Intermediate Models of Magidor-Radin Forcing-Part I

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Abstract

We continue the work done in [3],[1]. We prove that for every set A in a Magidor-Radin generic extension using a coherent sequence such that $o^{\vec{U}}(\kappa) < \kappa$, there is a subset C' of the Magidor club such that $V[A] = V[C']$. Also we classify all intermediate ZFC transitive models $V \subseteq M \subseteq V[G]$.

1 Introduction

In this paper we consider the version of Magidor-Radin forcing for $o^{\vec{U}}(\kappa) \leq \kappa$, but prove results for $o^{\vec{U}}(\kappa) < \kappa$. Section (2), will also be relevant to the forcing in Part II.

In [1], we assumed that $o^{\vec{U}}(\kappa) < \delta_0 := \min(\alpha \mid 0 < o^{\vec{U}}(\alpha))$. When we let $o^{\vec{U}}(\kappa) \geq \delta_0$, we might loss completness for some of the pairs in a condition p. For example, if $p = \langle\langle\delta_0, A_0\rangle, \langle\kappa, A_1\rangle\rangle$, we wont be able to take in account all the measures on κ, since there are δ_0 many of them and only δ_0-completness. The proof is by induction on κ. We will be to split $\mathbb{M}[\vec{U}]$ to the part below $o^{\vec{U}}(\kappa)$ and above it, then some but not all of the arguments of [1] generalizes.

The main result we obtain in this paper is:

Theorem 1.1 *Let $\vec{U}$ be a coherent sequence such that $o^{\vec{U}}(\kappa) < \kappa$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$, and every $A \in V[G]$, there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

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In the theorem, C_G denotes the generic Magidor-Radin club derived from G.

Note that the classification we had in [1] for models of the form $V[C']$, do not extend, even if $o^{\vec{U}}(\kappa) = \delta_0$.

Example 1.2 Consider C_G such that $C_G(\omega) = \delta_0$ and $o^{\vec{U}}(\kappa) = \delta_0$. Then in $V[G]$ we have the following sequence $C' = \langle C_G(C_G(n)) \mid n < \omega\rangle$ of points of the generic C_G which is determine by the first Prikry sequence at δ_0.

Then $I(C', C_G) = \langle C_G(n) \mid n < \omega\rangle \notin V$, where $I(X, Y)$ is the indices of $X \subseteq Y$ in the increasing enumeration of Y.

The forcing $\mathbb{M}_I[\vec{U}]$ which was defined in [1], is no longer defined in V since $I \notin V$.

In this case, we will add points to C', which are simply $\langle C_G(n) \mid n < \omega\rangle$, then the forcing will be a two step iteration. The first will be to add the Prikry sequence $\langle C_G(n) \mid n < \omega\rangle$, then the second will be a Diagonal Prikry forcing adding point from the measures $\langle U(\kappa, C_G(n)) \mid n < \omega\rangle$, which is of the form $M_I[\vec{U}]$.

Generally, we will define forcing $\mathbb{M}_f[\vec{U}]$, which are not subforcing of $\mathbb{M}[\vec{U}]$, but are a natural diagonal generalization of $\mathbb{M}[\vec{U}]$ and a bit closer to Magidor's original formulation in [5].

The classification of models is given by the following theorem:

Theorem 1.3 *Assume that for every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every V-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and every transitive ZFC intermediate model $V \subseteq M \subseteq V[G]$, there is a closed subset $C_{fin} \subseteq C_G$ such that:*

1. *$M = V[C_{fin}]$.*
2. *There is a finite iteration $\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_2}}[\vec{U}] ... * \mathbb{M}_{\underset{\sim}{f_n}}[\vec{U}]$, and a V-generic H^* filter for $\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_2}}[\vec{U}] ... * \mathbb{M}_{\underset{\sim}{f_n}}[\vec{U}]$ such that $V[H^*] = V[C_{fin}] = M$.*

2 Basic Definitions and Preliminaries

We will follow the description of Magidor forcing as presented in [2].

Let $\vec{U} = \langle U(\alpha, \beta) \mid \alpha \leq \kappa \ , \beta < o^{\vec{U}}(\alpha)\rangle$ be a coherent sequence. For every $\alpha \leq \kappa$, denote

$$\cap\vec{U}(\alpha) = \bigcap_{i < o^{\vec{U}}(\alpha)} U(\alpha, i)$$

Definition 2.1 $\mathbb{M}[\vec{U}]$ consist of elements p of the form $p = \langle t_1, ..., t_n, \langle\kappa, B\rangle\rangle$. For every $1 \leq i \leq n$, t_i is either an ordinal κ_i if $o^{\vec{U}}(\kappa_i) = 0$ or a pair $\langle\kappa_i, B_i\rangle$ if $o^{\vec{U}}(\kappa_i) > 0$.

1. $B \in \cap\vec{U}(\kappa)$, $\min(B) > \kappa_n$.
2. For every $1 \leq i \leq n$.
   (a) $\langle\kappa_1, ..., \kappa_n\rangle \in [\kappa]^{<\omega}$ (increasing finite sequence below κ).
   (b) $B_i \in \cap\vec{U}(\kappa_i)$.
   (c) $\min(B_i) > \kappa_{i-1}$ $(i > 1)$.

Definition 2.2 For $p = \langle t_1, t_2, ..., t_n, \langle\kappa, B\rangle\rangle, q = \langle s_1, ..., s_m, \langle\kappa, C\rangle\rangle \in \mathbb{M}[\vec{U}]$, define $p \leq q$ (q extends p) iff:

1. $n \leq m$.
2. $B \supseteq C$.
3. $\exists 1 \leq i_1 < ... < i_n \leq m$ such that for every $1 \leq j \leq m$:
   (a) If $\exists 1 \leq r \leq n$ such that $i_r = j$ then $\kappa(t_r) = \kappa(s_{i_r})$ and $C(s_{i_r}) \subseteq B(t_r)$.
   (b) Otherwise $\exists$ $1 \leq r \leq n+1$ such that $i_{r-1} < j < i_r$ then
      i. $\kappa(s_j) \in B(t_r)$.
      ii. $B(s_j) \subseteq B(t_r) \cap \kappa(s_j)$.
      iii. $o^{\vec{U}}(s_j) < o^{\vec{U}}(t_r)$.

We also use "p directly extends q", $p \leq^* q$ if:

1. $p \leq q$
2. $n = m$

Let us add some notation, for a pair $t = \langle\alpha, X\rangle$ we denote by $\kappa(t) = \alpha$, $B(t) = X$. If $t = \alpha$ is an ordinal then $\kappa(t) = \alpha$ and $B(t) = \emptyset$.

For a condition $p = \langle t_1, ..., t_n, \langle\kappa, B\rangle\rangle \in \mathbb{M}[\vec{U}]$ we denote $n = l(p)$, $p_i = t_i$, $B_i(p) = B(t_i)$ and $\kappa_i(p) = \kappa(t_i)$ for any $1 \leq i \leq l(p)$, $t_{l(p)+1} = \langle\kappa, B\rangle$, $t_0 = 0$. Also denote

$$\kappa(p) = \{\kappa_i(p) \mid i \leq l(p)\} \text{ and } B(p) = \bigcup_{i \leq l(p)+1} B_i(p)$$

Remark 2.3 Condition 3.b.iii is not essential, since the set

$$\left\{p \in \mathbb{M}[\vec{U}] \mid \forall i \leq l(p)+1.\forall\alpha \in B_i(p).o^{\vec{U}}(\alpha) < o^{\vec{U}}(\kappa_i(p))\right\}$$

is a dense subset of $\mathbb{M}[\vec{U}]$ and the order between any two elements of this dense subsets automatically satisfy 3.b.iii.

Definition 2.4 Let $p \in \mathbb{M}[\vec{U}]$. For every $i \leq l(p)+1$, and $\alpha \in B_i(p)$ with $o^{\vec{U}}(\alpha) > 0$, define

$$p^\frown\langle\alpha\rangle = \langle p_1, ..., p_{i-1}, \langle\alpha, B_i(p)\cap\alpha\rangle, \langle\kappa_i(p), B_i(p)\setminus(\alpha+1)\rangle, p_{i+1}, ..., p_{l(p)+1}\rangle$$

If $o^{\vec{U}}(\alpha) = 0$, define

$$p^\frown\langle\alpha\rangle = \langle p_1, ..., p_{i-1}, \alpha, \langle\kappa_i(p), B_i(p)\setminus(\alpha+1)\rangle, ..., p_{l(p)+1}\rangle$$

For $\langle\alpha_1, ..., \alpha_n\rangle \in [\kappa]^{<\omega}$ define recursively,

$$p^\frown\langle\alpha_1, ..., \alpha_n\rangle = (p^\frown\langle\alpha_1, ..., \alpha_{n-1}\rangle)^\frown\langle\alpha_n\rangle$$

Proposition 2.5 *Let $p \in \mathbb{M}[\vec{U}]$. If $p^\frown\vec{\alpha} \in \mathbb{M}[\vec{U}]$, then it is the minimal extension of p with stem*

$$\kappa(p) \cup \{\vec{\alpha}_1, ..., \vec{\alpha}_{|\vec{\alpha}|}\}$$

Moreover, $p^\frown\vec{\alpha} \in \mathbb{M}[\vec{U}]$ iff for every $i \leq |\vec{\alpha}|$ there is $j \leq l(p)$ such that:

1. $\vec{\alpha}_i \in (\kappa_j(p), \kappa_{j+1}(p))$.
2. $o^{\vec{U}}(\vec{\alpha}_i) < o^{\vec{U}}(\kappa_{j+1})$.
3. $B_{j+1}(p) \cap \vec{\alpha}_i \in \cap\vec{U}(\vec{\alpha}_i)$.

■

Note that if we add a pair of the form $\langle\alpha, B\cap\alpha\rangle$ then in $B\cap\alpha$ there might be many ordinals which are irrelevant to the forcing. Namely, ordinals β with $o^{\vec{U}}(\beta) \geq o^{\vec{U}}(\alpha)$, such ordinals cannot be added to the sequence.

Definition 2.6 Let $p \in \mathbb{M}[\vec{U}]$, define for every $i \leq l(p)$

$$p \restriction \kappa_i(p) = \langle p_1, ..., p_i\rangle \text{ and } p \restriction (\kappa_i(p), \kappa) = \langle p_{i+1}, ..., p_{l(p)+1}\rangle$$

Also, for λ with $o^{\vec{U}}(\lambda) > 0$ define

$$\mathbb{M}[\vec{U}] \restriction \lambda = \{p \restriction \lambda \mid p \in \mathbb{M}[\vec{U}] \text{ and } \lambda \text{ apears in } p\}$$

$$\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa) = \{p \restriction (\lambda, \kappa) \mid p \in \mathbb{M}[\vec{U}] \text{ and } \lambda \text{ apears in } p\}$$

Note that $\mathbb{M}[\vec{U}] \restriction \lambda$ is just Magidor forcing on λ and $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ is a subset of $\mathbb{M}[\vec{U}]$. The following decomposition is straight forward.

Proposition 2.7 *Let $p \in \mathbb{M}[\vec{U}]$ and $\langle \lambda, B \rangle$ a pair in p. Then*

$$\mathbb{M}[\vec{U}]/p \simeq \Big(\mathbb{M}[\vec{U}] \restriction \lambda\Big)/\Big(p \restriction \lambda\Big) \times \Big(\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)\Big)/\Big(p \restriction (\lambda, \kappa)\Big)$$

Proposition 2.8 *Let $p \in \mathbb{M}[\vec{U}]$ and $\langle \lambda, B \rangle$ a pair in p. Then the order $\leq^*$ in the forcing $\Big(\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)\Big)/\Big(p \restriction (\lambda, \kappa)\Big)$ is δ-directed where $\delta = \min(\nu > \lambda \mid o^{\vec{U}}(\nu) > 0)$. Meaning that for every $X \subseteq \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ such that $|X| < \delta$ and for every $q \in X$, $p \leq^* q$, there is an $\leq^*$-upper bound for X.*

Lemma 2.9 $\mathbb{M}[\vec{U}]$ *satisfy k^+-c.c.*

The following is known as the Prikry condition:

Lemma 2.10 $\mathbb{M}[\vec{U}]$ *satisfy the Prikry condition i.e. for any statement in the forcing language σ and any $p \in \mathbb{M}[\vec{U}]$ there is $p \leq^* p^*$ such that $p^* || \sigma$ i.e. either $p^* \Vdash \sigma$ or $p \Vdash \neg\sigma$.*

The next lemma can be found in [5]:

Lemma 2.11 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic and suppose that $A \in V[G]$ is such that $A \subseteq V_\alpha$. Let $p \in G$ and $\langle \lambda, B \rangle$ a pair in p such that $\alpha < \lambda$, then $A \in V[G \restriction \lambda]$.*

Proof. Consider the decomposition 2.7 $p = \langle q, r \rangle$, where $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and $r \in \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ Work in $V[G \restriction \lambda]$, Let $\underaccent{\sim}{A}$ be a $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$-name for A. For every $x \in V_\alpha$ use the Prikry condition 2.10, to find $r \leq^* r_x$ such that r_x decide the statement $r \in \underaccent{\sim}{A}$. By definition of λ and proposition 2.14, the forcing $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ is $|V_\alpha|^+$-directed with the $\leq^*$ order. Hence there is $r \leq^* r^*$ such that $p_x \leq^* p^*$ for every $x \in V_\alpha$. By density, we can find such $r^* \in G \restriction (\lambda, \kappa)$. It follows that $A = \{x \in V_\alpha \mid r^* \Vdash x \in \underaccent{\sim}{A}\}$ is definable in $V[G \restriction \lambda]$.$\blacksquare$

Corollary 2.12 $\mathbb{M}[\vec{U}]$ *preserves all cardinals.*

Definition 2.13 Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic, define the *Magidor club*

$$C_G = \{\nu \mid \exists\ A \exists p \in G \ s.t.\ \langle \nu, A \rangle \in p\}$$

We will abuse notation by sometimes considering C_G as a the canonical enumeration of the set C_G. The set C_G is closed and unbounded in κ, therefore, the order type of C_G determines the cofinality of κ in $V[G]$. The next propositions can be found in [2].

Proposition 2.14 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic. Then G can be reconstructed from C_G as follows*

$$G = \{p \in \mathbb{M}[\vec{U}] \mid (\kappa(p) \subseteq C_G) \wedge (C_G \setminus \kappa(p) \subseteq B(p))\}$$

In particular $V[G] = V[C_G]$.

Proposition 2.15 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic.*

1. *C_G is a club at κ.*
2. *For every $\delta \in C_G$, $o^{\vec{U}}(\delta) > 0$ iff $\delta \in Lim(C_G)$.*
3. *For every $\delta \in Lim(C_G)$, and every $A \in \cap\vec{U}(\delta)$, there is $\xi < \delta$ such that $C_G \setminus \xi \subseteq A$.*
4. *If $\langle \delta_i \mid i < \theta\rangle$ is an increasing sequence of elements of C_G, let $\delta^* = \sup_{i<\theta}\delta_i$, then $o^{\vec{U}}(\delta^*) \geq \limsup_{i<\theta} o^{\vec{U}}(\delta_i) + 1$.*[1]
5. *Let $\delta \in Lim(C_G)$ and let A be a positive set, $A \in (\cap\vec{U}(\delta))^+$. i.e. $\kappa \setminus A \notin \cap\vec{U}(\kappa)$.* [2] *Then,* $\sup(A \cap C_G) = \delta$.
6. *If $A \subseteq V_\alpha$, then $A \in V[C_G \cap \lambda]$, where $\lambda = \max(Lim(C_G) \cap \alpha + 1)$.*
7. *For every V-regular cardinal α, if $cf^{V[G]}(\alpha) < \alpha$ then $\alpha \in Lim(C_G)$.*

Proof. $(1), (2), (3)$ can be found in [2].

