Abstract
We present a method of constructing models with $Q$-point ultrafilters which have a Galvin property but are not sums of P-points. This answers a question of Tom Benhamou [1].

1 Some general facts

Our basic setting will be the following:

Let $W$ be a $\kappa$-complete ultrafilter over $\kappa$, $U = \{X \subseteq \kappa \mid \kappa \in j_W(X)\}$ and $k : M_U \rightarrow M_W$ the corresponding embedding. Let $\kappa_1 = j_U(\kappa)$. Suppose that $\kappa_1 = [id]_W$ and $\kappa_1 = \text{crit}(k)$.

The following lemma is well known:

Lemma 1.1 $W \supseteq \text{Cub}_\kappa$.

Lemma 1.2 Suppose that $\{A_\alpha \mid \alpha < \kappa\} \subseteq W$ and $\bigcap_{\alpha < \kappa} A_\alpha \in W$.
Then for every $B \in j_U(\{A_\alpha \mid \alpha < \kappa\}), \kappa_1 \in k(B)$.

Proof. Follows from elementarity and since $j_W = k \circ j_U$.

Lemma 1.3 For every $B \in j_U''W$, $\kappa_1 \in k(B)$.

Proof. Let $B = j_U(A)$, for some $A \in W$. Then

$$\kappa_1 \in j_W(A) = k(j_U(A)) = k(B).$$
Lemma 1.4 There is $B \in j_U(W)$ such that $\kappa_1 \notin k(B)$.

Proof. Let $f : \kappa \to \kappa$ be a function that represents $\kappa$ in $M_W$, i.e. $j_W(f)(\kappa_1) = \kappa$. It is a regressive function which is not constant on a set in $W$. Then, for every $\eta < \kappa$, 

$$A_\eta = \{ \nu \in X \mid f(\nu) \neq \eta \} \in W.$$

Let

$$\langle A_\eta^1 \mid \eta < \kappa_1 \rangle = k(\langle A_\eta \mid \eta < \kappa \rangle).$$

Then, $\kappa_1 \notin k(A_1^\kappa)$, since $j_W(f)(\kappa_1) = \kappa$.

□

The next lemma will be crucial for our further constructions.

Lemma 1.5 Suppose that $\{A_\alpha \mid \alpha < \kappa^+\}, \{B_\alpha \mid \alpha < \kappa^+\} \subseteq W$ are such that

1. $\{A_\alpha \mid \alpha < \kappa^+\} \subseteq U$;
2. for every $A \in j_U(\{A_\alpha \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(A)$;
3. for every $B \in j_U(\{B_\alpha \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(B)$.

Then there is $I \subseteq \kappa^+, |I| = \kappa$ such that

1. $\bigcap_{\alpha \in I} A_\alpha \in U \cap W$,
2. $\bigcap_{\alpha \in I} B_\alpha \in W$.

Proof. We repeat basically the Galvin proof simultaneously for $\{A_\alpha \mid \alpha < \kappa^+\}$ and $\{B_\alpha \mid \alpha < \kappa^+\}$.

Thus, define

$$H_{\alpha\xi} = \{ \beta < \kappa^+ \mid A_\alpha \cap \xi = A_\beta \cap \xi \text{ and } B_\alpha \cap \xi = B_\beta \cap \xi \},$$

for every $\alpha < \kappa^+, \xi < \kappa$.

Then, as in the Galvin proof, there will be $\alpha^* < \kappa^+$ such that for every $\xi < \kappa$, $|H_{\alpha^*\xi}| = \kappa^+$.

Define by induction a sequence $\langle \eta_\xi \mid \xi < \kappa \rangle$ such that

$$\eta_\xi \in H_{\alpha^*\xi + 1} \setminus \{ \eta_{\xi'} \mid \xi' < \xi \}.$$
Set $I = \{ \eta_\xi \mid \xi < \kappa \}$. Let us argue that such $I$ is as desired.

Apply $j_U$ and continue the inductive definition of the sequence $\langle \eta_{\kappa} \mid \xi < \kappa \rangle$ in $M_U$. Let $\langle \eta_{\kappa} \mid \xi < \kappa_1 \rangle$ be the resulting sequence. Denote $j_U(\{A_\alpha \mid \alpha < \kappa^+\})$ by $\{A_{\kappa}^1 \mid \alpha < j_U(\kappa^+)\}$ and $j_U(\{B_\alpha \mid \alpha < \kappa^+\})$ by $\{B_{\kappa}^1 \mid \alpha < j_U(\kappa^+)\}$.

