More on uniform ultrafilters over a singular cardinal.

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Abstract

We would like to show some additional results related to character of uniform ultrafilters over a singular cardinal and the ultrafilter number.

1 Some general observation.

Let us start with few simple well known observation:

Proposition 1.1 Suppose that $U, W$ are two ultrafilters and $U \geq_{R-K} W$. Then $\text{ch}(U) \geq \text{ch}(W)$.

Proof. Let $\pi$ be a projection of $U$ to $W$.
Let $\mathcal{U}$ be a generating family for $U$.
Then
$$W = \{ \pi'' A \mid A \in \mathcal{U} \}$$
will be a generating family for $W$.
\qed

The following follows:

Corollary 1.2 Suppose that $U$ is an ultrafilter over $\mu$, $W \leq_{R-K} U$ and $\text{ch}(W) = 2^\mu$.
Then $\text{ch}(U) = 2^\mu$, as well.

Proposition 1.3 Suppose that $U = F - \lim_{i \in I} U_i$ for an ultrafilter $F$ over $I$ and ultrafilters $U_i, i \in I$.
Suppose that $\langle U_i \mid i \in I \rangle$ are $F$–discrete, i.e. there are $X \in F$ and disjoint sets $\langle A_i \mid i \in X \rangle$

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such that $A_i \in U_i$, for every $i \in X$.

Assume that for almost every (mod $F$) $i \in I$, $U_i \geq_{R-K} W_i$. Let $W = F - \lim_{i \in I} W_i$.

Then $U \geq_{R-K} W$.

Proof. Let $X \in F$ and disjoint sets $\langle A_i \mid i \in X \rangle$ such that $A_i \in U_i$, for every $i \in X$.

Assume, in addition, that for every $i \in X$, $U_i \geq_{R-K} W_i$.

Set $A = \bigcup_{i \in X} A_i$. Then, clearly, $A \in U$.

For every $i \in X$, fix a projection $\pi_i$ of $U_i$ to $W_i$.

Set $\pi = \bigcup_{i \in X} \pi_i$.

Then $\pi$ projects $U$ to $W$.

□

In sixties C. Chang and J. Keisler formulated the following notions:

**Definition 1.4** Let $U$ be an ultrafilter on a set $I$.

1. $U$ is called $(\kappa, \lambda)$ regular iff there is subset of $U$ of cardinality $\lambda$ such that any $\kappa$–members of it have empty intersection.

2. $U$ is called $\lambda$–descendingly incomplete iff there are $\{X_\alpha \mid \alpha < \lambda\} \subseteq U$ such that $\alpha < \beta \rightarrow X_\alpha \supseteq X_\beta$ and $\bigcup_{\alpha < \lambda} X_\alpha = \emptyset$.

3. $U$ is $\lambda$–decomposable iff there is a partition of $I$ into disjoint $\langle I_\alpha \mid \alpha < \lambda\rangle$, so that whenever $S \subseteq \lambda$ and $|S| < \lambda$, $\bigcup_{\alpha \in S} I_\alpha \notin U$.

This subject was intensively investigated see for example [2],[9],[10],[11]. Let state some known propositions which are relevant for us here:

**Proposition 1.5** $U$ is $\lambda$–decomposable, then $U$ is $\lambda$–descendingly incomplete.

If $\lambda$ is regular, then the converse holds as well.

**Proposition 1.6** An ultrafilter $U$ over $I$ is $\lambda$–decomposable iff it Rudin-Keisler above a uniform ultrafilter over $\lambda$.

**Proposition 1.7** If $U$ is $(\kappa, \lambda)$–regular ultrafilter and $\nu$ is a regular cardinal so that $\kappa \leq \nu \leq \lambda$, then $U$ is $\nu$–descendingly incomplete, and so, $\nu$–decompossible.

Proof. Let $\{X_\alpha \mid \alpha < \lambda\} \subseteq U$ be a family such that the intersection of any $\kappa$–members of it is empty.
Set $Y_\gamma = \bigcup\{X_\alpha \mid \gamma \leq \alpha < \nu\}$.
Then each $Y_\gamma \in U$ and $\beta < \gamma < \nu \rightarrow Y_\beta \supseteq Y_\gamma$.

We have

$$\bigcap_{\gamma < \nu} Y_\gamma = \bigcup_{\gamma < \nu} \bigcap_{\alpha < \nu} X_{f(\alpha)} \cap f : \nu \rightarrow \nu \text{ and } \forall \alpha < \nu (f(\alpha) \geq \alpha).$$

The last union is the union of empty sets, by regularity of $\nu$ and $\kappa \leq \nu$.

