On ultrafilters in ZF models and indecomposable ultrafilters

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Abstract

We use indecomposable ultrafilters to answer some questions from Hayut, Karagila [4]. It is shown that the bound on the strength of Usuba [8] is optimal.

1 On indecomposable ultrafilters

In sixties C. Chang and J. Keisler formulated the following notions:

Definition 1.1 Let U be an ultrafilter on a set I.

1. U is called $(\kappa, \lambda)-$regular iff there is subset of U of cardinality λ such that any $\kappa-$members of it have empty intersection.

2. U is called $\lambda-$descendingly incomplete iff there are $\{X_\alpha \mid \alpha < \lambda\} \subseteq U$ such that $\alpha < \beta \to X_\alpha \supseteq X_\beta$ and $\bigcap_{\alpha<\lambda} X_\alpha = \emptyset$.

3. U is $\lambda-$decomposable iff there is a partition of I into disjoint sets $\langle I_\alpha \mid \alpha < \lambda\rangle$, so that whenever $S \subseteq \lambda$ and $|S| < \lambda$, $\bigcup_{\alpha \in S} I_\alpha \notin U$.

4. Suppose $\delta < \lambda$ are cardinals. U is called $(\delta, \lambda)-$indecomposable if any partition $\langle I_\nu \mid \nu < \alpha\rangle$ of I with $\alpha < \lambda$ has a subsequence $\langle I_{\nu_\xi} \mid \xi < \beta\rangle$ with $\beta < \delta$ whose union belongs to U.

Let state some known facts which are relevant for us here:

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Fact 1.2 U is $\lambda-$decomposable, then U is $\lambda-$descendingly incomplete.
If λ is regular, then the converse holds as well.

Fact 1.3 An ultrafilter U over I is $\lambda-$decomposable iff it Rudin-Keisler above a uniform ultrafilter over λ.

Fact 1.4 If U is $(\kappa,\lambda)-$regular ultrafilter and ν is a regular cardinal so that $\kappa \leq \nu \leq \lambda$, then U is $\nu-$descendingly incomplete, and so, $\nu-$decompossible.

Fact 1.5

1. U is $\gamma-$indecomposable if and only if U is $(\gamma,\gamma^+)-$indecomposable.
2. U is $(\delta,\lambda)-$indecomposable if and only if U is not $\gamma-$decomposable for any cardinal γ such that $\delta \leq \gamma < \lambda$.

We will relay on the following theorem of J. Silver:

Theorem 1.6 *Let δ and κ be cardinals with $2^\delta < \kappa$. Suppose that U is a $(\delta,\kappa)-$indecomposable ultrafilter over a set I. Then $j_U = j_W^{M_D} \circ j_D$ where D is an ultrafilter over a cardinal less than δ and W is an M_D-ultrafilter on $j_D(I)$ that is $j_D(\gamma)-$complete over M_D, for all $\gamma < \kappa$.*

Let us show the following:

Theorem 1.7 *Let δ and κ be cardinals with $2^\delta < \kappa$. Assume that κ is a limit cardinal. Suppose that U is a $(\delta,\kappa)-$indecomposable ultrafilter over a set I. Let P be a $\delta-$closed forcing of cardinality $\rho < \kappa$. Let $G \subseteq P$ be a generic.*
Then, in $V[G]$, $U^* = \{A \subseteq I \mid \exists B \in U(B \subseteq A)\}$ is a $(\delta,\kappa)-$indecomposable ultrafilter over a set I.

Proof. Consider $j_U : V \to M_U = \mathrm{Ult}(V,U)$. Note that M_U may be ill-founded. By the Silver theorem, $j_U = j_W^{M_D} \circ j_D$ where D is an ultrafilter over a cardinal less η than δ and W is an M_D-ultrafilter over $j_D(I)$ that is $j_D(\gamma)-$complete over M_D, for all $\gamma < \kappa$.

Claim 1 $j_U{}''G$ generates a generic subset of $j_U(P)$ over M_U.

