

# The Power Set Function

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## Abstract

We survey old and recent results on the problem of finding a complete set of rules describing the behavior of the power function, i.e. the function which takes a cardinal  $\kappa$  to the cardinality of its power  $2^\kappa$ .

## 1 Introduction

One of the central topics of Set Theory since Cantor was the study of the power function . The basic problem is to determine all the possible values of  $2^\kappa$  for a cardinal  $\kappa$ . Paul Cohen [1] proved the independence of the Continuum Hypothesis and invented the method of forcing. Shortly after, Easton [3] building on Cohen's results showed the function  $\kappa \longrightarrow 2^\kappa$ , for regular  $\kappa$ , can behave in any prescribed way consistent with König's Theorem. This reduces the study to singular cardinals. It turned out that the situation with powers of singular cardinals is much more involved. Thus, for example, a remarkable theorem of Silver [22] states that a singular cardinal of uncountable cofinality cannot be first to violate GCH. The Singular Cardinal Problem is the problem of finding a complete set of rules describing the behavior of the power function on singular cardinals. There are three main tools for dealing with the problem: *pcf*-theory, inner models theory and forcing involving large cardinals.

## 2 Classical results and Basic definitions

In 1938 Gödel proved the consistency of the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH) with the rest axioms of set theory. In 1963 Cohen proved the

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independence of AC and GCH. He showed, in particular, that  $2^{\aleph_0}$  can be arbitrary large. Shortly after Solovay proved that  $2^{\aleph_0}$  can take any value  $\lambda$  with  $cf(\lambda) > \aleph_0$ . The *cofinality* of a limit ordinal  $\alpha$  ( $cf(\alpha)$ ) is the least ordinal  $\beta \leq \alpha$  so that there is a function  $f : \beta \rightarrow \alpha$  with  $rng(f)$  unbounded in  $\alpha$ . A cardinal  $\kappa$  is called a *regular* if  $\kappa = cf(\kappa)$ . Otherwise a cardinal is called a *singular* cardinal. Thus, for example,  $\aleph_\delta$  is regular and  $\aleph_\omega$  is singular of cofinality  $\omega$ .

By a result of Easton [3], if we restrict ourselves to regular cardinals, then every class function

$F : \text{Regulars} \rightarrow \text{Cardinals}$

*satisfying*

(i)  $\kappa \leq \lambda$  implies  $F(\kappa) \leq F(\lambda)$

(ii)  $cf(F(\kappa)) > \kappa$  ( König's Theorem)

can be realized as a power function in a generic extension.

From this point we restrict ourselves to singular cardinals.

### 3 Restrictions on the power of singular cardinals

The Singular Cardinal Problem (SCP) is the problem of finding a complete set of rules describing the behavior of the power function on singular cardinals.

For singular cardinals there are more limitations. Thus

(c) (Bukovsky - Hechler) If  $\kappa$  is a singular and there is  $\gamma_0 < \kappa$  such that  $2^\gamma = 2^{\gamma_0}$  for every  $\gamma, \gamma_0 \leq \gamma < \kappa$ , then  $2^\kappa = 2^{\gamma_0}$ .

(d) (Silver) If  $\kappa$  is a singular strong limit cardinal of uncountable cofinality and  $2^\kappa > \kappa^+$  then  $\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$  contains a closed unbounded subset of  $\kappa$ .

A set  $C \subset \kappa$  is called a *closed unbounded* subset of  $\kappa$  iff

(1)  $\forall \alpha < \kappa \exists \beta \in C (\beta > \alpha)$  (*unbounded*)

(2)  $\forall \alpha < \kappa (C \cap \alpha \neq \emptyset \Rightarrow \sup(C \cap \alpha) \in C)$  (*closed*).

Subsets of  $\kappa$  containing a closed unbounded set form a filter over  $\kappa$  which is  $\kappa$  complete.

A positive for this filter sets are called *stationary*.

(e) (Galvin - Hajnal, Shelah) If  $\aleph_\delta$  is strong limit and  $\delta < \aleph_\delta$  then  $2^{\aleph_\delta} < \aleph_{2^{|\delta|}+}$

(f) (Shelah) It is possible to replace  $2^{|\delta|+}$  in (e) by  $|\delta|^{+4}$ .

(g) (Shelah) Let  $\aleph_\delta$  be the  $\omega_1$ -th fixed point of the  $\aleph$ -function. If it is a strong limit, then  $2^{\aleph_\delta} < \min((2^{\omega_1})^+, \omega_4\text{-th fixed point})$ .

A cardinal  $\kappa$  is called a *fixed point* of the  $\aleph$  function if  $\kappa = \aleph_\kappa$ .

