar arguments show that for every $\delta<\kappa^+$, if $\text{length}(\vec{V}) = \delta$ then the forcing $R_{\vec{V}}$ changes the cofinality of κ. If δ is a successor ordinal, then to $\aleph_0$; if δ is limit and $\text{cf}(\delta)\neq\kappa$ then to $\text{cf}\delta$ and, finally, if $\text{cf}(\delta)=\kappa$ then to $\aleph_0$.

Let us now show that if $\vec{V}$ is long enough then $R_{\vec{V}}$ can preserve measurability of κ. Later it will be shown that $\text{length}(\vec{V}) = \kappa^+$ suffices to keep κ regular and so inaccessible. The ability of keeping κ regular turned out to be very important in applications to the cardinal arithmetic. Thus a basic common theme used there is to arrange some particular pattern of the power function over C_G, sometimes adding Cohen subsets or collapsing cardinals in between and then to cut the universe at κ. This type of constructions were used by Foreman-Woodin [12], Cummings [9] and recently by Merimovich [39].

5.14 Definition. An ordinal $\gamma < \mathrm{length}(\vec{V})$ is called a *repeat point* for $\vec{V}$ if for every δ with $\gamma \le \delta < \mathrm{length}(\vec{V})$ and for every $A \in U(\delta)$, there is a $\delta' < \gamma$ such that $A \in U(\delta')$. Equivalently, $\bigcup \vec{V} = \bigcup \vec{V} \restriction \gamma$.

Note that if $2^\kappa = \kappa^+$ and our sequence has length κ^{++}, then there will be κ^{++} repeat points between κ^+ and κ^{++}. This implies that also $\vec{V} = \vec{U} \restriction \alpha$ will have a repeat point for unboundedly many α's below κ^{++}.

5.15 Theorem. *If γ is a repeat point for $\vec{V}$ and $G \subseteq R_{\vec{V}}$ is generic, then κ remains measurable in $V[G]$.*

Proof. Recall that $\vec{V} = \langle U(\alpha) \mid \alpha < \mathrm{length}(\vec{V})\rangle$ is a j-sequence for some elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$. By the definition of a repeat point, the forcing $R_{\vec{V}}$ and $R_{\vec{V}\restriction\gamma}$ are basically the same (we need only to replace $\langle \kappa, \vec{V}\rangle$ in each condition of $R_{\vec{V}}$ by $\langle \kappa, \vec{V}\restriction\gamma\rangle$ in order to pass to $R_{\vec{V}\restriction\gamma}$). So we can view G as a generic subset of $R_{\vec{V}\restriction\gamma}$. Define now an ultrafilter F over κ in $V[G]$. Let $\underset{\sim}{X}$ be a name of a subset of κ. Set $\underset{\sim}{X}[G] \in F$ iff for some $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle \in G$ the following holds in M: For some $B \in \bigcap j(\vec{V})$,

$$\langle d_1, \ldots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B\rangle \Vdash_{R_{j(\vec{V})}} \check{\kappa} \in j(\underset{\sim}{X}) \ .$$

First note that F is well defined. Thus, let some $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle \in G$ forces "$\underset{\sim}{X} = \underset{\sim}{Y}$". Then, in M

$$\langle d_1, \ldots, d_n, \langle j(\kappa), j(\vec{V}), j(A)\rangle\rangle \Vdash j(\underset{\sim}{X}) = j(\underset{\sim}{Y}) \ .$$

But $A \in \bigcap \vec{V}$. In particular, $A \in U(\gamma)$. Hence, $\langle \kappa, \vec{V}\restriction\gamma\rangle \in j(A)$. Also, $j(A) \cap V_\kappa = A$. So, $\langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle$ is addible to $\langle d_1, \ldots, d_n, \langle j(\kappa), j(\vec{V}), j(A)\rangle$. But if for some $B \in \bigcap j(\vec{V})$,

$$\langle d_1, \ldots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B\rangle \Vdash_{R_{j(\vec{V})}} \check{\kappa} \in j(\underset{\sim}{X}) \ ,$$

then

$$\begin{aligned}&\langle d_1, \ldots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B \cap j(A)\rangle \\ &\qquad\qquad \Vdash_{R_{j(\vec{V})}} \check{\kappa} \in j(\underset{\sim}{X}) \ \wedge\ j(\underset{\sim}{X}) = j(\underset{\sim}{Y}) \ .\end{aligned}$$

Let us establish normality for F. Suppose $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle\rangle \in G$ and $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle\rangle \Vdash (\{\nu < \kappa \mid \underset{\sim}{f}(\nu) < \nu\} \in \underset{\sim}{F})$. Then, in M, for some $B \in \bigcap j(\vec{V})$

$$\langle d_1, \ldots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B\rangle \Vdash_{R_{j(\vec{V})}} j(\underset{\sim}{f})(\check{\kappa}) < \check{\kappa} \ .$$

Working in M, construct $B' \in \bigcap j(\vec{V})$ such that: If for some $\nu < \kappa$, we have a condition $\langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle, \langle j(\kappa), j(\vec{V})\rangle, E\rangle$ forcing "$j(\underset{\sim}{f})(\check{\kappa}) = \check{\nu}$", then $\langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle, \langle j(\kappa), j(\vec{V})\rangle, B'\rangle$ forces the same.

Back in V, the set

$$D = \{\langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle\rangle \mid \text{for some } \nu < \kappa, \\ \langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle, \langle j(\kappa), j(\vec{V})\rangle, B'\rangle \Vdash_{R_{j(\vec{V})}} j(\underset{\sim}{f})(\check{\kappa}) = \check{\nu}\}$$

will be dense in $R_{\vec{V}\restriction\gamma}$ above $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$. Thus, if some $p \in R_{\vec{V}\restriction\gamma}$ with $p \geq \langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$ has no extension in D, then we consider the statement

$$\varphi \equiv \text{“There is a } q \in R_{\vec{V}} \text{ stronger than } p, \text{ a } \nu < \kappa, \text{ and} \\ \text{an } r \text{ in } \underset{\sim}{G}(R_{j(\vec{V})\setminus\kappa+1}) \text{ such that } \langle q, r\rangle \Vdash_{R_{j(\vec{V})}} j(\underset{\sim}{f})(\check{\kappa}) = \check{\nu}\}\text{”},$$

where $\underset{\sim}{G}(R_{j(\vec{V})\setminus\kappa+1})$ denotes the canonical name of a generic subset of $R_{j(\vec{V})\setminus k+1}$. Let, in M, $s \geq^* \langle\langle j(\kappa), j(\vec{V})\rangle, B'\rangle$ deciding φ. Then s must force φ. Find some $s_1 \geq^* s$ deciding the values of ν and q in φ. This leads to the contradiction.

So, pick some $\langle e_1, \ldots, e_m, \langle \kappa, \vec{V}\rangle, A'\rangle \geq \langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$ in $G \cap D$. There is a $\delta < \kappa$ such that

$$\langle\langle e_1, \ldots, e_m\rangle, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A'\rangle, \langle j(\kappa), j(\vec{V})\rangle, B'\rangle \Vdash j(\underset{\sim}{f})(\check{\kappa}) = \check{\delta}\ .$$

Then $\{\nu < \kappa \mid f(\nu) = \delta\} \in F$, by the definition of F. ⊣

Similar arguments show that it is possible to preserve the degree of strongness and even of supercompactness of j. Notice also that F defined above extends $U(0)$, but the elementary embedding of F does not extend that of $U(0)$. Instead, it extends a certain iterated ultrapower embedding using ultrafilters of $\mathrm{Ult}(V, U(0))$ between κ and $i_{U(0)}(\kappa)$.

We now want to show that κ remains regular in $V^{R_{\vec{V}}}$ when we have $\mathrm{cf}(\mathrm{length}(\vec{V})) \geq \kappa^+$. But first we need to extend a bit the Prikry condition lemma (5.8) in the spirit of 2.18. This will allow us to deal with dense sets. The situation here is more involved due to the possibility of extending a given condition by adding to it elements from different ultrafilters $U(\alpha)$'s. We start with the following definition.

5.16 Definition. Let $\vec{F}$ be a sequence of ultrafilters over some $\nu \leq \kappa$. A tree $T \subseteq [V_\nu]^{\leq n}$ with $n < \omega$ levels is called $\vec{F}$*-fat* iff

(1) For every $\langle \nu_1, \ldots, \nu_k\rangle \in T$, $\kappa(\nu_1) < \kappa(\nu_2) < \cdots < \kappa(\nu_k)$.

(2) For every $\langle \nu_1, \ldots, \nu_k\rangle \in T$ with $k < n$, there is an $\alpha < \mathrm{length}(\vec{F})$ so that $\mathrm{Suc}_T(\langle \nu_1, \ldots, \nu_k\rangle) \in F(\alpha)$.

Let T be as in 5.16 and η a maximal branch in T. A sequence $\vec{A} = \langle \vec{A}(1), \ldots, \vec{A}(n) \rangle \in [V_\nu]^n$ will be called a *sequence of η-measure one* if, for every i, $1 \leq i \leq n$ with $\eta(i)$ of form $\langle \tau_i, \vec{G}_{\tau_i} \rangle$ we have $\vec{A}(i) \in \bigcap \vec{G}_{\tau_i}$. Let $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$ and $d_i = \langle \langle \nu_i, \vec{F}_i \rangle, A_i \rangle$ or $d_i = \nu_i < \kappa$ for each i, $1 \leq i \leq n$. Here also denote $\langle \langle \kappa, \vec{V} \rangle, A \rangle$ by $\langle \langle \nu_{n+1}, \vec{F}_{n+1} \rangle, A_{n+1} \rangle$. Let $1 \leq i_1 < \cdots < i_m \leq n+1$ be some elements of the set $\{ i \mid 1 \leq i \leq n+1, d_i = \langle \langle \nu_i, \vec{F}_i \rangle, A_i \rangle \}$.

Let for each k with $1 \leq k \leq m$ and some $n_k < \omega$, $T_k \subseteq [V_{\nu_{i_k}}]^{n_k}$ be a $\vec{F}_{\nu_{i_k}}$-fat tree, η_k a maximal branch in T_k, and $\vec{A}_k \in [V_{\nu_{i_k}}]^{\leq n_k}$ a sequence of η_k-measure one. Let $q = \langle t_1, \ldots, t_\ell, t_{\ell+1} \rangle$ be obtained from p by adding to it between $d_{i_k - 1}$ and d_{i_k}, for each k, $1 \leq k \leq m$, the following n_k-sequence $\langle s_j \mid 1 \leq j \leq n_k \rangle$, where $s_j = \eta_k(j)$, if $\eta_k(j)$ is an ordinal, or $s_j = \langle \tau_j, \vec{G}_{\tau_j}, \vec{A}_k(j) \rangle$, if $\eta_k(j) = \langle \tau_i, G_{\tau_i} \rangle$. Denote by $p ^\frown \langle \eta_1, \vec{A}_1 \rangle ^\frown \cdots ^\frown \langle \eta_m, \vec{A}_m \rangle$ the condition in $R_{\vec{V}}$ obtained from q by the obvious shrinking of sets of measure one needed in order to satisfy 5.2, i.e. for every i with $1 < i \leq \ell + 1$, if $t_i = \langle \delta_i, \vec{H}_i, B_i \rangle$, then we replace B_i by $B_i \setminus V_{\kappa(t_{i-1})+1}$.

5.17 Lemma. *Let D be a dense open subset of $R_{\vec{V}}$ and $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$. Then there are $p^* = \langle d_1^*, \ldots, d_n^*, \langle \kappa, \vec{V} \rangle, A^* \rangle \geq^* p$; $1 \leq i_1 < \cdots < i_m \leq n+1$; and for $1 \leq k \leq m$, $T_k \subseteq [V_{\nu_{i_k}}]^{n_k}$ $\vec{F}_{\nu_{i_k}}$–fat trees so that the following holds:*

For every sequence $\langle \eta_k \mid 1 \leq k \leq m \rangle$ such that η_k is a maximal branch in T_k, there exists a sequence $\langle \vec{A}_k \mid 1 \leq k \leq m \rangle$ such that

(1) $\vec{A}_k \in [V_{i_k}]^{n_k}$ *is a sequence of η_k-measure one, and*

(2) $p^* {}^\frown \langle \eta_1, \vec{A}_1 \rangle ^\frown \cdots ^\frown \langle \eta_m, \vec{A}_m \rangle \in D$.

5.18 Remark. Roughly, the meaning of this is that in order to get into D we need to specify certain $U(\alpha)$'s (or $F(\alpha)$'s, if below κ) and sets A_α's in these ultrafilters. Then any choice of elements in A_α's will put us into D.

Proof. The proof is very similar to that of 5.8. Suppose for simplicity that $p = \langle \langle \kappa, \vec{V} \rangle, A \rangle$. We need to find a direct extension $p^* = \langle \langle \kappa, \vec{V} \rangle, A^* \rangle$ of p and a $\vec{V}$–fat tree T of some finite height m such that the following holds: for every maximal branch $\eta = \langle f_1, \ldots, f_m \rangle$ through T there are sets $\vec{A} = \langle a_1, \ldots, a_m \rangle$ of η-measure one (i.e. for every i with $1 \leq i \leq m$, if $f_i = \langle \tau_i, \vec{G}_{\tau_i} \rangle$ then $a_i \in \bigcap \vec{G}_{\tau_i}$) such that $p^* {}^\frown \langle \eta, \vec{A} \rangle \in D$, where

$$p^* {}^\frown \langle \eta, \vec{A} \rangle = \langle f_1', \ldots, f_m', \langle \kappa, \vec{V} \rangle, A^* \setminus V_{\kappa(f_m)} \rangle$$

and for every i with $1 \leq i \leq m$, either

(α) f_i is an ordinal and then $f_i' = f_i$, or

(β) $f_i = \langle \tau_i, \vec{G}_{\tau_i} \rangle$ and then $f'_i = \langle \tau_i, \vec{G}_{\tau_i}, a_i \rangle$.

If p already has a direct extension in D, then we take such an extension and set $T = \{\langle\rangle\}$. Suppose that this is not the case. Define $\widetilde{A}(\vec{d})$ as in 5.8. Here we split it only into two sets $A_0(\vec{d}) = \{d \in \widetilde{A}(\vec{d}) |$ either (i) or (ii)$\}$ and $A_1(d) = \widetilde{A}(\vec{d}) \setminus A_0(\vec{d})$, where:

(i) d is an ordinal and then there is $B_{\vec{d}}$ such that

$$\vec{d}^\frown d^\frown p \leq^* \langle \vec{d}^\frown d, \langle \kappa, \vec{V} \rangle, B_{\vec{d}} \rangle \in D\ .$$

(ii) d is of form $\langle \nu, \vec{F}_\nu \rangle$ and then there are $B_{\vec{d}}$ and $b_{\vec{d}}$ such that

$$\vec{d}^\frown \langle \nu, \vec{F}_\nu, A \cap V_\nu \rangle^\frown p \leq^* \langle \vec{d}^\frown \langle \nu, \vec{F}_\nu, b_{\vec{d}} \rangle, \langle \kappa, \vec{V} \rangle, B_{\vec{d}} \rangle \in D\,.$$

As in 5.8, define $A(\alpha, \vec{d})$'s and $A(\alpha) \in U(\alpha)$ for $\alpha < \text{length}(\vec{V})$. Set $A^1 = \bigcup \{A(\alpha) \mid \alpha < \text{length}(\vec{V})\}$ and $p^1 = \langle \langle \kappa, \vec{V} \rangle, A^1 \rangle$. Then p^1 satisfies the following:

$(*)_1$ If $p^1 \leq q = \langle e_0, \ldots, e_m, \langle \kappa, \vec{V} \rangle, B \rangle \in D$, then there is an $\alpha < \text{length}(\vec{V})$ such that for every $e'_m \in A(\alpha) \setminus V_{\kappa(e_{m-1})+1}$, $\langle e_0, \ldots, e_{m-1}, e'_m, \langle \kappa, \vec{V} \rangle, A^1 \rangle$ has a direct extension of form $\langle e_0, \ldots, e_{m-1}, e''_m, \langle \kappa, \vec{V} \rangle, A'' \rangle$ in D .

Just pick α with $e_m \in A(\alpha)$ (more precisely, only $\langle \nu, \vec{F}_\nu \rangle$ if $e_m = \langle \nu, \vec{F}_\nu, B_\nu \rangle$). Then $e_m \in A(\alpha, \langle e_0, \ldots, e_{m-1} \rangle)$ and so by the choice of $A(\alpha, \langle e_0, \ldots, e_{m-1} \rangle)$ for every $e'_m \in A(\alpha, \langle e_0, \ldots, e_{m-1} \rangle)$ a direct extension of $\langle e_0, \ldots, e_{m-1}, e'_m, \langle \kappa, \vec{V} \rangle, A^1 \rangle$ will be in D. But if $e \in A(\alpha) \setminus V_{\kappa(e_{m-1})+1}$ then $e \in A(\alpha, \langle e_0, \ldots, e_{m-1} \rangle)$, by the definition of the diagonal intersection.