Then, by the first assumptions of the lemma, $\kappa \in A_{\alpha}^1$, for every $\alpha \in j_U'' \kappa^+$. In particular, $\kappa \in A_{\eta_{\kappa}}^1$, for every $\xi < \kappa$.

For every $\xi, \kappa \leq \xi < \kappa_1$ we will have $A_{\eta_{\kappa}}^1 \cap \xi + 1 = A_{\eta_{\kappa}}^1 \cap \xi + 1$.

We have, $\kappa \in j_U(A_{\alpha^*})$, and so, $\kappa \in j_U(A_{\alpha^*}) \cap \xi + 1 = A_{\eta_{\kappa}}^1 \cap \xi + 1$.

So, $\kappa \in A_{\eta_{\kappa}}^1$, for every $\xi < \kappa_1$, and then,

$$\kappa \in \bigcap_{\xi < \kappa_1} A_{\eta_{\kappa}}^1 = j_U(\bigcap_{\xi < \kappa} A_{\eta_{\xi}}).$$

Hence, $\bigcap_{\xi < \kappa} A_{\eta_{\kappa}} \in U$.

By the second and the third assumptions of the lemma, $\kappa_1 \in k(A_{\eta_{\kappa}}^1)$ and $\kappa_1 \in k(B_{\eta_{\kappa}}^1)$, for every $\xi < \kappa_1$.

Apply $k$ and continue the inductive definition of the sequence $\langle \eta_{\kappa}^2 \mid \xi < \kappa_2 \rangle$ be the resulting sequence. Then for every $\xi, \kappa_1 \leq \xi < \kappa_2$ we will have $A_{\eta_{\kappa}}^2 \cap \xi + 1 = A_{\eta_{\kappa}}^2 \cap \xi + 1$ and $B_{\eta_{\kappa}}^2 \cap \xi + 1 = B_{\eta_{\kappa}}^2 \cap \xi + 1$,

where $\langle A_{\eta_{\kappa}}^2 \mid \xi < \kappa_2 \rangle = j_W(\langle A_{\eta_{\kappa}} \mid \xi < \kappa \rangle)$ and $\langle B_{\eta_{\kappa}}^2 \mid \xi < \kappa_2 \rangle = j_W(\langle B_{\eta_{\kappa}} \mid \xi < \kappa \rangle)$.

We have $A_{\eta_{\kappa}}^2 = j_W(A_{\alpha^*})$ and $A_{\alpha^*} \in W$. The same holds with $B_{\alpha^*}$. Hence, $\kappa_1 \in j_W(A_{\alpha^*})$, and so, $\kappa_1 \in j_W(A_{\alpha^*}) \cap \xi + 1 = A_{\eta_{\kappa}}^2 \cap \xi + 1$.

So, $\kappa_1 \in A_{\eta_{\kappa}}^2$, for every $\xi < \kappa_2$, and then,

$$\kappa_1 \in \bigcap_{\xi < \kappa_2} A_{\eta_{\kappa}}^2 = j_W(\bigcap_{\xi < \kappa} A_{\eta_{\xi}}).$$

The same is true with $B$'s instead of $A$'s.

Hence, $\bigcap_{\xi < \kappa} A_{\eta_{\kappa}} \in W$ and $\bigcap_{\xi < \kappa} B_{\eta_{\kappa}} \in W$.

$\square$

The next lemma is a slight generalization of 1.5.

**Lemma 1.6** Suppose that $\{A_{\alpha n} \mid \alpha < \kappa^+, n < n^*\}, \{B_{\alpha m} \mid \alpha < \kappa^+, m < m^*\} \subseteq W$, for some $n^*, m^* < \omega$, are such that, for every $n < n^*, m < m^*$,

1. $\{A_{\alpha n} \mid \alpha < \kappa^+\} \subseteq U$;
2. for every $A \in j_U(\{A_{an} \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(A)$;
3. for every $B \in j_U(\{B_{am} \mid \alpha < \kappa^+\})$, $\kappa_1 \in k(B)$.