To see (4), use closure of C_G, and find $q \in G$ such that δ^* appears in q. Since there are only finitely many ordinals in q, there is some $i < \theta$ such that for every $j > i$, δ_j does not appear in q. By 2.2, since every such δ_j appear in some $q_j \in G$ which is compatible with q, $o^{\vec{U}}(\delta_j) < o^{\vec{U}}(\delta^*)$. Hence

$$\limsup_{j<\theta} o^{\vec{U}}(\delta_j) + 1 \leq \sup(\limsup_{i<j<\theta} o^{\vec{U}}(\delta_j) + 1 \leq o^{\vec{U}}(\delta^*)$$

For (5), let $\rho < \delta$. Each condition p, such that $\delta = \kappa_i(p)$ for some $i \leq l(p) + 1$, must satisfy that $\sup(A \cap B_i(p)) = \delta$. Hence we can extend p using an element of $A \cap B_i(p)$ above ρ. By density, $\sup(A \cap C_G) \geq \rho$. Since ρ is general, $\sup(A \cap C_G) = \delta$.

(6) is a direct corollary of 2.11. As for (7), assume that $cf^{V[G]}(\alpha) < \alpha$, and let $X \subseteq \alpha$ be a club such that $\mathrm{otp}(X) = cf^{V[G]}(\alpha)$. Then $X \in V[G] \setminus V$. Let $\lambda = \max(Lim(C_G) \cap \alpha + 1)$, then $\lambda \leq \alpha$. By (6), $X \in V[C_G \cap \lambda]$. Toward a contradiction, assume that $\lambda < \alpha$, The the forcing $\mathbb{M}[\vec{U}] \restriction \lambda$ is α-c.c., but $cf^{V[C_G \cap \lambda]}(\alpha) < \alpha$, contradiction. ■

The Mathias-like criteria for Magidor forcing is due to Mitchell [6]:

[1] For a sequence of ordinals $\langle \rho_j \mid j < \gamma\rangle$, $\limsup_{j<\gamma} \rho_j = \min(\sup_{i<j<\gamma}\rho_j \mid i < \gamma)$.

[2] Equivalently, if there is some $i < o^{\vec{U}}(\kappa)$ such that $A \in U(\kappa, i)$.

Theorem 2.16 *Let U be a coherent sequence and assume that $c : \alpha \to \kappa$ is an increasing function. Then c is $\mathbb{M}[\vec{U}]$ generic iff:*

1. *c is continuous.*
2. *$c \restriction \beta$ is $\mathbb{M}[\vec{U} \restriction \beta]$ generic for every $\beta \in \mathrm{Lim}(\alpha)$.*
3. *$X \in \cap\vec{U}(\kappa)$ iff $\exists \beta < \kappa\ c \setminus \beta \subseteq X$.*

An equivalent formulation of the Mathias criteria is to require that for every $\beta \in \mathrm{Lim}(\alpha)$, and for every $X \in \cap\vec{U}(c(\beta))$, there is $\xi < \beta$ such that $c''(\xi, \beta) \subseteq X$.

For an additional proof of 2.16, We refer the reader to the last section, where we define a forcing notion $\mathbb{M}_f[\vec{U}]$, which generalizes $\mathbb{M}[\vec{U}]$, and prove in 5.9 a Mathias-like criteria for it.

Proposition 2.17 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be V-generic filter and C_G the corresponding Magidor sequence. Let $p \in G$, then for every $i \leq l(p) + 1$*

1. *If $o^{\vec{U}}(\kappa_i(p)) \leq \kappa_i(p)$,*
$$\mathrm{otp}([\kappa_{i-1}(p), \kappa_i(p)) \cap C_G) = \omega^{o^{\vec{U}}(\kappa_i(p))}$$
2. *If $o^{\vec{U}}(\kappa_i(p)) \geq \kappa_i(p)$, then*
$$\mathrm{otp}([\kappa_{i-1}(p), \kappa_i(p)) \cap C_G) = \kappa_i(p)$$

Proof. we prove (1) by induction on $\kappa_i(p)$. If $\kappa_i(p) = 0$, then $C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) = \{\kappa_{i-1}(p)\}$. Hence
$$\mathrm{otp}(C_G \cap [\kappa_{i-1}(p), \kappa_i(p))) = 1 = \omega^0 = \omega^{o^{\vec{U}}(\kappa_i(p))}$$

Assume the lemma holds for any $\delta < \kappa_i(p)$. If $o^{\vec{U}}(\kappa_i(p)) = \alpha + 1 \leq \kappa_i(p)$, then the set $X = \{\beta < \kappa_i(p) \mid o^{\vec{U}}(\beta) = \alpha\} \in U(\kappa_i(p), \alpha)$, hence by proposition 2.15,
$$\sup(X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p))) = \kappa_i(p)$$

We claim that $\mathrm{otp}(X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) = \omega$. Otherwise, let $\rho < \kappa_i(p)$ be such that ρ is a limit point of $X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p))$. Again by proposition 2.15,
$$o^{\vec{U}}(\rho) \geq \limsup(o^{\vec{U}}(\xi) \mid \xi \in X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p))) = \alpha + 1$$

Contradicting 2.2. Let $\langle \delta_n \mid n<\omega\rangle$ be the increasing enumeration of $X\cap C_G\cap[\kappa_{i-1}(p),\kappa_i(p))$. By induction hypothesis, for every $n<\omega$, $\mathrm{otp}(C_G\cap[\delta_n,\delta_{n+1}))=\omega^\alpha$. Hence,

$$\mathrm{otp}(C_G\cap[\kappa_{i-1}(p),\kappa_i(p))=\omega^{\alpha+1}$$

For limit $o^{\vec{U}}(\kappa_i(p))$, use proposition 2.15.5, to see that the sequence $\langle \delta_\alpha \mid \alpha< o^{\vec{U}}(\kappa_i(p))\rangle$ where

$$\delta_\alpha=\min(\rho\in C_G\cap[\kappa_{i-1}(p),\kappa_i(p))\mid o^{\vec{U}}(\rho)=\alpha)$$

is well defined. $x=\sup(\delta_\alpha\mid\alpha<\theta)\leq\kappa_i(p)$ is an element of C_G, and by proposition 2.15.4, $o^{\vec{U}}(x)\geq o^{\vec{U}}(\kappa_i(p))$, hence $x=\kappa_i(p)$. For every $\alpha<o^{\vec{U}}(\kappa_i(p))$, $\mathrm{otp}(C_G\cap[\kappa_i(p),\delta_\alpha))=\omega^\alpha$, since $p^\frown\langle\delta_\alpha\rangle\in G$ and by induction hypothesis. It follows that

$$\mathrm{otp}(C_G\cap[\kappa_{i-1}(p),\kappa_i(p))=\sup_{\alpha<o^{\vec{U}}(\kappa_i(p))}(\mathrm{otp}(C_G\cap[\kappa_{i-1}(p),\delta_\alpha))=\sup_{\alpha<o^{\vec{U}}(\kappa_i(p))}\omega^\alpha=\omega^{o^{\vec{U}}(\kappa_i(p))}$$

For (2), use (1), and the limit stage to conclude that if $o^{\vec{U}}(\kappa_i(p))=\kappa_i(p)$, then

$$\mathrm{otp}(C_G\cap[\kappa_{i-1}(p),\kappa_i(p))=\kappa_i(p)$$

If $o^{\vec{U}}(\kappa_i(p))>\kappa_i(p)$, then $\{\alpha<\kappa_i(p))\mid o^{\vec{U}}(\alpha)=\alpha\}\in U(\kappa_i(p),\kappa_i(p))$, hence by proposition 2.15, there are unboundedly many $\alpha\in C_G\cap[\kappa_{i-1}(p),\kappa_i(p))=:Y$ such that $o^{\vec{U}}(\alpha)=\alpha$. Hence

$$\kappa_i(p)=\sup(Y)=\sup(\mathrm{otp}(C_G\cap[\kappa_{i-1}(p),\alpha)\mid\alpha\in Y)\leq\kappa_i(p)$$

So equality holds.■

Proposition 2.17 suggest a connection between the index in C_G of ordinals appearing in p and Cantor normal form.

Definition 2.18 Let $p\in G$. For each $i\leq l(p)$ define

$$\gamma_i(p)=\sum_{j=1}^{i}\omega^{o^{\vec{U}}(\kappa_j(p))}$$

Corollary 2.19 *Let G be $\mathbb{M}[\vec{U}]$-generic and C_G the corresponding Magidor sequence. Let $p\in G$, then for every $1\leq i\leq l(p)$*

$$p\Vdash \underset{\sim}{C}_G(\gamma_i(p))=\kappa(t_i)$$

Proof. This is directly from 2.17.■

For more details and basic properties of Magidor forcing see [5],[2] or [1].

We are going to handle subsequences of the generic club, the following simple definition will turn out being usefull.

Definition 2.20 Let X, X' be sets of ordinals such that $X' \subseteq X \subseteq On$. Let $\alpha = otp(X, \in)$ be the order type of X and $\phi : \alpha \to X$ be the order isomorphism witnessing it. The indices of X' in X are

$$I(X', X) = \phi^{-1''} X' = \{\beta < \alpha \mid \phi(\beta) \in X'\}$$

In the last part of the proof we will need the definition of quotient forcing.

Definition 2.21 Let $\underset{\sim}{C}'$ be a $\mathbb{M}[\vec{U}]$-name such that $\underset{\sim}{C}'_G = C'$. Define $\mathbb{P}_{\underset{\sim}{C}'}$, the complete subalgebra of $RO(\mathbb{M}[\vec{U}])$ generated by the conditions $X = \{||\alpha \in \underset{\sim}{C}'|| \mid \alpha < \kappa\}$.

By [4, 15.42], $V[C'] = V[H]$ for some V-generic filter H of $\mathbb{P}_{\underset{\sim}{C}'}$. In fact

$$C' = \{\alpha < \kappa \mid ||\alpha \in \underset{\sim}{C}'|| \in X \cap H\}$$

Definition 2.22 Define the function $\pi : \mathbb{M}[\vec{U}] \to \mathbb{P}_{\underset{\sim}{C}'}$ by

$$\pi(p) = \inf(b \in \mathbb{P}_{\underset{\sim}{C}'} \mid b \geq p)$$

It not hard to check that π is a projection i.e.

1. π is order preserving.
2. $\forall p \in \mathbb{M}[\vec{U}] \forall \pi(p) \leq q \exists p' \geq p. \pi(p') \geq q$.
3. $Im(\pi)$ is dense in $\mathbb{P}_{\underset{\sim}{C}'}$.

Definition 2.23 Let $\pi : \mathbb{P} \to \mathbb{Q}$ be any projection, let $H \subseteq \mathbb{Q}$ be V-generic, define

$$\mathbb{P}/H = \pi^{-1''} H$$

We abuse notation by defining $\mathbb{M}[\vec{U}]/C' = \mathbb{M}[\vec{U}]/H$, where H is some generic for $\mathbb{P}_{\underset{\sim}{C}'}$ such that $V[H] = V[C']$. It is known that if G is $V[C']$-generic for $\mathbb{M}[\vec{U}]/C'$ then G is V generic for $\mathbb{M}[\vec{U}]$ and $\pi'' G = H$, hence $V[G] = V[C'][G]$.

3 Magidor forcing with $o^{\vec{U}}(\kappa) \leq \kappa$

Proposition 3.1 *Assume that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic filter, and let $p \in G$. Then $otp(C_G \cap (\kappa_{l(p)}(p), \kappa)) = \omega^{o^{\vec{U}}(\kappa)}$. Hence, $cf^{V[G]}(\kappa) = cf^{V[G]}(\omega^{o^{\vec{U}}(\kappa)})$.*

Corollary 3.2 *1. If $o^{\vec{U}}(\kappa)<\kappa$, then κ is singular in $V[G]$.*

2. If $o^{\vec{U}}(\kappa)=\kappa$, then $cf^{V[G]}(\kappa)=\omega$.

Proof. (1) is direct from proposition 3.1. For (2), The set $E=\{\alpha<\kappa\mid o^{\vec{U}}(\alpha)<\alpha\}\in\cap\vec{U}(\kappa)$. Hence, by proposition 2.15 find $\rho<\kappa$ such that $C_G\setminus\rho\subseteq E$. In $V[G]$ consider the sequence: $\alpha_0=\min(C_G\setminus\rho)$, then $\alpha_{n+1}=C_G(\alpha_n)$. This is a well defined sequence of ordinals below κ since $\mathrm{otp}(C_G)=\kappa$. Also, since $\{\alpha<\kappa\mid\omega^\alpha=\alpha\}\in\cap\vec{U}(\kappa)$, there is $n<\omega$, such that for every $m\geq n$, $o^{\vec{U}}(\alpha_{m+1})=\alpha_m$.

To see that $\alpha^*:=\sup_{n<\omega}\alpha_n=\kappa$, assume otherwise, then by closure of C_G, $\alpha^*\in C_G$. Also $\alpha^*>\rho$, hence $o^{\vec{U}}(\alpha^*)<\alpha^*$. By proposition 2.15.4,

$$o^{\vec{U}}(\alpha^*)\geq\limsup_{n<\omega}o^{\vec{U}}(\alpha_n)+1=\sup_{n<\omega}\alpha_n=\alpha^*$$

contradiction.$\blacksquare$

If $o^{\vec{U}}(\kappa)\leq\kappa$. We can decompose every set $A\in\cap\vec{U}(\kappa)$ in a very canonical way:

Proposition 3.3 *Assume that $o^{\vec{U}}(\kappa)\leq\kappa$. Let $A\in\cap\vec{U}(\kappa)$.*

1. *For every $i<\kappa$ define $A_i=\{\nu\in A\mid o^{\vec{U}}(\nu)=i\}$. Then $A=\biguplus_{i<\kappa}A_i$ and $A_i\in U(\kappa,i)$.*
2. *There exists $A^*\subseteq A$ such that:*
   (a) *$A^*\in\cap\vec{U}(\kappa)$*
   (b) *For every $0<j<o^{\vec{U}}(\kappa)$ and $\alpha\in A^*_j$, $A^*\cap\alpha\in\cap\vec{U}(\alpha)$.*

Proof. 1. Note that $X_i:=\{\nu<\kappa\mid o^{\vec{U}}(\nu)=i\}\in U(\kappa,i)$ and $A_i=X_i\cap A\in U(\kappa,i)$. Moreover, every $\alpha<\kappa$ must satisfy $o^{\vec{U}}(\alpha)<\kappa$, since there are at most $2^{2^\alpha}<\kappa$ measures on α.
2. For any $i<o^{\vec{U}}(\kappa)$,

$$Ult(V,U(\kappa,j))\models A=j_{U(\kappa,j)}(A)\cap\kappa\in\bigcap_{i<j}U(\kappa,i)$$

Coherency of the sequence imply that $A':=\{\alpha<\kappa\mid A\cap\alpha\in\cap\vec{U}(\alpha)\}\in U(\kappa,j)$, this is for every $j<o^{\vec{U}}(\kappa)$.
Define inductively $A^{(0)}=A$, $A^{(n+1)}=A'^{(n)}$. By definition, $\forall\alpha\in A^{(n+1)}_j$, $A^{(n)}\cap\alpha\in\cap\vec{U}(\alpha)$.
Define $A^*=\bigcap_{n<\omega}A^{(n)}\in\cap\vec{U}(\kappa)$, this set has the required property. $\blacksquare$

3.1 Extention Type

Definition 3.4 Let $p \in \mathbb{M}[\vec{U}]$. Define

1. For every $i \leq l(p)+1$, let $B_{i,j}(p) = B_i(p) \cap X_j$, where $X_j := \{\alpha < \kappa \mid o^{\vec{U}}(\alpha) = j\}$ are the sets defined in 3.3.