Proof. Let $E \subseteq j_U(P)$ be a dense open subset in M_U.
We have $|P| = \rho$, so without loss of generality, assume that $P \subseteq \rho$. W is $j_D(\rho^+)-$complete, so $j_U(P) = j_D(P)$.
Pick $f_E : I \to V$ which represents E. We can assume that f_E depends on η only. Then we

will have $|\mathrm{rng}(f_E)| \leq \eta$. Use $\delta-$closure of P and find $E^* \subseteq P$ which is a dense open and is contained in each dense open subset of P in $\mathrm{rng}(f_E)$. Then, $j_U(E^*) \subseteq E$.
$\square$ of the claim.

Now, exactly as in a well-founded case the elementary embeddings extend. Denote extensions by j^*, j_D^*, j_W^*.
We have, for every $X \subseteq I$,

$$X \in U \Leftrightarrow [id]_U \in j_U(X).$$

Now, in $V[G]$, let $A \subseteq I$ and $\underset{\sim}{A}$ be its name.
Set

$$A \in U^* \Leftrightarrow \exists p \in G(j_U(p) \Vdash [id]_U \in j_U(\underset{\sim}{A})).$$

Then, $U^* \supseteq U$.

Claim 2 $U^* = \{A \subseteq I \mid \exists B \in U(B \subseteq A)\}$.

Proof. Let $A \in U^*$. Pick $p \in G$ such that $j_U(p) \Vdash [id]_U \in j_U(\underset{\sim}{A})$.
Set $B = \{\nu \in I \mid p \Vdash \nu \in \underset{\sim}{A}\}$. Then $B \in U$ and $B \subseteq A$, since $p \in G$.
$\square$ of the claim.

The next claim completes the proof.

Claim 3 U^* is a $(\delta, \kappa)-$indecomposable ultrafilter in $V[G]$.

Proof. Let $\langle I_\nu \mid \nu < \alpha\rangle$ be a partition of I with $\alpha < \kappa$. We need to show that there is a subsequence $\langle I_{\nu_\xi} \mid \xi < \beta\rangle$ with $\beta < \delta$ whose union belongs to U^*.
Apply j_D^* to the partition. Let $\langle I'_\nu \mid \nu < j_D(\alpha)\rangle$ be the result. Note that $\alpha^+ < \kappa$, so W is $j_D(\alpha^+)-$complete, and hence, the further embedding j_{W^*} will not move $\alpha' = j_D(\alpha)$. Let $j_{W^*}(\langle I'_\nu \mid \nu < \alpha'\rangle) = \langle I''_\nu \mid \nu < \alpha'\rangle$. Then, for every $\nu < \alpha'$, $I''_\nu = j_{W^*}(A'_\nu)$. There must be some $\nu^* < \alpha'$ such that $[id]_U \in I''_{\nu^*}$.
Let $[f]_D$ be a function that represents ν^*. We can assume that $\mathrm{rng}(f) \subseteq \{I_\nu \mid \nu < \alpha\}$.
Then $\bigcup_{\nu \in \mathrm{rng}(f)} I_\nu \in U^*$. We are done, since $|\mathrm{rng}(f)| \leq |\mathrm{dom}(f)| < \delta$.
$\square$ of the claim.
$\square$

A similar, and a simpler argument gives the following:

Theorem 1.8 *Let δ and κ be cardinals with $2^\delta < \kappa$. Suppose that U is a $(\delta, \kappa)-$indecomposable ultrafilter over a set I. Let P be a forcing of cardinality less than*

the critical point of j_U. Let $G \subseteq P$ be a generic.
Then, in $V[G]$, $U^* = \{A \subseteq I \mid \exists B \in U(B \subseteq A)\}$ is a $(\delta, \kappa)-$indecomposable ultrafilter over a set I.

2 On existence of indecomposable ultrafilters on non-measurable cardinals

Clearly, if U is a $\kappa-$complete uniform ultrafilter over κ, then U is $\lambda-$indecomposable for every $\lambda < \kappa$. By D. Donder [2], if $\delta < \lambda$ and λ carries a uniform $\delta-$indecomposable ultrafilter, then there is an inner model of a measurable cardinal.
However, a cardinal which carries such ultrafilters need not be a measurable or even large. Thus, K. Prikry [6] showed $(\omega_1, \kappa)-$indecomposable uniform ultrafilter can exist over a singular cardinal κ. M. Sheard [7] produced such ultrafilter over a regular κ which is not a weakly compact. S. Ben David and M. Magidor [1] used a supercompact to construct a model in which there is a uniform ultrafilter over $\aleph_{\omega+1}$ which is $(\omega_1, \aleph_\omega)-$indecomposable.
H. Woodin, starting with a measurable and building on similar ideas constructed a GCH model in which there is a uniform ultrafilter over $\aleph_\omega$ which is $(\omega_1, \aleph_\omega)-$indecomposable.
In [3], starting with two measurables a GCH model in which there are regular cardinals $\omega < \kappa < \lambda$ such that λ is not measurable and carries a uniform $\sigma-$complete ultrafilter which is $\kappa-$indecomposable.