If for some $d \in A^1$, $d^\frown p^1$ has a direct extension in D, then we are done. Thus choose $\alpha < \text{length}(\vec{V})$ with $d \in A(\alpha)$. By the choice of $A(\alpha)$, then for every $d' \in A(\alpha)$ some direct extension of $d'^\frown p^1$ will be in D. Let us fix for every $d \in A(\alpha)$ a direct extension $\langle \widetilde{d}, \langle \kappa, \vec{V} \rangle, B_d \rangle$ of $d^\frown p^1$ in D, where $\widetilde{d}$ is either d, if d is an ordinal or $\langle \nu, \vec{F}_\nu, b_d \rangle$ if $d = \langle \nu, \vec{F}_\nu \rangle$. Set $A^* = \{e \in A^1 \mid \forall e' \in V_e (e \in B_{e'})\}$. Clearly, $A^* \in \bigcap \vec{V}$ and for every $d \in A^*$, $A^* \setminus V_{\kappa(d)+1} \subseteq B_d$. So, for every $d \in A(\alpha) \cap A^*$, $\langle \widetilde{d}, \langle \kappa, \vec{V} \rangle, B_d \rangle \leq^* \langle \widetilde{d}, \langle \kappa, \vec{V} \rangle, A^* \setminus V_{\kappa(d)+1} \rangle$. Hence, also $\langle \widetilde{d}, \langle \kappa, \vec{V} \rangle, A^* \setminus V_{\kappa(d)+1} \rangle$ is in D. Then we can take $p^* = \langle \langle \kappa, \vec{V} \rangle, A^* \rangle$ and T to be a one level tree which level consists of $A(\alpha) \cap A^*$.

Suppose now that there is no $d \in A^1$ with $d^\frown p^1$ having a direct extension in D. We continue to two steps extensions. Replacing A by A^1 we define $\widetilde{A}(\vec{d})$ as above. Let $A_0(\vec{d}) = \{d \in \widetilde{A}(\vec{d}) \mid$ there are $\alpha(\vec{d}) < \text{length}(\vec{V})$ and $C(\vec{d}) \subseteq \widetilde{A}(d) \setminus \kappa(d)$, $C(\vec{d}) \in U(\alpha(\vec{d}))$ such that for every $c \in C(\vec{d})$ there is in D a direct extension of the condition $\vec{d}^\frown d^\frown c^\frown p^1$ (i.e. the one obtained by adding $\vec{d}$, d and c to p^1)} and $A_1(\vec{d}) = A^1 \setminus A_0(\vec{d})$. Define $A(\alpha, \vec{d})$'s, $A(\alpha)$'s, A^2 and p^2 as was done above. Now, if for some $d_1, d_2 \in A^2$ some direct extension of $d_1{}^\frown d_2{}^\frown p^2$ is in D, then by $(*)_1$ for some $\beta < \text{length}(\vec{V})$, for every $d_2' \in A^1(\beta) \setminus V_{\kappa(d_1)+1}$, $d_1{}^\frown d_2'{}^\frown p^2$ will have a direct extension in D. But then for $\alpha < \text{length}(\vec{V})$ with $d_1 \in A(\alpha)$ we will have that $d_1 \in A_0(\langle\ \rangle)$, i.e. for every $d_1' \in A(\alpha)$ for some $\beta' < \text{length}(\vec{V})$ for every $d_2' \in A^1(\beta') \setminus V_{\kappa(d_1')+1}$, $d_1'{}^\frown d_2'{}^\frown p'$ will have a direct extension in D. In this case we can define p^* and two levels tree T. The definition is similar to those given above. Otherwise we consider $(*)_2$ the two steps analogue of $(*)_1$. Continue in a similar fashion. Thus at stage n we will have sets $A^n(\alpha) \in U(\alpha)$, $A^n = \bigcup\{A^n(\alpha) \mid \alpha < \text{length}(\vec{V})\}$ and $p^n = \langle\langle \kappa, \vec{V}\rangle, A^n\rangle$. Also the following n-dimension version of $(*)_1$ will hold:

$(*)_n$ If $p^n \leq q = \langle e_0, \ldots, e_{m-1}, d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, B\rangle \in D$, then there is an n-levels $\vec{V}$-fat tree T_q such that for every maximal branch $\eta = \langle f_1, \ldots, f_n\rangle$ of T_q there are sets $\vec{A} = \langle a_1, \ldots, a_n\rangle$ of η-measure one and $B_\eta \in \bigcap \vec{V}$ such that

$$\langle e_0, \ldots, e_{m-1}\rangle^\frown \langle \eta, \vec{A}\rangle^\frown \langle\langle \kappa, \vec{V}\rangle, B_\eta\rangle \in D\,.$$

Again, if for some $d_1, \ldots, d_n \in A^n$, a direct extension q of $\langle d_1, \ldots, d_n\rangle^\frown p^n$ is in D, then we can easily finish. Just use T_q given by $(*)_n$ as T and let $A^* = \{e \in A^n \mid \forall \eta \in V_{\kappa(e)}(e \in B_\eta)\}$.

Suppose the process does not stop at any $n < \omega$. Set

$$p^* = \langle\langle \kappa, \vec{V}\rangle, \textstyle\bigcap_{n<\omega} A_n\rangle\ .$$

Then $p^* \geq^* p$. By our assumption, no direct extension of p (and so of p^*) is in D. Pick some q, $q = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, B\rangle \geq p^*$ and $q \in D$. Then $q \geq^* \langle d_1, \ldots, d_n\rangle^\frown p^n$. So, by the choice of p^n, we were supposed to stop at stage n. Contradiction. ⊣

We are now ready to show the following:

5.19 Theorem. *If* $\text{cf}(\text{length}(\vec{V})) \geq \kappa^+$ *then* κ *remains regular (and hence inaccessible) in* $V^{R_{\vec{V}}}$.

5.20 Remark. In view of 5.15 the converse of 5.19 is false.

Proof. Suppose that $\delta < \kappa$ and $\underset{\sim}{f}$ is a $R_{\vec{V}}$-name so that the weakest condition forces

$$\underset{\sim}{f} : \check{\delta} \longrightarrow \check{\kappa}\ .$$

Let $t = \langle \mu_1, \ldots, \mu_s, \langle \kappa, \vec{V} \rangle, E \rangle \rangle \in R_{\vec{V}}$. We find $p \geq t$ forcing "ran$\underset{\sim}{f}$ is bounded in κ". Let $\xi < \delta$. Consider the set

$$D_\xi = \{p \in R_{\vec{V}} \mid \\ \text{for some } d \in V_\kappa \setminus V_{\mu_3+1} \text{ appearing in } p,\ (p \Vdash \underset{\sim}{f}(\check{\xi}) < \check{\kappa}(d))\}.$$

Clearly, D_ξ is a dense subset of $R_{\vec{V}}$. For every $\vec{d} = \langle d_1, \ldots, d_n \rangle \in V_\kappa$ with $\vec{d}^\frown \langle \langle \kappa, \vec{V} \rangle, V_\kappa \setminus V_{\kappa(d_n)+1} \rangle \in R_{\vec{V}}$ apply 5.17 to $\vec{d}^\frown \langle \langle \kappa, \vec{V} \rangle, V_\kappa \setminus V_{\kappa(d_n)+1} \rangle$ and to D_ξ. We are interested only in the last T_m and only if $i_m = n+1$ there. Such a T_m is a $\vec{V}$–fat tree of the height $n_m < \omega$. Denote T_m further as $T(\xi, \vec{d})$. By 5.16, for every $\eta \in T_m \setminus \mathrm{Lev}_{n_m}(T_m)$ there is an $\alpha(\eta) < \mathrm{length}(\vec{V})$ such that $\mathrm{Suc}_{T_m}(\eta) \in U(\alpha(\eta))$. Define $\alpha(\vec{d}) = \bigcup \{\alpha(\eta) \mid \eta \in T_m \setminus \mathrm{Lev}_{n_m}(T_m)\}$. Then $\alpha(\vec{d}) < \mathrm{length}(\vec{V})$, since $\mathrm{cf}(\mathrm{length}(\vec{V})) = \kappa^+$. Pick $\alpha(\xi) < \mathrm{length}(\vec{V})$ to be larger than each $\alpha(\vec{d})$ with $\vec{d}$ as above. Finally let $\alpha < \mathrm{length}(\vec{V})$ be above each $\alpha(\xi)$. Consider the following set:

$$B = \{\langle \nu, \vec{F}_\nu \rangle \in V_\kappa \mid \forall \xi < \delta \forall \vec{d} \in V_\nu (T(\xi, \vec{d}) \cap V_\nu \text{ is } \vec{F}_\nu\text{-fat})\}\ .$$

By the choice of $\alpha, B \in U(\alpha)$. For every $\xi < \delta$, let $A^*_\xi \in \bigcap \vec{V}$ be the set given by 5.17 applied to D_ξ and t. Let $A^* = \bigcap_{\xi<\delta} A^*_\xi$. Every condition of $R_{\vec{V}}$ can be extended to one containing elements of $B \setminus V_{\mu_s+1}$. Hence the following will conclude the proof:

Claim 5.1.19. *Let $p \geq \langle \mu_1, \ldots, \mu_s, \langle \kappa, \vec{V} \rangle, A^* \setminus V_{\mu_s+1} \rangle$ and some $\langle \nu, \vec{F}_\nu \rangle \in B \setminus V_{\mu_s+1}$ appears in p. Then*

$$p \Vdash \forall \xi < \check{\delta}\ (\underset{\sim}{f}(\xi) < \check{\nu})\ .$$

Proof. Suppose otherwise. Let

$$p \geq \langle \mu_1, \ldots, \mu_s, \langle \kappa, \vec{V} \rangle, A^* \setminus V_{\mu_s+1} \rangle\ ,$$

some $\langle \nu, \vec{F}_\nu \rangle \in B \setminus V_{\mu_s+1}$ appears in p and for some $\xi < \delta$ $p \Vdash \underset{\sim}{f}(\check{\xi}) \geq \check{\nu}$. Let $p = \langle d_1, \ldots, d_\ell, \langle \langle \nu, \vec{F}_\nu \rangle, a_\nu \rangle, d_{\ell+2}, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle$. Consider

$$p' = \langle d_1, \ldots, d_\ell, \langle \kappa, \vec{V} \rangle, A^* \setminus V_{\kappa(d_\ell)+1} \rangle\ .$$

We would like to apply 5.18. By the definition of B, $T(\xi, \langle d_1, \ldots, d_\ell \rangle) \cap V_\nu$ is $\vec{F}_\nu$-fat. Since $a_\nu \in \cap \vec{F}_\nu$, we can find a maximal branch $\langle f_1, \ldots, f_m \rangle$ through $T(\xi, \langle d_1, \ldots, d_\ell \rangle)$ inside a_ν. By 5.17, there is $q \geq p'$, $q \in D_\xi$ of form

$$\langle e_1, \ldots, e_i, \widetilde{f}_1, \ldots, \widetilde{f}_m, A^* \setminus V_{\kappa(f_m)+1} \rangle$$

where $\kappa(e_i) = \kappa(d_\ell)$ and for every j, $1 \leq j \leq m$, $\widetilde{f}_j$ is f_j, if f_j is an ordinal, or $\widetilde{f}_j = \langle f_j, b_j \rangle$ for some b_j, otherwise. $q \in D_\xi$ implies that $q \Vdash \underset{\sim}{f}(\check{\xi}) < \check{\kappa}(f_m)$.

Obviously, $\kappa(f_m) < \nu$, since $f_m \in a_\nu \subseteq V_\nu$. On the other hand, q and p are compatible, since $p \geq \langle \mu_1, \ldots, \mu_s, \langle \kappa, \vec{V}\rangle, A^* \setminus V_{\mu_s+1}\rangle$,

$$p = \langle d_1, \ldots, d_\ell, \langle\langle \nu, \vec{F}_\nu\rangle, a_\nu\rangle, d_{\ell+2}, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$$

and, hence $\langle \nu, \vec{F}_\nu\rangle, d_{\ell+2}, \ldots, d_n$ come from A^*. So they are addible to q. Hence

$$\langle e_1, \ldots, e_i, \widetilde{f}_1, \ldots, \widetilde{f}_m\ ,\ \langle\langle \nu, \vec{F}_\nu\rangle, a_\nu \setminus V_{\kappa(f_m)+1}\rangle, d_{\ell+2}, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$$

is a common extension of q and p. But this is impossible since $p \Vdash \underset{\sim}{f}(\check{\xi}) \geq \check{\nu}$ and $q \Vdash \underset{\sim}{f}(\check{\xi}) < \check{\nu}$. Contradiction. $\dashv$

5.2. Magidor Forcing and Coherent Sequences of Measures

Magidor [37] invented a forcing for changing the cofinality of a cardinal κ to an uncountable value $\delta < \kappa$. As an initial assumption, his forcing uses a coherent sequence of measures of length δ. Coherent sequences of measures were introduced by Mitchell [43]. In [42] Mitchell showed that it is possible to do the Radin forcing with coherent sequences of measures replacing an elementary embedding $j : V \to M$. The main advantage of this approach is reducing initial assumptions to weaker ones that in turn also provide equiconsistency results. This allows the simultaneous treatment of both the Magidor and the Radin forcings.

5.21 Definition. A *coherent sequence of measures* (ultrafilters) $\vec{U}$ is a function with domain of form

$$\{(\alpha, \beta) \mid \alpha < \ell^{\vec{U}} \text{ and } \beta < o^{\vec{U}}(\alpha)\}$$

for an ordinal $\ell^{\vec{U}}$, the length of $\vec{U}$, and a function $o^{\vec{U}}(\alpha)$, called the order of $\vec{U}$ at α. For each pair $(\alpha, \beta) \in \operatorname{dom}(\vec{U})$,

(1) $U(\alpha, \beta)$ is a normal ultrafilter over α, and

(2) if $j^\alpha_\beta : V \longrightarrow N^\alpha_\beta \simeq \operatorname{Ult}(V, (\alpha, \beta))$ is the canonical embedding, then

$$j^\alpha_\beta(\vec{U}) \restriction \alpha + 1 = \vec{U} \restriction (\alpha, \beta)\ ,$$

where
$$\vec{U} \restriction \alpha = \vec{U} \restriction \{(\alpha', \beta') \mid \alpha' < \alpha \text{ and } \beta' < o^{\vec{U}}(\alpha')\}$$

and

$$\vec{U} \restriction (\alpha, \beta) = \vec{U} \restriction \{(\alpha', \beta') \mid (\alpha' < \alpha \text{ and } \beta' < o^{\vec{U}}(\alpha')) \text{ or } (\alpha' = \alpha \text{ and } \beta' < \beta)\}\ .$$

Suppose that $\vec{U}$ is a coherent sequence of measures with $\ell^{\vec{U}} = \kappa + 1$ and $o^{\vec{U}}(\kappa) = \delta > 0$. Now we will use $\vec{U}$ as a replacement for $\vec{V}$ of the previous section. Thus, over κ, $\vec{U}(\kappa) = \langle \vec{U}(\kappa, \alpha) \mid \alpha < \delta \rangle$ is used. Let $A \in \bigcap \vec{U}(\kappa) = \bigcap_{\alpha<\delta} U(\kappa, \alpha)$. Elements of A are ordinals only, no more pairs of form $\langle \nu, \vec{F}_\nu \rangle$ with ν an ordinal and $\vec{F}_\nu$ a sequence of ultrafilters over V_ν. But actually, if $\nu \in A$ and $o^{\vec{U}}(\nu) > 0$, then we have a sequence of measures $\vec{U}(\nu) = \langle \vec{U}(\nu, \alpha) \mid \alpha < o^{\vec{U}}(\nu) \rangle$ over ν. And it can be used exactly as $\vec{F}_\nu$ of the previous section. Note that here $\vec{U}(\nu)$ is determined uniquely from ν and $\vec{U}$. Also, because of coherence, namely 5.21(2), there is no need to define the set $\overline{A}$ as it was done in the previous section before the definition of $R_{\vec{V}}$ (5.1).

Let us denote for an ordinal $d = \nu$ or pair $d = \langle \nu, B \rangle$, ν by $\kappa(d)$. Using the above observations we define $\mathcal{P}_{\vec{U}}$ a coherent sequences analogue of $R_{\vec{V}}$.

5.22 Definition. Let $\mathcal{P}_{\vec{U}}$ be the set of finite sequences $\langle d_1, \ldots, d_n, \langle \kappa, A \rangle \rangle$ such that:

(1) $A \in \bigcap \vec{U}(\kappa)$.

(2) $\min(A) > \kappa(d_n)$.

(3) For every m with $1 \le m \le n$, either

(3a) d_m is an ordinal and then $o^{\vec{U}}(d_m) = 0$, or

(3b) $d_m = \langle \nu, A_\nu \rangle$ for some ν with $o^{\vec{U}}(\nu) > 0$ and $A_\nu \in \bigcap_{\alpha < 0^{\vec{U}}(\nu)} U(\nu, \alpha)$.

(4) For every $1 \le i \le j \le m$,

(4a) $\kappa(d_i) < \kappa(d_j)$, and

(4b) If d_j is of form $\langle \nu, A_\nu \rangle$ then $\min(A_\nu) > \kappa(d_i)$.

