Spectrum of the club filter and another way of building master condition sequenses

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Abstract

We study a spectrum of Cub_κ for a measurable cardinal κ. The Woodin method of blowing up power of a measurable cardinal is extended.

1 Introduction

Which forcing notions can be used to blow up the power of a measurable cardinal preserving its measurability?

Starting with a Laver indestructible supercompact cardinal κ, basically any $\kappa-$directed closed forcing can be used.

The situation changes drastically if instead of a supercompacts we work under weaker assumptions, say no inner model with a strong.

H. Woodin was the first to show that it is possible, see [5]. He used the Cohen forcing for this. The main difficulty here is to obtain a generic object over the ultrapower with an extender which is not closed enough.

Later S. Friedman, K. Thompson [6] and S. Friedman, L. Zdomskii [7] showed that it is possible to use Sacks and Miller forcings for this purpose, as well. O. Ben-Neria and the author [3] used a non-stationary Cohen forcing. In this constructions, in contrast to Woodin's, a generic object over the ultrapower with an extender is already generated by pointwise image of those over V. C. Merimovich [9] used the extender based Radin forcing.

Our purpose here will be to give an additional example of such forcing. This forcing is due to Tom Benhamou and reminds the Mathias forcing. In order to show that it works, a certain replacement of the Woodin arguments introduced.

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Let κ be a measurable. Assume GCH. Let E be a $(\kappa, \kappa^{++})-$extender. Iterate the Cohen forcing $Cohen(\nu, \nu^{++})$ for blowing up the power of every inaccessible $\nu \leq \kappa$ to ν^{++} with the Easton support. Then by H. Woodin [5], j_E extends to a generic extension. Actually, H. Woodin used a further forcing $Cohen(\kappa^+, \kappa^{++})$ for this and later Y. Ben Shalom [4], showed that this additional forcing is unneeded. One of the crucial points in the Woodin argument was changing values of a generic set over M_E in order to insure $j''_E G \subseteq G^*$. This works fine for the Cohen forcing, but breaks down for a wide range of other forcing notions.

Here we would like to present a certain replacement of this argument.
As an application, we deal with the depth spectrum $Sp_{dp}(Cub_\kappa)$ for a measurable cardinal κ. This notion was introduced by Tom Benhamou [1]:

Definition 1.1

Let U be a filter. The *depth spectrum of* U, $Sp_{dp}(U)$ is the set of all regular cardinals θ such that there is a sequence $\langle A_i \mid i < \theta \rangle$ satisfying the following:

1. $A_i \in U$,
2. $i < i'$ implies $A_{i'} \subset^* A_i$,
3. there is no $A \in U$ such that $A \subseteq^* A_i$, for every $i < \theta$.

T. Benhamou [1] showed that the Cohen forcing $Cohen(\kappa, \kappa^{++})$ did not add κ^{++} to the depth spectrum. He showed also, using [2], that it is possible to have a measurable κ, $2^\kappa = \kappa^{++}$ and Cub_κ has $\subset^*$ $-$decreasing generating sequence. In particular, $Sp_{dp}(Cub_\kappa) = \{\kappa^{++}\}$.

Our application uses a variation of Mathias type forcing suggested by T. Benhamou.

Theorem 1.2 *Starting with* $o(\kappa) = \kappa^{++}$*, it is consistent to have a measurable* $\kappa, 2^\kappa = \kappa^{++}$ *and* $Sp_{dp}(Cub_\kappa) = \{\kappa^+, \kappa^{++}\}$.

2 Main construction

Let κ be a measurable cardinal, assume GCH and let E be a $(\kappa, \kappa^{++})-$extender. Let $j = j_E : V \to M_E$ be the ultrapower embedding by E. Consider $i = j_{E_\kappa} : V \to M_{E_\kappa}$ the ultrapower embedding by the normal measure of E and $k : M_{E_\kappa} \to M_E$ the connecting embedding.

We would like to force $2^\kappa = \kappa^{++}$ preserving a measurability of κ. Instead of using Cohen forcings, we would like to iterate a forcing notion suggested by Tom Benhamou.

For each inaccessible $\nu \leq \kappa$ define a forcing notion Q_ν. It will be $< -\nu-$support iteration of forcings that add a club which is contained mod bounded in the previously added clubs of the length ν^{++}, i.e. at stage $\alpha < \nu^{++}$ of such iteration over ν we have a decreasing mod bounded sequence $\langle C^\nu_\beta \mid \beta < \alpha\rangle$ of clubs of ν.