Hence, $\bigcap_{\gamma < \nu} Y_\gamma = \emptyset$.

The following corollaries follows now:

**Corollary 1.8** Let $U$ be a $(\kappa, \lambda)$-regular ultrafilter. Then for every regular $\nu, \kappa \leq \nu \leq \lambda$, $\text{ch}(U) \geq u_\nu$.

**Corollary 1.9** Let $U$ be an ultrafilter over $\mu$ which is a $(\kappa, \lambda)$-regular.

Suppose that for some regular $\nu, \kappa \leq \nu \leq \lambda$, $u_\nu = 2^\mu$.

Then $\text{ch}(U) = 2^\mu$.

## 2 Strongly uniform ultrafilters.

Let us define some strengthening of uniformity of an ultrafilter over a singular cardinal.

**Definition 2.1** Suppose that $\kappa$ is a singular cardinal of cofinality $\eta$ and $D$ is a uniform ultrafilter over $\kappa$.

(a) Let $\bar{\tau} = \langle \tau_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of regular cardinals converging to $\kappa$.

Let $F$ be an uniform ultrafilter over $\eta$.

$D$ is called $(\vartheta, F)$-uniform iff for every $A \in D$,

$$\{\alpha < \eta \mid |A \cap \tau_\alpha| = \tau_\alpha\} \in F.$$  

(b) $D$ is called strongly uniform iff $D$ is $(\vartheta, F)$-uniform for some $(\vartheta, F)$, as in (a).

Define the corresponding ultrafilter numbers:

**Definition 2.2** (a) Let $(\vartheta, F)$ be as above.

$u(\kappa, \vartheta, F) = \min(\{\text{ch}(D) \mid D \text{ is } (\vartheta, F) \text{ - uniform}\}).$

(b) $u^{str}(\kappa) = \min(\{\text{ch}(D) \mid D \text{ is strongly uniform ultrafilter over } \kappa\}).$

Clearly, $u(\kappa) \leq u^{str}(\kappa)$. 

**Proposition 2.3** Suppose that \( \kappa \) is a singular cardinal of cofinality \( \eta \). Let \( \langle \kappa_\alpha \mid \alpha < \eta \rangle \) be an increasing sequence of cardinals converging to \( \kappa \).

Suppose that \( \delta \) is a regular cardinal such that

1. \( \kappa < \delta \leq 2^\kappa \)

2. there is an increasing sequence of regular cardinals \( \tilde{\delta} = \langle \delta_\alpha \mid \alpha < \eta \rangle \) such that
   
   (a) \( \kappa_\alpha < \delta_\alpha \leq \kappa_{\alpha+1} < \delta_{\alpha+1} \), for every \( \alpha < \eta \),
   
   (b) \( \text{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta \), for some ultrafilter \( F \) on \( \eta \) which extends the filter of co-bounded subsets of \( \eta \).

Let \( D \) be a \( (\tilde{\delta}, F) \)-uniform ultrafilter over \( \kappa \).

Then \( \text{ch}(D) \geq \delta \).

**Proof.** Let us argue that \( \text{ch}(D) \geq \delta \).

Suppose otherwise. Let \( W \) be a generating family for \( D \) of cardinality less than \( \delta \).

Let \( \langle f_\xi \mid \xi < \delta \rangle \) be a scale witnessing \( \text{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta \).

For every \( \xi < \delta \) and \( i < \eta \) set \( A_{\xi,i} = \delta_i \setminus f_\xi(i) \).

Let \( A_\xi = \bigcup_{i<\eta} A_{\xi,i} \).

Then, \( A_\xi \in D \), since otherwise \( B := \kappa \setminus A_\xi \in D \) and, so, by \( (\tilde{\delta}, F) \)-uniformity, the set

\[ X := \{ i < \eta \mid |B \cap \delta_i| = \delta_i \} \in F. \]

But, each \( \delta_i \) is a regular cardinal, hence, if \( i \in X \), then \( B \cap \delta_i \) is unbounded in \( \delta_i \). In particular, \( (B \cap \delta_i) \cap A_{\xi,i} \neq \emptyset \). Which is impossible, since \( B \) is a complement of \( A_\xi \supseteq A_{\xi,i} \).

We assumed that \( |W| < \delta \), so there is a single \( A \in W \) such that for \( \delta \)-many \( \xi \)'s we have \( A \subseteq^* A_\xi \).

Set \( A_i = A \cap \delta_i \), for every \( i < \eta \).

Without loss of generality, using \( (\tilde{\delta}, F) \)-uniformity, we can assume that \( |A_i| = \delta_i \), for every \( i < \eta \). Define, for every \( i < \eta \), \( \rho_i \) to be the \( \kappa_i \)-th element of \( A_i \).