Then there is $I \subseteq \kappa^+, |I| = \kappa$ such that, for every $n < n^*, m < m^*$,
1. $\bigcap_{\alpha \in I} A_{an} \in U \cap W$,
2. $\bigcap_{\alpha \in I} B_{am} \in W$.

**Proof.** Similar to those of 1.5 only define $H_{\alpha\xi}$ as follows:

$$H_{\alpha\xi} = \{\beta < \kappa^+ \mid \forall n < n^* (A_{an} \cap \xi = A_{\beta n} \cap \xi) \text{ and } \forall m < m^* (B_{am} \cap \xi = B_{\beta m} \cap \xi)\},$$

for every $\alpha < \kappa^+, \xi < \kappa$.

□

The next lemma follows from Lemma 1.5.

**Lemma 1.7** Suppose that there are a family $D \subseteq W$ and a normal filter $V \subseteq W$ such that
1. for every $A \in W$ there is $B \in D$ which is contained in $A \mod V$,
2. for every $C \in j_U(D)$, $\kappa_1 \in k(C)$.

Then $W$ has the Galvin property.

## 2 Construction

Assume GCH and let $\kappa$ be a measurable cardinal. Let $U$ be a normal ultrafilter over $\kappa$.

Define an Easton support iteration

$$\langle P_\alpha, Q_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle.$$

Let $Q_\beta$ be trivial unless $\beta$ is an inaccessible cardinal.

If $\beta < \kappa$ is an inaccessible cardinal then set $Q_\beta = Cohen(\beta)$.

Let $G$ be generic subset of $P_{\kappa+1}$. The embedding $j_U : V \to M_U$ extends to $j^* : V[G] \to M_U[G^*]$ in a standard fashion.

Set

$$U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}.$$

Then
1. $U^* \supseteq U$

2. $j_{U^*} = j^*$

3. $M_{U^*} = M_U[G^*]$

We have $j_U(P) = P_{\kappa+1} \ast P_{\kappa,j_U(U)}$.

Consider now $U \times U$. We have that Denote $j_U(\kappa)$ by $\kappa_1$ and $j_{U \times U}(\kappa) = j_{j_U(U)}(\kappa_1)$ by $\kappa_2$. Then $j_{j_U(U)} : M_U \to M_{U \times U}$ and $\kappa_1$ is its critical point.

Extend, in $V[G]$, $j_{U \times U}$ to $j^{**} : V[G] \to M_{U \times U}[G^{**}]$ as follows:

Set $G^{**} \cap P_{\kappa_1+1} = G^*$. Continue to define $G^{**} \cap P_{(\kappa_1,j_{U \times U}(\kappa))}$ in the standard fashion in order to insure that $j^{**}$ is $j_{U \times U^*}$.

The main issue will be to define a Cohen function $f_{\kappa_2}$, where $\kappa_2 = j_{U \times U}(\kappa)$.

Set $f_{\kappa_2} \upharpoonright \kappa_1 = f_{\kappa_1}$. Also, set $f_{\kappa_2}(\kappa_1) = \kappa$. This will insure $U^*$ will be the normal ultrafilter Rudin-Keisler below the one which we will define.

Namely, define the continuation of $f_{\kappa_2}$ arbitrary, but meeting the relevant dense sets.

Then, in $V[G]$, let

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j^{**}(X)\}.$$  

Then $W$ is a $\kappa-$complete ultrafilter over $\kappa$ and $j_W = j^{**}$.

Also, $k = j_{j_W(U)} : M_U \to M_{U \times U}$ extends to $k^* : M_U[G^*] \to M_{U \times U}[G^{**}]$.

**Lemma 2.1** $W \not\subseteq_{R-K} U^*$.

**Proof.** This follows since $W$ includes Cub$_\kappa$ and $f_{\kappa_2}$ is a regressive function which is not constant mod $W$. Actually, it projects $W$ to $U^*$.

\qed

The main issue thus will be to choose such $W$ which is not a product, but still has a Galvin property Gal$(W, \kappa, \kappa^+)$.

Proceed as follows.