Definition 2.1 A condition in the forcing Q^α_ν will be of the form $\langle a, F\rangle$, where a is a closed subset of ν of cardinality $< \nu$, $F \subseteq \{C^\nu_\beta \mid \beta < \alpha\}$ of cardinality $< \nu$.
Set $\langle a, F\rangle \leq \langle a', F'\rangle$ iff a' is an end extension of a, $F \subseteq F'$ and for every $C \in F$, $a' \setminus a \subseteq C$.

Q_ν is a $< \nu-$closed forcing which satisfies ν^+-c.c.
Consider a subset Q'_ν of Q_ν which consists of all $\{r = \{\langle r_i, R_i\rangle \mid i \in \mathrm{supp}(r)\}$ such that

1. $\mathrm{supp}(r) \in V$,

2. for every $i \in \mathrm{supp}(r)$, the pair $\langle r_i, R_i\rangle$ is already decided by $r \restriction i$

It is not hard to see that such Q'_ν is dense in Q_ν. We will identify between them further.

If $\nu < \kappa$ is an accessible, then set Q_ν to be the trivial forcing.

Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle$ be the Easton support iteration of this forcing notions.

Let $G_{\kappa+1} \subseteq P_{\kappa+1}$ be a generic. Denote by $G(Q_\kappa)$ the generic subset of Q_κ induced by $G_{\kappa+1}$.

The issue will be to find an $M_E[G_{j(\kappa)}]-$generic set $G^* \subseteq Q_{j(\kappa)}$ such that $j''G(Q_\kappa) \subseteq G^*$.

We will go through $M_{E_\kappa}[G_{i(\kappa)}]$ and use k to transfer things to $M_E[G_{j(\kappa)}]$.
Consider $Q_{i(\kappa)}$ and define in $V[G_{\kappa+1}]$ a subset of it Q^* as follows.
Let $p = \langle p_\xi \mid \xi \in \mathrm{supp}(p)\rangle \in Q^*$ iff for every $\rho_1 < \rho_2 < \kappa^{++}$, if $i(\rho_1), i(\rho_2) \in \mathrm{supp}(p)$, then $p_{i(\rho_1)} \setminus \kappa \supseteq p_{i(\rho_2)} \setminus \kappa$.

Recall that $j = k \circ i$, so for every ξ (and, in particular, for $\xi = \kappa^{++}$ of V) $j(\xi) = k(i(\xi))$. In addition, $\bigcup i''\kappa^{++} = i(\kappa^{++}) = \kappa^{++}$.

We force with Q^* over $V[G_{\kappa+1}]$. Let G^* be a generic.
Let us argue that such G^* is $M_{E_\kappa}[G_{i(\kappa)}]-$generic for $i(Q_\kappa)$.

Let A be a maximal antichain of $i(Q_\kappa)$ in $M_{E_\kappa}[G_{i(\kappa)}]$. Pick, in $M_{E_\kappa}[G_{i(\kappa)}]$, an elementary submodel N of cardinality $i(\kappa)$ with $A \in N$ and ${}^{<i(\kappa)}N \subseteq N$.
Then $A \subseteq N$, by $i(\kappa)-$c.c. of the forcing. Also note that for every condition $p \in N$, $p \subseteq N$, since its cardinality $< i(\kappa)$ and N is closed enough.

Now, in $V[G_{\kappa+1}, G^*]$, consider $a = i''\kappa^{++} \cap N$.
Then $|a| \leq \kappa$. So, $a \in M_{E_\kappa}[G_{i(\kappa)}]$, and hence, $a \in N$.
Consider the following condition - $p = \{\langle \emptyset, a \cap i\rangle \mid i \in a\}$. Clearly, it is in $Q^* \cap N$ and $p \in G^*$.

Note that for every $q \in N \cap Q_{i(\kappa)}$, if $q \geq p$, then $q \in Q^*$.
In N, let $D \subseteq Q_{i(\kappa)}$ to be the set of all $q \geq p$ such that q is stronger than some element of A. Then such D is a dense open above p subset of $Q_{i(\kappa)}$ in $M_{E_\kappa}[G_{i(\kappa)}]$. Let, in N, $A' \subseteq D$ to be a maximal antichain above p.

Lemma 2.2 *A' is a maximal antichain in Q^* above p.*

Proof. Suppose otherwise. Then there is $t \in Q^*, t \geq p$ which is incompatible with every element of A'.
Let $t = \{\langle t_i, T_i \rangle \mid i \in i \in \mathrm{supp}(t)\}$.
Set $b = \mathrm{supp}(t) \cap N$. Then $b \in N$, since $|\mathrm{supp}(t)| < i(\kappa)$.
Find some $c \in N$ which is order isomorphic to $\mathrm{supp}(t)$ over b. Let $\pi : c \leftrightarrow \mathrm{supp}(t)$ be the isomorphism.
Now we copy the information from missing in N coordinates of t to c. Proceed as follows: let $t' = \{\langle t'_i, T'_i \rangle \mid i \in c\}$, where

1. for every $i \in b$, $\langle t'_i, T'_i \rangle = \langle t_i, T_i \rangle$,
2. for every $i \in c \setminus b$, $\langle t'_i, T'_i \rangle = \langle t_{\pi(i)}, \pi'' T_i \rangle$.