Then there is \( \xi^* < \delta \) such that for every \( \xi, \xi^* \leq \xi \leq \delta \), the set

\[ \{ i < \eta \mid f_\xi(i) > \rho_i \} \in F. \]

Now we pick any \( \xi, \xi^* \leq \xi < \delta \) with \( A \subseteq^* A_\xi \). Then, for most (mod \( F \)) \( i \)'s, \( |A_i \setminus A_{\xi,i}| \geq \kappa_i \).

Hence, \( |A \setminus A_\xi| = \kappa \), which is impossible.
Contradiction.

Let present an other condition that prevents the character of being too small.

**Proposition 2.4** Suppose that $\kappa$ is a singular cardinal of cofinality $\eta$. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to $\kappa$.

Suppose that $\delta$ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$

2. there is an increasing sequences of regular cardinals $\vec{\tau} = \langle \tau_\alpha \mid \alpha < \eta \rangle$ such that
   
   (a) $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, for every $\alpha < \eta$,
   
   (b) $\text{tcf}(\prod_{\alpha < \eta} \delta_\alpha, <_F) = \delta$, where $\delta_\alpha = 2^{\tau_\alpha}$ and $F$ is an ultrafilter on $\eta$ which extends the filter of co-bounded subsets of $\eta$,
   
   (c) $r(\tau_\alpha) = \delta_\alpha$ (non-splitting number), i.e. whenever $S \subseteq [\tau_\alpha]^{\tau_\alpha}$ of cardinality $< \delta_\alpha$, then there is $a \in [\tau_\alpha]^{\tau_\alpha}$ such that for every $s \in S$, $|s \cap a| = |s \setminus a| = \tau_\alpha$. The meaning is that $a$ splits $s$. In particular, if $2^{\tau_\alpha} = \tau_\alpha^+$, then $r(\tau_\alpha) = \tau_\alpha^+ = \delta_\alpha$.

Let $D$ be a $(\vec{\tau}, F)$–uniform ultrafilter over $\kappa$.

Then $\text{ch}(D) \geq \delta$.

**Proof.** Let us argue that $\text{ch}(D) \geq \delta$.

Suppose otherwise. Let $\mathcal{W}$ be a generating family for $D$ of cardinality less than $\delta$.

Let $i < \eta$. Using $s(\tau_i) = \delta_i = 2^{\tau_i}$, we define a sequence $\langle A_{i\beta} \mid \beta < \delta_i \rangle$ of subsets of $\tau_i$ such that

1. for every $a \in [\tau_i]^{\tau_i}$ there is $\beta < \delta_i$ with $a = A_{i\beta}$,

2. each set $A_{i\beta}$ appears $\delta_i$–many times in the sequence,

3. for every $\beta < \delta_i$ there is $\gamma, \beta \leq \gamma < \delta_i$ such that $A_{i\gamma}$ splits $\langle A_{i\beta'} \mid \beta' < \beta \rangle$.

Let $\langle f_\xi \mid \xi < \delta \rangle$ be a scale witnessing $\text{tcf}(\prod_{\alpha < \eta} \delta_\alpha, <_F) = \delta$.

Let $\langle B_\zeta \mid \zeta < \rho < \delta \rangle$ be an enumeration of $\mathcal{W}$.

For every $\zeta < \rho$ and $i < \eta$ set $B_{\zeta i} = B_\zeta \cap \tau_i$.

Then there is $X_\zeta \in F$ such that for every $i \in X_\zeta, |B_{\zeta i}| = \tau_i$.  

Pick $\alpha_{\zeta} < \delta_i$ to be such that $B_{\zeta_i} = A_{i \alpha_{\zeta_i}}$.

Define a function $g_{\zeta} \in \prod_{i < \eta} \delta_i$ by setting $g_{\zeta}(i) = \alpha_{\zeta_i}$, if $i \in X_{\zeta}$ and $g_{\zeta}(i) = 0$, otherwise.

Consider $\langle g_{\zeta} \mid \zeta < \rho \rangle$. We have $\rho < \delta$ and $\langle f_{\zeta} \mid \xi < \delta \rangle$ a scale in $(\prod_{\alpha < \eta} \delta_\alpha, <_F)$.

Consider $\langle g_{\zeta} \mid \zeta < \rho \rangle$. We have $\rho < \delta$ and $\langle f_{\zeta} \mid \xi < \delta \rangle$ a scale in $(\prod_{\alpha < \eta} \delta_\alpha, <_F)$.