The definition of orders $\le, \le^*$ on $\mathcal{P}_{\vec{U}}$ repeats those of $R_{\vec{V}}$ (5.2), only ultrafilter sequences $\vec{F}_\nu$'s and $\vec{V}$ are removed from the conditions there.

5.23 Definition. Let $p = \langle d_1, \ldots, d_n, \langle \kappa, A \rangle \rangle, q = \langle e_1, \ldots, e_m, \langle \kappa, B \rangle \rangle \in \mathcal{P}_{\vec{U}}$. We say that p *is stronger than* q and denote this by $p \ge q$ iff

(1) $A \subseteq B$.

(2) $n \ge m$.

(3) There are $1 \le i_1 < i_2 < \cdots < i_m \le n$ such that for every k with $1 \le k \le m$, either

(3a) $e_k = d_{i_k}$, or

(3b) $e_k = \langle \nu, B_\nu \rangle$ and then $d_{i_k} = \langle \nu, C_\nu \rangle$ with $C_\nu \subseteq B_\nu$.

(4) Let $i_1, \ldots, i_m$ be as in (3). Then the following holds for every $j with 1 \leq j \leq n$ and $j \notin \{i_1, \ldots, i_k\}$:

(4a) If $j > i_m$, then $d_j \in B$ or d_j is of form $\langle \nu, C_\nu \rangle$ with $\nu \in B$ and $C_\nu \subseteq B \cap \nu$.

(4b) If $j < i_m$, then for the least k with $j < i_k$, e_k is of form $\langle \nu, B_\nu \rangle$ so that

(i) if d_j is an ordinal then $d_j \in B_\nu$, and

(ii) if $d_j = \langle \rho, S \rangle$ then $\rho \in B_\nu$ and $S \subseteq B_\nu$.

5.24 Definition. Let $p = \langle d_1, \ldots, d_n, \langle \kappa, A \rangle \rangle, q = \langle e_1, \ldots, e_m, \langle \kappa, B \rangle \rangle \in \mathcal{P}_{\vec{U}}$. We say that p is a *direct extension* of q and denote this by $p \geq^* g$ iff

(1) $p \geq q$, and

(2) $n = m$.

Now all the results of the previous section are valid in the present context with $\mathcal{P}_{\vec{U}}$ replacing $R_{\vec{V}}$. Also their proofs require only trivial changes.

If $\delta < \kappa$, then $\langle U(\kappa, \alpha) \mid \alpha < \delta \rangle$ can be split. Thus for every $\alpha < \delta$ $U(\kappa, \alpha)$ concentrates on the set $Y_\alpha = \{\nu < \kappa \mid o^{\vec{U}}(\nu) = \alpha\}$. $\mathcal{P}_{\vec{U}}$, above the condition $\langle \langle \kappa, \bigcup_{\alpha < \delta} Y_\alpha \rangle \rangle$ is then the Magidor forcing for changing cofinality of κ to $\mathrm{cf}(\delta)$.

5.3. Extender-based Radin Forcing

In this section we give a brief description of the extender-based Radin forcing developed by C. Merimovich [40]. Previously, the extender-based Magidor forcing was introduced by M. Segal [49]. The basic idea will be to combine the forcing of Section 3 with those of Section 5.1.

Assume GCH and let $j : V \longrightarrow M \supseteq V_{\kappa+4}$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$. First, as in Section 3, but with $\lambda = \kappa^{++}$, for every $\alpha < \kappa^{++}$, we consider U_α an ultrafilter over κ defined by:

$$X \in U_\alpha \quad \text{iff} \quad \alpha \in j(X) \ .$$

Define a partial order $\leq_j$ on λ:

$$\alpha \leq_j \beta \text{ iff } \alpha \leq \beta \text{ and for some } f \in {}^\kappa\kappa,\ j(f)(\beta) = \alpha \ .$$

Let $\langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \kappa^{++},\ \alpha \geq_j \beta \rangle$ be the sequence of projections defined in Section 3. The whole system (i.e. the extender)

$$\langle \langle U_\alpha \mid \alpha < \kappa^{++} \rangle, \langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \kappa^{++}, \alpha \geq_j \beta \rangle \rangle$$

is in M, as ${}^{\kappa^{++}}V_{\kappa+3} \subseteq V_{\kappa+3} \subseteq M$. Denote this system by $E(0)$ and U_α by $E_\alpha(0)$ for every $\alpha < \kappa^{++}$. Now, as in 5.1, we use the fact that $E(0) \in M$ in order to define $E(1)$. Thus for every $\alpha < \kappa^{++}$, we define over V_κ the following ultrafilter:

$$A \in E_{\langle \alpha, E(0)\rangle}(1) \quad \text{iff} \quad \langle \alpha, E(0)\rangle \in j(A) \ .$$

It is possible to use only α as an index instead of $\langle \alpha, E(0)\rangle$, but it turns out that the latter notation is more convenient. Note that $E_{\langle \alpha, E(0)\rangle}(1)$ concentrates on elements of form $\langle \xi, e(0)\rangle$, where $e(0)$ is an extender over ξ^0 (recall, that in the notation of Section 3, ξ^0 denotes the projection of ξ to the normal ultrafilter by $\pi_{\alpha\kappa}$) of length $(\xi^0)^{++}$ including projections between its measures. Also note that σ_α defined by $\sigma_\alpha(\xi, e(0)) = \xi$ projects $E_{\langle \alpha, E(0)\rangle}(1)$ onto $E_\alpha(0) = U_\alpha$.

We define projections $\pi_{\langle \alpha, E(0)\rangle, \langle \beta, E(0)\rangle}$ for $\kappa^{++} > \alpha \geq \beta$ with $\alpha \geq_j \beta$ as follows:

$$\pi_{\langle \alpha, E(0)\rangle, \langle \beta, E(0)\rangle}(\langle \xi, e(0)\rangle) = \langle \pi_{\alpha\beta}(\xi), e(0)\rangle \ .$$

Then, in M,

$$j(\pi_{\langle \alpha, E(0)\rangle, \langle \beta, E(0)\rangle})(\langle \alpha, E(0)\rangle) = \langle \beta, E(0)\rangle \ .$$

This defines an extender

$$\begin{aligned} E(1) = \langle \langle E_{\langle \alpha, E(0)\rangle}(1) \mid \alpha < \kappa^{++}\rangle, & \\ & \langle \pi_{\langle \alpha, E(0)\rangle, \langle \beta, E(0)\rangle} \mid \kappa^{++} > \alpha \geq \beta,\ \alpha \geq_j \beta\rangle\rangle \ . \end{aligned}$$

Continue by recursion. Suppose that $\tau < \kappa^{+4}$ and a sequence of extenders $\langle E(\tau') \mid \tau' < \tau\rangle$ is already defined. Again, as ${}^{\kappa^{++}}V_{\kappa+4} \subseteq V_{\kappa+4} \subseteq M$, $\langle E(\tau') \mid \tau' < \tau\rangle \in M$. So, for every $\alpha < \kappa^{++}$ we can define an ultrafilter over V_κ as follows

$$\begin{aligned} A \in E_{\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau\rangle}(\tau) \text{ iff} & \\ & \langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau\rangle \in j(A) \ . \end{aligned}$$

Define projections:

$$\pi_{\langle \alpha, E(0), \dots, E(\tau), \dots \mid \tau' < \tau\rangle, \langle \beta, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau\rangle}(\langle \xi, d\rangle) = \langle \pi_{\alpha\beta}(\xi), d\rangle \ ,$$

for every α, β, with $\kappa^{+} > \alpha \geq \beta$ and $\alpha \geq_j \beta$. Further, let us suppress these long indexes and use only α and β, i.e. the above projection will be denote by $\pi_{\alpha\beta}$ and $E_{\langle \alpha, E(0), \dots, E(\tau'), \dots \mid \tau' < \tau\rangle}(\tau)$ by $E_\alpha(\tau)$. Define

$$E(\tau) = \langle \langle E_\alpha(\tau) \mid \alpha < \kappa^{++}\rangle, \langle \pi_{\alpha\beta} \mid \kappa^{++} > \alpha \geq \beta \ ,\ \alpha \geq_j \beta\rangle \ .$$

Fix some $\tau^* \leq \kappa^{+4}$. Let $\vec{E} = \langle E(\tau) \mid \tau < \tau^*\rangle$.

In [24], [25], Merimovich used such $\vec{E}$ to define the extender-based Radin forcing. The general definition is quite complicated and we will not reproduce it here. Instead let us concentrate on the case length$(\vec{E}) = 2$. This explains the idea of the Merimovich construction. So let $\vec{E} = \langle E(0), E(1)\rangle$. For each $\alpha < \kappa^{++}$ let $\overline{\alpha} = \langle \alpha, E(0), E(1)\rangle$. Set $\overline{E} = \{\langle \alpha, E(0), E(1)\rangle \mid \alpha < \kappa^{++}\}$.

5.25 Definition. A *basic condition* in $\mathcal{P}_{\vec{E}}$ over κ is one of form

$$p = \{\langle \overline{\gamma}, p^{\overline{\gamma}}\rangle \mid \overline{\gamma} \in s\} \cup \{\langle \overline{\alpha}, p^{\overline{\alpha}}\rangle, T\}$$

so that

(1) $s \subseteq \overline{E}$, $|s| \leq \kappa$ and $\overline{\kappa} \in s$.

This s is the support of the condition and here, instead of just ordinals used as supports in the extender-based Prikry forcing of Section 3, its elements are of form $\overline{\gamma} = \langle \gamma, E(0), E(1)\rangle$.

(2) $p^{\overline{\gamma}} \in V_\kappa$ is a finite sequence of elements of form an ordinal ν or a pair $\langle \nu, e_\nu(0)\rangle$ with $e_\nu(0)$ and extender of length $(\nu^0)^{++}$ over ν^0 (recall that, as in Section 3, ν^0 denotes the projection of ν by $\pi_{\gamma,\kappa}$, i.e. to the normal measure). We require that the ν^0's of elements of $p^{\overline{\gamma}}$ are increasing. Denote the ν of the last element of $p^{\overline{\gamma}}$ by $\kappa(p^{\overline{\gamma}})$, if $p^{\overline{\gamma}}$ is nonempty and let $\kappa(p^{\overline{\gamma}}) = 0$ otherwise.

(3) $\overline{\alpha}$ is above every $\overline{\gamma} \in s$ in the $\leq_j$ order (i.e. $\gamma \leq_j \alpha$).

(4) $\kappa(p^{\overline{\alpha}}) \leq \kappa(p^{\overline{\gamma}})$.

(5) $T \in E_\alpha(0) \cap E_\alpha(1) \setminus V_{\kappa(p^{\overline{\kappa}})+1}$.

(6) For every $\overline{\nu} \in T$,

$$|\{\overline{\gamma} \in s \mid (\kappa(p^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0\}| \leq (\kappa(\overline{\nu}))^0 \ .$$

(7) For every $\overline{\nu} \in T$, $\overline{\beta}, \overline{\gamma} \in \overline{s}$, if $(\kappa(p^{\overline{\beta}}))^0, (\kappa(p^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0$ and $\overline{\beta} \neq \overline{\gamma}$ then

$$\pi_{\overline{\alpha},\overline{\beta}}(\overline{\nu}) \neq \pi_{\overline{\alpha},\overline{\gamma}}(\overline{\nu}) \ .$$

As in Section 3, we write $T^p, mc(p), \mathrm{supp}(p)$ for $T, \overline{\alpha}$ and $s \cup \{\overline{\alpha}\}$ respectively.

5.26 Definition. For basic conditions p, q of $\mathcal{P}_{\vec{E}}$ over κ, define $p \geq^* q$ iff

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.

(2) For every $\overline{\gamma} \in \mathrm{supp}(q)$, $p^{\overline{\gamma}} = q^{\overline{\gamma}}$.

(3) $T^p \subseteq \pi^{-1}_{mc(p),mc(q)}$ "T^q.

(4) For every $\overline{\gamma} \in \operatorname{supp}(q)$ and $\overline{\nu} \in T^p$, if $(\kappa(p^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0$ then

$$\pi_{mc(p),\overline{\gamma}}(\overline{\nu}) = \pi_{mc(q),\overline{\gamma}}(\pi_{mc(p),mc(q)}(\overline{\nu})) \ .$$

Now let p_0 be a basic condition over κ and $\overline{\nu} \in T^{p_0}$. We define $p_0 {}^\frown \overline{\nu}$, a one-element extension of p_0 by $\overline{\nu}$.

5.27 Definition. $p_0 {}^\frown \langle \overline{\nu} \rangle$ will be of form $p'_1 {}^\frown p'_0$ where

(1) $\operatorname{supp}(p'_0) = \operatorname{supp}(p_0)$.

(2) For every $\overline{\gamma} \in \operatorname{supp}(p'_0)$

$$p'^{\overline{\gamma}}_0 = \begin{cases} \pi_{mc(p_0),\overline{\gamma}}(\overline{\nu}), & \text{if } (\kappa(p_0^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0 \text{ and} \\ & \quad \overline{\nu} \text{ is of form } \langle \nu, e_\nu(0) \rangle, \\ p_0^{\overline{\gamma}} {}^\frown \pi_{mc(p_0),\overline{\gamma}}(\overline{\nu}), & \text{if } (\kappa(p_0^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0 \text{ and} \\ & \quad \overline{\nu} \text{ is an ordinal,} \\ p_0^{\overline{\gamma}}, & \text{otherwise.} \end{cases}$$

(3) $T^{p'_0} = T^{p_0} \setminus V_{(\kappa(\overline{\nu}))^0+1}$.

If $\overline{\nu}$ is an ordinal then p'_1 is empty, otherwise the following holds:

(4) $mc(p'_1) = \overline{\nu}$.

(5) $\operatorname{supp}(p'_1) =$
$\{\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu}) \mid \overline{\gamma} \in \operatorname{supp}(p_0) \text{ and } (\kappa(p_0^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0\} \cup \{\overline{\nu}\}$.

(6) $p_1'^{\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu})} = p_0^{\overline{\gamma}}$.

(7) $T^{p'_1} = T^{p_0} \cap V_{(\kappa(\overline{\nu}))^0}$.

5.27 is the crucial step of the definition of $\mathcal{P}_{\vec{E}}$. If $\overline{\nu}$ was an ordinal then $p {}^\frown \langle \overline{\nu} \rangle = p'_0$ is generated as in Section 3. But if $\overline{\nu}$ is of form $\langle \nu, e_\nu(0) \rangle$ then after adding $\overline{\nu}$, p_0 splits into two blocks p'_0 and p'_1. p'_0 is still a basic condition over κ generated in the fashion of Section 3. But p'_1 is a new block. We just separate and move to the new block every $p_0^{\overline{\gamma}}$ to which $\overline{\nu}$ can be added. The actual addition, $\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu})$, is kept both in the support of p'_1 and on the new $p'^{\overline{\gamma}}_0$. T^{p_0} is moved down to ν and p'_1 is a basic condition over ν^0. We can extend it further using measures of the extender $e_\nu(0)$. It acts from now autonomously and as a condition in the extender-based Prikry forcing of Section 3. Note that we still keep some connection with the upper block p'_0. Thus $\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu})$'s appear in both $\operatorname{supp}(p'_1)$ and p'_0, as $p'^{\overline{\gamma}}_0$. See the figure below which gives an example of such p_0, p'_0, p'_1.

$$
\begin{array}{lllll}
 & & \overline{\mu}_{32} & & \\
 & \overline{\mu}_{11} & \overline{\mu}_{31} & & \\
\overline{\mu}_0 & \overline{\mu}_{10} & \overline{\mu}_{30} & \overline{\mu}_4 & T \cap V_\kappa(\overline{\nu}) \\
\hline
\overline{\nu}^0 & \pi_{\alpha_4\alpha_1}(\overline{\nu}) & \pi_{\alpha_4\alpha_3}(\overline{\nu}) & \overline{\nu} &
\end{array}
\quad p'_1
$$

$$
\begin{array}{llllll}
 & & \overline{\mu}_{21} & & & \\
\overline{\nu}^0 & \pi_{\alpha_4\alpha_1}(\overline{\nu}) & \overline{\mu}_{20} & \pi_{\alpha_4\alpha_3}(\overline{\nu}) & \overline{\nu} & T \setminus V_{\kappa(\overline{\nu})+1} \\
\hline
\overline{\kappa} & \overline{\alpha}_1 & \overline{\alpha}_2 & \overline{\alpha}_3 & \overline{\alpha}_4 = mc &
\end{array}
\quad p'_0
$$

$$
\begin{array}{llllll}
 & & & \overline{\mu}_{32} & & \\
 & \overline{\mu}_{11} & \overline{\mu}_{21} & \overline{\mu}_{31} & & \\
\overline{\mu}_0 & \overline{\mu}_{10} & \overline{\mu}_{20} & \overline{\mu}_{30} & \overline{\mu}_4 & T \\
\hline
\overline{\kappa} & \overline{\alpha}_1 & \overline{\alpha}_2 & \overline{\alpha}_3 & \overline{\alpha}_4 = mc &
\end{array}
\quad p_0
$$

Once we have a two block condition $p_1{}^\frown p_0$ we can extend it further in the same way by adding either $\overline{\nu} \in T_{p_0}$ or $\overline{\nu} \in T_{p_1}$. In the first case this will generate a new block between p_1 and p_0 and the second below p_1. We are allowed to repeat this any finite number of times. Thus a general condition in $P_{\vec{E}}$ will be of form $p = p_n{}^\frown p_{n-1}{}^\frown \cdots {}^\frown p_0$ where p_0 is a basic condition over κ, p_1 over some $\nu_0 < \kappa, \ldots$ and, p_n over some $\nu_{n-1} < \nu_{n-2}$.