Let us briefly recall Woodin's construction and those of [3].

A sketch of Woodin's construction.

Start with a measurable cardinal κ. Let F be a normal ultrafilter over κ. Force with the Prikry forcing with F.[1] Let $\langle \kappa_n \mid n < \omega \rangle$ be a resulting Prikry sequence. Let $V_1 = V[\langle \kappa_{2n} \mid n < \omega \rangle]$. Define a V_1-filter U_1 over κ by setting

$$X \in U_1 \Leftrightarrow \exists n_0 < \omega(X \supseteq \{\kappa_{2n+1} \mid n_0 \leq n < \omega\}).$$

Using the homogeneity of the Prikry forcing, it is possible to argue that $U_1 \in V_1$.
Pick a non-principal ultrafilter D on ω and define U:

$$X \in U \Leftrightarrow \exists A \in D(X \supseteq \{\kappa_{2n+1} \mid n \in A\}).$$

Again, such U will be in V_1 and it will be a uniform $(\omega_1, \kappa)-$indecomposable there.
Clearly, U is a uniform ultrafilter over κ.

[1] Actually, Woodin combines this with collapses in order to turn κ into $\aleph_\omega$, which is unneeded for our purposes.

Let us show that U is $(\omega_1, \kappa)-$indecomposable.
Let $\langle I_\nu \mid \nu < \alpha\rangle$ be a partition of κ into $\alpha < \kappa$ many pieces. For every $n < \omega$, pick $\nu_n < \alpha$ such that $\kappa_{2n+1} \in I_{\nu_n}$. Then $\bigcup_{n<\omega} I_{\nu_n} \supseteq \{\kappa_{2n+1} \mid n < \omega\}$. Note that $\alpha < \kappa$ and so, the Prikry forcing does not add new subsets to α. So, $\langle I_{\nu_n} \mid n < \omega\rangle \in V_1$, and hence, $\bigcup_{n<\omega} I_{\nu_n} \in U$.

A sketch of the constructioin from [3].

Start with two measurable cardinals $\kappa < \lambda$.
Fix normal ultrafilters U_κ, U_λ over κ and λ respectively. The final ultrafilter U will extend $U_\kappa \times U_\lambda$. In order to destroy measurability of λ a type of forcing adding Suslin trees is iterated below λ and at λ itself. Below λ branches are added to such Suslin trees and nothing is done over λ (in Kunen's fashion). The iteration is arranged in a special way which allows to extend the embedding $j_{U_\kappa \times U_\lambda}$. This will give a uniform $(\kappa^+, \lambda)-$indecomposable ultrafilter over a non-measurable cardinal λ. In addition, such ultrafilter will be $\kappa-$complete. Let us show $(\kappa^+, \lambda)-$indecomposability. Let $\alpha < \lambda$ and $\langle I_\nu \mid \nu < \alpha\rangle$ be a partition of λ. Apply the ultrapower embedding. It extends $j_{U_\kappa \times U_\lambda}$. Let $\langle I'_\nu \mid \nu < \alpha'\rangle$ be the image. Then $\alpha' = j_{U_\kappa \times U_\lambda}(\alpha) = j_{U_\kappa}(\alpha)$, since $\alpha < \lambda$ and $crit(U_\lambda) = \lambda$. Find $\nu^* < \alpha'$ such that $\langle \kappa, \lambda\rangle \in I'_{\nu^*}$. Let $\langle I''_\nu \mid \nu < \alpha'\rangle$ be the image of $\langle I_\nu \mid \nu < \alpha\rangle$ under the extension of j_{U_κ}. Let f be a function on κ which represents I''_{ν^*}. Consider $\langle I_{f(i)} \mid i < \kappa\rangle$. Let $I^* = \bigcup_{i<\kappa} I_{f(i)}$. Then $I^* \in U$.