So, there is $\xi^* < \delta$, such that for every $\zeta < \rho$, the set

$$Z = \{ i < \eta \mid g_{\zeta}(i) < f_{\zeta^*}(i) \} \in F.$$

Suppose for simplicity that $Z = \eta$. Let $i < \eta$. Consider the sequence $\langle A_{i \beta} \mid \beta < f_{\zeta^*}(i) \rangle$. We have $s(\tau_i) = \delta_i > f_{\zeta^*}(i)$, so there is $\gamma_i < \delta_i$ such that $A_{\gamma_i}$ splits $\langle A_{i \beta} \mid \beta < f_{\zeta^*}(i) \rangle$.

Let $\tilde{A}_{\gamma_i}$ denotes $\kappa_i \setminus (A_{\gamma_i} \cup \delta_{i-1})$.

Set $A = \bigcup_{i < \eta} A_{\gamma_i}$ and $\tilde{A} = \bigcup_{i < \eta} \tilde{A}_{\gamma_i}$.

$D$ is an ultrafilter, hence $A \in D$ or $\tilde{A} \in D$.

Suppose, for example, that $A \in D$. Then there is $\zeta < \rho$ such that $B_{\zeta} \subseteq^* A$.

We have $A \cap B_{\zeta} \in D$, and so, by $(\tau, F)$—uniformity, the set

$$X = \{ i < \omega \mid A \cap B_{\zeta} \cap \tau_i \text{ is unbounded in } \tau_i \}$$

is infinite. Clearly, $X \subseteq X_\zeta$.

Now, $|B_{\zeta} \setminus A| < \kappa$ will imply that for all but boundedly many $i \in X$, $B_{\zeta_i} = B_{\zeta} \cap \tau_i \subseteq^* A \cap \tau_i$.

This is impossible, since $B_{\zeta_i}$ appears in $\langle A_{i \beta} \mid \beta < f_{\zeta^*}(i) \rangle$ and $A_{\gamma_i}$ splits this family, for every $i < \eta$.

Contradiction.

\[\square\]

3 On character of uniform ultrafilters of the form $F - \lim_{\alpha < \eta} U_{\alpha}$.

Let us combine now regularity properties with the results of the previous section in order to produce lower bounds on the characters of ultrafilters of the form $F - \lim_{\alpha < \eta} U_{\alpha}$ over singular cardinals.

**Proposition 3.1** Suppose that $\kappa$ is a singular cardinal of cofinality $\eta$. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to $\kappa$.

Suppose that $\delta$ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$
2. there is an increasing sequence of regular cardinals \( \langle \delta_\alpha \mid \alpha < \eta \rangle \) such that

(a) \( \kappa_\alpha < \delta_\alpha \leq \kappa_{\alpha+1} \), for every \( \alpha < \eta \),

(b) \( \text{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta \), for some ultrafilter \( F \) on \( \eta \) which extends the filter of co-bounded subsets of \( \eta \).

Suppose that \( U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle \) is such that for every \( \alpha < \eta \)

1. \( U_\alpha \) is a uniform ultrafilter over a cardinal \( \mu_\alpha \),

2. \( \delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1} \),

3. \( U_\alpha \) is \( (\delta_\alpha, \mu_\alpha) \)-regular or just \( \delta_\alpha \)-decomposable.

Then \( U \) is a uniform ultrafilter over \( \kappa \) and \( \text{ch}(U) \geq \delta \).

**Proof.** Let \( \alpha < \eta \). By Proposition 1.7, \( U_\alpha \) is \( \delta_\alpha \)-decomposable. Then, by Proposition 1.6, \( U_\alpha \geq_{R-K} D_\alpha \), for some uniform ultrafilter \( D_\alpha \) over \( \delta_\alpha \).

Set \( D = F - \lim \langle D_\alpha \mid \alpha < \eta \rangle \). Then, by Proposition 1.3, \( U \geq_{R-K} D \) and by Proposition 2.3, \( \text{ch}(D) \geq \delta \). Now, by Proposition 1.1, \( \text{ch}(U) \geq \delta \).

\( \square \)

The next proposition is similar:

**Proposition 3.2** Suppose that \( \kappa \) is a singular cardinal of cofinality \( \eta \). Let \( \langle \kappa_\alpha \mid \alpha < \eta \rangle \) be an increasing sequence of cardinals converging to \( \kappa \).

Suppose that \( \delta \) is a regular cardinal such that

1. \( \kappa < \delta \leq 2^\kappa \),

2. there is an increasing sequences of regular cardinals \( \langle \tau_\alpha \mid \alpha < \eta \rangle \) such that

(a) \( \kappa_\alpha < \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1} \), for every \( \alpha < \eta \),

(b) \( \text{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta \), where \( \delta_\alpha = 2^{\tau_\alpha} \) and \( F \) is an ultrafilter on \( \eta \) which extends the filter of co-bounded subsets of \( \eta \),

(c) \( \text{r}(\tau_\alpha) = \delta_\alpha \).