An example of a condition in $P_{\vec{E}}$:

$$
\begin{array}{lllllll}
 & \overline{\tau}_{12} & & & & & \\
 & \overline{\tau}_{11} & & & & & \\
\overline{\tau}_0 & \overline{\tau}_{10} & \overline{\tau}_2 & \overline{\tau}_3 & \overline{\tau}_4 & \overline{\tau}_5 & R \\
\hline
\overline{\mu}_0 & \overline{\mu}_1 & \overline{\mu}_2 & \overline{\mu}_3 & \overline{\mu}_4 & \overline{\mu}_5 = mc &
\end{array}
\quad p_2
$$

$$
\begin{array}{lllllll}
 & & \overline{\mu}_{71} & & & & \\
\overline{\mu}_0 & \overline{\mu}_1 & \overline{\mu}_{70} & \overline{\mu}_8 & \overline{\mu}_6 & \overline{\mu}_5 & S \\
\hline
\overline{\nu}_0 & \overline{\nu}_1 & \overline{\nu}_2 & \overline{\nu}_3 & \overline{\nu}_4 & \overline{\nu}_5 = mc &
\end{array}
\quad p_1
$$

$$
\begin{array}{llllll}
 & & & \overline{\nu}_{71} & & \\
\overline{\nu}_0 & \overline{\nu}_1 & \overline{\nu}_6 & \overline{\nu}_{70} & \overline{\nu}_4 & T \\
\hline
\overline{\kappa} & \overline{\alpha}_1 & \overline{\alpha}_2 & \overline{\alpha}_3 & \overline{\alpha}_4 = mc &
\end{array}
\quad p_0
$$

Each block may grow separately. Thus in the example the maximal coordinate of p_1 changed from $\overline{\nu}_4$, corresponding to $\overline{\alpha}_4$, to a new value $\overline{\nu}_5$. New coordinates $\overline{\nu}_2$, $\overline{\nu}_3$ were added in p_1 and $\overline{\mu}_2, \overline{\mu}_3, \overline{\mu}_4$ in p_2.

The following is a straightforward generalization of 5.27.

5.28 Definition. Let $p, q \in P_{\overline{E}}$. We say that p is a *one-point extension* of q and denote this by $p \geq_1 q$ iff p and q are of form

$$
\begin{aligned}
p &= p_{n+1}{}^\frown p_n{}^\frown \cdots {}^\frown p_0 \\
q &= q_n{}^\frown \cdots {}^\frown q_0
\end{aligned}
$$

and there is a k with $0 \le k \le n$ such that

(1) p_i and q_i are basic conditions over some ν_i with $p_i \ge^* q_i$ for $i < k$.

(2) p_{i+1} and q_i are basic conditions over some ν_i with $p_{i+1} \ge^* q_i$ for each $k < i \le n$.

(3) There is an $\overline{\nu} \in T^{q_k}$ such that $p_{k+1}{}^\frown p_k \ge^* q_k{}^\frown\langle\overline{\nu}\rangle$.

We now define n-extension for every $n < \omega$.

5.29 Definition. Let $p, q \in P_{\vec{E}}$. We say that p is an *n-point extension* of q and denote this by $p \ge_n q$ iff either $n = 0$ and $p \ge^* q$, or else $n > 0$ and there are $p^n, \dots, p^0$ such that

$$p = p^n \ge_1 \cdots \ge_1 p^0 = q \ .$$

Finally, we can define the order $\le$ on $P_{\vec{E}}$.

5.30 Definition. Let $p, q \in P_{\vec{E}}$. Define $p \ge q$ iff there is $n < \omega$ such that $p \ge_n q$.

Let G be a generic subset of $\langle \mathcal{P}_{\vec{E}}, \le\rangle$. For every α with $\kappa \le \alpha < \kappa^{++}$ we want to collect together all the ordinals corresponding to α into a set which we call G^α. Define

$$G^\alpha = \{\kappa(p^{\vec{E}_\alpha}) \mid \exists p \in G\,(p \text{ is a basic condition over } \kappa \text{ with } \vec{E}_\alpha \in \operatorname{supp}(p) \text{ and } p^{\vec{E}_\alpha} \ne \emptyset)\} \ .$$

It is not hard to see using the definition of the order on $P_{\vec{E}}$ that G^α will be unbounded in κ sequence of order type ω^2. Also $\alpha \ne \beta$ will imply $G^\alpha \ne G^\beta$. In addition, the sequence G^κ (the one corresponding to the normal ultrafilter) will be closed.

Now let length$(\vec{E})$ be any ordinal $\le \kappa^{+4}$. Merimovich [40] showed that his forcing $P_{\vec{E}}$ shares all the properties of the Radin forcing of 5.1, only κ^+-c.c. should be replaced by κ^{++}-c.c.. This causes a new problem to show that κ^+ is preserved in cases of regular κ. In order to preserve measurability of κ the following variation of repeat point is used:

$\tau <$ length$(\vec{E})$ is called a *repeat point* of $\vec{E}$ if for every $\xi <$ length$(\vec{E})$ and $\alpha < \kappa^{++}$, $A \in E_\alpha(\xi)$ implies that for some $\xi' < \tau$ $A \in E_\alpha(\xi')$.

That is, τ acts simultaneously as a repeat point of the sequence of ultrafilters $\langle E_\alpha(\xi') \mid \xi <$ length$(\vec{E})\rangle$ for each $\alpha < \kappa^{++}$. Clearly, there will be lots of repeat points below κ^{+4}. The κ^{++} sets G^α defined above for a generic $G \subseteq P_{\vec{E}}$ will witness $2^\kappa = \kappa^{++}$; G^κ will be a club in κ.

In further work [39], Merimovich added collapses to the extender-based Radin forcing. This allowed him to reprove results of Foreman-Woodin [12], and Woodin and obtain new interesting patterns of global behaviour of the power function.

6. Iterations of Prikry-type Forcing Notions

In this section we present two basic techniques for iterating Prikry-type forcing notions. The first one is called the Magidor or full support iteration and the second, Easton support iteration.

A set with two partial orders $\langle \mathcal{P}, \leq, \leq^* \rangle$ is called a *Prikry-type forcing notion* iff

(a) $\leq \supseteq \leq^*$.

(b) (The Prikry condition) For every $p \in \mathcal{P}$ and statement σ of the forcing language of $\langle \mathcal{P}, \leq \rangle$ there is a $p^* \geq^* p$ deciding σ.

Notice that any forcing $\langle \mathcal{P}, \leq \rangle$ can be turned into a Prikry-type by defining $\leq^* = \leq$. In this case the iterations below coincide with the usual iterations with full or Easton support.

6.1. Magidor Iteration

The presentation below follows [16] and is a bit different from Magidor's original version [34].

Let ρ be an ordinal. We define an iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\alpha \mid \alpha < \rho \rangle$. For every $\alpha < \rho$ define by recursion $\mathcal{P}_\alpha$ to be the set of all p of form $\langle \underset{\sim}{p}_\gamma \mid \gamma < \alpha \rangle$ so that for every $\gamma < \alpha$

(a) $p \restriction \gamma = \langle \underset{\sim}{p}_\beta \mid \beta < \gamma \rangle \in \mathcal{P}_\gamma$, and

(b) $p \restriction \gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma$ is a condition in the forcing $\langle \underset{\sim}{Q}_\gamma, \underset{\sim}{\leq}_\gamma, \underset{\sim}{\leq}^*_\gamma \rangle$ of the Prikry type".

Define two orderings $\leq_{\mathcal{P}_\alpha}$ and $\leq^*_{\mathcal{P}_\alpha}$ on $\mathcal{P}_\alpha$.

6.1 Definition. Let $p = \langle\, \underset{\sim}{p}_\gamma \mid \gamma < \alpha \rangle$, $q = \langle \underset{\sim}{q}_\gamma \mid \gamma < \alpha \rangle \in \mathcal{P}_\alpha$. Then $p \geq_{\mathcal{P}_\alpha} q$ iff

(1) For every $\gamma < \alpha$, $p \restriction \gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \geq_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

(2) There exists a finite $b \subseteq \alpha$ such that for every $\gamma \in \alpha \setminus b$, $p \restriction \gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \geq^*_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

If the set b in (2) is empty, then we call p a *direct extension* of q and denote this by $p \geq^*_{\mathcal{P}_\alpha} q$.

Thus here we use full support iteration, but in order to pass from a condition $q \in \mathcal{P}_\alpha$ to a stronger one, we are allowed to take nondirect extensions only at finitely many places. A typical example and the one originally used by Magidor in [34], is iteration of Prikry forcings at each measurable below α. Here in order to extend a condition we may shrink sets of measure one at each measurable $\beta < \alpha$ but only for finitely many β's is it allowed to add new elements of the Prikry sequence. We further discuss this important example in detail. Let us now show that $\langle \mathcal{P}_\alpha, \leq, \leq^* \rangle$ is itself of the Prikry type.

6.2 Lemma. *Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma < \alpha \rangle \in \mathcal{P}_\alpha$ and σ be a statement of the forcing language of $\langle \mathcal{P}_\alpha, \leq \rangle$. Then there is a direct extension of p deciding σ.*

Proof. We deal first with the successor case. Let $\alpha = \alpha' + 1$. Assume that $\mathcal{P}_{\alpha'}$ has the Prikry property, $\mathcal{P}_\alpha = \mathcal{P}_{\alpha'} * \underset{\sim}{Q}_{\alpha'}$, and $\Vdash_{\mathcal{P}_{\alpha'}} (\langle \underset{\sim}{Q}_{\alpha'}, \underset{\sim}{\leq}_{\alpha'}, \underset{\sim}{\leq}^*_{\alpha'} \rangle$ has the Prikry property). Let $G_{\alpha'} \subseteq \mathcal{P}_{\alpha'}$ be generic for $\langle \mathcal{P}_{\alpha'}, \leq \rangle$ with $p \restriction \alpha' = \langle \underset{\sim}{p}_\gamma \mid \gamma < \alpha' \rangle \in G_{\alpha'}$. Find $p^*_{\alpha'} \geq^*_{\alpha'} p_{\alpha'}$ in $Q_{\alpha'}$ which decides $\sigma[G_{\alpha'}]$. Back in V, let $\underset{\sim}{p}^*_{\alpha'}$ be a name of such $p^*_{\alpha'}$ so that

$$p \restriction \alpha' \Vdash_{\mathcal{P}_{\alpha'}} \underset{\sim}{p}^*_{\alpha'} \text{ decides } \sigma \,.$$

Use the Prikry property of $\langle \mathcal{P}_{\alpha'}, \leq, \leq^* \rangle$ and find $q \geq^* p \restriction \alpha'$ such that $q \Vdash_{\mathcal{P}_{\alpha'}} (\underset{\sim}{p}^*_{\alpha'} \Vdash_{\underset{\sim}{Q}^*_{\alpha'}} {}^i\sigma)$, for some $i < 2$, where ${}^o\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$. Then, with $r = q ^\frown \underset{\sim}{p}^*_{\alpha'}$, we have $r \Vdash_{\mathcal{P}_\alpha} {}^i\sigma$.

Suppose now that α is a limit ordinal. Assume that there is no direct extension of p deciding σ. We define by recursion on $\beta < \alpha$

$$p(\beta) = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \beta \rangle ^\frown \langle \underset{\sim}{p}_\gamma \mid \beta \leq \gamma < \alpha \rangle \geq^* p$$

so that $p(\beta) \restriction \beta = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \beta \rangle \Vdash_{\mathcal{P}_\beta} \neg\sigma_\beta$ where $\sigma_\beta \equiv (\exists \underset{\sim}{q} \in \mathcal{P}_\alpha \setminus \beta (\underset{\sim}{q} \geq^* p \setminus \beta$ and $\underset{\sim}{q} \parallel \sigma))$.

Suppose that $\langle p(\gamma) \mid \gamma < \beta \rangle$ are defined and $\leq^*$-increasing. Define $p(\beta)$:

Case 1. $\beta = \beta' + 1$.

Force with $\mathcal{P}_{\beta'} = \mathcal{P}_\alpha \restriction \beta'$, i.e. with $\langle \mathcal{P}_{\beta'}, \leq \rangle$. Let $G_{\beta'} \subseteq \mathcal{P}_{\beta'}$ be generic with $p(\beta') \restriction \beta' \in G_{\beta'}$. At stage β' we use $\langle Q_{\beta'}, \leq_{\beta'}, \leq^*_{\beta'} \rangle$. It satisfies the Prikry condition. So there is a $p^*_{\beta'} \geq^*_{\beta'} p_{\beta'}$ deciding σ_β.

Claim 6.2.1. $p^*_{\beta'} \Vdash_{Q_{\beta'}} \neg\sigma_\beta$.

Proof. Suppose otherwise. Then there is a $p^{**}_{\beta'} \geq^* p^*_{\beta'}$ with

$$p^{**}_{\beta'} \Vdash_{Q_{\beta'}} (\exists \underset{\sim}{q} \in \mathcal{P}_\alpha \setminus \beta (\underset{\sim}{q} \geq^* p \setminus \beta \text{ and } \underset{\sim}{q} \Vdash_{\mathcal{P}_\alpha \setminus \beta} {}^i\sigma))$$

for some $i < 2$, where ${}^o\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$. Without loss of generality assume $i = 0$. Then there are $r = \langle \underset{\sim}{r}_\gamma \mid \gamma < \beta' \rangle \in G_{\beta'}$ and $\underset{\sim}{q}$ such that $p(\beta') \restriction \beta' \leq r$ and

$$r \Vdash_{\mathcal{P}_{\beta'}} (\underset{\sim}{p}_{\beta'} \leq^*_{\beta'} \underset{\sim}{p}^{**}_{\beta'} \Vdash_{\underset{\sim}{Q}_{\beta'}} (\underset{\sim}{q} \geq^* p \setminus \beta \text{ and } \underset{\sim}{q} \Vdash_{\mathcal{P}_\alpha \setminus \beta} \sigma)) \,.$$

Hence, $r \Vdash_{\mathcal{P}_{\beta'}} (\underset{\sim}{p}^{**}_{\beta'} {}^\frown \underset{\sim}{q} \geq^* p \setminus \beta'$ and $\underset{\sim}{p}^{**}_{\beta'} {}^\frown \underset{\sim}{q} \Vdash_{\mathcal{P}_\alpha \setminus \beta'} \sigma)$. In particular, $p(\beta') \restriction \beta' \leq r \Vdash_{\mathcal{P}_{\beta'}} \sigma_{\beta'}$ which contradicts the choice of $p(\beta')$. $\dashv$

Now, since $G_{\beta'}$ was arbitrary, we can take a name $\underset{\sim}{p}^*_{\beta'}$ of $p^*_{\beta'}$ such that $p(\beta')\restriction\beta' \Vdash (\underset{\sim}{p}^*_{\beta'} \parallel_{\underset{\sim}{Q}_{\beta'}} \neg\sigma_\beta)$. Set $p(\beta) = p(\beta')\restriction\beta' {}^\frown \underset{\sim}{p}^*_{\beta'} {}^\frown \langle \underset{\sim}{p}_\gamma \mid \beta \leq \gamma < \alpha\rangle$.

Case 2. β is a limit ordinal.

Then we need to show that

$$p(\beta) = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \beta\rangle^\frown \langle \underset{\sim}{p}_\gamma \mid \beta \leq \gamma < \alpha\rangle$$

is as desired, i.e. $p(\beta)\restriction\beta \Vdash \neg\sigma_\beta$. Suppose otherwise; then there is an $r = \langle \underset{\sim}{r}_\gamma \mid \gamma < \beta\rangle \in \mathcal{P}_\beta$ such that $r \geq p(\beta)\restriction\beta$ and $r \Vdash \sigma_\beta$. Extend it, if necessary, so that for some $\underset{\sim}{q}$ and $i < 2$

$$r \Vdash (\underset{\sim}{q} \geq^* p \setminus \beta \text{ and } \underset{\sim}{q} \parallel_{\mathcal{P}_\alpha \setminus \beta} {}^i\sigma)$$

where ${}^0\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$. Let us assume that $i = 0$. By the definition of order on $\mathcal{P}_\beta$ (6.1(2)), there is a $\beta^* < \beta$ such that for every γ with $\beta^* \leq \gamma < \beta$, $r\restriction\gamma \Vdash \underset{\sim}{r}_\gamma \geq^*_\gamma \underset{\sim}{p}^*_\gamma$. Consider a $\mathcal{P}_{\beta^*}$-name $\underset{\sim}{q}' = \langle \underset{\sim}{r}_\gamma \mid \beta^* \leq \gamma < \beta\rangle^\frown \underset{\sim}{q}$. Then, $r\restriction\beta^* \Vdash (\underset{\sim}{q}' \geq^* p \setminus \beta^*$ and $\underset{\sim}{q}' \parallel_{\mathcal{P}_\alpha \setminus \beta^*} \sigma)$. But $r\restriction\beta^* \geq p(\beta^*)\restriction\beta^* \Vdash \neg\sigma_{\beta^*}$. Contradiction.