Extend t' inside N to some $t'' = \{\langle t''_i, T''_i \rangle \mid i \in \mathrm{supp}(t'')\}$. which is stronger than an element of A'.
Finally, let $t^* = \{\langle t^*_i, T^*_i \rangle \mid i \in i \in \mathrm{supp}(t^*)\}$ be obtained from t and t'' as follows.
First set $\mathrm{supp}(t^*) = \mathrm{supp}(t) \cup \mathrm{supp}(t'')$. Then, for every $i \in \mathrm{supp}(t'')$, let $\langle t^*_i, T^*_i \rangle = \langle t''_i, T''_i \rangle$.
Finally, for every $i \in \mathrm{supp}(t) \setminus \mathrm{supp}(t^*)$, let $\langle t^*_i, T^*_i \rangle = \langle t''_{\pi^{-1}(i)}, \pi^{-1\prime\prime} T^*_i \rangle$.
Then $t^* \geq t, t''$ and $t^* \in Q^*$, since $a = i''\kappa^{++} \cap N, t'' \in N$, and so, no new coordinates of from $i''\kappa^{++}$ are added to t.
Contradiction.
□

Applying the lemma, $G^* \cap A' \neq \emptyset$, and hence $G^* \cap A \neq \emptyset$.
So, G^* is $M_{E_\kappa}[G_{i(\kappa)}]-$generic subset of $i(Q_\kappa)$.

Now, apply the embedding k and move G^* to M_E. Let
$G^{**} = \{r \in k(Q_\kappa) \mid \exists s \in G^*(r \leq s\}$.
G^{**} then will be $M_E[G_{j(\kappa)}]-$generic subset of $j(Q_\kappa)$. Denote by $G^{**}_\xi, \xi < j(\kappa^{++})$ the club added by G^{**} at the coordinate ξ. Then, for every $\alpha < \beta < \kappa^{++}$, we will have $G^{**}_{j(\beta)} \setminus \kappa \subseteq G^{**}_{j(\alpha)} \setminus \kappa$.
Finally, as in the Woodin original argument, we use the completeness of the forcing to alter

$G^{**} \restriction \kappa$ in order to insure that for every $\alpha < \kappa^{++}$, $G^{**}_{j(\alpha)} \cap \kappa = G(Q_\kappa)$, where $G(Q_\kappa)$ is a generic subset of Q_κ induced by $G_{\kappa+1}$.
Let G^{***} be the resulting $M_E[G_{j(\kappa)}]-$generic subset of $j(Q_\kappa)$.
So, j extends to $j^* : V[G_{\kappa+1}] \to M_E[G_{j(\kappa)}, G^{***}]$.

3 Applications

The following result follows directly from the construction.

Theorem 3.1 *Starting with $o(\kappa) = \kappa^{++}$, it is consistent to have a measurable $\kappa, 2^\kappa = \kappa^{++}$ and $\kappa^{++} \in Sp_{dp}(Cub_\kappa)$.*

Proof. Let $\langle C_\alpha \mid \alpha < \kappa^{++}\rangle$ be the generic $\subseteq^*$ $-$decreasing sequence of clubs added by $G(Q_\kappa)$. We need only to argue that there is no club C such that $C \subseteq^* C_\alpha$, for every $\alpha < \kappa^{++}$. This follows by κ^+-c.c. of the forcing. Namely, if there was such a club C, then it will be added at some bounded step β of the iteration Q_κ. But, then, by density arguments, already $C \subseteq^* C_\beta$ fails.
□

Let us prove now the theorem stated in the introduction. It will rely on the following general proposition:

Proposition 3.2 *Suppose that there is an elementary embedding $j : V \to M$ with a critical point κ such that*

1. *there is $\delta < j(\kappa)$ with no generators in the interval $(\delta, j(\kappa))$,*[1]
2. *$\mathrm{cof}(j(\kappa))$, in V, is λ,*

Then there is a sequence of clubs $\langle C_\alpha \mid \alpha < \lambda\rangle$ such that there is no unbounded C, $C \subseteq^* C_\alpha$, for every $\alpha < \lambda$.