It is unknown if 4 in (f) and in (g) can be reduced or just replaced by 1. One of the major questions in Cardinal Arithmetic asks if  $2^{\aleph_\omega}$  can be bigger than  $\aleph_{\omega_1}$  provided it is a strong limit.

We refer to the books by Jech [12] and by Shelah [23] for the proofs of the above results.

## 4 Inner models and large cardinals

There are other restrictions which depend on large cardinals. Thus the celebrated Covering Theorem of Jensen [2] implies that

For every singular strong limit cardinal  $\kappa$   $2^\kappa = \kappa^+$ , provided the universe is close to Gödel's model  $L$  ( precisely, if  $o\#$  does not exists, or ,equivalently , there is no elementary embedding from  $L$  into  $L$ ).

On the other hand, using large cardinals (initially supercompact cardinals were used [14]) it is possible to have the following.

(Prikrý-Silver, see [12]) :

$\kappa$  is a strong limit of cofinality  $\omega$  and  $2^\kappa > \kappa^+$ .

(Magidor [15],[16],[17]) :

(1) the same with  $\kappa$  of any uncountable cofinality.

(2) the same with  $\kappa = \aleph_\omega$ .

So, the answer to SCP may depend on presence of particular large cardinals. Hence, it is reasonable to study the possibilities for the power function level by level according to existence of particular large cardinals. There are generalizations of the Gödel model  $L$  which may include bigger and bigger large cardinals have nice combinatorial properties, satisfy GCH and are invariant under set forcing extensions. This models are called *Core Models*. We refer to the book by Zeman [25] for a recent account on this fundamental results.

The Singular Cardinals Problem can now be reformulated as follows:

Given a core model  $K$  with certain large cardinals. Which functions can be realized in extensions of  $K$  as power set functions , i.e. let  $F : Ord \longrightarrow Ord$  be a class function in  $K$ , is there an extension (generic) of  $K$  satisfying  $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$  for all ordinals  $\alpha$  ?

We will need few definitions.

An uncountable cardinal  $\kappa$  is called a *measurable* cardinal iff there is  $\mu : P(\kappa) \longrightarrow \{0, 1\}$  such that

(1)  $\forall \alpha < \kappa \mu(\{\alpha\}) = 0$

$$(2) \mu(\kappa) = 1$$

$$(3) A \subseteq B \implies \mu(A) \leq \mu(B)$$

$$(4) \forall \delta < \kappa \forall \{A_\nu | \nu < \delta\} \text{ subsets of } \kappa \text{ with } \mu(A_\nu) = 0 \mu(\cup A_\nu | \nu < \delta) = 0$$

If  $\kappa$  is a measurable, then it is possible always to find  $\mu$  with an additional property called *normality*:

(5) If  $\mu(A) = 1$  and  $f : A \longrightarrow \kappa, f(\alpha) < \alpha$  then there is a subset of  $A$  of measure one on which  $f$  is constant.

Further by measure we shall mean a normal measure, i.e. one satisfying (1)-(5).

A cardinal  $\kappa$  has the *Mitchell order*  $\geq 1$  ( $o(\kappa) \geq 1$ ) iff  $\kappa$  is a measurable.

A cardinal  $\kappa$  has the *Mitchell order*  $\geq 2$  ( $o(\kappa) \geq 2$ ) iff there is a measure over  $\kappa$  concentrating on measurable cardinals,

$$\text{i.e. } \mu(\{\alpha < \kappa | o(\alpha) \geq 1\}) = 1.$$

In a similar fashion we can continue further, but up to  $\kappa^{++}$  only. Just the total number of ultrafilters over  $\kappa$  under GCH is  $\kappa^{++}$ . In order to continue above this point, directed systems of ultrafilters called extenders are used. This way we can reach  $\kappa$  with  $o(\kappa) = Ord$ . Such  $\kappa$  is called a *strong cardinal*. Core models are well developed to the level of strong cardinal and much further. Almost all known consistency results on the Singular Cardinals Problem require large cardinals below the level of a strong cardinal.

## 5 Finite gaps

By results of Jensen [2], Dodd- Jensen [13], Mitchell [20], Shelah [23] and Gitik [5] nothing interesting in sense of SCP happens bellow the level of  $o(\kappa) = \kappa^{++}$ .

If there is  $n < \omega$  such that for every  $\alpha, o(\alpha) \leq \alpha^{+n}$ , then we have the following additional restrictions:

(1) (Gitik-Mitchell [10]) If  $\kappa$  is a singular strong limit and  $2^\kappa = \kappa^{+m}$  for some  $m > 1$ , then, in  $K, o(\kappa) \geq \kappa^{+m}$ . In particular,  $m \leq n$ .