This completes the construction. Consider $p(\alpha) = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \alpha\rangle$. Pick some $r \geq p(\alpha)$ deciding σ. Now we obtain a contradiction as in Case 2. This completes the proof of the lemma. $\dashv$

Let us now use this type of iteration to prove the following result of Magidor [34]:

6.3 Theorem. *Let κ be a strongly compact cardinal. Then there is a cardinal preserving extension in which κ is the least strongly compact and also the least measurable.*

Proof. We use the Magidor iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ defined by recursion on α as follows:

(a) If $\parallel_{\mathcal{P}_\alpha}$ (α is not measurable), then take $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ to be the trivial forcing

(b) If $\parallel_{\mathcal{P}_\alpha}$ (α is a measurable cardinal), then let $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ be the Prikry forcing over α with some normal ultrafilter.

(c) If $\neg(a)$ and $\neg(b)$, then we pick a maximal antichain $\langle p^i \mid i < \tau\rangle$ of elements of $\mathcal{P}_\alpha$ so that each p^i decides measurability of α. Above each p^i forcing (α is not measurable) we take $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ to be the trivial forcing. Above every p^i forcing measurability of α let $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ be the Prikry forcing over α with some normal ultrafilter.

This means that $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is a $\mathcal{P}_\alpha$-name such that $\mathcal{P}_\alpha$ forces: "if α is a measurable then $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is the Prikry forcing, and otherwise $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is trivial.

Let us now force with $\langle \mathcal{P}_\kappa, \leq\rangle$. Let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic. Then, in $V[G_\kappa]$, all measurable cardinals below κ are destroyed. Note that for $\alpha < \kappa$ the iteration past stage $\alpha + 1$ does not add measurables below α, since it is itself a Prikry-type iteration with $\leq^*$-order more than 2^α-closed. So, no new subsets are added to α. We need only show that κ remains strongly compact. This will follow from the next more general statement. $\dashv$

Note that the above proof is a simplification of Magidor's proof, which showed that the measures to be killed are exactly the unique normal extensions of measures of order 0 in V.

6.4 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is the Magidor iteration of Prikry-type forcing notions such that $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$ for unboundedly many α's. Then κ is strongly compact in $V^{\mathcal{P}_\kappa}$ provided it was such in V and for every $\alpha < \kappa$, $\Vdash_{\mathcal{P}_\alpha}$ ((a) $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leqslant}_\alpha{}^*\rangle$ is $|\alpha|$-closed, and (b) for all $p, q, r \in \underset{\sim}{Q}_\alpha$, if $p, q \geq^* r$ there is a $t \in \underset{\sim}{Q}_\alpha$ such that $t \geq^* p, q$).*

6.5 Remark. The requirement (a) holds for most of the Prikry-type forcing notions. But we may refer the reader to [16] and [46] doing without closure but still preserving measurability. The requirement (b) is much more restrictive. For example extender-based Prikry forcings of Section 2,3 do not satisfy it. Also the Easton support iteration that will be defined later fails to satisfy (b). It will be shown in 6.8 in non trivial cases (a)+(b) imply existence of a measurable cardinal $\geq |\alpha|$.

Proof. Let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic, i.e. generic for $\langle \mathcal{P}_\kappa, \leq\rangle$. Let $\lambda \geq \kappa$. We want to establish the λ-strong compactness of κ. In V pick a κ-complete fine ultrafilter U over $\mathcal{P}_\kappa(\lambda)$ (recall that U is called fine, if for every $\alpha < \lambda$ the set $\{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in P\}$ is in U). Let $j : V \longrightarrow M \simeq \mathrm{Ult}(V, U)$. Back in $V[G_\kappa]$, let us define $U^* \supseteq U$ over $\mathcal{P}_\kappa(\lambda)$ as follows:

$$\begin{aligned}&X \in U^* \text{ iff for some } p \in G_\kappa, \text{ in } M \text{ there is a } q \in \mathcal{P}_{j(\kappa)} \setminus \kappa \text{ with}\\ &\quad q \geq^* j(p) \setminus \kappa \text{ so that } p^\frown q \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{id}]_U \in j(\underset{\sim}{X}) \text{ for some name } \underset{\sim}{X} \text{ of } X.\end{aligned}$$

Note that $\mathcal{P}_\kappa = j(\mathcal{P}_\kappa){\restriction}\kappa$, since, in M, $j(\mathcal{P}_\kappa){\restriction}\kappa \subseteq {}^\kappa V_\kappa$ and $j``V_\kappa = V_\kappa$. So G_κ is an M-generic subset of $\mathcal{P}_\kappa$. Also $j(p){\restriction}\kappa = p$ for every $p \in \mathcal{P}_\kappa$. We need to first check that U^* is well defined. By (b) any two q's as above are compatible. Note also that if p, $p' \in \mathcal{P}_\kappa$ are $\leq$-compatible, then, in M, $j(p) \setminus \kappa$ and $j(p') \setminus \kappa$ are $\leq^*$-compatible. To see this, let $r \in \mathcal{P}_\kappa$, $r \geq p, p'$. Then there is a $\beta < \kappa$ such that for $\beta \leq \alpha < \kappa$, $r{\restriction}\beta \Vdash_{\mathcal{P}_\beta} \underset{\sim}{r}_\beta \geq^*_\beta \underset{\sim}{p}_\beta, \underset{\sim}{p}'_\beta$, where $r = \langle \underset{\sim}{r}_\gamma \mid \gamma < \kappa\rangle$, $p = \langle \underset{\sim}{p}_\gamma \mid \gamma < \kappa\rangle$, and $p' = \langle \underset{\sim}{p}'_\gamma \mid \gamma < \kappa\rangle$. So, in M, the same is true for $j(r)$, $j(p)$ and $j(p')$. Hence, r forces $\leq^*$-compatibility of

$j(p) \setminus \kappa$ and $j(p') \setminus \kappa$ witnessed by $j(r) \setminus \kappa$. In particular, this shows using (b) that $q \geq^* j(p) \setminus \kappa$ is $\leq^*$-compatible with every $j(p') \setminus \kappa$ with p, $p' \in G$.

Now applying above, if $p \in G$ forces "$\underset{\sim}{X} \in \underset{\sim}{U}^*$ and $\underset{\sim}{X} = \underset{\sim}{Y}$", then for some $q \in \mathcal{P}_{j(\kappa)} \setminus \kappa$, $q \geq^* j(p) \setminus \kappa$ we have

$$j(p){\restriction}\kappa^\frown q \ \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{id}]_U \in j(\underset{\sim}{X}) \ .$$

But by elementarity, $j(p) \Vdash j(\underset{\sim}{X}) = j(\underset{\sim}{Y})$. Also, $j(p){\restriction}\kappa^\frown q \geq^* j(p)$. Hence

$$j(p){\restriction}\kappa^\frown q \ \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{id}]_U \in j(\underset{\sim}{Y}) \ .$$

Clearly, $U^* \supseteq U$, and so it is fine. Let $\langle X_\nu \mid \nu < \delta < \kappa \rangle$ be a partition of $\mathcal{P}_\kappa(\lambda)$. We need to show that then for some $\nu < \delta$, $X_\nu \in U^*$. Pick some $p \in G_\kappa$ and names $\langle \underset{\sim}{X}_\nu \mid \nu < \delta \rangle$ such that $p \Vdash \langle \underset{\sim}{X}_\nu \mid \nu < \delta \rangle$ is a partition of $\mathcal{P}_\kappa(\lambda)$.

Then in M, $j(p) \Vdash (\langle j(\underset{\sim}{X}_\nu) \mid \nu < \delta \rangle$ is a partition of $\mathcal{P}_{j(\kappa)}(j(\lambda))$. Now we use κ-completeness of $\langle \mathcal{P}_{j(\kappa)} \setminus \kappa, \leq^* \rangle$ in order to find $\nu^* < \delta$ and $q \in \mathcal{P}_{j(\kappa)} \setminus \kappa$ with $q \geq^* j(p) \setminus \kappa$ such that for some $r \in G_\kappa$,

$$r^\frown q \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{id}] \in j(\underset{\sim}{X}_{\nu^*}) \ .$$

Hence $X_{\nu^*} \in U^*$ and we are done. $\dashv$

Note that once the ultrafilter U (in the proof above) is normal and the forcing $\langle \mathcal{P}_{j(\kappa)} \setminus \kappa, \leq^* \rangle$ is λ^+-closed, then the ultrafilter U^* extending U will be normal as well. Just use a regressive function instead of a partition in the proof of 6.4.

In particular, if we change the cofinality of each measurable cardinal below a measurable cardinal κ using the Magidor iteration of Prikry forcings, then the normal measure U over κ in V extends to a normal measure in the extension, provided $\langle \mathcal{P}_{j(\kappa)} \setminus \kappa, \leq^* \rangle$ is κ^+-closed. In order to insure this degree of closure, we may take U which concentrates on non-measurables, i.e.

$$\{\alpha < \kappa \mid \alpha \text{ is not a measurable } \} \in U.$$

It is still necessary to check that the iteration $\mathcal{P}_\kappa$ does not turn κ into a measurable in M (the ultrapower by U). This will follow from the following general statement. The proof of it is based on [34].

6.6 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is the Magidor iteration of Prikry-type forcing notions such that*

(a) *$\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$ for unboundedly many α's.*

(b) *For every $\alpha < \kappa$, $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leqslant}_\alpha{}^* \rangle$ is $|\alpha|$-closed, and: for all $p, q, r \in \underset{\sim}{Q}_\alpha$, if $p, q \geq^* r$ there is a $t \in \underset{\sim}{Q}_\alpha$ such that $t \geq^* p, q)$.*

(c) *The forcing in the interval* $[\alpha, (2^\alpha)^+]$ *is trivial for stationary many* α*'s.*

If κ *is measurable in* $V^{\mathcal{P}_\kappa}$ *then it was measurable in* V.

6.7 Remark. We do not know if there is a nontrivial Prikry-type forcing $\langle Q, \leq, \leq^* \rangle$ satisfying the clause 2 for a non-measurable cardinal α, assuming that $\langle Q, \leq^* \rangle$ is not α^+- closed. So, the clause 3 may hold automatically.

Proof. Let G be a generic subset of $\mathcal{P}_\kappa$ and W a κ-complete ultrafilter over κ in $V[G]$. Then, clearly, κ is at least a Mahlo cardinal in V. So, the following set is stationary in V:

$$S = \{\alpha < \kappa \mid \mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha, |V_\alpha| = \alpha, \text{ the forcing is trivial in the interval } [\alpha, (2^\alpha)^+]\}.$$

Suppose for simplicity that $0_{\mathcal{P}_\kappa} = \langle 0_{\underset{\sim}{Q}_\gamma} \mid \gamma < \kappa \rangle \in G$ and it forces that W is a κ-complete ultrafilter over κ in $V[G]$; otherwise, just work above a condition forcing this. Note that in our setting $0_{\mathcal{P}_\kappa}$ need not be weaker than every other condition in $\mathcal{P}_\kappa$: We may have a $t = \langle \underset{\sim}{t}_\gamma \mid \gamma < \kappa \rangle \in \mathcal{P}_\kappa$ such that for infinitely many γ's $\underset{\sim}{t}_\gamma$ is a non-direct extension of $\underset{\sim}{0}_\gamma$ in $\underset{\sim}{Q}_\gamma$; such a t would be incompatible with $0_{\mathcal{P}_\kappa}$.

Let $\alpha \in S$. Define an ultrafilter U_α over κ in $V[G \restriction \alpha]$ as follows:

$X \in U_\alpha$ iff for some $p \in G \restriction \alpha$ there is a $q \in \mathcal{P}_\kappa \setminus \alpha$ with $q \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha$ so that $p^\frown q \Vdash_{\mathcal{P}_\kappa} \underset{\sim}{X} \in \underset{\sim}{W}$ for some name $\underset{\sim}{X}$ of X.

Trivially, U_α is well-defined. $\alpha \in S$ implies that U_α is at least a $(2^\alpha)^+$-complete ultrafilter over κ in $V[G \restriction \alpha]$ (just use the $\leq^*$-completeness of the forcing $\mathcal{P}_\kappa \setminus \alpha$ to deal with partitions of κ into $\leq (2^\alpha)^+$ many pieces).

We use now the argument of Levy-Solovay [33] to find a condition $t(\alpha) \in \mathcal{P}_\alpha$ with $t(\alpha) \geq 0_{\mathcal{P}_\alpha}$ so that for every set $X \in V$ with $X \subseteq \kappa$, either

$$t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha \text{ or } t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \notin \underset{\sim}{U}_\alpha\,.$$

Thus, suppose that there is no such $t(\alpha)$. Work in V. For each $q \in \mathcal{P}_\alpha$ with $q \geq 0_{\mathcal{P}_\alpha}$, we pick a set $A_q \subseteq \kappa$ such that q does not decide whether $A_q \in \underset{\sim}{U}_\alpha$. Define a function from κ into a set of cardinality at most 2^α as follows:

$$F(\nu) = \langle \langle q, i \rangle \mid q \in \mathcal{P}_\alpha, i < 2, \text{ and: } i = 0 \text{ if } \nu \in A_q, i = 1 \text{ otherwise } \rangle.$$

Now, in $V[G \restriction \alpha]$, U_α is $(2^\alpha)^+$-complete ultrafilter, hence there is $X \in V \cap U_\alpha$ such that $F(\nu) = F(\mu)$, for any $\nu, \mu \in X$. Pick some $q \in G \restriction \alpha$ forcing this. Finally, back in V, there is an $i < 2$ such that for each $\nu \in X$ the pair $\langle q, i \rangle$ appears in $F(\nu)$. Then, $i = 0$ implies $X \subseteq A_q$ and $i = 1$ implies $X \subseteq \kappa \setminus A_q$. But $q \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha$. Hence, either $q \Vdash_{\mathcal{P}_\alpha} A_q \in \underset{\sim}{U}_\alpha$ or $q \Vdash_{\mathcal{P}_\alpha} \kappa \setminus A_q \in \underset{\sim}{U}_\alpha$, which contradicts the choice of A_q.

Set now (in V)

$$U(\alpha) = \{X \subseteq \kappa \mid t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha\}\,.$$

Clearly, $U(\alpha)$ is a $(2^\alpha)^+$-complete ultrafilter over κ.

We shall find a stationary subset S' of S such that for every $\alpha < \beta \in S', U(\alpha) = U(\beta)$. Then, $\alpha \in S'$ will imply that $U(\alpha)$ is a κ-complete ultrafilter over κ.

Thus, consider the sequence of conditions $\langle t(\alpha) \mid \alpha \in S\rangle$. For each $\alpha \in S$ we have $t(\alpha) \geq 0_{\mathcal{P}_\alpha}$. Hence, by the definition of the order $\leq$, there is a finite set $b(\alpha) \subseteq \alpha$ such that for each $\gamma \in \alpha \setminus b(\alpha)$,

$$t(\alpha){\restriction}\gamma \Vdash_{\mathcal{P}_\gamma} \underset{\sim}{t(\alpha)}_\gamma \geq^* \underset{\sim}{0}_\gamma \text{ in the forcing } \underset{\sim}{Q}_\gamma\,.$$

Now, we shrink S to a stationary set S_1 such that for each $\alpha, \beta \in S_1$, $b(\alpha) = b(\beta)$. Denote $b(\alpha)$ for $\alpha \in S_1$ by b. Let $\delta = \max(b) + 1$. The cardinality of the forcing $\mathcal{P}_\delta$ is less than α, for each $\alpha \in S_1$, since $\alpha = |V_\alpha|$ and $\mathcal{P}_\delta \in V_\alpha$. Hence, there are a stationary $S' \subseteq S_1$ and $t \in \mathcal{P}_\delta$ such that for each $\alpha \in S'$ we have $t(\alpha){\restriction}\delta = t$. It follows that $t(\alpha)$ and $t(\beta)$ are compatible in the order $\leq^*$, for any $\alpha, \beta \in S'$. We claim that $U(\alpha) = U(\beta)$, for each $\alpha, \beta \in S'$.

Recall the definition of $U(\alpha)$. Thus,

$$\begin{aligned} X \in U(\alpha) \text{ iff } & t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha \\ & \text{iff } \exists q \in \mathcal{P}_\kappa \setminus \alpha \text{ with } q \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha \text{ such that } t(\alpha)^\frown q \Vdash_{\mathcal{P}_\kappa} X \in \underset{\sim}{W}. \end{aligned}$$

Suppose for a moment that there is $X \in U(\alpha) \setminus U(\beta)$. Find $q_\alpha \in \mathcal{P}_\kappa \setminus \alpha$ with $q_\alpha \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha$ such that $t(\alpha)^\frown q_\alpha \Vdash_{\mathcal{P}_\kappa} X \in \underset{\sim}{W}$ and $q_\beta \in \mathcal{P}_\kappa \setminus \alpha, q_\beta \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha$ such that $t(\beta)^\frown q_\beta \Vdash_{\mathcal{P}_\kappa} \kappa \setminus X \in \underset{\sim}{W}$. But $t(\alpha)^\frown q_\alpha$ and $t(\beta)^\frown q_\beta$ are $\leq^*$-compatible, which is impossible since they force contradictory information. $\dashv$

The next simple observation shows that the conditions (a) and (b) of 6.4 already imply some strength.