Fix in $V$ an enumeration $\langle D_i \mid i < \kappa^+ \rangle$ of all dense open subsets of Cohen($\kappa_2$) of $M_{U \times U}[G^{**} \cap j_{U \times U}(P_{\kappa})]$.

We define a master condition sequence $\langle \mathcal{S}_j \mid i < \kappa^+ \rangle$ as follows:

if $i < \kappa^+$ and $\mathcal{S}_j$ is defined, then let $\mathcal{S}_{j+1}$ be an element of $D_{i+1}$ which is stronger than $\mathcal{S}_j$ and dom($\mathcal{S}_{j+1}$) is of the form $j_{U \times U}(h)(\kappa_1)$, for some $h : \kappa \to \kappa$, i.e. depends only on the second coordinate. Also require that it strictly includes those of $\mathcal{S}_j$. 

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This is possible since for every \( g : [\kappa]^2 \to \kappa \) there is \( g' : \kappa \to \kappa \) such that for every \( \alpha < \beta < \kappa, g(\alpha, \beta) < g'(\beta) \). Just define \( g'(\beta) = \bigcup_{\alpha < \beta} g(\alpha, \beta) + 1 \).

If \( i \) is a limit ordinal of cofinality \( < \kappa \), then set \( s_j = \bigcup_{j < i} s_j \) and then let \( s_j \) be an element of \( D_i \), which is stronger than \( s_j \) and dom(\( s_j \)) is of the form \( j_U \times U(h)(\kappa) \), for some \( h : \kappa \to \kappa \), i.e. depends only on the second coordinate.

Finally, let us deal with the main case when \( i \) is a limit ordinal of cofinality \( \kappa \).

We set first \( s_j' = \bigcup_{j < i} s_j \).

For every \( \alpha < i \), pick a function \( t_\alpha : \kappa \times \kappa \to V_\alpha, t_\alpha(\mu, \nu) \in Cohen(\kappa), \) in \( V \), which represents \( s_\alpha \) in the ultrapower \( M_{U \times U} \). Also, we can assume that dom(\( t_\alpha \))(\( \mu, \nu \)) depends only on \( \nu \).

We have, for every \( \alpha < \beta < i \),

\[
\{(\mu, \nu) \in [\kappa]^2 \mid t_{\alpha+1}(\mu, \nu) \upharpoonright \text{dom}(t_\alpha(\mu, \nu)) = t_\alpha(\mu, \nu), \text{dom}(t_{\alpha+1}(\mu, \nu)) > \text{dom}(t_\alpha(\mu, \nu))\} \in U \times U,
\]

and so,

\[
\{\mu < \kappa \mid \{\nu < \kappa \mid t_{\alpha+1}(\mu, \nu) \upharpoonright \text{dom}(t_\alpha(\mu, \nu)) = t_\alpha(\mu, \nu), \text{dom}(t_{\alpha+1}(\mu, \nu)) > \text{dom}(t_\alpha(\mu, \nu))\} \in U \}
\]

Using the dependence on the second coordinate only, we may assume that for every \( \mu < \kappa \),

\[
\{\nu < \kappa \mid t_{\alpha+1}(\mu, \nu) \upharpoonright \text{dom}(t_\alpha(\mu, \nu)) = t_\alpha(\mu, \nu), \text{dom}(t_{\alpha+1}(\mu, \nu)) > \text{dom}(t_\alpha(\mu, \nu))\} \in U.
\]

Set \( \langle t^1_\alpha \mid \alpha < \kappa_1 \rangle = j_U(\langle t_\alpha \mid \alpha < \kappa \rangle). \) Then, in \( M_U \), for every \( \mu < \kappa_1 \),

\[
\{\nu < \kappa_1 \mid t^1_{\alpha+1} \upharpoonright \text{dom}(t^1_\alpha(\mu, \nu)) = t^1_\alpha(\mu, \nu), \text{dom}(t^1_{\alpha+1}(\mu, \nu)) > \text{dom}(t^1_\alpha(\mu, \nu))\} \in j_U(U).
\]

In particular, for \( \mu = \kappa \),

\[
\{\nu < \kappa_1 \mid t^1_{\alpha+1} \upharpoonright \text{dom}(t^1_\alpha(\kappa, \nu)) = t^1_\alpha(\kappa, \nu), \text{dom}(t^1_{\alpha+1}(\kappa, \nu)) > \text{dom}(t^1_\alpha(\kappa, \nu))\} \in j_U(U).
\]