2. $Ex(p) = \prod_{i=1}^{l(p)+1}[o^{\vec{U}}(\kappa_i(p))]^{[<\omega]}$ ($[\lambda]^{[<\omega]}$ is the set of finite, not necessarily increasing sequences in λ).

3. If $X \in Ex(p)$, then X is of the form $\langle X_1, ..., X_{n+1}\rangle$. Denote $x_{i,j}$, the j-th element of X_i, for $1 \leq j \leq |X_i|$ and $mc(X)$ is the last element of X.

4. Let $X \in Ex(p)$, then

$$\vec{\alpha} = \langle \vec{\alpha_1}, ..., \vec{\alpha_{l(p)+1}}\rangle \in \prod_{i=1}^{l(p)+1}\prod_{j=1}^{|X_i|} B_{i,x_{i,j}}(p) =: X(p)$$

   call X an *extension-type of* p and $\vec{\alpha}$ is of *type* X, note that $\vec{\alpha}$ is an increasing sequence of ordinals.

The idea of extension types is simply to classify extensions of p according to the measures from which the ordinals added to the stem of p are chosen. Note that if $o^{\vec{U}}(\kappa) = \lambda < \kappa$ then there is a bound on the number of extension types, $|Ex(p)| < \min(\nu > \lambda \mid o^{\vec{U}}(\nu) > 0)$.

By proposition 3.3 any $p \in \mathbb{M}[\vec{U}]$ can be extended to $p \leq^* p^*$ such that for every $X \in Ex(p)$ and any $\vec{\alpha} \in X(p)$, $p^\frown\vec{\alpha} \in \mathbb{M}[\vec{U}]$. Let us move to this dense subset of $\mathbb{M}[\vec{U}]$.

Proposition 3.5 *Let $p \in \mathbb{M}[\vec{U}]$ be any condition and $p \leq q \in \mathbb{M}[\vec{U}]$. Then there exists unique $X \in Ex(p)$ and $\vec{\alpha} \in X(p)$ such that $p^\frown\vec{\alpha} \leq^* q$. Moreover, for every $X \in Ex(p)$ the set $\{p^\frown\vec{\alpha} \mid \vec{\alpha} \in X(p)\}$ form a maximal antichain above p.*

Proof. The first part is trivial. We will prove that $\{p^\frown\vec{\alpha} \mid \vec{\alpha} \in X(p)\}$ form a antichain above p, by induction on $|X|$. For $|X| = 1$, we merely have some $X(p) = B_{i,\xi}(p) \in U(\kappa_i(p), \xi)$. To see it is an antichain, let $\beta_1 < \beta_2$ are in $X(p)$. Toward a contradiction, assume that $p^\frown\beta_1, p^\frown\beta_2 \leq q$, then β_1 appear in a pair in q and is added between $\kappa_{i-1}(p)$ and β_2, so by definition 2.2, it must be that $\xi = o^{\vec{U}}(\beta_1) < o^{\vec{U}}(\beta_2) = \xi$ contradiction.

To see it is maximal, fix $q \geq p$ and let $\vec{\alpha}$ be such that $p^\frown\vec{\alpha} \leq^* q$. Consider the type of $\vec{\alpha}$,

$$Y \in Ex(p)$$

, then $\vec{\alpha} \in Y(p)$. In Y_i let j be the minimal such that $y_{i,j} \geq \xi$. If $y_{i,j} = \xi$ then $q \geq p^\frown\langle \alpha_{i,j} \rangle \in X(p)$ and we are done. Otherwise, $y_{i,j} > \xi$, then one of the pairs in q is of the form $\langle \alpha_{i,j}, B \rangle$ where $B \in \cap\vec{U}(\alpha_{i,j})$ and $B \subseteq B_i(p)$. Any $\alpha \in B \cap B_{i,\xi}(p)$, will satisfy that $p^\frown\langle \alpha \rangle \in X(p)$ and $p^\frown\langle \alpha \rangle, q \leq q^\frown\langle \alpha \rangle$.

Assume that the claim holds for n, and let $X \in Ex(p)$ be such that $|X| = n+1$. Let $\vec{\alpha}, \vec{\beta} \in X(p)$ be distinct, if for some $x_{i,j} \neq mc(X)$ we have $\alpha_{i,j} \neq \beta_{i,j}$ apply the induction to $X \setminus mc(X)$ to see that $p^\frown\vec{\alpha} \setminus \alpha^*, p^\frown\vec{\beta} \setminus \beta^*$ are incompatible, hence $p^\frown\vec{\alpha}, p^\frown\vec{\beta}$ are incompatible. If $\vec{\alpha} \setminus \alpha^* = \vec{\beta} \setminus \beta^*$, then $\alpha^* \neq \beta^*$ and by the case $n = 1$ we are done. To see it is maximal, let $q \geq p$ apply the induction to $X \setminus mc(X)$ to find $\vec{\alpha} \in [X \setminus mc(X)](p)$ such that $p^\frown\vec{\alpha}$ is compatible with q and let q' be a common extension. Again by the case $n = 1$, there is $\langle \alpha \rangle \in mc(X)(p^\frown\vec{\alpha})$ such that $p^\frown\vec{\alpha}^\frown\langle \alpha \rangle$ and q' are compatible. ■

Definition 3.6 Let $U_1, ..., U_n$ be ultrafilters on a $\kappa_1 \leq ... \leq \kappa_n$ respectively, define recursively the ultrafilter $\prod_{i=1}^n U_i$ over $\prod_{i=1}^n \kappa_i$, as follows: for $B \subseteq \prod_{i=1}^n \kappa_i$

$$B \in \prod_{i=1}^n U_i \leftrightarrow \{\alpha_1 < \kappa_1 \mid B_{\alpha_1} \in \prod_{i=2}^n U_i\} \in U_1$$

where $B_\alpha = B \cap \Big(\{\alpha\} \times \prod_{i=2}^n \kappa_i\Big)$.

Proposition 3.7 *If $U_1, ..., U_n$ are normal θ-complete ultrafilter, then $\prod_{i=1}^n U_i$ is generated by sets of the form $A_1 \times ... \times A_n$ (increasing sequences of the product) such that $A_i \in U_i$.*

Proof. Directly from the definition of normality.■

Every $X \in Ex(p)$ defines an ultrafilter

$$\vec{U}(X,p) = \prod_{i=1}^{n+1} \prod_{j=1}^{|X_i|} U(\kappa_i(p), x_{i,j})$$

Note that $X(p) \in \vec{U}(X,p)$ by the definition of the product. Fix an extension type X of p, every extension of p of type X correspond to some element in the set $X(p)$ which is just a product of large sets.

Let us state here some combinatorical properties, the proof can be found in [1].

Lemma 3.8 *Let $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_n$ be a non descending finite sequence of measurable cardinals and let $U_1, ..., U_n$ be normal measures*[3] *over them respectively. Assume $F : \prod_{i=1}^n A_i \longrightarrow$*

[3]A measure over a measurable cardinal λ is a λ-complete non trivial ultrafilter over λ.

ν where $\nu < \kappa_1$ and $A_i \in U_i$. Then there exists $H_i \subseteq A_i$, $H_i \in U_i$ such that $\prod_{i=1}^{n} H_i$ is homogeneous for F i.e. $|Im(F \restriction \prod_{i=1}^{n} H_i)| = 1$.

$\blacksquare$

Let $F : \prod_{i=1}^{n} A_i \to X$ be a function, and $I \subseteq \{1, ..., n\}$. Let

$$(\prod_{i=1}^{n} A_i)_I = \{\vec{\alpha} \restriction I \mid \vec{\alpha} \in \prod_{i=1}^{n} A_i\}$$

For $\vec{\alpha}' \in (\prod_{i=1}^{n} A_i)_I$, define $F_I(\vec{\alpha}') = F(\vec{\alpha})$ where $\vec{\alpha} \restriction I = \vec{\alpha}'$. With no further assumption, F_I is not a well define function.

Lemma 3.9 *Let $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_n$ be a non descending finite sequence of measurable cardinals and let $U_1, ..., U_n$ be normal measures over them respectively. Assume $F : \prod_{i=1}^{n} A_i \longrightarrow B$ where B is any set, and $A_i \in U_i$. Then there exists $H_i \subseteq A_i$, $H_i \in U_i$ and set $I \subseteq \{1, ..., n\}$ such that $F_I \restriction (\prod_{i=1}^{n} H_i)_I : (\prod_{i=1}^{n} H_i)_I \to B$ is well defined and injective.*

Definition 3.10 Let $F : \prod_{i=1}^{n} A_i \to X$ be a function. An important coordinate is an index $r \in \{1, ..., n\}$, such that for every $\vec{\alpha}, \vec{\beta} \in \prod_{i=1}^{n} A_i$, $F(\vec{\alpha}) = F(\vec{\beta}) \to \vec{\alpha}(r) = \vec{\beta}(r)$.

Proposition 3.9 insures the existence of a set I of important coordinates, such that I is ideal in the sense that removing any coordinate defect definition of F_I as a function, and any coordinate outside of I is redundant.

We will need here another property that does not appear in [1].

Lemma 3.11 *Let $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_n$ and $\theta_1 \leq \theta_2 ... \leq \theta_m$ be a non descending finite sequences of measurable cardinals with coresponding normal measures $U_1, ..., U_n, W_1, ..., W_m$. Let*

$$F : \prod_{i=1}^{n} A_i \to X,\ G : \prod_{j=1}^{m} B_j \to X$$

be functions such that X is any set, $A_i \in U_i$ and $B_j \in W_j$. Assume that $I \subseteq \{1, ..., n\}$ and $J \subseteq \{1, ..., m\}$ are sets of important coordinates for F, G respectively obtained by lemma 3.9. Then there exists $A'_i \in U_i$ and $B'_j \in W_j$. such that one of the following holds

1. *$Im(F \restriction \prod_{i=1}^{n} A'_i) \cap Im(G \restriction \prod_{j=1}^{m} B'_j) = \emptyset$.*
2. *$(\prod_{i=1}^{n} A'_j)_I = (\prod_{j=1}^{m} B'_j)_J$ and $F_I \restriction (\prod_{i=1}^{n} A'_i)_I = G_J \restriction (\prod_{j=1}^{m} B'_j)_J$.*

Proof. Fix F, G. without loss of generality, assume that $\kappa_1 \leq \theta_1$. If $\kappa_1 < \theta_1$ shrink the sets so that $\min(B_1) > \kappa_1$. By induction on $\langle n, m\rangle \in \mathbb{N}^2_+$.

Case 1: Assume that $n = m = 1$, define

$$H_1 : A_1 \times B_1 \to \{0, 1\}, \quad H(\alpha, \beta) = 1 \Leftrightarrow F(\alpha) = G(\beta)$$

By 3.8, shrink A_1, B_1 to A'_1, B'_1 so that H_1 are constant with colors c_1. If $c_1 = 1$ by fixing α we see that G is constant on B'_1 with some value γ. It follows that $J = \emptyset$. Also F is constant since for every $\alpha \in A'_1$ we can take $\beta > \alpha$ and $F(\alpha) = G(\beta) = \gamma$. Hence $I = \emptyset$ and $F_\emptyset \restriction (A'_1)_\emptyset = G_\emptyset \restriction (B'_1)_\emptyset = \{\langle\rangle\}$. Assume that $c_1 = 0$, then for every $\alpha \in A_1, \beta \in B_1$ if $\alpha < \beta$ then $H_1(\alpha, \beta) = 0$, this suffices for the case $\kappa_1 < \theta_1$. If $\kappa_1 = \theta_1$, define

$$H_2 : B_1 \times A_1 \to \{0, 1\} \quad H_2(\beta, \alpha) = 1 \Leftrightarrow F(\alpha) = G(\beta)$$

Again shrink the sets so that H_2 is constantly $c_2 \in \{0, 1\}$. The case $c_2 = 1$ is similar to $c_1 = 1$. Assume that $c_2 = 0$, hence if $\beta < \alpha$ then $H_2(\beta, \alpha) = 0$, it follows that $F(\alpha) \neq G(\beta)$. If $U_1 \neq W_1$ then we are done since we can separate A'_1, B'_1 and conclude that

$$Im(F \restriction A'_1) \cap Im(G \restriction B'_1) = \emptyset$$

If $U_1 = W_1$ then define

$$H_3 : A'_1 \cap B'_1 \to \{0, 1\}, \quad H_3(\alpha) = 1 \Leftrightarrow F(\alpha) = G(\alpha)$$

Again by 3.8 we can assume that H_3 is constant on A^*, if that constant is 1 then we have $F \restriction A^* = G \restriction A^*$ (in particular $I = J = \{1\}$ and $F_I \restriction (A^*)_I = G_J \restriction (A^*)_J$) otherwise,

$$Im(F \restriction A^*) \cap Im(G \restriction A^*) = \emptyset$$

Case 2: Assume $\langle n, m\rangle >_{LEX} \langle 1, 1\rangle$ If $n = 1$, define

$$H_1 : A_1 \times \prod_{j=1}^{m} B_j \to \{0, 1\}, \quad H_1(\alpha, \vec{\beta}) = 1 \Leftrightarrow F(\alpha) = G(\vec{\beta})$$

Shrink the sets so that H_1 is constantly c_1. As before, if $c_1 = 1$ then F, G are constant on large sets, thus $I = J = \emptyset$ and we are done. Assume that $c_1 = 0$. If $n > 1$, for $\alpha \in A_1$ define the functions

$$F_\alpha : \prod_{i=2}^{n} A_i \setminus (\alpha + 1) \to X, \quad F_\alpha(\vec{\alpha}) = F(\alpha, \vec{\alpha})$$

By the induction hypothesis applied to F_α, G and $I \setminus \{1\}, J$, we obtain

$$A^\alpha_i \in U_i \text{ for } 2 \leq i \leq n, \quad B^\alpha_j \in W_j \text{ for } 1 \leq j \leq m$$

such that one of the following holds:

1. $(\prod_{i=2}^n A_i^\alpha)_{I\setminus\{1\}} = (\prod_{j=1}^m B_j^\alpha)_J$, and $(F_\alpha)_{I\setminus\{1\}} \restriction (\prod_{i=2}^n A_i^\alpha)_{I\setminus\{1\}} = G_J \restriction (\prod_{j=1}^m B_j^\alpha)_J$.
2. $Im(F_\alpha \restriction \prod_{i=2}^n A_i^\alpha) \cap Im(G \restriction \prod_{j=1}^m B_j^\alpha) = \emptyset$.