In particular, if \( 2^{\tau_\alpha} = \tau_\alpha^+ \), then \( \text{r}(\tau_\alpha) = \tau_\alpha^+ = \delta_\alpha \).

Suppose that \( U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle \) is such that for every \( \alpha < \eta \)

1. \( U_\alpha \) is a uniform ultrafilter over a cardinal \( \mu_\alpha \),
2. \( \delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1} \).

3. \( U_\alpha \) is \((\tau_\alpha, \mu_\alpha)\)-regular or just \(\tau_\alpha\)-decompossible.

Then \( U \) is a uniform ultrafilter over \( \kappa \) and \( \text{ch}(U) \geq \delta \).

**Proof.** Let \( \alpha < \eta \). By Proposition 1.7, \( U_\alpha \) is \( \delta_\alpha \)-decompossible. Then, by Proposition 1.6, \( U_\alpha \geq R - K D_\alpha \), for some uniform ultrafilter \( D_\alpha \) over \( \tau_\alpha \).

Set \( D = F \lim (D_\alpha \mid \alpha < \eta) \). Then, by Proposition 1.3, \( U \geq R - K D \) and by Proposition 2.4, \( \text{ch}(D) \geq \delta \). Now, by Proposition 1.1, \( \text{ch}(U) \geq \delta \).

\( \square \)

**Corollary 3.3** Let \( \kappa, U, \delta \) be as in Propositions 3.1 or 3.2. Suppose that \( \delta = 2^\kappa \).

Then \( \text{ch}(U) = 2^\kappa \).

Assume as above that \( \kappa \) is a singular cardinal of cofinality \( \eta \). Define now a cardinal invariant of \( \kappa \) which corresponds to ultrafilters of the form \( F \lim (U_\alpha \mid \alpha < \eta) \).

**Definition 3.4** Let \( u'(\kappa) \) be the smallest possible cardinality of \( \text{ch}(U) \), such that \( U \) is a uniform ultrafilter over \( \kappa \) of a form \( F \lim (U_\alpha \mid \alpha < \eta) \), where \( F \) is a uniform ultrafilter over \( \eta \) and \( U_\alpha \) is a uniform ultrafilter over a regular cardinal \(< \kappa \), for every \( \alpha < \eta \).

Clearly, \( u(\kappa) \leq u^{\text{str}}(\kappa) \leq u'(\kappa) \). Note that in models of [3], [4], \( u(\kappa) = u^{\text{str}}(\kappa) = u'(\kappa) = \kappa^+ \). However, \( \kappa \) in this models is limit of measurables. In [5], a model with \( u(\aleph_\omega) = \aleph_{\omega+1} < 2^{\aleph_\omega} \) was constructed. It turns out that \( u(\kappa) = u^{\text{str}}(\kappa) < u'(\kappa) \) in this model. Namely, the following always holds:

**Proposition 3.5** Assume that \( \aleph_\omega \) is a strong limit cardinal and \( 2^{\aleph_\omega} < \aleph_{\omega+1} \).

Then \( u'(\aleph_\omega) = 2^{\aleph_\omega} \).

**Proof.** If \( 2^{\aleph_\omega} = \aleph_{\omega+1} \), then the statement is obvious.

So, suppose that \( 2^{\aleph_\omega} > \aleph_{\omega+1} \).

Then \( 2^{\aleph_\omega} \) is a regular cardinal, since \( 2^{\aleph_\omega} < \aleph_{\omega+1} \), by S. Shelah [13] and by König, \( \text{cof}(2^{\aleph_\omega}) > \aleph_\omega \).

Again, by S. Shelah [13], Ch.IX, 1.8,1.9 there is an increasing sequence \( \langle n_i \mid i < \omega \rangle \) such that

\[
\text{tcf}(\prod_{i<\omega} \aleph_{n_i}) = 2^{\aleph_\omega}.
\]
Let now $U = F - \lim \langle U_i \mid i < \omega \rangle$ be as in Definition 3.4. Suppose that $U_i$ is a uniform ultrafilter over $\aleph_{m_i}$ for every $i < \omega$. Let $i < \omega$. By K. Kunen and K. Prikry [10], $U_i$ is $\aleph_k$-descendingly incomplete for every $k \leq m_i$. Hence, it is $\aleph_k$-decomposable, for every $k \leq m_i$. Now we can apply Proposition 3.1 and to conclude that $u'(\aleph_\omega) = 2^{\aleph_\omega}$.