Apply \( k \) and move to \( M_{U \times U} \). Let \( \langle t^2_\alpha \mid \alpha < \kappa_2 \rangle = k(\langle t^1_\alpha \mid \alpha < \kappa_1 \rangle). \) Then, in \( M_{U \times U}, \)

\[
t^2_{\alpha+1}(\kappa, \kappa_1) \upharpoonright \text{dom}(t^2_\alpha(\kappa, \kappa_1)) = t^2_\alpha(\kappa, \kappa_1), \text{dom}(t^2_{\alpha+1}(\kappa, \kappa_1)) > \text{dom}(t^2_\alpha(\kappa, \kappa_1)).
\]

Consider \( \langle t^2_\alpha \mid \alpha < \kappa_1 \rangle. \) They are compatible. Take an upper bound for them in \( D_i \) and set it to be \( s_i \).

The above construction allows to satisfy conditions of Lemma 1.5. Thus define

\[
B_\alpha = \{(\mu, \nu) \in [\kappa]^2 \mid t_\alpha(\mu, \nu) \in G \cap Cohen(\kappa)\}.
\]
Set \((B^1_\alpha | \alpha < \kappa_1) = j^*((B_\alpha | \alpha < \kappa))\).

Then, by elementarity of \(j^*\),

\[
B^1_\alpha = \{(\mu, \nu) \in [\kappa_1]^2 | t^1_\alpha(\mu, \nu) \in G^* \cap \text{Cohen}(\kappa_1)\}.
\]

Then \((\kappa, \kappa_1) \in k(B^1_\alpha)\), for every \(\alpha < \kappa_1\).

This completes the definition of the master condition sequence, and so, \(G^*\) and \(f_{\kappa_2}\).

Define \(W\) using it in the usual fashion.

Remember that \(W\) not supposed be a sum. However, we think that \(W\) as defined above is indeed a sum.

Let assume that \(f_{\kappa_2}\), and so, \(G^*\) are in \(M_U[G^*]\).

We redefine \(f_{\kappa_2}\) in order to prevent this, but still to keep the Galvin property.

Do the following:

at each limit stage \(i\) of cofinality \(\kappa\) such that \(i\) is of the form \(\delta + \kappa\), for some \(\delta \geq \kappa\), we replace the values on \(\text{dom}(s_i) \setminus \bigcup_{r < i} \text{dom}(s_r)\) from 1 to 0 and 0 to 1.

The rest is kept unchanged, only in the previous construction of \(s_i\)’s we take care that such switches between 0’s and 1’s still keep conditions in the corresponding dense sets from the list.

Denote the resulting Cohen function by \(\tilde{f}_{\kappa_2}\).

**Lemma 2.2** \(\tilde{f}_{\kappa_2}\) cannot be in \(M_U[G^*]\).

**Proof.** Otherwise, compare \(\tilde{f}_{\kappa_2}\) with \(f_{\kappa_2}\). It will decode a cofinal in \(\kappa_2\) sequence of order type \(\kappa^+\), which is impossible since the cofinality of \(\kappa_2\) in \(M_U\) is \(\kappa^+_1 > \kappa^+\).

\(\square\)

Let \(G^{***} = (G^* \cap P_{\kappa_2}) * \tilde{f}_{\kappa_2}\). Define \(\tilde{W}\) using \(G^{***}\).

We would like now to argue that \(\tilde{W}\) satisfies the conditions of Lemma 1.5, and so the Galvin property.

Let us specify relevant subsets of \(\tilde{W}\).

First we deal with elements of \(P_{\kappa_2}\).