Denote by $i_\alpha \in \{1,2\}$ the relevant case. There is $A_1' \subseteq A_1$, $A_1' \in U_1$, and $i^* \in \{1,2\}$ such that for every $\alpha \in A_1'$, $i_\alpha = i^*$. Let

$$A_i' = \underset{\alpha\in A_1}{\Delta} A_i^\alpha,\ B_j' = \underset{\alpha\in A_1}{\Delta} B_j^\alpha \text{ (Since } \theta_1 \geq \kappa_1 \text{ we can take the diagonal intersection)}$$

If $i^* = 1$, then $(\prod_{i=2}^n A_i^\alpha)_{I\setminus\{1\}} = (\prod_{j=1}^m B_j^\alpha)_J$, denote by $I \setminus \{1\} = \{i_1, ..., i_k\}$, $J = \{j_1, ..., j_k\}$. Then necessarily, $U_{i_r} = W_{j_r}$ for every $1 \leq r \leq k$. Define

$$A^*_{i_r} = B^*_{j_r} := A'_{i_r} \cap B'_{j_r}$$

If $i \notin I$ or $j \notin J$ then keep $A_i^* = A_i'$ and $B_j^* = B_j'$. Then $(\prod_{i=1}^n A_i^*)_{I\setminus\{1\}} = (\prod_{j=1}^m B_j')_J$. Let $\alpha, \alpha' \in A_1'$, $\vec{\alpha} \in \prod_{i=2}^n A_i'$ with $\min(\vec{\alpha}) > \alpha, \alpha'$, then

$$F_\alpha(\vec{\alpha}) = (F_\alpha)_{I\setminus\{1\}}(\vec{\alpha} \restriction I) = G_J(\vec{\alpha} \restriction I) = (F_{\alpha'})_{I\setminus\{1\}}(\vec{\alpha} \restriction I) = F_{\alpha'}(\vec{\alpha})$$

From this it follows that $1 \notin I$ and $F_I = F_{I\setminus\{1\}} = G_J$. Assume $i^* = 2$. If $\theta_1 = \kappa_1$, we repeat the same process, if $m = 1$ we define H_2 as above, if $c_2 = 1$ again we are done, so we assume that $c_2 = 0$. If $m > 1$ we use G_β and fix F, denoting j_β the relevant case, shrink the sets so that j^* is constant. In case $j^* = 1$ the proof is the same as $i^* = 1$. So we assume that $i^* = j^* = 2$, meaning that for every $\langle \alpha, \vec{\alpha} \rangle \in \prod_{i=1}^n A_i'$ and every $\langle \beta, \vec{\beta} \rangle \in \prod_{j=1}^m B_j'$ if $\alpha < \beta$ then $\langle \beta, \vec{\beta} \rangle \in \prod_{j=1}^m B_j^\alpha$ and by $i^* = 2$ (or $c_1 = 0$ if $n = 1$)

$$F(\alpha, \vec{\alpha}) = F_\alpha(\vec{\alpha}) \neq G(\beta, \vec{\beta})$$

Similarly, if $\beta < \alpha$ then $\langle \alpha, \vec{\alpha} \rangle \in \prod_{i=1}^n A_i^\beta$ and by $j^* = 2$ (or $c_2 = 0$), $F(\alpha, \vec{\alpha}) \neq G(\beta, \vec{\beta})$. Hence we are left with the case $\alpha = \beta$.

Case 2a: Assume that $U_1 \neq W_1$ Then we can just shrink the sets A_1', B_1' so that $A_1' \cap B_1' = \emptyset$. Together with the construction of case 2, conclude that

$$Im(F \restriction \prod_{i=1}^n A_i') \cap Im(G \restriction \prod_{j=1}^m B_j') = \emptyset$$

Case 2b: Assume that $U_1 = W_1$, then we shrink the sets so that $A_1' = B_1'$. If $n = 1$ (the case $m = 1$ is similar) let

$$T_1 : A_1' \times \prod_{j=2}^m B_j' \to \{0,1\},\quad T_1(\alpha, \vec{\beta}) = 1 \Leftrightarrow F(\alpha) = G(\alpha, \vec{\beta})$$

We shrink A'_1 and B'_j so that T_1 is constantly d_1. If $d_1 = 0$ then we have eliminated the possibility of $\alpha = \beta$ and $F(\alpha) = G(\beta, \vec{\beta})$ and so we are done again we conclude that

$$Im(F \restriction \prod_{i=1}^{n} A'_i) \cap Im(G \restriction \prod_{j=1}^{m} B'_j) = \emptyset$$

If $d_1 = 1$ then $F \restriction A'_1 = G_{\{1\}} \restriction (A'_1 \times \prod_{j=2}^{m} B'_j)_{\{1\}}$. In particular $J \subseteq \{1\}$, it follows that $F_I \restriction (A'_1)_I = G_J \restriction (A'_1 \times \prod_{j=2}^{m} B'_j)_J$. If $n, m > 1$, for every $\alpha \in A'_1$ we apply the induction hypothesis to the functions F_α, G_α, this time denoting the cases by r^*. If $r^* = 2$, then we have eliminated the possibility of $F(\alpha, \vec{\alpha}) = G(\alpha, \vec{\beta})$, together with $i^* = 2, j^* = 2$ we are done. Finally, assume $r^* = 1$, namely that for

$$I^* := I \setminus \{1\} \subseteq \{2, ..., n\}, \;\; J^* := J \setminus \{1\} \subseteq \{2, ..., m\}$$

We have

$$(\prod_{i=2}^{n} A'_i)_{I^*} = (\prod_{j=2}^{m} B'_j)_{J^*} \text{ and } (F_\alpha)_{I^*} \restriction (\prod_{i=2}^{n} A'_i)_{I^*} = (G_\alpha)_{J^*} \restriction (\prod_{j=2}^{m} B'_j)_{J^*}$$

Since $A'_1 = B'_1$ it follows that

$$(*) \qquad (\prod_{i=1}^{n} A'_i)_{I^* \cup \{1\}} = (\prod_{j=1}^{m} B'_j)_{\in J^* \cup \{1\}} \text{ and } (F_\alpha)_{I^* \cup \{1\}} \restriction (\prod_{i=2}^{n} A'_i)_{I^*} = (G_\alpha)_{J^*} \restriction (\prod_{j=2}^{m} B'_j)_{J^* \cup \{1\}}$$

Since if $\langle \alpha \rangle^\frown \vec{\alpha} \in (\prod_{i=1}^{n} A'_i)_I$,

$$F_{I^* \cup \{1\}}(\alpha, \vec{\alpha}) = (F_\alpha)_{I^*}(\vec{\alpha}) = (G_\alpha)_{J^*}(\vec{\alpha}) = G_{J^* \cup \{1\}}(\alpha, \vec{\alpha})$$

We claim that $1 \in I$ if and only if $1 \in J$. By symmetry, it suffices to prom one implication, for example, if $1 \in I$, then $I = I^* \cup \{1\}$, take $\vec{\alpha} \restriction I, \vec{\alpha}' \restriction I \in (\prod_{i=1}^{n} A'_i)_I$ which differs only at the first coordinate, therefore $F(\vec{\alpha}) \neq F(\vec{\alpha}')$. By $(*)$, there are $\vec{\beta}, \vec{\beta}' \in \prod_{i=1}^{m} B'_i$ such that

$$\vec{\beta} \restriction (J^* \cup \{1\}) = \vec{\alpha} \restriction I \text{ and } \vec{\beta}' \restriction (J^* \cup \{1\}) = \vec{\alpha}' \restriction I$$

It follows that from $(*)$ that $G(\vec{\beta}) = F(\vec{\alpha}) \neq F(\vec{\alpha}') = G(\vec{\beta}')$, therefore $1 \in J$.

In any case, $F_I \restriction (\prod_{i=1}^{n} A'_i)_I = G_J \restriction (\prod_{i=1}^{m} B'_i)_J$. ■

4 The main result

Let us turn to prove the main result (theorem 1.1) for Magidor forcing with $o^{\vec{U}}(\kappa) < \kappa$. The proof presented here is based on what was done in [1] and before that in [3], it is a proof by induction of κ.

4.1 Short Sequences

In this section we prove the theorem for sets A of small cardinality.

Proposition 4.1 *Let $p \in \mathbb{M}[\vec{U}]$ be any condition, X an extension type of p. For every $\vec{\alpha} \in X(p)$ let $p_{\vec{\alpha}} \geq^* p^\frown\vec{\alpha}$. Then there exists $p \leq^* p^*$ such that for every $\vec{\beta} \in X(p^*)$, every $p^{*\frown}\vec{\beta} \leq q$ is compatible with $p_{\vec{\beta}}$.*

Proof. By induction of $|X|$. $X = \langle\xi\rangle$, then $\vec{U}(X,p) = U(\kappa_i(p),\xi)$ and $X(p) = B_{i,\xi}(p)$. For each $\beta \in B_{i,\xi}(p)$

$$p_\beta = \langle\langle\kappa_1(p), A_1^\beta\rangle, ..., \langle\kappa_{i-1}(p), A_{i-1}^\beta\rangle, \langle\beta, B_\beta\rangle, \langle\kappa_i(p), A_i^\beta\rangle, ..., \langle\kappa, A_\beta\rangle\rangle$$

For $j > i$ let $A_j^* = \cap_{\beta\in B_{i,\xi}(p)} A_j^\beta$. For $j < i$ we can find A_j^* and shrink $B_{i,\xi}(p)$ to E_ξ so that for every $\beta \in E_\xi$ and $j < i$ $A_j^\beta = A_j^*$. For i, first let $E = \Delta_{\alpha\in B_{i,\xi}(p)} A_i^\beta$. By ineffability of $\kappa_i(p)$ we can find $A_\xi^* \subseteq E_\xi$ and a set $B^* \subseteq \kappa_i(p)$ such that for every $\beta \in A_\xi^*$ $B^* \cap \beta = B_\beta$. Claim that $B^* \in U(\kappa_i(p),\gamma)$ for every $\gamma < \xi$,

$$Ult(V, U(\kappa_i(p),\xi)) \models B^* = j_{U(\kappa_i(p),j)}(B^*) \cap \kappa_i(p)$$

and since

$$\{\beta < \kappa \mid B^* \cap \beta \in \cap\vec{U}(\beta)\} \in U(\kappa_i(p),\xi)$$

it follows that $B^* \in \cap j_{U(\kappa_i(p),\xi)}(\vec{U})(\kappa_i(p))$. By coherency $B^* \in \cap_{\gamma<\xi} U(\kappa_i(p),\gamma)$. Define

$$A_i^* = B^* \uplus A_\xi^* \uplus (\bigcup_{\xi<i} E_i) \in \cap\vec{U}(\kappa_i(p))$$

Let $q \geq p^{*\frown}\beta$ and suppose that $q \geq^* (p^{*\frown}\beta)^\frown\vec{\gamma}$. Then every $\gamma \in \vec{\gamma}$ such that $\gamma > \beta$ belong to some $A_j^* \setminus \beta$ for $j \geq i$, and by the definition of these sets $\gamma \in A_j^\beta$. If $\gamma < \kappa_{i-1}$ then also $\gamma \in A_j^*$ for some $j < i$. Since $\beta \in E_\xi$ it follows that $A_j^\beta = A_j^*$ so $\gamma \in A_j^\beta$. For $\gamma \in (\kappa_{i-1},\beta)$, by definition of the order we have $o^{\vec{U}}(\gamma) < o^{\vec{U}}(\beta) = \xi$ and therefore $\gamma \in A_{i,\eta}^* \cap \beta$ for some $\eta < \xi$, but

$$A_{i,\eta}^* \cap \beta \subseteq B^* \cap \beta = B_\beta$$

it follows that q, p_β are compatible. For general X, fix $\min(\vec{\beta}) = \beta$. Apply the induction hypothesis to $p^\frown\beta$ and $p_{\vec{\beta}}$ to find $p_\beta^* \geq^* p^\frown\beta$. Next apply the case $n = 1$ to p_β^* and p, find $p^* \geq p$. Let $q \geq p^{*\frown}\vec{\beta}$ and denote $\beta = \min(\vec{\beta})$ then q is compatible with p_β^* thus let $q' \geq q, p_\beta^*$. Since $q' \geq p_\beta^*$ and $q' \geq p^{*\frown}\vec{\beta}$ it follows that $q' \geq p_\beta^{*\frown}\vec{\beta}$. Therefore there is $q'' \geq q', p_{\vec{\beta}}$. ■

Lemma 4.2 *Let $\lambda < \kappa$, $p \in \mathbb{M}[\vec{U}] \restriction (\lambda,\kappa)$, $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and $X \in Ex(p)$. Also. let $\underset{\sim}{x}$ be an ordinal $\mathbb{M}[\vec{U}]$-name. There is $p \leq^* p^*$ such that*

$$\textit{If } \exists\vec{\alpha} \in X(p^*)\ \exists p' \geq^* p^{*\frown}\vec{\alpha}\ \langle q, p'\rangle || \underset{\sim}{x} \quad \textit{Then } \forall\vec{\alpha} \in X(p^*)\langle q, p^{*\frown}\vec{\alpha}\rangle || \underset{\sim}{x}$$

Proof. Fix p, λ, q, X as in the lemma. Consider the set

$$B_0 = \{\vec{\beta} \in X(p) \mid \exists p' \, {}^*{\geq} p^\frown\vec{\beta} \ s.t. \ \langle q, p'\rangle || \underset{\sim}{x}\}$$

One and only one of B_0 and $X(p) \setminus B_0$ is in $\vec{U}(X, P)$. Denote this set by A'. By proposition 3.7, we can find $A'_{i,j} \in U(\alpha_i, x_{i,j})$ such that $\prod_{i=1}^{l(p)+1} \prod_{j=1}^{|X_i|} A'_{i,j} \subseteq A'$, let $p \leq^* p'$ be the condition obtained by shrinking $B_{i,j}(p)$ to $A'_{i,j}$ so that $X(p') = \prod_{i=1}^{n+1} \prod_{j=1}^{|X_i|} A'_{i,j}$. If

$$\exists \vec{\beta} \in X(p') \ \exists p'' \, {}^*{\geq} p'^\frown\vec{\beta} \ \langle q, p''\rangle || \ \underset{\sim}{x}$$

Then $\vec{\beta} \in B_0 \cap A'$ and therefore $B_0 = A'$, we conclude that

$$\forall \vec{\beta} \in X(p') \ \exists p_{\vec{\beta}} \, {}^*{\geq} p'^\frown\vec{\beta} \ \langle q, p_{\vec{\beta}}\rangle || \ \underset{\sim}{x}$$

By proposition 4.1 we can amalgamate all these $p_{\vec{\beta}}$ to find $p' \leq^* p^*$ such that for every $\vec{\beta} \in X(p^*)$, $p^{*\frown}\vec{\beta}$ decides $\underset{\sim}{x}$, then p^* is as wanted. ■

Lemma 4.3 *Consider the decomposition of 2.7 at some $\lambda \geq o^{\vec{U}}(\kappa)$ and let $\underset{\sim}{x}$ be a $\mathbb{M}[\vec{U}]$-name for an ordinal. Then for every $p \in \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$, there exists $p \leq^* p^*$ such that for every $X \in Ex(p)$ and $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ the following holds:*

$$If \ \exists \vec{\alpha} \in X(p^*) \ \exists p' \geq^* p^{*\frown}\vec{\alpha} \ \langle q, p'\rangle || \ \underset{\sim}{x} \ Then \ \forall \vec{\alpha} \in X(p^*) \ \langle q, p^{*\frown}\vec{\alpha}\rangle || \underset{\sim}{x}$$

Proof. Fix $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and and $X \in Ex(p)$. Use 4.2, to find $p \leq^* p_{q,X}$ such that

$$If \ \exists \vec{\alpha} \in X(p_{q,X}) \ \exists p' \geq^* (p_{q,X})^\frown\vec{\alpha} \ s.t. \ \langle q, p'\rangle || \ \underset{\sim}{x} \ Then \ \forall \vec{\alpha} \in X(p_{q,X}) \ \langle q, (p_{q,X})^\frown\vec{\alpha}\rangle || \underset{\sim}{x}$$

By the definition of λ, the forcing $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ is $\leq^*$-max$(|Ex(p)|^+, |\mathbb{M}[\vec{U}] \restriction \lambda|^+)$-directed. Hence we can find $p \leq^* p^*$ so that for every X, q, $p_{q,X} \leq^* p^*$. ■

Lemma 4.4 *Let $A \in V[G]$ be a set of ordinals such that $|A| < \kappa$. Then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$.*