6.8 Lemma. *Let $\langle Q, \leq, \leq^*\rangle$ be a non-trivial Prikry-type forcing notion and κ be an uncountable cardinal such that*

(1) *$\langle Q, \leq^*\rangle$ is κ-closed.*

(2) *For all $p, q, r \in Q$, if $p, q \geq^* r$ there is a $t \in Q$ such that $t \geq^* p, q$.*

Then there is a measurable cardinal $\geq \kappa$.

Proof. Let λ be a cardinal which contains a new subset. Fix a name $\underset{\sim}{a}$ of such a subset of λ. We assume that 0_Q already forces this.

Set

$$A = \{\rho < \lambda \mid \exists t \geq^* 0_Q \ t \Vdash_Q \check{\rho} \in \underset{\sim}{a}\}.$$

Then

$$0_Q \Vdash_Q \underset{\sim}{a} \neq \check{A}$$

just since A is old but a is new. Define now U to be the set of all $X \subseteq \lambda$ such that

$$\exists t \geq^* 0_Q \ t \Vdash_Q (\rho \in \check{X} \text{ for the least } \rho \text{ such that } \rho \in \underset{\sim}{a} \Delta \check{A}).$$

Then, clearly, U is a κ-complete ultrafilter over λ. Let us show that it is a non-principal one. Suppose otherwise. Then, for some $\rho < \lambda$ we will have $\{\rho\} \in U$. Hence there is a $t \geq^* 0_Q$ such that $t \Vdash \check{\rho} \in \underset{\sim}{a} \Delta \check{A}$. Extend t to some $s \geq^* t$ such that

$$s \Vdash_Q \check{\rho} \in \underset{\sim}{a} \ \text{ or } \ s \Vdash_Q \check{\rho} \in \check{A}.$$

The former possibility implies that $\rho \in A$, by the definition of A, which is impossible. If the later possibility occurs, then, again by the definition of A, we will have an $r \geq^* 0_Q$ such that $r \Vdash_Q \check{\rho} \in \underset{\sim}{a}$. But r is compatible with s, so we arrive to a contradiction. Hence, U is non-principal and we are done. $\dashv$

6.9 Example. Let us show how the Magidor iteration may destroy stationarity. Fix a regular cardinal κ, and set $Z = \{\alpha < \kappa \mid \alpha \text{ is a measurable}\}$. Assume that Z is stationary. Change the cofinality of each measurable cardinal below κ to ω using the Magidor iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ of Prikry forcings. By 6.6, only the members of Z change their cofinality. Let G be a generic subset of $\mathcal{P}_\kappa$ with $0_{\mathcal{P}_\kappa} \in G$. Let C_α denote the Prikry sequence for α deduced from G, where $\alpha \in Z$. Define a function $f : Z \to \kappa$ by setting $f(\alpha) = \min(C_\alpha)$.

6.10 Claim. *There is a finite $b \subseteq \kappa$ such that the elements of the sequence $\langle C_\alpha \mid \alpha \in Z \setminus b \rangle$ are pairwise disjoint. In particular, f is one-to-one on $Z \setminus b$ and, so Z is not stationary in $V[G]$.*

Proof. Work in V. Let $t \in \mathcal{P}_\kappa$ with $t \geq 0_{\mathcal{P}_\kappa}$. Suppose for simplicity that $t \geq^* 0_{\mathcal{P}_\kappa}$; otherwise, we work only with the coordinates where the extension is direct. Let $t = \langle \underset{\sim}{t}_\gamma \mid \gamma < \kappa \rangle$ and for each $\gamma \in Z$ we have $\underset{\sim}{t}_\gamma = \langle \langle \rangle, , \underset{\sim}{A}_\gamma \rangle$, where $\underset{\sim}{A}_\gamma$ is a $\mathcal{P}_\gamma$-name of a set in the normal ultrafilter $\underset{\sim}{U}^*_\gamma$ over γ which extends a normal ultrafilter U_γ, as in 6.4. Note that by 6.6, the forcing at each $\gamma \in \kappa \setminus Z$ is trivial.

Fix $\gamma \in Z$. Let G_γ be a generic subset of $\mathcal{P}_\gamma$ with $r = t \restriction \gamma \in G_\gamma$. Turn to $V[G_\gamma]$. Let us show that the set

$$B_\gamma = \{\nu \in A_\gamma \mid \forall \delta \in Z \cap \gamma \, (\nu \notin C_\delta)\}$$

must be in U^*_γ. Consider $j(r)\setminus\kappa$ in M, where $j:V\to M$ is the canonical embedding into the ultrapower of V by U_γ. Let $q\geq^* j(r)\setminus\kappa$ be obtained from $j(r)\setminus\kappa$ by replacing each set of measure one $\underset{\sim}{A}_\delta$ (for $\delta\in j(Z)\setminus(\kappa+1)$) of $j(r)\setminus\kappa$ by $\underset{\sim}{A}_\delta\setminus(\kappa+1)$. Then

$$r^\frown q \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{\gamma}]\in j(\underset{\sim}{B}_\gamma)\ .$$

Hence, $B_\gamma\in U^*_\gamma$.

Finally, back in V, we define $t^*\geq^* t$ by replacing each $\underset{\sim}{A}_\gamma$ by $\underset{\sim}{B}_\gamma$. Then t^* will force that C_γ's are pairwise disjoint. $\dashv$

Suppose now that κ above was a measurable and there was a measure U on κ concentrating on measurables. Then $Z\in U$. But in $V[G]$, Z is not stationary any more. Hence U does not extend to a normal ultrafilter.

6.2. Leaning's Forcing

J. Leaning [32] suggested a new and interesting way to put together Prikry forcings over different cardinals avoiding iteration. Below, we will briefly describe his forcing.

Fix a set Z of measurable cardinals, and set $\kappa=\sup(Z)$. For each $\delta\in Z$ pick a normal ultrafilter U_δ over δ. Set $\vec{U}=\langle U_\delta\mid\delta\in Z\rangle$.

6.11 Definition. Let the *filter of long measure one sets* be

$$\mathfrak{L}(\vec{U})=\{X\subseteq\kappa\mid X\cap\delta\in U_\delta \text{ for all } \delta\in Z\}.$$

6.12 Definition. Let $\mathbf{D}(\vec{U})$ be the set of all the pairs $\langle s,X\rangle$ such that

(1) $s\in[\kappa]^{<\omega}$.

(2) $X\in\mathfrak{L}(\vec{U})$.

6.13 Definition. Let $\langle s,X\rangle,\langle t,Y\rangle\in\mathbf{D}(\vec{U})$. Then $\langle s,X\rangle\geq\langle t,Y\rangle$ iff

(1) $s\subseteq t$.

(2) $X\subseteq Y$.

(3) $s\setminus t\subseteq Y$.

If $s=t$ then $\langle s,X\rangle\geq^*\langle t,Y\rangle$.

In [32] Leaning showed that $\langle\mathbf{D}(\vec{U}),\leq,\leq^*\rangle$ satisfies the Prikry condition, and so it is a Prikry-type forcing notion. He found a very interesting application of this forcing. Thus, starting from an assumption weaker than $o(\kappa)=2$, Leaning constructed a forcing extension in which the first measurable cardinal κ may have any number $\lambda\leq\kappa$ normal measures.

Note that if Z does not include its limit points (for example, if there is no measurable which is a limit of measurables), then this forcing is equivalent to the Magidor iteration of Prikry forcings for elements of Z. Crucially for each $\delta \in Z$, the forcing $\mathcal{P}_\delta$ below δ has cardinality less than δ. Hence, it is not hard to replace a name of a set of measure one by an actual set in U_δ; see [33] or just apply the corresponding argument from 6.6. Also, for each $\delta \in Z$ the set $A_\delta = \delta \setminus \sup(Z \cap \delta)$ is in U_δ and these sets are disjoint. Hence, we can link between finite sequences s and measurable cardinals in Z.

Leaning's forcing is equivalent for a while to a kind of the Magidor "iteration" of Prikry forcings, where instead of names of sets of measure one actual sets of measure one (i.e. those from U_δ's) are used. But once the set Z includes δ such that

$$\text{for all } X \in U_\delta, \text{ there is a } \mu < \kappa \text{ such that } X \cap \mu \in U_\mu\,,$$

the forcing $\langle \mathbf{D}(\vec{U}), \leq, \leq^* \rangle$ is different. Namely, at this stage the Magidor "iteration" of Prikry forcings without names fails to satisfy the Prikry condition. Thus, for example, there is no direct extension of the condition $\langle \langle \langle \rangle, \gamma \rangle \mid \gamma \in Z \rangle$ which can decide the following statement: "The first element of the Prikry sequence for δ belongs to the Prikry sequence of some $\mu < \delta$."

6.3. Easton-support Iteration

In many applications of iterated forcings it is important to have the κ-c.c. at stage κ of an iteration. The Magidor iteration or full support iteration fails to have this property, as well as usually does a full support iteration in different contexts. The common approach is to replace a full support by an Easton one. In the present section we show how to realize this dealing with iterations of Prikry-type forcing notions. The method was introduced in [13] and simplified in [16]. Shelah [51] found generalizations and applied them to small cardinals.

Let us give one example that illuminates the difference between full and Easton-support iteration.

6.14 Example. Suppose that κ is inaccessible and the limit of a set A of measurable cardinals. Assume for simplicity that A does not contain any of its limit points. Either iteration can be used to add a Prikry sequence C_γ for each $\gamma \in A$. In case of the full support iteration this sequence is uniform (below a certain condition) in the sense that if $\langle X_\gamma \mid \gamma \in A \rangle$ is any sequence in V such that X_γ is in a normal ultrafilter U_γ over γ, then $\bigcup_{\gamma \in A} (C_\gamma \setminus X_\gamma)$ is finite. Just the definition of the Magidor iteration and an easy density argument imply this. Thus let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in A \rangle$ be a condition in this iteration. A does not contain its limit points, so we can assume that each $\underset{\sim}{p}_\gamma$ is in V. Then p_γ is just a condition in the Prikry forcing with U_γ. Hence $p_\gamma = \langle t_\gamma, A_\gamma\rangle$, where $t_\gamma \in [\gamma]^{<\omega}$ and $A_\gamma \in U_\gamma$. Suppose now that we force only with extensions of the condition $\{\langle \emptyset, \gamma\rangle \mid \gamma \in A\}$. Then all but finitely many t_γ's are empty. Let $t_{\gamma_1}, \ldots, t_{\gamma_n}$ be the only nonempty t_γ's. Extend p to a condition $q = \{\langle t_\gamma, X_\gamma \cap A_\gamma\rangle \mid \gamma \in A\}$. Then $q \Vdash (\bigcup_{\gamma\in A} \underset{\sim}{C}_\gamma \setminus \check{X}_\gamma) \subseteq \bigcup_{i=1}^n t_{\gamma_i}$.

In the case of Easton-support iteration this will not be true: for example the set $\{\min(C_\gamma) \mid \gamma \in A\}$ will be essentially an Easton-support Cohen subset of κ, and in fact $V[\langle C_\gamma \mid \gamma \in A\rangle]$ will not have uniform sequence of Prikry sequences as in the full support iteration.

Let us now turn to the definition of the Easton iteration of Prikry-type forcing notions.

Let ρ be an ordinal. We define an iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\alpha \mid \alpha < \rho\rangle$ with Easton support. For every $\alpha < \rho$ define by recursion $\mathcal{P}_\alpha$ to be the set of all elements p of form $\langle \underset{\sim}{p}_\gamma \mid \gamma \in g\rangle$, where

(1) $g \subseteq \alpha$.

(2) g has an Easton support, i.e. for every inaccessible $\beta \leq \alpha$, $\beta > |g \cap \beta|$, provided that for every $\gamma < \beta$, $|\mathcal{P}_\gamma| < \beta$.

(3) For every $\gamma \in \operatorname{dom}(g)$,

$$p{\restriction}\gamma = \langle \underset{\sim}{p}_\beta \mid \beta \in g \cap \gamma\rangle \in \mathcal{P}_\gamma$$

and $p{\restriction}\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma$ is a condition in the forcing $\langle \underset{\sim}{Q}_\gamma, \underset{\sim}{\leq}_\gamma, \underset{\sim}{\leq}^*_\gamma\rangle$ of Prikry type".

Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g\rangle$ and $q = \langle \underset{\sim}{q}_\gamma \in f\rangle$ be elements of $\mathcal{P}_\alpha$. Then $p \geq q$ iff

(1) $g \supseteq f$.

(2) For every $\gamma \in f$, $p{\restriction}\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \geq_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

(3) There exists a finite subset b of f so that for every $\gamma \in f \setminus b$, $p{\restriction}\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \geq^*_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

If the set b in (3) is empty, then we call p a *direct extension* of q, and denote this by $p \geq^* q$.

Notice that in contrast to 6.1, we are allowed to take nondirect extensions in both $\leq$ and $\leq^*$ orderings for infinitely many coordinates $\gamma < \alpha$ provided that they are outside of the support (i.e. outside of f for extensions of $q = \langle \underset{\sim}{q}_\gamma \mid \gamma \in f\rangle$). Inside the support, as in 6.1, only for finitely many γ's can a nondirect extension be taken.

Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g\rangle \in \mathcal{P}_\alpha$ and $\beta < \alpha$. Consider $p{\restriction}\beta = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g \cap \beta\rangle$. Let $G_\beta \subseteq \mathcal{P}_\beta$ be generic with $p{\restriction}\beta \in G_\beta$. Then $p \setminus \beta = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g \setminus \beta\rangle \in \mathcal{P} \setminus \beta = \mathcal{P}_\alpha / G_\beta$. Let $t = \langle \underset{\sim}{t}_\gamma \mid \gamma \in f\rangle \in \mathcal{P}_\alpha/G_\beta$ be an extension of $p \setminus \beta$.

The support f of t need not be in V. But we can always find $f^* \in V$, $f \subseteq f^* \subseteq \alpha \setminus \beta$ satisfying (2) of the definition of the conditions. Thus let $\underset{\sim}{t}, \underset{\sim}{f}$ be a $\mathcal{P}_\beta$-names of t, f so that

$$p{\restriction}\beta \Vdash \underset{\sim}{t} = \langle \underset{\sim}{t_\gamma} \mid \gamma \in \underset{\sim}{f} \rangle \geq p \setminus \beta \ .$$

Work in V. Define $f^* \subseteq \alpha$ covering $\underset{\sim}{f}$ and satisfying (2) of the definition of the conditions. The construction of f^* is recursive. Let $f^* \cap \beta = \emptyset$. Suppose that $\beta < \gamma < \alpha$ and $f^* \cap \delta$ is already defined for each $\delta < \gamma$. If γ is a limit ordinal then let $f^* \cap \gamma = \bigcup_{\delta<\gamma} f^* \cap \delta$. If $\gamma = \gamma' + 1$, then we include γ' in f^* only in the case if some extension of $p{\restriction}\beta$ forces (in $\mathcal{P}_\beta$) "$\check{\gamma}' \in \underset{\sim}{f}$". This completes the definition of f^*. It is easy to check that for every $\gamma \leq \alpha$,

$$p{\restriction}\beta \Vdash_{\mathcal{P}_\beta} (\check{f}^* \supseteq \underset{\sim}{f} \text{ and } |\check{f}^* \cap \check{\gamma}| \leq |\underset{\sim}{f} \cap \check{\gamma}| + |\mathcal{P}_\beta|) \ .$$

Now, if γ with $\beta < \gamma \leq \alpha$ is inaccessible and for every $\delta < \gamma$, $|\mathcal{P}_\delta| < \gamma$, then $|f^* \cap \gamma| < \gamma$, since, back in $V[G_\beta]$ we have $|f \cap \gamma| < \gamma$ and $|\mathcal{P}_\beta| < \gamma$. So γ remains inaccessible and $|f^* \cap \gamma| \leq |f \cap \gamma| + |\mathcal{P}_\beta| < \gamma$. Clearly, in V, $|f^* \cap \gamma| < \gamma$ holds then as well.

Using the observation above, we can establish the Prikry condition for $\langle \mathcal{P}_\alpha, \leq, \leq^* \rangle$ repeating the argument of 6.2.

6.15 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q_\beta} \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. Then κ is measurable in $V^{\mathcal{P}_\kappa}$, provided:*

(a) *κ is measurable in V.*

(b) $V \models 2^\kappa = \kappa^+$.