Proof. Let $\langle \lambda_\alpha \mid \alpha < \lambda\rangle$ be a cofinal in λ sequence with $\lambda_0 > \delta$. For every $\alpha < \lambda$ there is $h_\alpha : \kappa \to \kappa$ such that for some $\rho \le \delta$, $j(h_\alpha)(\rho) = \lambda_\alpha$. Set

$$C_\alpha = \{\nu < \kappa \mid \forall \nu' < \nu (h_\alpha(\nu') < \nu)\}.$$

Let us argue that the sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$ is as desired.

[1] $\rho < j(\kappa)$ is called a generator, if for every $f : \kappa \to \kappa$, for every $\xi < \rho$, $j(f)(\xi) \neq \rho$.

Suppose that there is an unbounded $C \subseteq \kappa$ such that $C \subseteq^* C_\alpha$, for every $\alpha < \lambda$. Then , for every $\alpha < \lambda$, $j(C) \setminus \kappa \subseteq j(C_\alpha) \setminus \kappa$.
Note that $j(C_\alpha) \cap (\delta, \lambda_\alpha] = \emptyset$, for every $\alpha < \lambda$, by elementarity of j. Then, $j(C) \cap (\delta, \lambda_\alpha] = \emptyset$. But this holds for every for every $\alpha < \lambda$. Hence, $j(C) \cap (\delta, j(\kappa)) = \emptyset$, and so, $j(C)$ is a bounded subset of $j(\kappa)$ which implies that C is a bounded subset of κ. Contradiction.
□

Corollary 3.3 *Under the assumption of 3.2,* $\mathrm{cof}(j(\kappa))$ *in* V *must be at least* κ^+.

Proof. Otherwise we can take C to be the intersection or diagonal intersection of C_α's.
□

Corollary 3.4 *Let* U *be a* $\kappa-$*complete ultrafilter over* κ. *Then there is a sequence of clubs* $\langle C_\alpha \mid \alpha < \mathrm{cof}(j_U(\kappa))\rangle$ *such that there is no unbounded* C, $C \subseteq^* C_\alpha$, *for every* $\alpha < \mathrm{cof}(j_U(\kappa))$.

Proof. Note that there is no generators of j_U above $[id]_U$.
□

Corollary 3.5 *Assume* $2^\kappa = \kappa^+$. *Let* U *be a* $\kappa-$*complete ultrafilter over* κ. *Then there is a sequence of clubs* $\langle C_\alpha \mid \alpha < \kappa^+\rangle$ *such that there is no unbounded* C, $C \subseteq^* C_\alpha$, *for every* $\alpha < \kappa^+$.

Proof. It follows from 3.4, since $2^\kappa = \kappa^+$ implies $\mathrm{cof}(j_U(\kappa)) = \kappa^+$.
□

Theorem 3.6 *Starting with* $o(\kappa) = \kappa^{++}$, *it is consistent to have a measurable* $\kappa, 2^\kappa = \kappa^{++}$ *and* $Sp_{dp}(Cub_\kappa) = \{\kappa^+, \kappa^{++}\}$.

Proof. Start with a model with an extender E constructed in [8]. Then $\mathrm{cof}(j_E(\kappa)) = \kappa^+$ and all the generators of j_E are below κ^{++}.
Use now the construction of the previous section.
We will obtain a generic extension in which $\kappa^{++} \in Sp_{dp}(Cub_\kappa)$ and j_E extends. By 3.2, $\kappa^+ \in Sp_{dp}(Cub_\kappa)$ as well, since $j_E(\kappa)$ was not changed and generators cannot be added.
□

The method extends to longer chains. For example:

Theorem 3.7 *Starting with* $o(\kappa) = \kappa^{+3}$, *it is consistent to have a measurable* $\kappa, 2^\kappa = \kappa^{+3}$ *and* $Sp_{dp}(Cub_\kappa) = \{\kappa^+, \kappa^{++}, \kappa^{+3}\}$.

Proof. We start with a $(\kappa,\kappa^{+3})-$extender produced as in [8], but now for the gap 3. This will insure, as in 3.6, that $\kappa^+ \in Sp_{dp}(Cub_\kappa)$.

Now use the following variation of the forcing of the previous section:

At each inaccessible $\nu \leq \kappa$ instead of adding an almost decreasing sequence of clubs of the length ν^{++}, we add two such sequences - one of the length ν^{++} and another of the length ν^{+3}. Let $Q_\nu(\nu^{++})$ and $Q_\nu(\nu^{+3})$ be the corresponding forcing notions. We will use $Q_\nu = Q_\nu(\nu^{++}) \times Q_\nu(\nu^{+3})$ at the stage ν. Using ν^+-c.c. and commutativity, it is not hard to argue that no pseudo intersection is added to any of two generic decreasing sequences of clubs added by Q_ν.