Proof. Assume that $|A| = \lambda' < \kappa$ and let $\delta = max(\lambda', \text{otp}(C_G)) < \kappa$. Split $\mathbb{M}[\vec{U}]$ as in proposition 2.7. Find $p \in G$ such that some $\delta \leq \lambda$ appears in p. The generic G also splits to $G = G_1 \times G_2$ where G_1 is the generic for Magidor forcing below λ and G_2 above it. Let $\langle \underset{\sim}{a}_i \mid i < \lambda'\rangle$ be a $\mathbb{M}[\vec{U}]$-name for A in V and $p \in \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$. For every $i < \lambda'$ find $p \leq^* p_i$ as in lemma 4.3, such that for every $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and $X \in Ex(p)$ we have:

$$If \ \exists \vec{\alpha} \in X(p_i) \ \exists p_i^\frown\vec{\alpha} \leq^* p' \ \langle q, p'\rangle \ || \ \underset{\sim}{a}_i \ Then \ \forall \vec{\alpha} \in X(p_i) \ \langle q, p_i^\frown\vec{\alpha}\rangle \ || \ \underset{\sim}{a}_i \quad (*)$$

Since we have λ'-closure for $\leq^*$ we can find $p_i \leq^* p_*$. Next, for every $i < \lambda'$, fix a maximal anti chain $Z_i \subseteq \mathbb{M}[\vec{U}] \restriction \lambda$ such that for every $q \in Z_i$ there is an extension type $X_{q,i}$ for which

$\forall\vec{\alpha} \in p_*^\frown X_{q,i}$ $\langle q, p_*^\frown\vec{\alpha}\rangle \parallel \underset{\sim}{a}_i$, these anti chains can be found using (*) and Zorn's lemma. Recall the sets $X_{q,i}(p_*)$ is a product of large sets. Define $F_{q,i} : X_{q,i}(p_*) \to On$ by

$$F_{q,i}(\vec{\alpha}) = \gamma \quad \Leftrightarrow \quad \langle q, p_*^\frown\vec{\alpha}\rangle \Vdash \underset{\sim}{a}_i = \check{\gamma}$$

By lemma 3.9 we can assume that there are important coordinates

$$I_{q,i} \subseteq \{1, ..., dom(X_{q,i}(p_*))\}$$

Fix $i < \lambda'$, for every $q, q' \in Z_i$ we apply lemma 3.11 to the functions $F_{q,i}, F_{q,i'}$ and find $p_* \leq^* p_{q,q'}$ for which one of the following holds:

1. $Im(F_{q,i} \restriction A(X_{q,i}, p_{q,q'})) \cap Im(F_{q',i} \restriction A(X_{q',i}, p_{q,q'})) = \emptyset$
2. $(F_{q,i})_{I_{q,i}} \restriction (A(X_{q,i}, p_{q,q'}))_{I_{q,i}} = (F_{q',i})_{I_{q',i}} \restriction (A(X_{q',i}, p_{q,q'}))_{I_{q',i}}$

Finally find p^* such that for every q, q', $p_{q,q'} \leq^* p^*$. By density, there is such $p^* \in G_2$. We use $F_{q,i}$ to translate information from C_G to A and vice versa, distinguishing from [1] this translation is made in $V[G_1]$ rather then V: For every $i < \lambda'$, $G_1 \cap Z_i = \{q_i\}$. Use lemma 3.5, to find $D_i \in X_{q_i,i}(p^*)$ be such that $p^{*\frown}D_i \in G_2$, define $C_i = D_i \restriction I_{q_i,i}$ and let $C' = \bigcup_{i<o^{\vec{U}}(\kappa)} C_i$. Define as in 2.20, $I(C_i, C') \in [\mathrm{otp}(\kappa)]^{<\omega}$, since $\mathrm{otp}(C') \leq \mathrm{otp}(C_G) \leq \lambda$ and $V[G_2]$ does not add sequences to λ we have that $\langle I(C_i, C') \mid i < \lambda'\rangle \in V[G_1]$. It follows that

$$(V[G_1])[A] = (V[G_1])[\langle C_i \mid i < \lambda'\rangle] = (V[G_1])[C']$$

In fact let us prove that $\langle C_i \mid i < \lambda'\rangle \in V[A]$. Indeed, define in $V[A]$ the sets

$$M_i = \{q \in Z_i \mid a_i \in Im(F_{q,i})\}$$

then, for any $q, q' \in M_i$ $a_i \in Im(F_{q_i}) \cap Im(F_{q',i}) \neq \emptyset$. Hence 2 must hold for $F_{q,i}, F_{q',i}$ i.e.

$$(F_{q,i})_{I_{q,i}} \restriction (X_{q,i}(p^*))_{I_{q,i}} = (F_{q',i})_{I_{q',i}} \restriction (X_{q',i}(p^*))_{I_{q',i}}$$

This means that no matter how we pick $q'_i \in M_i$, we will end up with the same function $(F_{q'_i,i})_{I_{q'_i,i}} \restriction (X_{q'_i,i}(p^*))_{I_{q'_i,i}}$. In $V[A]$, choose any $q'_i \in M_i$ and let $D'_i \in F^{-1}_{q'_i,i}(a_i)$, $C'_i = D_i \restriction I_{q'_i,i}$. Since $q_i, q'_i \in M_i$ we have $C_i = C'_i$, hence $\langle C_i \mid i < \lambda'\rangle \in V[A]$. We still have to determine what information A uses in the part of G_1, namely, $\{q'_i \mid i < \lambda'\}, \langle I(C_i, C') \mid i < \lambda'\rangle \in V[A]$. This sets can be coded as a subset of ordinals below $(2^\lambda)^+$, therefore,

$$\{q'_i \mid i < \lambda'\}, \langle I(C_i, C') \mid i < \lambda'\rangle \in V[G_1]$$

By the induction hypothesis, we can find $C'' \subseteq C_{G_1}$ such that

$$V[\{q'_i \mid i < \lambda'\}, \langle I(C_i, C') \mid i < \lambda'\rangle] = V[C'']$$

Since all the information needed to restore A is coded in $C' \uplus C''$, it is clear that $V[A] = V[C'' \uplus C']$. $\blacksquare$

4.2 General Subsets of κ

Assume that $A \in V[G]$ such that $A \subseteq \kappa$. For some A's, the proof is similar to the one in [1] works. This proof relays on the following lemma:

Lemma 4.5 *Assume that* $o^{\vec{U}}(\kappa) < \kappa$ *and let* $A \in V[G]$, $\sup(A) = \kappa$. *Assume that* $\exists C^* \subseteq C_G$ *such that*

1. $C^* \in V[A]$ *and* $\forall \alpha < \kappa \ A \cap \alpha \in V[C^*]$
2. $cf^{V[A]}(\kappa) < \kappa$

Then $\exists C' \subseteq C_G$ *such that* $V[A] = V[C']$.

Proof. Let $\langle \alpha_i \mid i < \lambda \rangle \in V[A]$ be cofinal in κ. Since $|C^*| < \kappa$, by 4.4, we can find $C'' \subseteq C_G$ such that

$$V[C''] = V[C', \langle \alpha_i \mid i < \lambda \rangle] \subseteq V[A]$$

In $V[C'']$ choose for every i, a bijection $\pi_i : 2^{\alpha_i} \to P^{V[C'']}(\alpha_i)$. Since $A \cap \alpha_i \in V[C'']$ there is δ_i such that $\pi_i(\delta_i) = A \cap \alpha_i$. Finally let $C' \subseteq C_G$ such that

$$V[C'] = V[C'', \langle \delta_i \mid i < \lambda \rangle]$$

We claim that $V[A] = V[C']$. Obviously, $C' \in V[A]$, for the other direction,

$$\langle A \cap \alpha_i \mid i < \lambda \rangle = \langle \pi_i(\delta_i) \mid i < \lambda \rangle \in V[C']$$

Thus $A \in V[C']$. $\blacksquare$

Definition 4.6 We say that $A \cap \alpha$ stabilizes, if

$$\exists \alpha^* < \kappa. \ \forall \alpha < \kappa. \ A \cap \alpha \in V[A \cap \alpha^*]$$

First we deal with A's such that $A \cap \alpha$ does not stabilize.

Lemma 4.7 *Assume* $o^{\vec{U}}(\kappa) < \kappa$, $A \subseteq \kappa$ *unbounded in* κ *such that* $A \cap \alpha$ *does not stabilizes, then there is* $C' \subseteq C_G$ *such that* $V[C'] = V[A]$.

Proof. Work in $V[A]$, define the sequence $\langle \alpha_\xi \mid \xi < \theta \rangle$:

$$\alpha_0 = \min(\alpha \mid V[A \cap \alpha] \supsetneq V)$$

Assume that $\langle \alpha_\xi \mid \xi < \lambda \rangle$ has been defined and for every ξ, $\alpha_\xi < \kappa$. If $\lambda = \xi + 1$ then set

$$\alpha_\lambda = \min(\alpha \mid V[A \cap \alpha] \supsetneq V[A \cap \alpha_\xi])$$

If $\alpha_\lambda = \kappa$, then α_λ satisfies that

$$\forall \alpha < \kappa \ \ A \cap \alpha \in V[A \cap \alpha_\xi]$$

Thus $A \cap \alpha$ stabilizes which by our assumption is a contradiction. If λ is limit, define

$$\alpha_\lambda = \sup(\alpha_\xi \mid \xi < \lambda)$$

if $\alpha_\lambda = \kappa$ define $\theta = \lambda$ and stop. The sequence $\langle \alpha_\xi \mid \xi < \theta \rangle \in V[A]$ is a continues, increasing unbounded sequence in κ. Therefore, $cf^{V[A]}(\kappa) = cf^{V[A]}(\theta)$. Let us argue that $\theta < \kappa$. Work in $V[G]$, for every $\xi < \theta$ pick $C_\xi \subseteq C_G$ such that $V[A \cap \alpha_\xi] = V[C_\xi]$. The map $\xi \mapsto C_\xi$ is injective from θ to $P(C_G)$, by the definition of α_ξ's. Since $o^{\vec{U}}(\kappa) < \kappa$, $|C_G| < \kappa$, and κ stays strong limit in the genenic extension. Therefore

$$\theta \leq |P(C_G)| = 2^{|C_G|} < \kappa$$

Hence κ changes cofinality in $V[A]$, according to lemma 4.5, it remains to find C^*. Denote $\lambda = |C_G|$ and work in $V[A]$, for every $\xi < \theta$, $C_\xi \in V[A]$ (Although the sequence $\langle C_\xi \mid \xi < \theta \rangle$ may not be in $V[A]$). C_ξ witnesses that

$$\exists d_\xi \subseteq \kappa. \ |d_\xi| \leq \lambda \text{ and } V[A \cap \alpha_\xi] = V[d_\xi]$$

Fix $d = \langle d_\xi | \xi < \theta \rangle \in V[A]$. It follows that d can be coded as a subset of κ of cardinality $\leq \lambda \cdot \theta < \kappa$. Finally, by 4.4, there exists $C^* \subseteq C_G$ such that $V[C^*] = V[d] \subseteq V[A]$ so

$$\forall \alpha < \kappa. \ A \cap \alpha \in V[d_\xi] \subseteq V[C^*]$$

$\blacksquare$

Next we assume that $A \cap \alpha$ stabilizes on some $\alpha^* < \kappa$. By lemma 4.4 There exists $C^* \subseteq C_G$ such that $V[A \cap \alpha^*] = V[C^*]$, if $A \in V[C^*]$ then we are done, assume that $A \notin V[C^*]$. To apply 4.5, it remains to prove that $cf^{V[A]}(\kappa) < \kappa$. The subsequence C^* must be bounded, denote $\kappa_1 = \sup(C^*) < \kappa$ and $\kappa^* = \max(\kappa_1, \mathrm{otp}(C_G))$. Find $p \in G$ that decides the value of κ^* and assume that κ^* appear in p (otherwise take some ordinal above it). As in lemma 2.7 we split

$$\mathbb{M}[\vec{U}]/p \simeq \Big(\mathbb{M}[\vec{U}] \restriction \kappa^*\Big)/\Big(p \restriction \kappa^*\Big) \times \Big(\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)\Big)/\Big(p \restriction (\kappa^*, \kappa)\Big)$$

There is a subforcing $\mathbb{P}$ of $RO(\Big(\mathbb{M}[\vec{U}] \restriction \kappa^*\Big)/\Big(p \restriction \kappa^*\Big)$ such that $V[C^*]$ is a generic for $\mathbb{P}$. Let

$$\mathbb{Q} = \Big[\Big(\mathbb{M}[\vec{U}] \restriction \kappa^*\Big)/\Big(p \restriction \kappa^*\Big)\Big]/C^*$$

be the quotient forcing completing $\mathbb{P}$ to $\Big(\mathbb{M}[\vec{U}] \restriction \kappa^*\Big)/\Big(p \restriction \kappa^*\Big)$. Finally note that G is generic over $V[C^*]$ for

$$\mathbb{S} = \mathbb{Q} \times \Big(\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)\Big)/\Big(p \restriction (\kappa^*, \kappa)\Big)$$

Lemma 4.8 $cf^{V[A]}(\kappa) < \kappa$

Proof. Let $G = G_1 \times G_2$ be the decomposition such that G_1 is generic for $\mathbb{Q}$ above $V[C^*]$ and G_2 is $\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$ generic over $V[C^*][G_1]$. Let $\underset{\sim}{A}$ be a $\mathbb{S}$-name for A in $V[C^*]$. and $\langle q_0, p_0\rangle \in G$ such that

$$\langle q_0, p_0\rangle \Vdash \text{"}\forall\alpha < \kappa\ \underset{\sim}{A} \cap \alpha \textit{ is old"}\ (\textit{i.e. in } V[C^*])$$

Proceed by a density argument in $\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa))/p \restriction (\kappa^*, \kappa)$, let $p_0 \le p$, as in 4.4 find $p \le^* p^*$ such that for all $q_0 \le q \in \mathbb{Q}$ and $X \in Ex(p^*)$:

$$\exists\vec{\alpha}^\frown\langle\alpha\rangle \in X(p^*)\exists p' \ge^* p^{*\frown}\vec{\alpha}^\frown\langle\alpha\rangle\ \langle q, p'\rangle \mid\mid \underset{\sim}{A} \cap \alpha \Rightarrow \forall\vec{\alpha}^\frown\langle\alpha\rangle \in X(p^*)\langle q, p^{*\frown}\vec{\alpha}^\frown\langle\alpha\rangle \mid\mid \underset{\sim}{A} \cap \alpha$$

Denote the consequent by $(*)_{X,q}$, since $\underset{\sim}{A} \cap \alpha$ is forced to be old, we will find Many q, X for which $(*)_{q,X}$ holds. For such q, X, for every $\vec{\alpha}^\frown\langle\alpha\rangle \in X(p^*)$ define the value forced for $\underset{\sim}{A}\cap\alpha$ by $a(q, \vec{\alpha}, \alpha)$. Fix q, X such that $(*)_{q,X}$ holds. Assume that the maximal measure which appears in X is $U(\kappa_i(p), mc(X))$ and fix $\vec{\alpha} \in (X \setminus \{mc(X)\})(p^*)$. For every $\alpha \in B_{i,mc(X)}(p) \setminus \max(\vec{\alpha})$ the set $a(q, \vec{\alpha}, \alpha) \subseteq \alpha$ is defined. By ineffability, we can shrink $B_{i,mc(X)}(p)$ to $A^{q,\vec{\alpha}}_{i,mc(X)}$ and find a set $A(q, \vec{\alpha}) \subseteq \kappa_i(p)$ such that for every $\alpha \in A^{q,\vec{\alpha}}_{i,mc(X)}$, $A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha)$ define

$$A'_{i,mc(X)} = \underset{\vec{\alpha},q}{\Delta} A^{q,\vec{\alpha}}_{i,mc(X)}$$

Let $p^* \le^* p'$ be the condition obtained by shrinking to those sets. p' has the property that whenever $(*)_{q,X}$ holds for some $q \in \mathbb{Q}$ and $X \in Ex(p')$, there exists sets $A(q, \vec{\alpha})$ for $\vec{\alpha} \in X \setminus \{mc(X)\}$ such that for every $\vec{\alpha}^\frown\langle\alpha\rangle \in X(p')$, $A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha)$. By density there is such $p' \in G_2$.