(c) *For every cardinal $\alpha < \kappa$ we have*

  (i) $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q_\alpha}, \underset{\sim}{\leq^*_\alpha} \rangle$ *is $|\alpha|$-closed).*

  (ii) *for every β with $\alpha < \beta < \alpha^+$, $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q_\alpha}, \underset{\sim}{\leq^*_\alpha} \rangle$ is α^+-closed).*

(d) *For a closed unbounded set of α's below κ,*
$\Vdash_{\mathcal{P}_\alpha}$ either

  (i) *$\langle \underset{\sim}{Q_\alpha}, \underset{\sim}{\leq^*_\alpha} \rangle$ is $|\alpha|^+$-closed, or*

  (ii) *for all $p, q \in \underset{\sim}{Q^*_\alpha}$, if $p, q \underset{\sim}{\geq^*_\alpha} 0_{\underset{\sim}{Q_\alpha}}$ there is a $t \in \underset{\sim}{Q_\alpha}$ such that $t \underset{\sim}{\geq^*_\alpha} p, q$*

  where $0_{\underset{\sim}{Q_\alpha}}$ is the weakest element of $\underset{\sim}{Q_\alpha}$.

6.16 Remark. (1) The requirement $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$ for unboundedly many $\alpha < \kappa$ easily implies here that $\mathcal{P}_\alpha \subseteq V_\alpha$, for every inaccessible α in a closed unbounded subset of κ, due to the Easton support of conditions.

(2) If at each $\alpha < \kappa$ $\leq_\alpha = \leq^*_\alpha$ then also $\leq = \leq^*$ for $\mathcal{P}_\kappa$ and the lemma is actually the Kunen-Paris [30] result on preservation of measurability. Also our argument is very close to the Kunen-Paris one.

(3) If κ was a supercompact then as in [4] it is possible to show that κ remains strongly compact. Clearly, the supercompactness may be lost by iterating the Prikry forcing at each measurable below κ.

(4) Even if the alternative (ii) of the conclusion holds for each $\alpha < \kappa$, $\langle \mathcal{P}_\kappa, \leq^* \rangle$ fails to satisfy it, i.e. in $\mathcal{P}_\kappa$ there are lots of incompatible direct extensions of a fixed condition.

Proof. Let U be a κ-complete ultrafilter over κ. Consider its elementary embedding

$$j : V \to M \simeq \mathrm{Ult}(V, U) \ .$$

Then ${}^\kappa M \subseteq M$.

Let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic. The set of $\alpha < \kappa$ such that $\mathcal{P}_\alpha \subseteq V_\alpha$ is a member of U. Hence $\mathcal{P}_\kappa \subseteq V_\kappa$, $\mathcal{P}_\kappa = \mathcal{P}_{j(\kappa)} \restriction \kappa$ and for every $p \in \mathcal{P}_\kappa$ we have $j(p) = p$. Using $2^\kappa = \kappa^+$, we chose an enumeration $\langle \underset{\sim}{A}_\alpha \mid \alpha < \kappa^+ \rangle$ of all canonical names of subsets of κ. In M, at κ either $\langle \underset{\sim}{Q}_\kappa, \underset{\sim}{\leq}^*_\kappa \rangle$ is κ^+-closed, or for every $p, q \in Q_\kappa$, if $p, q \geq^*_\kappa 0_{Q_\kappa}$ then there is a $t \in Q_\kappa$ with $t \geq^*_\kappa p, q$. Suppose first that $\langle Q_\kappa, \leq^*_\kappa \rangle$ is κ^+-closed. Define by recursion a $\leq^*$-increasing sequence $\langle r_\alpha \mid \alpha < \kappa^+ \rangle$ of conditions in $\mathcal{P}_{j(\kappa)} \setminus \kappa$ such that for every $\alpha < \kappa^+$ there is a $p \in G_\kappa$ satisfying

$$p^\frown r_\alpha \parallel \check{\kappa} \in j(\underset{\sim}{A}_\alpha) \ .$$

Let $U^* = \{A_\alpha \mid \alpha < \kappa^+$, for some $p \in G_\kappa$ $p^\frown r_\alpha \Vdash \check{\kappa} \in j(\underset{\sim}{A}_\alpha)\}$. It is routine to check that U^* is well-defined and is a normal ultrafilter over κ extending U.

We now turn to the second possibility, i.e. any two $\leq^*_\kappa$-extensions of $0_{\underset{\sim}{Q}_\kappa}$ in Q_κ are $\leq^*_\kappa$-compatible. Define by recursion an $\leq^*$-increasing sequence $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+ \rangle$ of conditions in $\mathcal{P}_{j(\kappa)} \setminus (\kappa + 1)$ such that for every $\alpha < \kappa^+$ there are $p \in G_\kappa$ and $\underset{\sim}{t}$ such that $p \Vdash_{\mathcal{P}_\kappa} \underset{\sim}{t} \underset{\sim}{\geq}^*_\kappa 0_{\underset{\sim}{Q}_\kappa}$ and

$$p^\frown \underset{\sim}{t}^\frown \underset{\sim}{r}_\alpha \parallel \kappa \in j(\underset{\sim}{A}_\alpha) \ .$$

Let

$$U^* = \{A_\alpha \mid \alpha < \kappa^+, \text{ and for some } p \in G_\kappa \text{ and } \underset{\sim}{t}, \\ p \Vdash_{\mathcal{P}_\kappa} \underset{\sim}{t} \underset{\sim}{\geq}^*_\kappa 0_{\underset{\sim}{Q}_\kappa} \text{ and } p^\frown \underset{\sim}{t}^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j(\underset{\sim}{A})\} \ .$$

Using the compatibility in $\langle Q_\kappa, \leq^* \rangle$ of any two extensions of 0_{Q_κ}, it is routine to check that U^* is well defined and is a κ-complete ultrafilter extending U. Note that U^* need not be normal anymore. ⊣

Using a similar idea a bit more general result can be shown.

6.17 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. Let U_1 be a κ-complete ultrafilter over κ and U_0 a normal ultrafilter over κ such that $U_0 \trianglelefteq U_1$ in the Mitchell order (i.e. $U_0 = U_1$ or $U_0 \in \mathrm{Ult}(V, U_1)$). Then U_1 extends to a κ-complete ultrafilter in $V^{\mathcal{P}_\kappa}$ provided:*

(a) $V \vDash 2^\kappa = \kappa^+$.

(b) *For every cardinal $\alpha < \kappa$ we have*

(i) $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ *is $|\alpha|$-closed)*

(ii) *For every β with $\alpha < \beta < \alpha^+$, $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is α^+-closed)*

(c) *The set of $\alpha < \kappa$ satisfying the condition below is in U_0:*
$\Vdash_{\mathcal{P}_\alpha}$ *either*

(i) $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ *is $|\alpha|^+$-closed, or*

(ii) *for all $p, q \in \underset{\sim}{Q}^*_\alpha$, if $p, q \underset{\sim}{\geq}^*_\alpha 0_{\underset{\sim}{Q}_\alpha}$ there is a $t \in \underset{\sim}{Q}_\alpha$ with $t \underset{\sim}{\geq}^*_\alpha p, q$.*

Proof. If $U_1 = U_0$, then this was proved in 6.15. Suppose then that $U_0 \in \mathrm{Ult}(V, U_1)$. Let $M_1 = \mathrm{Ult}(V, U_1)$ and $j_1 : V \to M_1$ be the corresponding elementary embedding. Consider $M = \mathrm{Ult}(M_1, U_0)$ and $j_{10} : M_1 \to M$ the corresponding elementary embedding. Set $j = j_{10} \circ j_1$. Clearly, $j : V \to M$ is an elementary embedding, $U_0 = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ and $U_1 = \{X \subseteq \kappa \mid j_{10}([id]_{U_1}) \in j(X)\}$. We use j, M as in the proof of 6.15 to define a $\leq^*$-increasing sequence $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+\rangle$ but now deciding statements "$j_{10}([id]_{U_1}) \in j(\underset{\sim}{A}_\alpha)$" and not "$\check{\kappa} \in j(\underset{\sim}{A}_\alpha)$" as it was in 6.15. The κ-complete ultrafilter defined using this sequence as in 6.15 will then be as desired. $\dashv$

The above lemma turned out to be useful for iterations of extender-based Prikry and Radin forcings for which the $\leq^*_\alpha$-compatibility condition (i.e. the alternative (ii) of the conclusion of the lemma) fails.

The next lemma is a basic tool our Easton-support iteration and has the same proof as that for the usual Easton-support iteration. See Baumgartner [5], Jech [25] or Shelah [54] for the proof.

6.18 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. If κ is a Mahlo cardinal, then $\mathcal{P}_\kappa$ satisfies the κ-c.c.*

Let us show now an analog of 6.6 that Easton iterations of Prikry type forcing notions do not create new measurable cardinals. The proof is based on an argument of Kimchi and Magidor [28]; see also Apter [3].

6.19 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. Let G be a generic subset of $\mathcal{P}_\kappa$. If κ is a measurable cardinal in $V[G]$, then it was measurable already in V.*

Proof. Let W be a normal ultrafilter over κ in $V[G]$. Fix $p \in G$ such that $p \Vdash_{\mathcal{P}_\kappa} (\underset{\sim}{W}$ is a normal ultrafilter over $\kappa)$. Work in V. Clearly, κ is a Mahlo cardinal. It is enough to find $q \geq p$ such that for every $X \subseteq \kappa$, q decides the statement "$\check{X} \in \underset{\sim}{W}$".

Suppose that there is no such q. We build a binary κ-tree T of height κ. Star with $\langle p, \kappa \rangle$.

Successor levels

Let the pair $\langle r, A \rangle$ be on the level α of T. We assume that $r \geq p$, $A \subseteq \kappa$ and $r \Vdash_{\mathcal{P}_\kappa} (\check{A} \in \underset{\sim}{W})$. Pick some partition A_0, A_1 of A and incompatible extensions r_0, r_1 of r such that $r_0 \Vdash_{\mathcal{P}_\kappa} (\check{A}_0 \in \underset{\sim}{W})$ and $r_1 \Vdash_{\mathcal{P}_\kappa} (\check{A}_1 \in \underset{\sim}{W})$. Place both $\langle r_0, A_0 \rangle$ and $\langle r_1, A_1 \rangle$ in T at the level $\alpha + 1$ to be the successors of $\langle r, A \rangle$.

Limit levels

Let $\alpha < \kappa$ be a limit ordinal. For each branch in T of the height α, we take the intersection of all second coordinates of elements along the branch. We thus obtain a partition of κ into at most 2^α many sets. But κ is Mahlo, hence $2^\alpha < \kappa$. Also,

$$p \Vdash_{\mathcal{P}_\kappa} (\underset{\sim}{W} \text{ is a normal ultrafilter over } \kappa)\,.$$

Hence, there are an element A of this partition and $r \geq p$ such that

$$r \Vdash_{\mathcal{P}_\kappa} \check{A} \in \underset{\sim}{W}\,.$$

For all such A, we place a pair of form $\langle r, A \rangle$ into T at level α as the successor of each element of the branch generating A.

This completes the construction of T.

Turn now to $V[G]$. κ is measurable and so weakly compact. Hence T must have a κ-branch. Let $\langle \langle r_\alpha, A_\alpha \rangle \mid \alpha < \kappa \rangle$ be such a branch. For each $\alpha < \kappa$ set $B_\alpha = A_\alpha \setminus A_{\alpha+1}$. By the construction of T, then there is s_α such that $\langle s_\alpha, B_\alpha \rangle$ is an immediate successor of $\langle r_\alpha, A_\alpha \rangle$. In addition, $s_\alpha \geq r_\alpha$ and the conditions $r_{\alpha+1}$, s_α are incompatible. Also, for each $\beta > \alpha$, we have $A_\beta \subseteq A_{\alpha+1}$. So, $A_\beta \cap B_\alpha = \emptyset$. But

$$r_\beta \Vdash_{\mathcal{P}_\kappa} \check{A}_\beta \in \underset{\sim}{W} \text{ and } s_\alpha \Vdash_{\mathcal{P}_\kappa} \check{B}_\alpha \in \underset{\sim}{W},$$

hence r_β and s_α are incompatible. This implies that s_β and s_α are incompatible as well, since $s_\beta \geq r_\beta$.

Hence, $\langle s_\alpha \mid \alpha < \kappa \rangle$ forms an antichain of size κ in $V[G]$. But this is impossible, since we can run the usual Δ-system argument for the Easton support iteration $(\mathcal{P}_\kappa)^V$ inside $V[G]$ and this will give the κ-c.c. Contradiction. $\dashv$

Let us conclude with two applications. The first one will be a construction of a κ^+-saturated ideal over an inaccessible κ concentrating on cardinals of cofinality $\aleph_0$. Such an ideal was first constructed by Woodin starting from a supercompact and using a beautiful construction involving passing to a model without AC and then restoring the choice by forcing. Mitchell in [44] gave another construction from the optimal assumptions. The construction below follows the lines of [13]. Let $U_0 \triangleleft U_1$ be normal ultrafilters over κ (i.e. $U_0 \in \mathrm{Ult}(V, U_1)$). Suppose GCH for simplicity. Fix a sequence of normal ultrafilters $\langle U(\beta) \mid \beta < \kappa \rangle$ representing U_0 in the ultrapower by U_1. Pick some $A \subseteq \kappa$, $A \in U_1 \setminus U_0$ such that for every $\beta \in A$ $A \cap \beta \notin U(\beta)$. We define $\langle \mathcal{P}_\kappa, \leq, \leq^* \rangle$ by taking the Easton iteration of Prikry forcings with $U(\beta)$ (or more precisely with the extension of $U(\beta)$ defined in 6.7) for every $\beta \in A$. Let $j : V \longrightarrow M_1 \simeq \mathrm{Ult}(V, U_1)$ and let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic. Fix an enumeration $\langle \underset{\sim}{A}_\alpha \mid \alpha < \kappa^+ \rangle$ of all canonical names of subsets of κ. As in 6.15, we define a $\leq^*$-increasing sequence $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+ \rangle$ of elements of $\mathcal{P}_{j(\kappa)} \setminus \kappa + 1$ such that for every $\alpha < \kappa^+$ there are $p \in G_\kappa$ and $t \in Q_\kappa$

$$p^\frown t^\frown \underset{\sim}{r}_\alpha \parallel \check{\kappa} \in j(\underset{\sim}{A}_\alpha) \ .$$

Define

$$F_1 = \{B \subseteq \kappa \mid \text{ there are } p \in G_\kappa \text{ and } \alpha < \kappa^+ \\ \text{such that } p^\frown 0_{\underset{\sim}{Q}_\kappa}{}^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j(\underset{\sim}{B})\} \ .$$

It is not hard to see that F_1 is a well defined normal filter over κ extending U_1.

Let us establish the normality. Suppose that $\langle B_\beta \mid \beta < \kappa \rangle$ is a sequence of elements of F_1. We need to show that $B = \Delta\{B_\beta \mid \beta < \kappa\} \in F_1$. By the definition of F_1, for each $i < \kappa$ there are $p_\beta \in G_\kappa$ and $\alpha_\beta < \kappa^+$ such that

$$p_\beta{}^\frown 0_{\underset{\sim}{Q}_\kappa}{}^\frown r_{\underset{\sim}{\alpha_\beta}} \Vdash \check{\kappa} \in j(\underset{\sim}{B_\beta}) \ .$$

Let $\alpha \geq \bigcup_{\beta<\kappa} \alpha_\beta$. We would like to show that for some $p \in G_\kappa$,

$$p^\frown 0_{\underset{\sim}{Q}_\kappa}{}^\frown \underset{\sim}{r}_\alpha \ \Vdash \ \check{\kappa} \in j(\underset{\sim}{B}) \ .$$

Suppose otherwise. Then for some $p \in G_\kappa$, $t \in Q_\kappa$ and $\underset{\sim}{r} \geq \underset{\sim}{r}_\alpha$,

$$p^\frown t^\frown \underset{\sim}{r} \ \Vdash \ \check{\kappa} \notin j(\underset{\sim}{B}) \ .$$

Then, by the definition of the diagonal intersection, there would be $\beta < \kappa$, $p' \in G$, $t' \in Q_\kappa$, and $\underset{\sim}{r}'$ such that

$$p^\frown t^\frown \underset{\sim}{r} \leqslant p'^\frown t'^\frown \underset{\sim}{r}' \ \Vdash \ \check{\kappa} \notin j(\underset{\sim}{B_\beta}) \ .$$

But this is impossible, since $p_\beta{}^\frown 0_{Q_\kappa}{}^\frown r_{\underset{\sim}{\alpha_\beta}} \Vdash \check{\kappa} \in j(\underset{\sim}{B_\beta})$, there is a $q \in G_\kappa$ which is stronger than both p and p_β, and so $q^\frown t^\frown \underset{\sim}{r} \geq p_\beta{}^\frown 0_{Q_\kappa}{}^\frown 0_{Q_\kappa}{}^\frown r_{\underset{\sim}{\alpha_\beta}}$. Contradiction.