This gives the desired conclusion.

$\square$

Deal now with the following question:

Is it possible to have a measurable κ such that $Sp_{dp}(Cub_\kappa) = \{\kappa^+, \kappa^{+3}\}$?

Tom Benhamou informed us that he with coauthors gave the affirmative answer using supercompacts. We will show that this can be done from optimal assumptions.

Theorem 3.8 *Starting with $o(\kappa) = \kappa^{+3}$, it is consistent to have a measurable $\kappa, 2^\kappa = \kappa^{+3}$ and $Sp_{dp}(Cub_\kappa) = \{\kappa^+, \kappa^{+3}\}$.*

Proof. Proceed as in 3.7, only use Q_ν's which add a $\subseteq^*$ $-$sequences of the length ν^{+3} only.

Let us argue that no $\subseteq^*$ $-$sequences of the length κ^{++} of clubs in κ without pseudo intersection was added.

Suppose otherwise. Let $\langle E_\alpha \mid \alpha < \kappa^{++}\rangle$ be such a sequence and let $\langle C_\alpha \mid \alpha < \kappa^{+3}\rangle$ is a generic sequence of clubs added by a generic $G \subseteq Q_\kappa$.

Pick nice names $\langle \underset{\sim}{E}_\alpha \mid \alpha < \kappa^{++}\rangle$ and $\langle \underset{\sim}{C}_\alpha \mid \alpha < \kappa^{+3}\rangle$ for such sequences. Suppose that the weakest condition already forces the above situation.

By κ^+-c.c. of the forcing, each $\underset{\sim}{E}_\alpha$ depends only on a set $A_\alpha \subseteq \kappa^{+3}, |A_\alpha| \leq \kappa$ of indexes of the forcing Q_κ. Without loss of generality assume that each A_α have a maximal element of cofinality $\geq \kappa^+$. Denote it by β_α.

Shrink to form a $\Delta-$system. Suppose for simplicity that A_α's already form a $\Delta-$system. Also, assume for simplicity that β_α's are strictly increasing and $A_\alpha \cap \beta_\alpha$ is the cornel.

We can assume, using niceness of $\underset{\sim}{E}_\alpha$'s and shrinking if necessary, that there is a single term s such that for every $\alpha < \kappa^{++}$, $s(C_\alpha) = E_\alpha$ and this is forced by the weakest condition. Just the number of isomorphism types of such nice names is κ^+.

Pick now some $\beta^*, \bigcup_{\alpha<\kappa^{++}} \beta_\alpha \leq \beta^* < \kappa^{+3}$ of cofinality bigger than κ.

Lemma 3.9 *For every $\alpha < \kappa^{++}$, C_{β^*} is $Q_\kappa^{\beta_\alpha}-$generic club.*

Proof. It is enough to show that each maximal antichain of $Q_\kappa^{\leq \beta_\alpha}$ remans such also in $Q_\kappa^{\leq \beta^*}$ Suppose otherwise. Let $A \subseteq Q_\kappa^{\leq \beta_\alpha}$ be a maximal antichain in $Q_\kappa^{\leq \beta_\alpha}$, but it is not a maximal in $Q_\kappa^{\leq \beta^*}$.

Then there is $t \in Q_\kappa^{\leq \beta^*}$ which is incompatible with every element of A.

Let $t = \{\langle t_i, T_i \rangle \mid i \in i \in \mathrm{supp}(t)\}$.

Set $b = \mathrm{supp}(t) \cap \beta_\alpha$.

Find some $c \subseteq \beta_\alpha$ which is order isomorphic to $\mathrm{supp}(t)$ over b. Let $\pi : c \leftrightarrow \mathrm{supp}(t)$ be the isomorphism.

Now we copy the information from missing in β_α coordinates of t to c. Proceed as follows: let $t' = \{\langle t'_i, T'_i \rangle \mid i \in c\}$, where

1. for every $i \in b$, $\langle t'_i, T'_i \rangle = \langle t_i, T_i \rangle$,
2. for every $i \in c \setminus b$, $\langle t'_i, T'_i \rangle = \langle t_{\pi(i)}, \pi'' T_i \rangle$.

Extend t' inside $Q_\kappa^{\leq \beta_\alpha}$ to some $t'' = \{\langle t''_i, T''_i \rangle \mid i \in i \in \mathrm{supp}(t'')\}$ which is stronger than an element of A.

Finally, let $t^* = \{\langle t^*_i, T^*_i \rangle \mid i \in i \in \mathrm{supp}(t^*)\}$ be obtained from t and t'' as follows.