Remark 3.6 It is possible to strengthen 3.5 a bit and to relax the requirement on $\aleph_\omega$ being a strong limit, since here $U = F - \lim \langle U_i \mid i < \omega \rangle$ implies that $U \geq_{R-K} F$, and so, by 1.1, $\text{ch}(U) \geq \text{ch}(F)$.

4 On character of uniform ultrafilters of the form $F - \lim_{\alpha < \eta} U_\alpha$, square principles and inner models.

The following crucial observation was made by D. Donder [1]:

**Theorem 4.1 (Donder)**

Let $\kappa > \omega$ be regular and assume that $\Box(\kappa)$ holds. Then every uniform ultrafilter $U$ on $\kappa$ is $(\omega, \tau)$-regular for every $\tau < \kappa$.

Let us combine this with the results of the previous section.

**Proposition 4.2** Suppose that $\kappa$ is a singular cardinal of cofinality $\eta$. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to $\kappa$. Suppose that $\delta$ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$

2. there is an increasing sequence of regular cardinals $\langle \delta_\alpha \mid \alpha < \eta \rangle$ such that
   
   (a) $\kappa_\alpha < \delta_\alpha \leq \kappa_{\alpha+1}$, for every $\alpha < \eta$,
   
   (b) $\text{tcf}(\prod_{\alpha < \eta} \delta_\alpha, <_F) = \delta$, for some ultrafilter $F$ on $\eta$ which extends the filter of co-bounded subsets of $\eta$,

Suppose that $U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle$ is such that for every $\alpha < \eta$

1. $U_\alpha$ is a uniform ultrafilter over a cardinal $\mu_\alpha$,

2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$.
3. $\Box(\mu_\alpha)$ holds.

Then $U$ is a uniform ultrafilter over $\kappa$ and $\text{ch}(U) \geq \delta$.

Proof. We have $\mu_\alpha$ is not weakly compact cardinal in $\mathcal{K}$, so $\Box(\mu_\alpha)$ holds in $\mathcal{K}$, by E. Schimmerling and M. Zeman [15].

In addition $(\mu_\alpha^+)^\mathcal{K} = \mu_\alpha^+$, hence the sequence which witnesses $\Box(\mu_\alpha)$ in $\mathcal{K}$ will witness it in $V$, as well.

By 4.1, $U_\alpha$ will be $(\omega, \mu_\alpha)$-regular. Now, 3.1 applies.

$\Box$

Similarly, using 3.2:

**Proposition 4.3** Suppose that $\kappa$ is a singular cardinal of cofinality $\eta$. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to $\kappa$.

Suppose that $\delta$ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$

2. there is an increasing sequences of regular cardinals $\langle \tau_\alpha \mid \alpha < \eta \rangle$ such that

   (a) $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, for every $\alpha < \eta$,

   (b) $\text{tcf}(\prod_{\alpha < \eta} \delta_\alpha, <_F) = \delta$, where $\delta_\alpha = 2^{\tau_\alpha}$ and $F$ is an ultrafilter on $\eta$ which extends the filter of co-bounded subsets of $\eta$,

   (c) $r(\tau_\alpha) = \delta_\alpha$.

   In particular, if $2^{\tau_\alpha} = \tau_\alpha^+$, then $r(\tau_\alpha) = \tau_\alpha^+ = \delta_\alpha$.

Suppose that $U = F - \lim \langle U_\alpha \mid \alpha < \eta \rangle$ is such that for every $\alpha < \eta$

1. $U_\alpha$ is a uniform ultrafilter over a cardinal $\mu_\alpha$,

2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$,

3. $\Box(\mu_\alpha)$ holds.

Then $U$ is a uniform ultrafilter over $\kappa$ and $\text{ch}(U) \geq \delta$.

**Corollary 4.4** Let $\kappa$ be a singular cardinal of cofinality $\eta$.

Suppose that there is an increasing sequence of regular cardinals $\langle \delta_\alpha \mid \alpha < \eta \rangle$ such that

1. $\kappa = \bigcup_{\alpha < \eta} \delta_\alpha$,
2. $\text{tcf}(\prod_{\alpha<\eta} \delta_{\alpha}, <_{\mu\alpha}) = 2^\kappa$, where $J^{\text{bd}}$ is the ideal of all bounded subsets of $\eta$.

Suppose that $U = F - \lim (U_\alpha | \alpha < \eta)$, for some ultrafilter $F$ over $\eta$ which includes all co-bounded subsets of $\eta$, is such that for every $\alpha < \eta$

1. $U_\alpha$ is a uniform ultrafilter over a cardinal $\mu_\alpha$,
2. $\delta_\alpha \leq \mu_\alpha < \kappa_{\alpha+1}$,
3. $\square(\mu_\alpha)$ holds.