(2) If  $\kappa$  is a singular cardinal of uncountable cofinality and for some  $m, 1 \leq m < \omega$   $\{\alpha < \kappa | 2^\alpha = \alpha^{+m}\}$  is stationry, then  $\{\alpha < \kappa | 2^\alpha = \alpha^{+m}\}$  contains a closed unbounded subset of  $\kappa$ .

By results of Merimovich [18] it looks like this are the only restrictions.

## 6 Uncountable cofinality case

Assume only that there is no inner model with a strong cardinal. Then we have the following restrictions:

(1) If  $\kappa$  is a singular strong limit cardinal of uncountable cofinality  $\delta$  and  $2^\kappa \geq \lambda > \kappa^+$ , where  $\lambda$  is not the successor of a cardinal of cofinality less than  $\kappa$ , then  $o(\kappa) \geq \lambda + \delta$ , if  $\delta > \omega_1$  or  $o(\kappa) \geq \lambda$ , if  $\delta = \omega_1$ .

(2) Let  $\kappa$  be a singular strong limit cardinal of uncountable cofinality  $\delta$  and let  $\tau < \delta$ . If  $A = \{\alpha < \kappa \mid cf\alpha > \omega, 2^\alpha = \alpha^{+\tau}\}$  is stationary, then  $A$  contains a closed unbounded subset of  $\kappa$ .

(3) If  $\delta < \aleph_\delta$ ,  $\aleph_\delta$  strong limit then  $2^{\aleph_\delta} < \aleph_{|\delta|^+}$

(This was improved recently by R.Schindler to a Woodin cardinal, provided there is a measurable above  $\delta$ ).

(4) Let  $\aleph_\delta$  be the  $\omega_1$ -th fixed point of the  $\aleph$ -function. If it is a strong limit cardinal then  $2^{\aleph_\delta} < \omega_2$ -th fixed point.

(5) If  $a$  is an uncountable set of regular cardinals with  $\min(a) > 2^{|a|^+ + \aleph_2}$ , then  $|pcf(a)| = |a|$ .

Where  $pcf(a) = \{cf(\prod a/D) \mid D \text{ is an ultrafilter over } a\}$ .

It is a major problem of Cardinal Arithmetic if it is possible to have a set of regular cardinals  $a$  with  $\min(a) > |a|$  such that  $|pcf(a)| > |a|$ . The results above were proved in Gitik-Mitchell [10], and in [7]. It is unknown if there is no further restrictions in this case (i.e. singulars of uncountable cofinality under the assumption that there is no inner model with a strong cardinal). Some local cases were checked by Segal [21] and Merimovich [19].

## 7 Countable cofinality case

In this section we review some more recent results dealing with countable cofinality. First suppose that

$$(\forall n < \omega \exists \alpha o(\alpha) = \alpha^{+n}), \text{ but } \neg(\exists \alpha o(\alpha) = \alpha^{+\omega}).$$

Then the following holds:

Let  $\kappa$  be a cardinal of countable cofinality such that for every  $n < \omega$   $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+n}\}$  is unbounded in  $\kappa$ . Then for every  $\lambda \geq \kappa^+$  there is cardinals preserving generic extension satisfying "  $\kappa$  is a strong limit and  $2^\kappa \geq \lambda$  ".

So the gap between a singular and its power can already be arbitrary large. But by [7]:

If  $2^\kappa \geq \kappa^{+\delta}$  for  $\delta \geq \omega_1$ , then GCH cannot hold below  $\kappa$ .

(Actually , GCH can hold if the gap is at most countable [8]).

We do not know if "pcf( $a$ ) uncountable for a countable  $a$ " is stronger than the assumption above. If we require also GCH below, then it is.

Once one likes to have uncountable gaps between a singular cardinal and its power together with GCH below , then the following results provide this and are sharp. The proofs are spread trough papers [8], [9], [10].

Suppose that  $\kappa > \delta \geq \aleph_0$ ,  $\delta$  is a cardinal,  $2^\kappa \geq \kappa^{+\delta}$ ,  $cf\kappa = \aleph_0$  and GCH below  $\kappa$ . Then

(i)  $cf\delta = \aleph_0$  implies (that in the core model) for every  $\tau < \delta$   $\{\alpha < \kappa | o(\alpha) \geq \alpha^{+\tau}\}$  is unbounded in  $\kappa$ .

(ii)  $cf\delta > \aleph_0$  implies (in the core model)  $o(\kappa) \geq \kappa^{+\delta+1} + 1$  or  $\{\alpha < \kappa | o(\alpha) \geq \alpha^{+\delta+1}\}$  is unbounded in  $\kappa$ .