First set $\mathrm{supp}(t^*) = \mathrm{supp}(t) \cup \mathrm{supp}(t'')$. Then, for every $i \in \mathrm{supp}(t'')$, let $\langle t^*_i, T^*_i \rangle = \langle t''_i, T''_i \rangle$.

Finally, for every $i \in \mathrm{supp}(t) \setminus \mathrm{supp}(t^*)$, let $\langle t^*_i, T^*_i \rangle = \langle t''_{\pi^{-1}(i)}, \pi^{-1\prime\prime} T^*_i \rangle$.

Then $t^* \geq t, t''$ and $t^* \in Q_\kappa^{\leq \beta^*}$.

Contradiction.

□

Denote now $s(C_{\beta^*})$ by E^*. Then, by Lemma 3.9, for every $\gamma < \kappa^{++}$, $E^* \subseteq^* E_\gamma$. Just pick some $\alpha < \kappa^{++}$ above γ and apply the lemma to $Q_\kappa^{\beta_\alpha}$.

This is impossible, since the sequence $\langle E_\gamma \mid \gamma < \kappa^{++} \rangle$ has no pseudointersection.

□

4 A remark on clubs over a measurable

Proposition 4.1 *Assume that $\neg O^\P$. Suppose that κ is a measurable and there is a club C such that $C \subseteq^* E$, for every club E in $\mathcal{K}$.*
Then there exists a repeat point over κ in $\mathcal{K}$ and C is a Radin type club.

Proof. Let $j : V \to M$ be an ultrapower embedding by a normal measure over κ.
Consider $i = j \restriction \mathcal{K}$. By Mitchell, it is an iterated ultrapower embedding by measures in $\mathcal{K}$.

By the assumption, $j(C) \setminus \kappa \subseteq i(E)$, for every club E in $\mathcal{K}$.

Let $\langle \rho_\alpha \mid \alpha < \delta \rangle$ be the increasing sequence of all principal generators of i, i.e. images of κ under the iteration.

Claim 1 $j(C) \setminus \kappa \subseteq \{\rho_\alpha \mid \alpha < \delta\}$.

Proof. Let $\tau < j(\kappa)$ and suppose that it is not a generator.
Then there are $n < \omega, \rho_{\alpha_1} < ... < \rho_{\alpha_n} < \tau$ and a function $f : \kappa^n \to \kappa, f \in \mathcal{K}$ such that

$$i(f)(\rho_{\alpha_1}, ..., \rho_{\alpha_n}) \geq \tau.$$

Consider in $\mathcal{K}$ a club

$$C_f = \{\nu < \kappa \mid f''\nu^n \subseteq \nu\}.$$

Then $\tau \notin i(C_f) \supseteq j(C)$.
$\square$ of the claim.

Note that ρ_i's are images of κ under the iterated ultrapower embeddings.
So, $j(C)$ satisfies a Mathias type criterion for Radin type genericity in M over $i(\mathcal{K})$. Then the same is true for C in V.

Let us argue that there must be a repeat point.
Namely, we show that

$$U = \{X \subset \kappa \mid X \in \mathcal{K}, \kappa \in i(X)\}$$

is a repeat point.
Otherwise, there is $A \in U$ which does not belong to any smaller measure over κ in $\mathcal{K}$. By intersecting A with the set

$$\{\nu \in A \mid \text{for every measure } W \text{ over } \nu \text{ in } \mathcal{K}, A \cap \nu \notin W\},$$

if necessary, we may assume that A does not belong to any smaller measure over κ in $\mathcal{K}$.

Now, $A \cap C$ is unbounded in κ, since $\kappa \in j(C) \cap j(A)$. So, $j(A \cap C)$ is an unbounded subset of $j(\kappa)$. Then, for every $B \in j(U), j(C \cap A) \subseteq^* B$, by the Radiness of $j(C)$ over $j(\mathcal{K})$ in M and the choice of A, since at each point of $j(A \cap C) \setminus \kappa$ an extender with normal measure which is an image of U was used.
By elementarity, in V, $C \cap A \subseteq^* B$, for every $B \in U$.

However, $C \cap A \in M$, and so, U can be defined there using it which is impossible. Contradiction.
$\square$

Proposition 4.2 *Suppose that U is a normal ultrafilter over κ. Let $j : V \to M$ be its ultrapower embedding. Assume that $\mathrm{cof}(j(\kappa))$, in V, is λ,*

Then there is a sequence of clubs $\langle C_\alpha \mid \alpha < \lambda \rangle$ such that there is no unbounded C, $C \subseteq^* C_\alpha$, for every $\alpha < \lambda$.
The same for any $\lambda-$many C_α's.