Then $U$ is a uniform ultrafilter over $\kappa$ and $\text{ch}(U) = 2^\kappa$.

Assume now that there is no inner model with a Woodin cardinal and then use the core model $K$ of R. Jensen and J. Steel [8].

Even under a weaker assumption that there is no inner model with class many strong cardinals, which handled by R. Schindler [12], there are plenty overlapping extenders relevant for consistency results of [3], [4].

By results of E. Schimmerling, M. Zeman [15] and M. Zeman [17], $\square_\kappa$ holds in $K$ for every $\kappa$ and $\square(\kappa)$ holds in $K$ for every regular $\kappa > \omega$ which is not weakly compact.

In particular, if $\kappa^+ = (\kappa^+)^K$, then $\square_\kappa$ holds.

E. Schimmerling proved in [14] that if both $\square(\kappa)$ and $\square_\kappa$ fail and $\kappa \geq 2^{\aleph_0}$, then there is an inner model with Woodin cardinal (and more). He showed also that if $\kappa$ is a limit cardinal and $\kappa^+ > (\kappa^+)^K$, then $\square(\kappa)$ (see 5.1.1, 4.7 of [14]).

5 A remark on $\mathfrak{r}(\kappa)$.

Note that if $U$ is a uniform ultrafilter over $\kappa$ and $\mathcal{W}$ is its bases, then $\mathcal{W}$ is a non-splitting family. Namely, if $B \in [\kappa]^\kappa$, then $B$ does not split $\mathcal{W}$, since $B \in U$ or $\kappa \setminus B \in U$, and so contains a member of $\mathcal{W}$.

This implies that $\mathfrak{r}(\kappa) \leq \mu(\kappa)$.

We have seen in the previous section that $\mu'(\kappa)$ is related to $\square(\tau)$’s below $\kappa$. Failure of such square principle implies weak compactness in the core model of the corresponding cardinal.

On the other hand T. Suzuki [16] observed that:

a regular uncountable cardinal $\tau$ is a weakly compact iff $\mathfrak{s}(\tau) \geq \tau^+$;

where $\mathfrak{s}(\tau)$ a splitting number of $\tau$ is

$$\min\{|S| \mid S \subseteq [\tau]^\tau, \text{for every } x \in [\tau]^\tau \text{ there is } s \in S, |x \setminus s| = |x \setminus s| = \tau\}.$$
The next proposition indicates the connection of $r(\kappa)$ to weak compactness below.

**Proposition 5.1** Suppose that $\kappa$ is a singular cardinal of cofinality $\eta$. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to $\kappa$. Suppose that $\delta$ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$

2. there is an increasing sequences of regular cardinals $\langle \tau_\alpha \mid \alpha < \eta \rangle$ such that
   
   (a) $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, for every $\alpha < \eta$,
   
   (b) $\text{tcf}(\prod_{\alpha<\eta} \tau_\alpha, <_{J^{\mathfrak{u}}}) = \delta$,
   
   (c) $\text{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_{J^{\mathfrak{u}}}) = \delta$, where $\delta_\alpha = 2^{\tau_\alpha}$,
   
   (d) $s(\tau_\alpha) = \delta_\alpha$.

In particular, $\tau_\alpha$ must be at least weakly compact here.

If $2^{\tau_\alpha} = \tau_\alpha^{+}$, then we can assume just that $\tau_\alpha$ is a weakly compact.$^1$

Then $r(\kappa) \leq \delta$.

**Proof.**

Let $\langle f_\xi \mid \xi < \delta \rangle$ be a scale which witnesses $\text{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_{\mathfrak{F}}) = \delta$ and $\langle h_\zeta \mid \zeta < \delta \rangle$ be a scale which witnesses $\text{tcf}(\prod_{\alpha<\eta} \tau_\alpha, <_{\mathfrak{F}}) = \delta$.

Let $i < \eta$. Fix an enumeration $\langle A^i_{\beta} \mid \beta < \delta_\alpha \rangle$ of all subsets of $\tau_\alpha$ of cardinality $\tau_\alpha$.

Define a sequence $\langle A_\alpha \mid \alpha < \delta \rangle$ of subsets of $\kappa$ of cardinality $\kappa$ by induction as follows:

Suppose that $\alpha < \delta$ and $A_{\alpha'}$ is defined for every $\alpha' < \alpha$.