Work $V[A]$, for every $\vec{\alpha}$ and q, if $A(q, \vec{\alpha})$ is defined, let

$$\eta(q, \vec{\alpha}) = \min(A\Delta A(q, \vec{\alpha}))$$

otherwise $\eta(q, \vec{\alpha}) = 0$. $\eta(q, \vec{\alpha})$ is well defined since $A \notin V[C^*]$ and $A \in V[C^*]$. Also let

$$\eta(\vec{\alpha}) = \sup(\eta(q, \vec{\alpha}) \mid q \in \mathbb{Q})$$

If $\eta(\vec{\alpha}) = \kappa$ then we are done (since $|\mathbb{Q}| < \kappa$). Define a sequence in $V[A]$: $\alpha_0 = \kappa^*$. Fix $\xi < \text{otp}(C_G)$ and assume that $\langle\alpha_i \mid i < \xi\rangle$ is defined. At limit stages take

$$\alpha_\xi = \sup(\alpha_i \mid i < \xi) + 1$$

Assume that $\xi = \lambda + 1$ and let

$$\alpha_\xi = \sup(\eta(\vec{\alpha}) + 1 \mid \vec{\alpha} \in [\alpha_\lambda]^{<\omega})$$

If at some point we reach κ we are done. If not, let us prove by induction on ξ that $C_G(\xi) < \alpha_\xi$ which will indicate that the sequence α_ξ is unbounded in κ. At limit ξ we have $C_G(\xi) = \sup(C_G(\beta) \mid \beta < \xi)$ since the Magidor sequence is a club. By the definition of the sequence α_ξ and the induction hypothesis, $\alpha_\xi > C_G(\xi)$. If $\xi = \lambda + 1$, use corollary 2.19 to find $\vec{\alpha}, \alpha$ and q such that

$$\langle q, p^{'\frown}\vec{\alpha}^\frown\langle\alpha\rangle\rangle \Vdash \check{\alpha} = \underset{\sim}{C}_G(\check{\xi})$$

Fix any $q' \geq q$, and split the forcing at α so that $\langle q', p'^\frown\vec{\alpha}, \alpha\rangle = \langle q', r_1, r_2\rangle$ where $r_1 \in \mathbb{M}[\vec{U}] \restriction (k^*, \alpha)$ and $r_2 \in \mathbb{M}[\vec{U}] \restriction (\alpha, \kappa)$. Let H_1 be some generic up to α with $\langle q, r_1\rangle \in H_1$ and work in $V[C^*][H_1]$, the name $\underset{\sim}{A}$ has a natural interpretation in $V[C^*][H_1]$ as a $\mathbb{M}[\vec{U}] \restriction (\alpha, \kappa)$-name, $(\underset{\sim}{A})_{H_1}$. Use the fact that $\mathbb{M}[\vec{U}] \restriction \alpha$ is $\leq^*$-closed and the prikry condition to find $r_2 \leq^* r_2'$ and X such that

$$r_2' \Vdash_{\mathbb{M}[\vec{U}]\restriction(\alpha,\kappa)} (\underset{\sim}{A})_{G_1} \cap \alpha = X$$

since it is forced that $\underset{sim}{A}$ is old, $X \in V[C^*]$ and therefore we can find $\langle q'', r_1'\rangle \geq \langle q', r_1\rangle$ such that

$$\langle q'', r_1'\rangle \Vdash \text{"}r_2' \Vdash \underset{\sim}{A} \cap \alpha = X\text{"} \Rightarrow \langle q'', r_1', r_2'\rangle \Vdash \underset{\sim}{A} \cap \alpha = X$$

and $\vec{\alpha}, \alpha$ such that

$$\langle q', p^{**\frown}\vec{\alpha}^\frown\langle\alpha\rangle\rangle \mid\mid \underset{\sim}{A} \cap \check{\alpha}$$

but then $\langle r_1', r_2'\rangle$ is of the form $p'^\frown\vec{\beta}, \alpha \leq^* p''$ for some $\vec{\beta}$. Let X be the extension type of $\vec{\beta}, \alpha$, by definition of p', $(*)_{q'',X}$ holds. Use density to find a q^* in the generic of $\mathbb{Q}$ such that for some X that decides the ξth element of C_G, $(*)_{X,q^*}$ holds. The set $\{p'^\frown\vec{\gamma} \mid \gamma \in X\}$ is a maximal antichain according to proposition 3.5, so let $\vec{C}, C_G(\xi)$ be the extension of p' of type X in C_G. By the construction of q^* and p^{**} we have that

$$\langle q^*, p'^\frown\langle\vec{C}, C_G(\xi)\rangle \Vdash \underset{\sim}{A} \cap \check{C_G}(\xi) = A(q^*, \vec{C}) \cap \check{C_G}(\xi)$$

Since $(\underset{\sim}{A})_G = A$, $A(q^*, \vec{C}) \cap C_G(\xi) = A \cap C_G(\xi)$ (otherwise we would've found compatible conditions forcing contradictory information). This imply that

$$\eta(q^*, \vec{C}) \geq C_G(\xi)$$

By the induction hypothesis $\alpha_\lambda > C_G(\lambda)$ and $\vec{C} \subseteq C_G(\lambda)$ thus $\vec{C} \in [\alpha_\lambda]^{<\omega}$ thus

$$\alpha_\xi > sup(\eta(\vec{\alpha}) \mid \vec{\alpha} \in [\alpha_\lambda]^{<\omega}) \geq \eta(\vec{C}) \geq \eta(q^*, \vec{C}) \geq C_G(\xi)$$

This proves that $\langle\alpha_\xi \mid \xi < \text{otp}(C_G) < \kappa\rangle \in V[A]$ is cofinal in κ indicating $cf^{V[A]}(\kappa) < \kappa$. ■

Thus we have proven the result for any subset of κ.

Corollary 4.9 *Let $A \in V[G]$ be a set of ordinals, such that $|A| = \kappa$ then there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

Proof. By κ^+-c.c. of $\mathbb{M}[\vec{U}]$, there is $B \in V$, $|B| = k$ such that $A \subseteq B$. Fix in V $\phi : \kappa \to B$ a bijection and let $B' = \phi^{-1''}A$. then $B' \subseteq \kappa$. By the theorem for subsets of κ there is $C' \subseteq C_G$ such that $V[C'] = V[B'] = V[A]$. ■

4.3 general sets of ordinals

In [1], we gave an explicit formulation of subforcings of $\mathbb{M}[\vec{U}]$ using the indices of subsequences of C_G. In the larger framework of this paper, these indices might not be in V. By example 1.2, subforcing of the Magidor forcing can be an iteration of Magidor type forcing.

Lemma 4.10 *Let $A \in V[G]$ be such that $A \subseteq \kappa^+$. Then there is $C^* \subseteq C_G$ closed such that*

1. *$\exists \alpha^* < \kappa^+$ such that $C^* \in V[A \cap \alpha^*] \subseteq V[A]$.*
2. *$\forall \alpha < \kappa^+$ $A \cap \alpha \in V[C^*]$.*

Proof. Work in $V[G]$, for every $\alpha < \kappa^+$ find subsequences $C_\alpha \subseteq C_G$ such that

$$V[C_\alpha] = V[A \cap \alpha]$$

using corollary 4.9. The function $\alpha \mapsto C_\alpha$ has range $P(C_G)$ and domain κ^+ which is regular in $V[G]$, and since $o^{\vec{U}}(\kappa) < \kappa$ then $|P(C_G)| < \kappa^+$. Therefore there exist $E \subseteq \kappa^+$ unbounded in κ^+ and $\alpha^* < \kappa^+$ such that for every $\alpha \in E$, $C_\alpha = C_{\alpha^*}$. Set $C^* = C_{\alpha^*}$, By lemma 4.12 we may assume that C^* is closed. Note that for every $\alpha < \kappa$ there is $\beta \in E$ such that $\beta > \alpha$ therefore

$$A \cap \alpha = (A \cap \beta) \cap \alpha \in V[A \cap \beta] = V[C^*]$$

■

Lemma 4.11 *Let C^* be as in the last lemma. If there is $\alpha < \kappa$ such that $A \in V[C_G \cap \alpha][C^*]$ then $V[A] = V[C^*]$.*

Proof. Consider the quotient forcing $\mathbb{M}[\vec{U}]/C^* \subseteq \mathbb{M}[\vec{U}]$ completing $V[C^*]$ to $V[C^*][G]$. Then the forcing

$$\mathbb{Q} = (\mathbb{M}[\vec{U}]/C^*) \restriction \alpha$$

completes $V[C^*]$ to $V[C^*][C_G \cap \alpha]$ and $|\mathbb{Q}| < \kappa$. By the assumption, $A \in V[C^*][C_G \cap \alpha]$, and for every $\alpha < \kappa^+$, $A \cap \alpha \in V[C^*]$. Let $\underset{\sim}{A} \in V[C^*]$ be a $\mathbb{Q}$-name for A and $q \in G \restriction \alpha$ be any condition such that

$$q \Vdash \forall \alpha < \kappa^+, \underset{\sim}{A} \cap \alpha \in V[C^*]$$

In $V[C^*]$, for every $\alpha < \kappa^+$ find $q_\alpha \geq q$ such that $q_\alpha ||_{\mathbb{Q}} \underset{\sim}{A} \cap \alpha$, there is $q^* \geq q$ and $E \subseteq \kappa^+$ of cardinality κ^+ such that for very $\alpha \in E$, $q_\alpha = q^*$. By density, find such $q^* \in G \restriction \alpha$ in the generic. In $v[C^*]$, consider the set

$$B = \{X \subseteq \kappa^+ \mid \exists \alpha \; q^* \Vdash X = \underset{\sim}{A} \cap \alpha = X\}$$

Let us argue that $\cup B = A$. Let $X \in B$ then there is $\alpha < \kappa^+$ such that $q^* \Vdash X = \underset{\sim}{A} \cap \alpha$ then $X = A \cap \alpha \subseteq A$, thus, $\cup B \subseteq A$. Let $\gamma \in A$, there is $\alpha \in E$ such that $\gamma < \alpha$, by the definition of E there is $X \subseteq \alpha$ such that $q^* \Vdash \underset{\sim}{A} \cap \alpha = X$ it must be that $X = A \cap \alpha$ otherwise would have found compatible conditions forcing contradictory information. but the $\gamma \in A \cap \alpha = X \subseteq \cup B$. We conclude that $A = \cup B \in V[C^*]$. ■

Eventually we will prove that there is $\alpha < \kappa$ such that $A \in V[C_G \cap \alpha][C^*]$ and by the last lemma we will be done.

We would like to change C^* so that it is closed. We can do that above $\alpha_0 := \text{otp}(C_G)$:

Lemma 4.12 *$V[C_G \cap \alpha_0][Cl(C^*)] = V[C_G \cap \alpha_0][C^*]$.*[4]

Proof. Consider $I(C^*, Cl(C^*)) \subseteq \text{otp}(C_G)$, by proposition 2.15.5, $I(C^*, Cl(C^*)) \in V[C_G \cap \alpha_0]$. Thus $V[C_G \cap \alpha_0][C^*] = V[C_G \cap \alpha_0][Cl(C^*)]$. ■

Work in $V[C_G \cap \alpha_0]$, since $C^* \cap \alpha_0 \in V[C_G \cap \alpha_0]$, we can assume $min(C^*) > \alpha_0$. Since $I = I(C^*, C_G \setminus \alpha_0) \subseteq \text{otp}(C_G)$, it follows that $I \in V[C_G \cap \alpha_0]$. Let $N = V[C_G \cap \alpha_0]$, consider the coherent sequence

$$\vec{W} = \vec{U}^* \restriction (\alpha_0, \kappa] = \langle U^*(\beta, \delta) \mid \delta < o^{\vec{U}}(\beta), \alpha_0 < \delta < \kappa \rangle$$

where $U^*(\beta, \delta)$ is the ultrafilter generated by $U(\beta, \delta)$ in N. Also denote $G^* = G \restriction (\alpha_0, \kappa)$.

Proposition 4.13 *$N[G^*]$ is a $\mathbb{M}[\vec{W}]$ generic extension of N.*

Proof. Let us argue that the Mathias criteria holds. Let $X \in \cap \vec{W}(\delta)$ where $\delta \in Lim(C_{G^*})$. By definition of $\vec{W}$, for every $i < o^{\vec{W}}(\delta)$, there is $X_i \in U(\delta, i)$, such that $X_i \subseteq X$. The choice of X_i's is done in N and the sequence $\langle X_i \mid i < o^{\vec{U}}(\delta) \rangle$ might not be in V. Fortunately, $\mathbb{M}[\vec{U}] \restriction \alpha_0$ is α_0^+-c.c. and $\alpha_0^+ < \delta$, so in V, we can find sets

$$E_i := \{X_{i,j} \mid j \leq \alpha_0\} \subseteq U(\delta, i)$$

such that $X_i \in E_i$ By δ-completness of $U(\delta, i)$, the set $X_i^* := \cap E_i \in U(\delta, i)$ and $X_i^* \subseteq X_i \subseteq X$. Note that $X^* := \cup_{i < o^{\vec{U}}(\delta)} X_i^* \in \cap \vec{U}(\delta)$ and therefore by genericity of G there is $\xi < \delta$ such that

$$C_G \cap (\xi, \delta) \subseteq X^* \subseteq X$$

Hence $C_{G^*} \cap (\max(\alpha_0, \xi), \delta) \subseteq X$.■

[4] For a set of ordinals X, $Cl(X) = X \cup \text{Lim}(X)\{\xi \mid \xi \in X \vee \sup(X \cap \xi) = \xi$

Note that $o^{\vec{W}}(\kappa) < \min(\nu \mid o^{\vec{W}}(\nu) = 1)$ and $I(C^*, C) \in N$, which is the situation dealt with in [1]. We state here the main results and definitions and refer the reader to [1] for the proofs:

We will define a Magidor type forcing that produces the sequence C^* above N. Thinking of C^* as a function with domain I, we would like to have a function similar to $\gamma(t_i, p)$ which tells us the coordinate we unveil. Given any sequence of pairs, $p = \langle t_1, ..., t_n, t_{n+1}\rangle$, define[5]

$$I(t_1, p) = \min(j \in I \mid o_L(j) = o^{\vec{W}}(t_i))$$

then recursively,

$$I(t_i, p) = \min(j \in I \setminus I(t_{i-1}, p) + 1 \mid o_L(j) = o^{\vec{W}}(t_i))$$

It is tacitly assumed that $\{j \in I \setminus I(t_{i-1}, p) + 1 \mid o_L(j) = o^{\vec{W}}(t_i)\} \neq \emptyset$. If at some point of the inductive definition we obtain $\emptyset$, leave $I(t_i, p)$ undefined, we will ignore such conditions p anyway.