Proof. Let $\langle \lambda_\alpha \mid \alpha < \lambda \rangle$ be a cofinal in λ sequence with $\lambda_0 > \kappa$. For every $\alpha < \lambda$ there is $h_\alpha : \kappa \to \kappa$ such that $j(h_\alpha)(\kappa) = \lambda_\alpha$. Set

$$C_\alpha = \{\nu < \kappa \mid \forall \nu' < \nu (h_\alpha(\nu') < \nu)\}.$$

Let us argue that the sequence $\langle C_\alpha \mid \alpha < \lambda \rangle$ is as desired.

Suppose that there is an unbounded $C \subseteq \kappa$ such that $C \subseteq^* C_\alpha$, for every $\alpha < \lambda$. Then , for every $\alpha < \lambda$, $j(C) \setminus \kappa \subseteq j(C_\alpha) \setminus \kappa$.
Note that $j(C_\alpha) \cap (\kappa, \lambda_\alpha] = \emptyset$, for every $\alpha < \lambda$, by elementarity of j. Then, $j(C) \cap (\kappa, \lambda_\alpha] = \emptyset$. But this holds for every for every $\alpha < \lambda$. Hence, $j(C) \cap (\kappa, j(\kappa)) = \emptyset$, and so, $j(C)$ is a bounded subset of $j(\kappa)$ which implies that C is a bounded subset of κ. Contradiction.
$\square$

Proposition 4.3 *Suppose that κ is a measurable, $\lambda > \kappa^+$ is a regular and there is a $\subseteq^*$ $-$decreasing sequence of clubs of the length λ that generates Cub_κ.*
Then

1. *$\mathrm{cof}(j_U(\kappa))$ in V is λ,*
2. *$U \cap \mathcal{K}$ is a repeat point,*
3. *there is a Radin type club in κ over $\mathcal{K}$.*

Proof. By the previous propositions, we need only to show that $\mathrm{cof}(j_U(\kappa)) = \lambda$.
Suppose otherwise. Let $\mathrm{cof}(j_U(\kappa)) = \mu \neq \lambda$. Let $\langle \lambda_\alpha \mid \alpha < \mu \rangle$ and $\langle C_\alpha \mid \alpha < \mu \rangle$ be as in 4.2. Let $\langle E_\beta \mid \beta < \lambda \rangle$ be a $\subseteq^*$ $-$decreasing sequence of clubs of the length κ^{++} that generates

Cub_κ.
Then for every $\alpha < \lambda$, there is $\beta_\alpha < \kappa^{++}$ such that for every $\beta, \beta_\alpha \leq \beta < \kappa^{++}$

$$E_\beta \subseteq^* C_\alpha.$$

If $\mu < \lambda$, then let $\beta^* = \bigcup_{\alpha<\mu} \beta_\alpha$. Then $E_{\beta^*} \subseteq^* C_\alpha$, for every $\alpha < \mu$.
If $\mu > \lambda$, then there must be $\beta^* < \kappa^{++}$ such that $E_{\beta^*} \subseteq^* C_\alpha$ for $\lambda-$many α's, which is still impossible.
$\square$

We do not know whether $o(\kappa) = \kappa^{++}$ suffices in order to have a $\subseteq^*$ $-$decreasing sequence of clubs of the length κ^{++} which generates Cub_κ over a measurable cardinal κ. T. Benhamou [1] proved the consistency starting with an assumption slightly below $o(\kappa) = \kappa^{+3}$. The next result is in a negative direction.

Proposition 4.4 *Assume that $o(\kappa) \leq \kappa^{++}$. Suppose that κ is a measurable, U is a normal ultrafilter over κ. Assume that:*

1. *there is a club C such that $C \subseteq^* E$, for every club E in $\mathcal{K}$,*
2. *$\mathrm{cof}(j_U(\kappa))$ in V is κ^{++}.*

Then

1. *$U \cap \mathcal{K}$ is a repeat point,*
2. *every normal ultrafilter $W \in \mathcal{K}$ over κ such that $U \cap \mathcal{K} \trianglelefteq W$ is a repeat point as well.*

Proof. Suppose otherwise. Let $W \in \mathcal{K}$ be a normal ultrafilter over κ such that $U \cap \mathcal{K} \trianglelefteq W$, but W is not a repeat point.
Pick $A \in W$ which does not belong to smaller measures. Define a set

$$B = \{\nu < \kappa \mid \exists F \text{ a normal ultrafilter over } \nu \text{ in } \mathcal{K} \text{ such that } A \cap \nu \in F\}.$$

Then for every normal ultrafilter $\mathcal{V} \rhd W$ in $\mathcal{K}$ over κ, $B \in \mathcal{V}$, by coherency.