Finally let us consider the following large cardinal:

$\kappa$  is singular of cofinality  $\omega$  and for every  $\tau < \kappa$   $\{\alpha < \kappa | o(\alpha) \geq \alpha^{+\tau}\}$  is unbounded in  $\kappa$ .

Under this assumption it is possible to blow up the power of  $\kappa$  arbitrary high preserving GCH below  $\kappa$ . Also, it is possible to turn  $\kappa$  into the first fixed point of the  $\aleph$  function, see [6]. This answers Question ( $\gamma$ ) from the Shelah book on cardinal arithmetic [23]. What are the possibilities for the power function under the assumption above? First in order to be able to deal with cardinals above  $\kappa$ , let us replace it by a global one :

For every  $\tau$  there is  $\alpha$   $o(\alpha) \geq \alpha^{+\tau}$ .

We do not know the status of " pcf of a countable set uncountable" , but other limitations like

(1)  $\aleph_\omega$  strong limit implies  $2^{\aleph_\omega} < \aleph_{\omega_1}$

(2) If  $\kappa$  is a singular of uncountable cofinality then either  $\{\alpha < \kappa | 2^\alpha \geq \alpha^+\}$  or  $\{\alpha < \kappa | 2^\alpha > \alpha^+\}$  contains a closed unbounded subset of  $\kappa$

are true below strong cardinal. By recent result [11] the negation of the second assumption implies initially unrelated statement - Projective Determinacy. We refer to the books by A. Kanamori [14] and H. Woodin [24] on this subject.

We conjecture that there is no other limitations, i.e. (1) with  $\aleph_\omega$  replaced by  $\aleph_\delta$  for  $\delta < \aleph_\delta$ , (2) and the classical ones.

## 8 One idea

Let us conclude with a sketch of one basic idea which is crucial for the forcing constructions in the countable cofinality case. Let  $U$  be a  $\kappa$  complete nontrivial ultrafilter over  $\kappa$  (say, in  $K$ ). A sequence  $\langle \delta_n | n < \omega \rangle$  is called a *Prikry sequence* for  $U$  iff for each  $A \in U \exists n_0 \forall n \geq n_0 \delta_n \in A$ .

Suppose now that  $\kappa$  is a strong limit singular cardinal of cofinality  $\omega$  and  $2^\kappa = \kappa^{++}$ . Then, usually (by [5], [10]), we will have a sequence  $\langle U_\alpha | \alpha < \kappa^{++} \rangle$  of ultrafilters in  $K$  and a sequence  $\langle \delta_{\alpha,n} | \alpha < \kappa^{++}, n < \omega \rangle$  so that

- (1)  $\alpha < \beta \implies \exists n_0 \forall n \geq n_0 \delta_{\alpha,n} < \delta_{\beta,n}$
- (2)  $\langle \delta_{\alpha,n} | n < \omega \rangle$  is a Prikry sequence for  $U_\alpha$ .

Ultrafilters  $U_\alpha$  are different here. So each sequence  $\langle \delta_{\alpha,n} | n < \omega \rangle$  relates to unique ultrafilter from the list. But once  $\kappa^{++}$  is replaced by  $\kappa^{+++}$ , the corresponding sequence of ultrafilters  $\langle U_\alpha | \alpha < \kappa^{+++} \rangle$  will have different  $\alpha$  and  $\beta$ ,  $\kappa^{++} < \alpha < \beta < \kappa^{+++}$  with  $U_\alpha = U_\beta$ . Then a certain Prikry sequence  $\langle \delta_n | n < \omega \rangle$  may pretend to correspond to both  $U_\alpha$  and  $U_\beta$ . In order to decide, we will need a Prikry sequence for some  $U_\gamma$  with  $\gamma < \kappa^{++}$  (more precisely, if  $f_\beta$  is the canonical one to one correspondence in  $K$  between  $\kappa^{++}$  and  $\beta$  then  $f_\beta(\gamma) = \alpha$ ). Dealing with  $\kappa^{+4}$  we will need go down twice, first to  $\kappa^{+3}$  and after that to  $\kappa^{++}$ . In general, for  $n, 3 \leq n < \omega$ ,  $n - 2$ -many times. Certainly, it is impossible to go down infinitely many times, but instead we replace the fixed  $\kappa$  by an increasing sequence  $\langle \kappa_n | n < \omega \rangle$  with each  $\kappa_n$  carrying  $\kappa_n^{+n+3}$  many ultrafilters. Now it turns out to be possible to add  $\omega$  - sequences with no assignment to ultrafilters. Just the number of steps needed to produce the assignment is  $\omega$  which is not enough for sequences of the length  $\omega$ .

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