The forcing with F_1-positive sets is equivalent to the forcing with $\langle Q_\kappa, \leq_\kappa\rangle$. The last forcing is just Prikry forcing with an extension of U_0. Hence it satisfies the κ^+-c.c. Clearly, F_1 concentrates on cardinals of cofinality $\aleph_0$, since each member of A is such a cardinal in $V[G_\kappa]$. [13] contains generalizations of the above construction for cofinalities different from $\aleph_0$ and to the nonstationary ideal. Thus it was shown there that $NS_\kappa{\restriction}S$ can be κ^+-saturated for a stationary set $S \subseteq \kappa$ so that for every regular cardinal $\delta < \kappa$ $S \cap \{\beta < \kappa \mid \operatorname{cf}(\beta) = \delta\}$ is stationary.

If we define a function $f : A \to \kappa$ by $f(\alpha) = \min(C_\alpha)$, where C_α is the Prikry sequence for α, then for every $\gamma < \kappa$ the set $\{\alpha < \kappa \mid f(\alpha) = \gamma\}$ will be F_1–positive. This is in contrast to a similar construction 6.6 with the Magidor iteration considered at the beginning of §6.2. There, f is one to one. Below, we will see that this f may be a projection function from a non-normal extension of U_1 to a normal extension of U_0.

Let us now turn to the second application. Consider $U_0 \lhd U_1$ as above. Perform the same iteration. Let $j_1 : V \longrightarrow M_1 \simeq \operatorname{Ult}(V, U_1)$. In $M_1[G_\kappa]$, at stage κ we are supposed to use the Prikry forcing with a normal ultrafilter U_0^* extending U_0. Clearly, U_0^* is such also in $V[G_\kappa]$. Obviously, any two direct extensions of the weakest condition in Prikry forcing are compatible. Hence, by 6.6 or 6.8, there is a κ-complete ultrafilter U_1^* extending U_1. We pick U_1^* as it was defined in 6.17 using the embedding j_1.

6.20 Lemma. $U_1^* >_{RK} U_0^*$.

Proof. Define the projection map $f : A \longrightarrow \kappa$ as follows: $f(\alpha) =$ the first element of the Prikry sequence of α, where $A \in U_1 \setminus U_0$ is as in the first application. In order to show that this f projects U_1^* onto U_0^*, it is enough to prove that for every $B \in U_0^*$ and $C \in U_1^*$

$$f^{-1}(B) \cap C \neq \emptyset\ .$$

So let $C \in U_1^*$ and $B \in U_0^*$. By the definition of U_1^* there are $p \in G_\kappa$, $t = \langle \emptyset, D\rangle \in Q_\kappa$ and $\alpha < \kappa^+$ so that $p^\frown t^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j_1(\underset{\sim}{C})$. Then also, $p^\frown\langle \emptyset, D \cap B\rangle^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j_1(\underset{\sim}{C})$ and in addition $\langle \emptyset, D \cap B\rangle \Vdash_{Q_\kappa}$ (the first element of the Prikry sequence of κ is in B). Hence,

$$p^\frown\langle \emptyset, D \cap B\rangle^\frown \underset{\sim}{r}_\alpha \ \Vdash \check{\kappa} \in j_1(\underset{\sim}{C}) \cap j_1(\underset{\sim}{f}^{-1})(j_1(\underset{\sim}{B}) = j_1(\underset{\sim}{C} \cap \underset{\sim}{f}^{-1}(\underset{\sim}{B}))\ .$$

So, $C \cap f^{-1}(B) \in U_1^*$. In particular, $C \cap f^{-1}(B) \neq \emptyset$.

Notice now that U_1^* cannot be isomorphic to U_0^* or in other words, f cannot be $1-1$ on a set in U_1^*. Thus, by the κ-c.c. every closed unbounded subset of κ in $V[G_\kappa]$ contains a closed unbounded subset of κ which is in V. U_1 was normal in V, hence U_1^* containing U_1 contains as well all closed unbounded subsets of κ. Clearly, f is regressive. So, if it is one to one on a set $E \in U_1^*$ then E is nonstationary which is impossible. Hence $U_1^* >_{RK} U_0^*$ and we are done. $\dashv$

The construction above turns the Mitchell order into the Rudin Keisler order for two ultrafilters. Longer sequences were dealt in [13], and the consistency correlation between these orderings was studied in [14]. In [15], the construction above was extended further in order to turn a Mitchell increasing sequence of length κ^{++} into a Rudin-Keisler increasing sequence of the same length. Such a sequence (with minor changes) can be used in the extender-based Prikry forcing of Section 3 for changing the cofinality of κ to $\aleph_0$ blowing simultaneously its power to κ^{++}. This way, the consistency strength of the negation of the Singular Cardinal Hypothesis is reduced to the optimal value $o(\kappa) = \kappa^{++}$, i.e. a measurable of the Mitchell order κ^{++}.

6.4. An Application to Distributive Forcing Notions

We would like to apply the iteration techniques of §6.1 and §6.2 to distributive forcing notions.

Let $\langle Q, \leq \rangle$ be (κ, ∞)-distributive, i.e. it does not add new sequences of ordinals of length less than κ or, equivalently, the intersection of any less than κ dense open subsets of Q is dense open. If κ is $2^{|Q|}$-supercompact (or $2^{|Q|}$-strongly compact) then it is possible to turn Q into a Prikry-type forcing $\langle Q, \leq, \leq^* \rangle$ with $\langle Q, \leq^* \rangle$ κ-closed.

Recall that a map $\pi : \mathcal{P}_1 \to \mathcal{P}_2$ between forcing notions is called a *projection* if

(a) $q \leq r$ implies $\pi(q) \leq \pi(r)$.

(b) $\pi(0_{\mathcal{P}_1}) = 0_{\mathcal{P}_2}$.

(c) If $p \geq \pi(q)$, then there is a $r \geq q$ with $\pi(r) \geq p$.

If $G_1 \subseteq \mathcal{P}_1$ is generic then π"G_1 generates a generic subset of $\mathcal{P}_2$. We say that in this case $\mathcal{P}_2$ is a subforcing of $\mathcal{P}_1$.

6.21 Lemma. *Assume that $\langle Q, \leq \rangle$ is a (κ, ∞)-distributive forcing notion where κ is $2^{|Q|}$-supercompact. Let $\langle \mathcal{P}, \leq, \leq^* \rangle$ be the supercompact Prikry forcing with a normal ultrafilter over $\mathcal{P}_\kappa(2^{|Q|})$. Then $\langle Q, \leq \rangle$ is a subforcing of $\langle \mathcal{P}, \leq \rangle$.*

Proof. Let $\lambda = 2^{|Q|}$. Fix $\langle D_\alpha \mid \alpha < \lambda \rangle$ a list of all dense open subsets of Q. Let G be a generic subset of $\mathcal{P}$ and $\langle P_n \mid n < \omega \rangle$ its Prikry sequence. Then, by 1.50, $\lambda = \bigcup_{n<\omega} P_n$. Each $P_n \in V$ and has cardinality less than κ. Hence, by distributivity, $D(n) = \bigcap \{D_\alpha | \alpha \in P_n\} \in V$ is dense open subset of Q. Also, $D(n+1) \subseteq D(n)$, since $P_{n+1} \supseteq P_n$. Now, we pick an increasing sequence $\langle q_n \mid n < \omega \rangle$ with $q_n \in D(n)$. It will generate a generic subset of Q. ⊣

Let $\pi : \mathcal{P} \to Q$ be a projection map, which exists by the previous lemma. Define now a forcing ordering (quasiorder) $\leq_Q$ over $\mathcal{P}$:

$$p \leq_Q r \text{ iff } \pi(p) \leq \pi(r) \ .$$

Then $\langle \mathcal{P}, \leq_Q \rangle$ is a forcing equivalent to $\langle Q, \leq \rangle$.

6.22 Lemma. *$\langle \mathcal{P}, \leq_Q, \leq^* \rangle$ is a Prikry-type forcing notion.*

Proof. Clearly, $\leq_Q \supseteq \leq \supseteq \leq^*$. So we need to check that for every $p \in \mathcal{P}$ and a statement σ of the forcing $\langle \mathcal{P}, \leq_Q \rangle$ there is $p^* \geq^* p$ deciding σ in $\langle \mathcal{P}, \leq_Q \rangle$. Set

$$A_0 = \{q \in \mathcal{P} \mid q \geq_Q p \text{ and } q \Vdash_{\langle \mathcal{P}, \leq_Q \rangle} \sigma\}, \text{ and}$$
$$A_1 = \{q \in \mathcal{P} \mid q \geq_Q p \text{ and } q \Vdash_{\langle \mathcal{P}, \leq_Q \rangle} \neg\sigma\} \ .$$

Note that any $q_0 \in A_0$ and $q_1 \in A_1$ are incompatible in $\langle \mathcal{P}, \leq \rangle$, since $\leq \subseteq \leq_Q$. Also, each $r \in \mathcal{P}$ has $\leq_Q$-extension in A_0 or in A_1. Thus, it must have $\leq$-extension in one of these sets. Let, for example, $r \leq_Q s \in A_0$. So, $\pi(r) \leq \pi(s)$ and by (3) of the definition of projection there is an $r' \geq r$ such that $\pi(r') \geq \pi(s)$. Hence, $r' \geq_Q s \in A_0$ and so $r' \in A_0$. The above means that $A_0 \cup A_1$ is dense $\langle \mathcal{P}, \leq \rangle$. The Prikry condition for $\langle \mathcal{P}, \leq, \leq^* \rangle$ implies then that there is a $p^* \geq^* p$ forcing in $\langle \mathcal{P}, \leq \rangle$ "$\underset{\sim}{G} \cap A_i \neq \emptyset$" for some $i \in 2$, where $\underset{\sim}{G}$ is the canonical name for a $\langle \mathcal{P}, \leq \rangle$-generic set. Without loss of generality suppose $i = 0$. Then, $p^* \Vdash_{\langle \mathcal{P}, \leq_Q \rangle} \sigma$. Otherwise, there will be $q \in A_1$, $q \geq_Q p^*$. But, then, using the property (3) of the projection, there will be $q' \geq p^*$ such that $q' \geq_Q q$. Hence $q' \in A_1$ which means $q' \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{G} \cap A_1 \neq \emptyset$. This contradicts to the choice of p^*. $\dashv$

Let us conclude with an example of iterating distributive forcing notions. We refer to [16, 13, 46] and [29] for more sophisticated applications.

A subset E of a regular $\kappa > \aleph_0$ is called *fat* if for every $\delta < \kappa$ and every closed unbounded subset C of κ there is a closed subset $s \subseteq E \cap C$ of order type δ. It is not hard to obtain a fat subset with fat complement. For example, just force a Cohen subset to κ. It will be as desired. Suppose now that $E \subseteq \kappa$ is fat. Consider the usual forcing for adding a club to E: $P[E] = \{d \mid d \text{ is a closed bounded subset of } E\}$ ordered by the end extension, i.e. $d_1 \geq d_2$ iff $d_1 \cap \max(d_2) + 1 = d_2$. By Abraham and Shelah [2], or just directly, the forcing $\langle P[E], \leq \rangle$ is (κ, ∞)–distributive.

Suppose now that for every $n < \omega$, κ_n is a κ_n^+-supercompact cardinal, $2^{\kappa_n} = \kappa_n^+$ and E_n is a fat subset of κ_n. We would like to produce a cardinal preserving extension in which every E_n will contain a club.

By 6.22, for every $n < \omega$ there is a Prikry-type forcing $\langle Q_n, \leq_n, \leq_n^* \rangle$ such that $\langle Q_n, \leq_n \rangle$ is equivalent to $\langle P[E_n], \leq \rangle$ and $\langle Q_n, \leq_n^* \rangle$ is κ_n–closed. Let $\langle \mathcal{P}_n, \underset{\sim}{Q}_n \mid n < \omega \rangle$, $\langle \mathcal{P}_\omega, \leq, \leq^* \rangle$ be the Magidor iteration (the Easton iteration is just the same in case of ω stages) of $\langle Q_n, \leq_n, \leq_n^* \rangle$'s. It certainly will add clubs to each E_n. We need to show only that cardinals are preserved. Let $m < \omega$. We use an obvious splitting $\mathcal{P}_\omega = \mathcal{P}_{\leq m} * \mathcal{P}_{>m}$ of $\mathcal{P}_\omega$ into the part of the iteration up to m and those above m. Then, $\langle \mathcal{P}_{>m}, \leq^* \rangle$ will be κ_{m+1}–closed. So the Prikry condition will imply that it does not add new bounded subsets to κ_{m+1}. $\mathcal{P}_{\leq m}$ is a finite iteration $P[E_0] * P[E_1] * \cdots * P[E_m]$. For every $k \leq m$, $|\mathcal{P}_{\leq k}| = \kappa_k$. So each E_{k+1} remains fat in $V^{\mathcal{P}_{\leq k}}$. Hence, $\mathcal{P}_{\leq m}$ preserves all the cardinals.

7. Some Open Problems

We conclude this chapter with several open problems on cardinal arithmetic. Some of them are well known; others are less so, but seem to us important for the further understanding of the power function.

The first and probably the most well known:

Problem 1

Suppose that $\aleph_\omega$ is strong limit or even $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$. Is it possible to have $2^{\aleph_\omega} > \aleph_{\omega_1}$?

By Shelah [53], an upper bound is $\min(\aleph_{\omega_4}, \aleph_{(2^{\aleph_0})^+})$. It is shown in [21] that "$2^{\aleph_\omega} > \aleph_{\omega_1}$" implies an inner model with overlapping extenders. Recently this was improved in [22] to PD.

The next problem is probably a bit less well known, but according to Shelah it is the crucial for cardinal arithmetic.

Problem 2

Let a be a set of regular cardinals with $|a| < \min(a)$. Can $|pcf(a)| > |a|$?

Recall that $pcf(a) = \{\mathrm{cf}(\prod a/D) \mid D \text{ an ultrafilter over } a\}$. By the basics of the pcf-theory, $|pcf(a)\ | \leq 2^{|a|}$ (see [53], [6] or [1]). It is unknown even if for countable a's "$|pcf(a)| > |a|$" implies an inner model with a strong cardinal. But in [18], it was shown that if for a set a of regular cardinals above $2^{|a|^+ + \aleph_2}$ we have $|pcf(a)| > |a| + \aleph_1$, then there is an inner model with a strong cardinal.

Recall that $pp(\kappa) = \sup\{\mathrm{cf}(\prod a/D) \mid a \subseteq \kappa$ is a set of at most $\mathrm{cf}(\kappa)$ of regular cardinals, unbounded in κ and D an ultrafilter over a including all cobounded subsets of $a\}$. The next problem was proposed by Shelah in [52] and deals with the following strengthening of "$|pcf(a)| = |a|$" called the Shelah Weak Hypothesis :

For every cardinal λ the number of singular cardinals $\kappa < \lambda$ with $pp(\kappa) \geq \lambda$ is at most countable.

Also, for uncountable cofinality an even stronger statement is claimed:

For every cardinal λ the number of singular cardinals $\kappa < \lambda$ of uncountable cofinality with $pp(\kappa) \geq \lambda$ is finite.

Problem 3

Is the negation of the Shelah Weak Hypothesis consistent?

In [23] was shown that much more complicated forcing notions than those of Sections 2,3 seem to be needed in order to deal with the negation of the Weak Hypothesis.

The general formulation of the Singular Cardinals Problem (SCP) is as follows: Find a complete set of rules describing the behavior of the power (or even more general – pseudo-power (pp)) function on singular cardinals. In terms of core models (see the inner model chapters of this handbook) we can reformulate SCP in a more concrete form: Given a core model K with certain large cardinals, which functions in K can be realized as the power set function in a set generic extension of K, i.e. if $F : \lambda \to \lambda \in K$ for some ordinal λ, is there a generic extension of K satisfying $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ for all $\alpha < \lambda$?

If we restrict ourselves to finite gaps between singular cardinals and its power then, at present, the most general results on possible behavior of the power function are due C. Merimovich [39]. They extend previous results by Foreman-Woodin [12], Woodin, Cummings [9] and M. Segal [49]. However lots of possibilities are still open. Let us state a few of the simplest:

Problem 4

Is it possible to have $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+2}$ and two stationary sets $S_1, S_2 \subseteq \omega_1$ with $S_1 \cup S_2 = \omega_1$ such that

$$\begin{aligned} &\alpha \in S_1 \quad \text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+2} \quad \text{and} \\ &\alpha \in S_2 \quad \text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+3} \text{ ?} \end{aligned}$$

Problem 5

Is it possible to have two stationary sets $S_1, S_2 \subseteq \omega_2$ with $S_1 \cup S_2 = \omega_2$ and $S_2 \cap \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\}$ stationary such that

$$\begin{aligned} &\alpha \in S_1 \quad \text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad \text{and} \\ &\alpha \in S_2 \quad \text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+2} \text{ ?} \end{aligned}$$

The usual approach via Magidor or Radin forcing produce a club with the same behavior and here we would like to have different ones on relatively big sets. The first of these two problems may be an easier one, since we need only GCH on S_1 and, so starting with the GCH in the ground model nothing special should be done on S_1. Note also that in view of the Silver Theorem (see [25], Sec. 1.8) we must have $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$ in models of Problem 5 and $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1+2}$ in those of Problem 4. Methods of [21] can be used to show that at least a strong cardinal is needed for constructing a model of Problem 4. By [22], the strength of at least PD is needed for constructing a model of Problem 5.

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