Definition 4.14 The conditions of $\mathbb{M}_I[\vec{W}]$ are of the form $p = \langle t_1, ..., t_{n+1}\rangle$ such that:

1. I is defined on p.
2. $\kappa(t_1) < ... < \kappa(t_n) < \kappa(t_{n+1}) = \kappa$
3. For $i = 1, ..., n + 1$
   (a) If $I(t_i, p) \in \mathrm{Succ}(I)$
      i. $t_i = \kappa(t_i)$
      ii. $I(t_{i-1}, p)$ is the predecessor of $I(t_i, p)$ in I
      iii. $I(t_{i-1}, p) + \sum\limits_{i=1}^{m}\omega^{\gamma_i} = I(t_i, p)$ is the Cantor normal form difference, then

      $$Y(\gamma_1) \times ... \times Y(\gamma_{m-1}) \bigcap [(\kappa(t_{i-1}), \kappa(t_i))]^{<\omega} \neq \emptyset$$

      where $Y(\gamma) = \{\alpha < \kappa \mid o^{\vec{U}}(\alpha) = \gamma\}$
   (b) If $I(t_i, p) \in \mathrm{Lim}(I)$
      i. $t_i = \langle \kappa(t_i), B(t_i)\rangle$, $B(t_i) \in \bigcap\limits_{\xi < o^{\vec{W}}(t_i)} U(t_i, \xi)$
      ii. $I(t_{i-1}, p) + \omega^{o^{\vec{W}}(t_i)} = I(t_i, p)$. (i.e. there are no elements of higher order then $o^{\vec{W}}(t_i)$ to add in the interval $(\kappa(t_{i-1}), \kappa(t_i))$.
      iii. $\min(B(t_i)) > \kappa(t_{i-1})$

[5] For an ordinal α, denote by $o_L(\alpha) = \gamma$ if the cantor normal form of $\alpha = \sum_{i=1}^{n} \omega^{\gamma_i} m_i$ and $\gamma = \gamma_n$.

Definition 4.15 Let $p = \langle t_1, ..., t_n, t_{n+1}\rangle, q = \langle s_1, ..., s_m, s_{m+1}\rangle \in \mathbb{M}_I[\vec{W}]$ be two conditions. Define $\langle t_1, ..., t_n, t_{n+1}\rangle \leq_I \langle s_1, ..., s_m, s_{m+1}\rangle$ iff $\exists 1 \leq i_1 < ... < i_n \leq m < i_{n+1} = m+1$ such that

1. For every $1 \leq r \leq n$ $\kappa(t_r) = \kappa(s_{i_r})$ and $B(s_{i_r}) \subseteq B(t_r)$
2. For $i_k < j < i_{k+1}$
   (a) $\kappa(s_j) \in B(t_{k+1})$
   (b) If $I(s_j, q) \in \mathrm{Succ}(I)$ then
   $$[(\kappa(s_{j-1}), \kappa(s_j))]^{<\omega} \cap B(t_{k+1}, \gamma_1) \times ... \times B(t_{k+1}, \gamma_{k-1}) \neq \emptyset$$
   where $I(s_{i-1}, q) + \sum_{i=1}^{k} \omega^{\gamma_i} = I(s_i, q)$ (Cantor normal form difference)
   (c) If $I(s_j, q) \in \mathrm{Lim}(I)$ then $B(s_j) \subseteq B(t_{k+1}) \cap \kappa(s_j)$

Lemma 4.16 *Let $G_I \subseteq \mathbb{M}_I[\vec{W}]$ be N-generic , define*

$$C_I = \bigcup \{\{\kappa(t_i) | i = 1, ..., n\} \mid \langle t_1, ..., t_n, t_{n+1}\rangle \in G_I\}$$

Then $N[G_I] = N[C_I]$

Lemma 4.17 *There is a projection $\pi : \mathbb{M}[\vec{W}] \to \mathbb{M}_I[\vec{W}]$.*

Corollary 4.18 *Let $C \subseteq C_G$ be closed, Assume that $I = I(C, C_G) \in N$ and consider $\pi_I, \mathbb{M}_I[\vec{W}]$, then $N[G_I] = N[C]$ where $G_I = \pi'' G \subseteq \mathbb{M}_I[\vec{W}]$.*

Lemma 4.19 *Let $G^* \subseteq \mathbb{M}[\vec{W}]$ be N-generic filter. Then the forcing $\mathbb{M}[\vec{W}]/G_I$ satisfies $\kappa^+ -$c.c. in $N[G^*]$.*

Theorem 4.20 $A \in N[C^*]$.

Proof. Let $I = I(Cl(C^*), C_G)$. Then

$$I, \mathbb{M}_I[\vec{W}], \pi_I \in N$$

Let G_I be the generic induced for $\mathbb{M}_I[\vec{W}]$ from G, it follows that $\mathbb{M}[\vec{W}]/G_I$ is defined in N. Toward a contradiction, assume that $A \notin N[C^*]$. By lemma 4.12, $N[C^*] = N[Cl(C^*)]$, hence $A \notin N[Cl(C^*)]$. Let $\underset{\sim}{A}$ be a name for A in $\mathbb{M}[\vec{U}]/G_I$ where $\pi_I'' G = G_I$. Work in $N[G_I]$, by corollary 4.18, $N[G_I] = N[Cl(C^*)]$. For every $\alpha < \kappa^+$ define

$$X_\alpha = \{B \subseteq \alpha \mid ||\underset{\sim}{A} \cap \alpha = B|| \neq 0\}$$

where the truth value is taken in $RO(\mathbb{M}[\vec{W}]/G_I)$- the complete boolean algebra of regular open sets for $\mathbb{M}[\vec{W}]/G_I$. Different B's in X_α yield incompatible conditions of $\mathbb{M}[\vec{W}]/G_I$ and we have κ^+-c.c by lemma 4.19 thus

$$\forall\alpha < \kappa^+ \ |X_\alpha| \leq \kappa$$

For every $B \in X_\alpha$ define

$$b(B) = ||\underset{\sim}{A} \cap \alpha = B||$$

Assume that $B' \in X_\beta$ and $\alpha \leq \beta$ then $B = B' \cap \alpha \in X_\alpha$. Moreover $b(B') \leq_B b(B)$ (we Switch to boolean algebra notation $p \leq_B q$ means p extends q). Note that for such B, B' if $b(B') <_B b(B)$, then there is

$$0 < p \leq_B (b(B) \setminus b(B')) \leq_B b(B)$$

Therefore

$$p \cap b(B') \leq_B (b(B) \setminus b(B')) \cap b(B') = 0$$

meaning $p \perp b(B')$. Work in $N[G^*]$, denote $A_\alpha = A \cap \alpha$. Recall that

$$\forall\alpha < \kappa^+ \ A_\alpha \in N[Cl(C^*)] = N[G_I]$$

thus $A_\alpha \in X_\alpha$. Consider the $\leq_B$-non-increasing sequence $\langle b(A_\alpha) \mid \alpha < \kappa^+\rangle$. If there exists some $\gamma^* < \kappa^+$ on which the sequence stabilizes, define

$$A' = \bigcup\{B \subseteq \kappa^+ \mid \exists\alpha \ b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha = B\} \in N[Cl(C^*)]$$

Claim that $A' = A$, notice that if B, B', α, α' are such that

$$b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha = B, \quad b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \alpha' = B'$$

WLOG $\alpha \leq \alpha'$ then we must have $B' \cap \alpha = B$ otherwise, the non zero condition $b(A_{\gamma^*})$ would force contradictory information. Consequently, for every $\xi < \kappa^+$ there exists $\xi < \gamma < \kappa^+$ such that

$$b(A_{\gamma^*}) \Vdash \underset{\sim}{A} \cap \gamma = A \cap \gamma$$

hence $A' \cap \gamma = A \cap \gamma$. This is a contradiction to $A \notin N[Cl(C^*)]$. We conclude that he sequence $\langle b(A_\alpha) \mid \alpha < \kappa^+\rangle$ does not stabilize. By regularity of κ^+, there exists a subsequence

$$\langle b(A_{i_\alpha}) \mid \alpha < \kappa^+\rangle$$

which is strictly decreasing. Use the observation we made to find $p_\alpha \leq_B b(A_{i_\alpha})$ such that $p_\alpha \perp b(A_{i_{\alpha+1}})$. Since $b(A_{i_\alpha})$ are decreasing, for any $\beta > \alpha$ $p_\alpha \perp b(A_{i_\beta})$ thus $p_\alpha \perp p_\beta$. This shows that $\langle p_\alpha \mid \alpha < \kappa^+\rangle \in N[G^*]$ is an antichain of size κ^+ which contradicts Lemma 4.19. $\blacksquare$

Sets of ordinals above κ^+: By induction on $\sup(A) = \lambda > \kappa^+$. It suffices to assume that λ is a cardinal.

case1: $cf^{V[G]}(\lambda)>\kappa$, the arguments for κ^+ works.

case2: $cf^{V[G]}(\lambda)\leq\kappa$ and since κ is singular in $V[G]$ then $cf^{V[G]}(\lambda)<\kappa$. Since $\mathbb{M}[\vec{U}]$ satisfies $\kappa^+-c.c.$ we must have that $\nu:=cf^V(\lambda)\leq\kappa$. Fix

$$\langle\gamma_i|\ i<\nu\rangle\in V$$

cofinal in λ. Work in $V[A]$, for every $i<\nu$ find $d_i\subseteq\kappa$ such that $V[d_i]=V[A\cap\gamma_i]$. By induction, there exists $C^*\subseteq C_G$ such that $V[\langle d_i\mid i<\nu\rangle]=V[C^*]$, therefore

1. $\forall i<\nu\ A\cap\gamma_i\in V[C^*]$
2. $C^*\in V[A]$

Work in $V[C^*]$, for $i<\nu$ fix

$$\langle X_{i,\delta}\mid\delta<2^{\gamma_i}\rangle=P(\gamma_i)$$

then we can code $A\cap\gamma_i$ by some δ_i such that $X_{i,\delta_i}=A\cap\gamma_i$. By 4.9, we can find $C''\subseteq C_G$ such that

$$V[C'']=V[\langle\delta_i\mid i<\nu\rangle]$$

Finally we can find $C'\subseteq C_G$ such that $V[C']=V[C^*,C'']$, it follows that $V[A]=V[C']$. ■

5 Classification of Intermediate Models

Let $G\subseteq\mathbb{M}[\vec{U}]$ be a V-generic filter. Assume that for every $\alpha\leq\kappa$, $o^{\vec{U}}(\alpha)<\alpha$. Let M be a transitive ZFC model such that $V\subseteq M\subseteq V[G]$. We would like to prove it is a generic extension of a "Magidor-like" forcing which we will define shortly. First, by [4], there is a set $A\in V[G]$ such that $V[A]=M$. By the results so far, there is $C'\subseteq C_G$ such that $M=V[A]=V[C']$.

Proposition 5.1 *Let $C,D\subseteq C_G$, then there is E, such that $C\cup D\subseteq E\subseteq C_G\cap\sup(C\cup D)$. such that $V[C,D]=V[E]$.*

Proof. By induction on $\sup(C\cup D)$. If $\sup(C\cup D)\leq C_G(\omega)$ then $|C|,|D|\leq\aleph_0$, we can take $E=C\cup D$, and

$$I(C,C\cup D),I(D,C\cup D)\subseteq\omega_1$$

and there fore in V. In the general case, consider $I(C,C\cup D),I(D,C\cup D)$. Since

$$o^{\vec{U}}(\sup(C\cup D))<\sup(C\cup D)$$

it follows that

$$\mathrm{otp}(C \cup D) \leq \mathrm{otp}(C_G \cap \sup(C \cup D)) < \sup(C \cup D)$$

Denote by $\lambda = \mathrm{otp}(C_G \cap \sup(C \cup D))$. By theorem 1.1, there is $F \subseteq C_G \cap \lambda$, such that

$$V[I(C, C \cup D), I(D, C \cup D)] = V[F]$$

We apply the induction hypothesis to $F, (C \cup D) \cap \lambda$ and find $E_* \subseteq \lambda$ such that

$$V[E_*] = V[F, (C \cup D) \cap \lambda]$$

Let $E = E_* \cup (D \cup C) \setminus \lambda$, then $E \in V[C, D]$ as the union of two sets in $V[C, D]$. In $V[E]$ we can find

$$E_* = E \cap \lambda \text{ and } (D \cup C) \setminus \lambda = E \setminus \lambda$$

Thus $F, (C \cup D) \cap \lambda \in V[E]$ and therefore also

$$D \cup C, I(C, C \cup D), I(D, C \cup D) \in V[E]$$

It follows that $C, D \in V[E]$.■

Corollary 5.2 *For every $C' \subseteq C_G$ there is $C^* \subseteq C_G \cap \sup(C')$, such that C^* is closed and $V[C'] = V[C^*]$.*

Proof. Again we go by induction on $\sup(C')$. If $\sup(C') = C_G(\omega)$ then $C^* = C'$ is already closed. For general C', consider $C' \subseteq Cl(C')$[6], then $I(C', Cl(C'))$ is bounded by some $\nu < \sup(C')$. So there is $D \subseteq C_G \cap \nu$ such that $V[D] = V[I(C', Cl(C'))]$. By the last proposition, we can find E such that

$$D \cup Cl(C') \cap \nu \subseteq E \subseteq C_G \cap \nu$$

and $V[E] = V[D, Cl(C')]$. By the induction hypothesis there is a closed E_*, such that $E \subseteq E^* \subseteq C_G \cap \nu$ such that $V[E] = V[E_*]$. Finally, let

$$C^* = E_* \cup \{\sup(E_*)\} \cup Cl(C') \setminus \nu$$

Then $C^* \in V[C']$, and also $Cl(C')$ and $I(C', Cl(C'))$ can be constructed in $V[C^*]$ so $C' \in V[C^*]$. Obviously, C^* is closed, hence, C^* is as desired.■

Definition 5.3 Let $\lambda < \kappa$ be any ordinal. A function $f : \lambda \to \kappa$ is said to be suitable for κ, if for every limit δ[7]

$$\limsup_{\alpha < \delta} f(\alpha) + 1 \leq f(\delta)$$

[6] For $A \subseteq On$, $Cl(A) = \{\alpha \mid \sup(A \cap \alpha) = \alpha\} \cup A$

[7] For a sequence of ordinals $\langle x_i \mid i < \rho \rangle$, define $\limsup_{i<\rho} x_i = \min(\{\sup_{\alpha<i<\rho} x_i \mid \alpha < \rho\})$

Proposition 5.4 *If $C^* \subseteq C_G$ is a closed subset, let $\lambda + 1 = \mathrm{otp}(C^* \cup \{\sup(C^*)\})$, and $\langle c^*_i \mid i \leq \lambda \rangle$ be the increasing continuous enumeration of C^*, then then function $f : \lambda+1 \to \kappa$, defined by $f(i) = o^{\vec{U}}(c^*_i)$ is suitable.*

Proof. Let $\delta < \lambda + 1$ be limit, then $c^*_\delta \in Lim(C_G \cup \{\kappa\})$ and therefore, there is $\xi < c^*_\delta$ such that for every $x \in C_G \cap (\xi, c^*_\delta)$, $o^{\vec{U}}(x) < o^{\vec{U}}(c^*_\delta)$. Let $\rho < \delta$ be such that $\xi < c^*_i < c^*_\delta$ for every $\rho < i < \delta$, then $\sup_{\rho<i<\delta} o^{\vec{U}}(c^*_i) + 1 \leq o^{\vec{U}}(c^*_\delta)$. Thus also

$$\min(\{\sup_{\alpha<i<\delta} o^{\vec{U}}(c^*_i) + 1 \mid \alpha < \delta\}) \leq o^{\vec{U}}(c^*_\delta)$$

■

We would like to define $\mathbb{M}_f[\vec{U}]$ for some suitable f, to be the forcing which construct a continuous sequence with orders as prescribed by f.