Let $i < \eta$. Consider $f_\alpha(i)$. It is an ordinal less than $\delta_i$. So, $\langle A^i_\beta \mid \beta < f_\alpha(i) \rangle$ is not a splitting family, since $s(\tau_i)) = \delta_i$. Hence, there is $\beta(\alpha, i), f_\alpha(i) < \beta(\alpha, i) < \delta_i$ such that $A^i_{\beta(\alpha, i)}$ cannot be split by any $A^i_{\beta}$ with $\beta < f_\alpha(i)$.

Set $A_\alpha = \bigcup_{i<\eta}(A^i_{\beta(\alpha, i)} \cap (h_\alpha(i), \tau_i))$.

This completes the induction.

For every $X \subseteq \eta, \alpha, \zeta < \delta$ set

$$A(\alpha, X, \zeta) = \bigcup_{i \in X}(A^i_{\beta(\alpha, i)} \cap (h_\zeta(i), \tau_i)).$$

In particular, $A_\alpha = A(\alpha, \eta, \alpha)$.

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$^1$Note that in [3], [4], measurability was used instead in order to get an upper bound for $u'(\kappa)$.
Consider now
\[ Z = \{ A(\alpha, X, \zeta) \mid \alpha, \zeta < \delta, X \subseteq \eta \}. \]
We claim that \( Z \) is an unsplittable family.
Suppose otherwise. Then there is \( B \subseteq \kappa, |B| = \kappa \) such that for every \( A \in Z \), both \( A \cap B \) and \( A \setminus B \) have cardinality \( \kappa \).
Note first that for unboundedly many \( i < \eta \), \( |B \cap \tau_i| = \tau_i \). Just otherwise, for all but boundedly many \( i \)'s, there is \( \rho_i < \tau_i \) such that \( B \cap \tau_i \subseteq \rho_i \).
Then there is \( \alpha < \delta \) such that for all but boundedly many \( i \)'s, \( \rho_i < h_\alpha(i) \). Hence, there is \( i^* < \eta \) such that for every \( i, i^* \leq i < \eta, B \cap A_i \cap \tau_i \subseteq \tau_i \).
This is impossible, since \( |B \cap A_\alpha| = \kappa \).
Assume now for simplicity that for every \( i < \eta, |B \cap \tau_i| = \tau_i \).
Then for every \( i < \eta \), there is \( \beta_i < \delta_i \) such that \( B \cap \tau_i = A^i_{\beta_i} \).
Find \( \alpha < \delta \) such that for all but boundedly many \( i \)'s, \( f_\alpha(i) > \beta_i \).
Again, assume for simplicity that this holds for every \( i < \eta \). Recall that by the choice of \( A^i_{\beta(\alpha,i)} \), it cannot be split by any \( A^j_\beta \) with \( \beta < f_\alpha(i) \). In particular, by \( B \cap \tau_i = A^i_{\beta_i} \).
So, either \( A^i_{\beta(\alpha,i)} \cap B \cap \tau_i \) is bounded in \( \tau_i \) or \( A^i_{\beta(\alpha,i)} \setminus (B \cap \tau_i) \) is bounded in \( \tau_i \).
Suppose for example that the set
\[ X = \{ i < \eta \mid A^i_{\beta(\alpha,i)} \cap B \cap \tau_i \text{ is bounded in } \tau_i \} \]
is cardinality \( \eta \).
Let for every \( i \in X, \gamma_i \leq \tau_i \) be a bound of \( A^i_{\beta(\alpha,i)} \cap B \cap \tau_i \). If \( i \in \eta \setminus X \), then set \( \gamma_i = 0 \).
There is \( \zeta < \delta \) and \( i^* < \eta \) such that for every \( i, i^* \leq i < \eta, h_\zeta(i) > \gamma_i \).
Then, for every \( i \in X \setminus i^* \), \( A^i_{\beta(\alpha,i)} \cap B \cap \tau_i \subseteq h_\zeta(i) \).
But then \( A(\alpha, X, \zeta) \cap B \subseteq \tau_i \) is bounded. Contradiction.

Define \( \tau^{str}(\kappa) \) to be
\[ \min(\{|X| \mid X \text{ is an unsplittable family,} \} \] such that for some increasing sequence of regular cardinals below \( \kappa \),
\[ \bar{\tau} = \langle \tau_\alpha \mid \alpha < \text{cof}(\kappa), \text{ for every } A \in X, \text{ for unboundedly many } \alpha < \text{cof}(\kappa), |A \cap \tau_\alpha| = \tau_\alpha \rangle \} \).
Clearly, \( \kappa^+ \leq \tau(\kappa) \leq \tau^{str}(\kappa) \).
The proposition above actually shows that \( \tau^{str}(\kappa) \leq \delta \).
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