We have $\mathrm{cof}(j_U(\kappa))$ in V is κ^{++}. Hence, for every $\alpha < \kappa^{++}$, a normal measure over $j_U(\kappa)$ of the Mitchell order $\beta_\alpha \geq j_U(\alpha)$ in $\mathcal{K}^{M_U}$ was applied in the iteration.
In particular, $j_U(B) \cap j_U(C)$ is unbounded in $j_U(\kappa)$. It is not hard then to define $j_U(W)$ from $j_U(C \cap B)$:

$$X \in j_U(W) \text{ iff for all but boundedly many } \nu \in j_U(C \cap B), X \cap \nu \in F_\nu, \text{ where } F_\nu \in \mathcal{K}^{M_U}$$

is the least in the Mitchell order normal ultrafilter over ν such that $j_U(A) \cap \nu \in F_\nu$.

Now, by elementarity, $C \cap B$ can be used in the same fashion to define W in V, and then, in M_U, which is impossible.
Contradiction.
$\square$

Remark 4.5 Starting with $o(\kappa) = \kappa^{++}$ it is possible to build a model which satisfies the assumptions of 4.4. We pick a measure U in K which satisfies (2) of the conclusion and with the corresponding Radin forcing. In such extension U and every $W \trianglerighteq U$ extends. Then we force as in [8] to blow up the power of κ to κ^{++} preserving measurability.

Let us extend the above more and show that even stronger repeat points are needed.

Proposition 4.6 *Assume that $o(\kappa) \leq \kappa^{++}$. Suppose that κ is a measurable, U is a normal ultrafilter over κ. Assume that:*

1. *there is a club C such that $C \subseteq^* E$, for every club E in $\mathcal{K}$,*
2. *$\mathrm{cof}(j_U(\kappa))$ in V is κ^{++}.*

Then $U \cap \mathcal{K}$ is not the first repeat point such that every normal ultrafilter $W \in \mathcal{K}$ over κ such that $U \cap \mathcal{K} \trianglelefteq W$ is a repeat point as well.

Proof. Suppose otherwise. Work in $\mathcal{K}$. For every measure $F \trianglelefteq U \cap \mathcal{K}$, pick the $\trianglelefteq$-least $W_F \trianglerighteq F$which is not a repeat point. Let $A_F \subseteq \kappa$ be the $\mathcal{K}-$ least such that $A_F \in W_F$, but it doest not belong to any smaller measure.
Let $\langle A_\alpha \mid \alpha < \kappa^+ \rangle$ be the canonical enumeration of all subsets of κ. Fix $I \subseteq \kappa^+$ such that $\langle A_\alpha \mid \alpha \in I \rangle$ enumerates all A_F's.

Now in V, for every $\alpha \in I$, let

$$C_\alpha = \{\nu < \kappa \mid o(\nu) > 0 \text{ and there is some } \mu < o(\nu) \text{ with } A_\alpha \cap \nu \text{ in a measure over } \nu \text{ in } \mathcal{K}\}.$$

We have $C \subseteq^* C_\alpha$, for every $\alpha \in I$.

Turn to the ultrapower M_U. It will satisfy $j_U(C) \setminus \kappa \subset j_U(C_\alpha)$, for every $\alpha \in I$.
Recall that members of $j_U(C) \setminus \kappa$ are images of κ under the iteration $j_U \restriction \mathcal{K}$.
Then for every $\xi \in j_U(C) \setminus \kappa$, each $j_U(A_\alpha) \cap \xi$ belongs to a measure over the corresponding image $\mathcal{K}_\xi$ of $\mathcal{K}$. By elementarity, this implies that the measure used in the iteration to move ξ must be the image of $U \cap \mathcal{K}$ or one which is above it.
But this allows us to define $j_U(U \cap \mathcal{K})$ in M_U using $j_U(C) \setminus \kappa$:

$$X \in j_U(U \cap \mathcal{K}) \text{ iff starting with some } \xi_0, \text{ for every } \xi \geq \xi_0 \text{ in } j_U(C) \setminus \kappa,$$

$$\text{either } \xi \in X, \text{ if there is no repeat points below } o(\xi),$$

$$\text{or there are repeat points in the sense of the theorem, and then,}$$

$$X \text{ belongs to the first such .}$$

By elementarity, we will have a club which defines $U \cap \mathcal{K}$ in V in the same fashion. But then it can be used to do define $U \cap \mathcal{K}$ also in M_U, which is impossible.
Contradiction.
$\square$

Remark 4.7 1. It is not hard to extend the previous argument from the first to second, third etc. repeat point. This goes all the way beyond κ^+.
2. Using methods of [2] and [8], it is possible to show the consistency of the assumptions of 4.6 from $o(\kappa) = \kappa^{++}$.

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