Let us conclude this section with a simple construction of a uniform $(\omega_1, \kappa)-$indecomposable ultrafilter over a cardinal κ of countable cofinality, and so, not measurable.

A simple construction.

Assume that κ is a limit of an increasing sequence of measurable cardinals $\langle\langle \kappa_n \mid n < \omega\rangle$. Fix a non-principal ultrafilter D over ω and let U_n be a normal ultrafilter over κ_n, for every $n < \omega$. Define U over κ as follows:

$$X \in U \Leftrightarrow \{n < \omega \mid X \cap \kappa_n \in U_n\} \in D.$$

Clearly, U is a uniform ultrafilter over κ. Let us show such U will be $(\omega_1, \kappa)-$indecomposable ultrafilter. Let $\alpha < \kappa$ and $\langle I_\nu \mid \nu < \alpha\rangle$ be a partition of κ. Pick $n_0 < \omega$ such that $\kappa_{n_0} > \alpha$. For every $n \geq n_0$, $\bigcup_{\nu<\alpha} I_\nu \cap \kappa_n = \kappa_n$. So, there is $\nu_n < \alpha$ such that $I_{\nu_n} \cap \kappa_n \in U_n$. Then $\bigcup_{n_0 \leq n<\omega} I_{\nu_n} \in U$.

3 Applications to ZF models

Y. Hayut and A. Karagila, in [4], introduced and studied the class $\mathcal{U}$ of all infinite cardinals which carry a uniform ultrafilter in ZF context. They asked whether the following:

Is it possible to have a situation when some cardinal κ does not carry a uniform ultrafilter, κ^+ does, but κ^+ is not measurable, and is this possible without using large cardinals? In particular, is it possible that $\aleph_0$ is the only measurable cardinal, while $\aleph_1 \notin \mathcal{U}$ and $\aleph_2 \in \mathcal{U}$?

T. Usuba [8] showed that large cardinals are needed. Namely, he proved the following:

Theorem 3.1 *(ZF) If there are cardinals $\kappa < \lambda$ with $\kappa \notin \mathcal{U}$ and $\lambda \in \mathcal{U}$, then there is an inner model with a measurable cardinal.*

Usuba argued that in an inner ZFC model there is a uniform ultrafilter over λ which is $\kappa-$indecomposible. Then by Donder [2], there exists an inner model with a measurable cardinal.

Our aim here will be to use indecomposable ultrafilters from ZFC models in order to provide affirmative answers to remaining parts of the above question, and also, to argue that it is impossible to improve Usuba's lower bound.

A model in which $\aleph_0$ is the only measurable cardinal, while $\aleph_1 \notin \mathcal{U}$ and $\aleph_2 \in \mathcal{U}$.

Let U be a uniform $(\omega_1, \kappa)-$indecomposable ultrafilter over a singular cardinal κ of cofinality ω, in a ZFC model.
Use symmetric extensions with collapses in a standard fashion, in order to turn κ into $\aleph_2$ by collapsing a cofinal in κ sequence to ω_1. By symmetry and 1.7, U will generate an ultrafilter in the extension. In addition, by standard arguments ω_1 will not carry an ultrafilter.
If V does not have measurable cardinals, then same will hold in such symmetric extension.

A model in which $\aleph_1$ is the measurable cardinal, while $\aleph_2 \notin \mathcal{U}$ and $\aleph_3 \in \mathcal{U}$, also it carries a $\sigma-$complete ultrafilter.

Start with two measurable cardinals $\kappa < \lambda$. Use [3] to construct a model with a uniform $(\kappa^+, \lambda)-$indecomposable $\kappa-$complete ultrafilter U over a non-measurable cardinal λ.
Use symmetric extensions with collapses in a standard fashion, in order to turn κ into $\aleph_1$ and λ into ω_3. By symmetry and 1.7,1.8, U will generate a $\sigma-$complete ultrafilter in the extension. In addition, by standard arguments ω_2 will not carry an ultrafilter.
Note that $\lambda = \aleph_3$ will remain non-measurable, since the collapses (their supports) used has small cardinality, and so, cannot add a branch to a Suslin tree over λ.

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