Definition 5.5 Let $f : \lambda+1 \to \kappa$ be suitable for κ, define the forcing $\mathbb{M}_f[\vec{U}]$, the conditions are functions F, such that:

1. F is finite partial function, with $Dom(F) \subseteq \lambda + 1$. such that $\lambda \in Dom(F)$.
2. For every $i \in Dom(F) \cap Lim(\lambda + 1)$:
   (a) $F(i) = \langle \kappa^{(F)}_i, A^{(F)}_i \rangle$.
   (b) $o^{\vec{U}}(\kappa^{(F)}_i) = f(i)$.
   (c) $A^{(F)}_i \in \cap\vec{U}(\kappa_i)$.
   (d) Let $j = \max(Dom(F) \cap i)$ or $j = -1$ if $i = \min(Dom(F))$, then for every $j < k < i$, $f(k) < f(i)$.
3. For every $i \in Dom(F) \setminus Lim(\lambda)$
   (a) $F(i) = \kappa^{(F)}_i$.
   (b) $o^{\vec{U}}(\kappa^{(F)}_i) = f(i)$.
   (c) $i - 1 \in Dom(F)$.
4. The map $i \mapsto \kappa^{(F)}_i$ is increasing.

Definition 5.6 The order of $\mathbb{M}_f[\vec{U}]$ is defined as follows $F \leq G$ iff

1. $Dom(F) \subseteq Dom(G)$.
2. For every $i \in Dom(G)$, let $j = \min(Dom(F) \setminus i)$.

(a) If $i \in Dom(F)$, then $\kappa_i^{(F)} = \kappa_i^{(G)}$, and $A_i^{(G)} \subseteq A_i^{(F)}$.

(b) If $i \notin Dom(F)$, then $\kappa_i^{(G)} \in A_j^{(F)}$, and $A_i^{(G)} \subseteq A_j^{(F)}$.

A straight forward verification shows that

Proposition 5.7 $\mathbb{M}_f[\vec{U}]$ *is a forcing notion.*

Note that if $f : \kappa + 1 \to \kappa$, defined by $f(\alpha) = o_L(\alpha)$ (see footnote 5). Then $\mathbb{M}_f[\vec{U}]$ is isomorphic to $\mathbb{M}[\vec{U}]$.[8]

Similar to $\mathbb{M}[\vec{U}]$, we have a decomposition $A_i^{(F)} = \biguplus_{j<o^{\vec{U}}(\kappa_i^{(F)})} A_{i,j}^{(F)}$. Also we have the notation $F^\frown\vec{\alpha}$ which we generalize from $\mathbb{M}[\vec{U}]$.

Proposition 5.8 *Let $H \subseteq \mathbb{M}_f[\vec{U}]$ be a V-generic filter. Let*

$$C_H^* = \{\kappa_i^{(F)} \mid i \in Dom(F), F \in H\}$$

Then

1. $\text{otp}(C_H^*) = \lambda + 1$ *and C_H^* is continuous.*
2. *For every $i < \lambda$, $o^{\vec{U}}(C_H^*(i)) = f(i)$.*
3. *$V[C_H^*] = V[H]$.*
4. *For every $\delta \in \text{Lim}(\lambda)$, and every $A \in \cap\vec{U}(\delta)$, there is $\xi < \delta$ such that $C^* \cap (\xi, \delta) \subseteq A$.*
5. *For every $\rho < \lambda$, $H \restriction \rho := \{F \restriction \rho \mid F \in H\}$ is V-generic for $\mathbb{M}_{f\restriction\rho}[\vec{U}]$.*

Proof. To see (1), let us argue by induction on $i < \lambda$ The set

$$E_i = \{F \in \mathbb{M}_f[\vec{U}] \mid i \in Dom(F)\}$$

is dense. Let $F \in \mathbb{M}_f[\vec{U}]$, if $i \in Dom(F)$ we are done. Otherwise, let

$$j_M := \min(Dom(F) \setminus i) > i > \max(Dom(F) \cap i) =: j_m$$

By condition 3, $j_M \in Lim(\lambda + 1)$. Split into two cases. First, if i is successor, then we can find $F \leq G$ such that $i - 1 \in Dom(G)$ by induction hypothesis. by condition 2.d and 2.b,

[8]Compare with proposition 2.19

$f(i) < o^{\vec{U}}(\kappa^{(F)}_{j_M})$. By condition 2.$c$, we can find $\alpha \in A^{(F)}_{j_M}$ such that $\alpha > \kappa^{i}_{j_m}$, $o^{\vec{U}}(\alpha) = f(i)$ and $A^{(F)}_{j_M} \cap \alpha \in \cap\vec{U}(\alpha)$. Then

$$G' = G \cup \{\langle i, \langle \alpha, A^{(F)}_{j_M} \cap \alpha\rangle\rangle\}$$

is as wanted. If i is limit, since f is suitable, there is $i' < i$, such that for every $i' < k < i$, $f(k) < f(i)$. Again by induction, find $F \leq G$ such that $i' \in Dom(G)$. Then the desired G' is construct as in successor step. Denote by F_H, the function with domain $\lambda + 1$, and $F_H(i) = \gamma$, be the unique γ such that for some $F \in H$, $i \in Dom(F)$ and $\kappa^{(F)}_i = \gamma$. Then it is clear that F_H is order preserving and $1-1$ from λ To C^*_H. By the same argument as for $\mathbb{M}[\vec{U}]$, we conclude also that F_H is continuous.

For (2), note that $C^*_H(i) = F_H(i)$, thus there is a condition $F \in H$ such that $F(i) = C^*_H(i)$. Hence $o^{\vec{U}}(C^*_H(i)) = f(i)$ by the definition of condition in $\mathbb{M}_f[\vec{U}]$.

For (3), as for $\mathbb{M}[\vec{U}]$, we note that H can be defined in terms of C^*_H as the filter $H_{C^*_H}$ of all the conditions $F \in \mathbb{M}_f[\vec{U}]$ such that for every $i \leq \lambda$,

1. If $i \in Dom(F)$, then $\kappa^{(F)}_i = C^*_H(i)$.
2. If $i \notin Dom(F)$, then $C^*_H(i) \in \bigcup_{i \in Dom(F)} A^{(F)}_i$.

(4) is again the standard density argument given for $\mathbb{M}[\vec{U}]$.

As for (5), note that the restriction function $\phi : \mathbb{M}_f[\vec{U}] \to \mathbb{M}_{f\restriction\rho}[\vec{U}]$ is a projection of forcings which suffices o conclude (5).■

The following theorem is a Mathias criteria for $\mathbb{M}_f[\vec{U}]$.

Theorem 5.9 *Let $f : \lambda \to \kappa$ be suitable, and let $C \subseteq \kappa$ be such that:*

1. $\mathrm{otp}(C) = \lambda$ *and C is continuous.*
2. *For every $i < \lambda$, $o^{\vec{U}}(C_i) = f(i)$.*
3. *For every $\delta \in \mathrm{Lim}(\lambda)$, and every $A \in \cap\vec{U}(C_\delta)$, there is $\xi < \delta$ such that $C \cap (\xi, \delta) \subseteq A$.*

Then There is a generic H for $\mathbb{M}_f[\vec{U}]$ such that $C^*_H = C$.

Proof.

Define H_C to consist of all the conditions $\langle F, A\rangle$ such that for every $i \in Dom(F)$:

1. $F(i) = (C)_i$.

2. $C \setminus \{\kappa_i^{(F)} \mid i \in Dom(F)\} \subseteq \bigcup_{i \in Dom(F)} A_i^{(F)}$.

We prove by induction on $\sup(C) = \kappa$ that H_C is V-generic. Assume for every $\rho < \kappa$ and any suitable function $g : \lambda \to \rho$, every C' satisfying $(1)-(3)$ the definition of $H_{C'}$ is generic. Let f, C as in the theorem. For every $\delta < \kappa$, by definition, $H_C \restriction \delta = H_{C \restriction \delta}$. Hence by the induction hypothesis $H_C \restriction \delta$ is generic. Obviously condition (1) insures that $C^*_{H_C} = C$. Also it is a straight forward verification that H_C is a filter. Let D be a dense open subset of $\mathbb{M}_f[\vec{U}]$.

Claim 1 For every $F \in \mathbb{M}_f[\vec{U}]$, there is $F \leq G_F$ such that

1. $\max(Dom(F) \cap \lambda)) = \max(Dom(G_F) \cap \lambda)$.

2. There is are $i_1^{(F)} < ... < i_k^{(F)}$ such that every $\langle \alpha_1, ..., \alpha_k \rangle \in \prod_{i=1}^k A_{\lambda,i}^{(F)}$, $G_F^\frown \langle \alpha_1, .., \alpha_n \rangle \in D$.

Proof. For every $i_1 < ... < i_k < o^{\vec{U}}(\kappa)$ and every $F \leq G$ such that

$$\max(Dom(F) \cap \lambda) = \max(Dom(G) \cap \lambda \text{ and } G(\lambda) = F(\lambda)$$

consider the set

$$B = \{\vec{\alpha} \in \prod_{j=1}^{k} A_{\lambda,i_j}^{(F)} \mid \exists R . G^\frown \vec{\alpha} \leq^* R \in D\}$$

Then

$$B \in \prod_{j=1}^{k} U(\kappa, i_j) \quad \vee \quad \prod_{j=1}^{k} A_{\lambda,i_j}^{(F)} \setminus B \in \prod_{j=1}^{k} U(\kappa, i_j)$$

Denote this set by B'. Find $B_{i_j} \in U(\kappa, i_j)$ such that $\prod_{j=1}^k B_{i_j} \subseteq B'$. Let $A^*_{G,i_1,..,i_n}$ be the set obtained by shrinking $A_{\lambda,i_j}^{(F)}$ to B_{i_j}. Since $o^{\vec{U}}(\kappa) < \kappa$ the possibilities for G and $i_1, ..., i_n$ is less than κ. So by κ-completness

$$A^* = \cap_{G,i_1,..,i_n} A^*_{G,i_1,...,i_n} \in \cap \vec{U}(\kappa)$$

Let $F \leq^* F^*$ be the condition obtained by shrinking $A_\lambda^{(F)}$ to A^*. By density, there is $G \geq F$ such that $G \in D$. So there is $\vec{\alpha} \in [A^*]^{<\omega}$ such that

$$(G \restriction \max(Dom(F) \cap \lambda) \cup \{\langle \lambda, \langle \kappa, A^* \}) ^\frown \vec{\alpha} \leq^* G$$

Hence for every $\vec{\beta}$ from the mesures of $\vec{\alpha}$, there is

$$G_{\vec{\beta}} \geq^* (G \restriction \max(Dom(F) \cap \lambda) \cup \{\langle \lambda, \langle \kappa, A^* \}\})^\frown \vec{\beta}$$

in D. Amalgamate all the $G_{\vec{\beta}}$'s to a single G^*. Then G^* is as wanted.■

For every F, pick G_F and A_F. Let $A^* = \Delta_F A_F$. There is $\xi < \kappa$ such that $C \cap (\xi, \kappa) \subseteq A^*$. Let F be a function in H_C such that for some $i \in Dom(F)$, $F(i) > \xi$. To see that there is such a condition, pick any $\delta \in C \setminus \xi$. Use the induction hypothesis, and find $F \in X_C$ such that $F \restriction \delta \in H_C \restriction \delta$.

By the claim, The set

$$E = \left\{ F \in \mathbb{M}_{f \restriction \xi}[\vec{U}] \mid \exists i_1 < ... < i_k.\ \forall \vec{\alpha} \in \prod_{j=1}^{k} A^*_{i_j}.\ G_F^\frown \vec{\alpha} \in D \right\}$$

is dense. Find $G^* \in H_C \restriction \xi \cap E$. We can find in the upper part $c_1 < c_2, ... < c_n \in C \cap A^*$ such that $c_j \in A^*_{i_j}$. Thus

$$(G^* \cup \{\langle \lambda, \langle \kappa, A^* \rangle \rangle\})^\frown \langle c_1, .., c_n \rangle \in H_C \cap D$$

And H_C is generic.

Theorem 5.10 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic and let $C^* \subseteq C_G$ be any closed subset. Let f be the suitable function derived from C^*. If $f \in V$, then there is a generic H for $\mathbb{M}_f[\vec{U}]$ such that $C^*_H = C^*$.*

Proof. since C_G satisfy the Mathias criteria, also does C^*.■

We will now prove that any transitive ZFC intermediate model $V \subseteq M \subseteq V[G]$ is a generic extension of a finite iteration of the form

$$\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_2}}[\vec{U}] ... * \mathbb{M}_{\underset{\sim}{f_n}}[\vec{U}]$$

We start with $M = V[C']$, then find a closed C^* such that $V[C'] = V[C^*]$. Let $\lambda_0 = \kappa$, recursively define $\lambda_{i+1} = \text{otp}(C_G \cap \lambda_i) < \lambda_i$. After finitely man steps we reach $\lambda_n \leq C_G(\omega)$, denote $\kappa_i = \lambda_{n-i}$. Consider

$$\langle o^{\vec{U}}(x) \mid x \in C * \cap (\kappa_{n-1}, \kappa_n) \rangle$$

This is added by a generic $E \subseteq C_G \cap \kappa_{n-1}$ Find a closed $C^*_{n-1} \in V[C^*]$ such that $V[C^*_{n-1}] = V[E, C^* \cap \kappa_{n-1}]$. Now consider

$$\langle o^{\vec{U}}(x) \mid x \in C^*_{n-1} \cap (\kappa_{n-2}, \kappa_{n-1}) \rangle$$

There is a closed generic $C^*_{n-2} \in V[C^*_{n-1}]$ such that

$$V[C^*_{n-2}] = V[C^*_{n-1}, \langle o^{\vec{U}}(x) \mid x \in C^*_{n-1} \cap (\kappa_{n-2}, \kappa_{n-1})\rangle]$$

In a similar fashion we find after finitely many steps, $\langle o^{\vec{U}}(x) \mid x \in C^*_0\rangle \in V$. Define

$$C_{fin} = C^*_0 \cup (C^*_1 \setminus \kappa_0) \cup (C^*_2 \setminus \kappa_1)....(C^* \setminus \kappa_{n-1})$$

Then C^*_{fin} is a closed, and have the property that for every $i \leq n$,

$$\langle o^{\vec{U}}(x) \mid x \in C^*_{fin} \cap [\kappa_{i-1}, \kappa_i)\rangle \in V[C^*_{fin} \cap \kappa_{i-1}]$$

Also $V[C^*_{fin}] = V[C^*] = M$.

Theorem 5.11 *Let f_i be the derived suitable function from $o^{\vec{U}''}[C^*_{fin} \cap (\kappa_{i-1}, \kappa_i)]$. Then:*

1. *$f_i \in V[C^*_{fin} \cap \kappa_{i-1}]$. Therefore $\mathbb{M}_{f_i}[\vec{U}]$ is defined in $V[C^*_{fin} \cap \kappa_{i-1}]$*
2. *There is a $V[C^*_{fin} \cap \kappa_{i-1}]$-generic filter $H \subseteq \mathbb{M}_{f_i}[\vec{U}]$ such that*
$$V[C^*_{fin} \cap \kappa_{i-1}][H] = V[C^*_{fin} \cap \kappa_{i-1}][C^*_{fin} \cap [\kappa_{i-1}, \kappa_i)] = V[C^*_{fin} \cap \kappa_i]$$
3. *Let $\underset{\sim}{f}_i$ be a $(\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f}_2}[\vec{U}]... * \mathbb{M}_{\underset{\sim}{f}_{i-1}}[\vec{U}])$-name for f_i, then there is a V-generic H^* for $\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f}_2}[\vec{U}]... * \mathbb{M}_{\underset{\sim}{f}_n}[\vec{U}]$ such that $V[H^*] = V[C^*_{fin}] = M$.*

Proof. (1) is clear by the construction of C_{fin}, and the fact that f_i is definable from $o^{\vec{U}''}[C^*_{fin} \cap (\kappa_{i-1}, \kappa_i)]$.

For (2), we use theorem 5.10.

(3) follows by (2) and by the definition of iteration.■

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