For every \(r \in P_{\kappa_2}\) pick a function \(h_r : [\kappa]^2 \rightarrow P_\kappa\) (in \(V\)) which represents \(r \mod U \times U\), i.e., \(r = (j_{U \times U}(h_r))(\kappa, \kappa_1)\). Set

\[
C_r = \{(\mu, \nu) \in [\kappa]^2 | h_r(\mu, \nu) \in G \cap P_\kappa\}.
\]

**Lemma 2.3** \(j_{\tilde{W}} \upharpoonright V[G \cap P_\kappa] = j_{U \times U\cdot} \upharpoonright V[G \cap P_\kappa]\) and

\[
k^* \upharpoonright M_U[G^* \cap P_{\kappa_1}] = k_{\tilde{W}} \upharpoonright M_U[G^* \cap P_{\kappa_1}],\] where \(k_{\tilde{W}} : M_U \rightarrow M_{\tilde{W}}\) is the canonical embedding.
Proof. This holds, since the ultrapowers $M_{U \times U}[G^{**}]$ by $U \times U$ and $M_{U \times U}[G^{**} \cap P_{\kappa_2}, \tilde{f}_{\kappa_2}]$ by $\tilde{W}$ agree about generic set up to the final step, i.e. where the Cohen function is added to $\kappa_2$.

□

**Lemma 2.4** $U^* \cap V[G \cap P_\kappa] = \tilde{W} \cap V[G \cap P_\kappa]$.

**Proof.** $A \in U^* \cap V[G \cap P_\kappa]$ iff $j U^*(A) \in j U^* M_{U^*}(G \cap P_\kappa_1)$ iff $\kappa_1 \in k^* (j U^*(A))$, $\kappa_1 \in j U^* \cap V[G \cap P_\kappa](A)$. By the previous lemma this is the same as $\kappa_1 \in j \tilde{W} \cap V[G \cap P_\kappa](A)$. So we are done.

□

The next lemma follows from Lemma 2.4:

**Lemma 2.5** If $A \in \tilde{W} \cap V[G \cap P_\kappa]$, then both $\kappa$ and $\kappa_1$ are in $j \tilde{W}(A)$.

**Lemma 2.6** If $X \in j U^*(U^* \cap V[G \cap P_\kappa])$, then $\kappa_1$ is in $k \tilde{W}(X)$.

**Proof.** By elementarity, $X \in j_U(U^* \cap M_U[G^* \cap P_{\kappa_1}])$. Then, $\kappa_1 \in k(X)$, since $X \in j_U(U^*)$.

By Lemma 2.3, $k(X) = k \tilde{W}(X)$, since $X \in M_U[G^* \cap P_{\kappa_1}]$.

□

**Lemma 2.7** $\tilde{W}$ satisfies the Galvin property.

**Proof.** Let $A \in \tilde{W}$ and $\tilde{A}$ be a name of it. Consider $x = ||\kappa_1 \in j_{U \times U}(\tilde{A}_\alpha)||$. By the definition of $\tilde{W}$, there are some $\langle r, s_\alpha \rangle \in (G^{**} \cap P_{\kappa_2}) * Cohen(\kappa_2)$ (in $M_{U \times U}$) which are stronger than $x$ (in the forcing sense, or alternatively, less than $x$ in the corresponding Boolean algebra), where $\alpha < \kappa^+$ and $s_\alpha$ is from the master condition sequence.

Split $r$ into $\langle r_1, r_2 \rangle$, where $r_1 \in P_{\kappa_1+1}, r_2 \in P_{\kappa_2}/P_{\kappa_1+1}$.

Pick in $V$ functions $h_{r_1}$ and $h_{r_2}$ which represent $r_1$ and $r_2$ in the ultrapower.

We can assume that $h_{r_1}$ is a function of the first coordinate only and, using $> \kappa_1$ completeness, $h_{r_2}$ is a function of the second coordinate only.

Strengthening if necessary, pick some $i(\alpha) < \kappa^+$ such that

1. $i(\alpha) > \alpha$,
2. $\text{cof}(i(\alpha)) = \kappa$,
3. $i(\alpha)$ is not of the form $\delta + \kappa$, for some $\delta \geq \kappa$,
Then, in particular, $s_i(\alpha) \geq s_\alpha$.

Let $B_{i(\alpha)}$ denotes the set in $\tilde{W}$ defined by $s_i(\alpha)$, i.e.

$$B_{i(\alpha)} = \{ (\mu, \nu) \in [\kappa]^2 \mid t_\alpha(\mu, \nu) \in G \cap Cohen(\kappa) \}.$$ 

Let

$$E_{r_1} = \{ \mu < \kappa \mid h_{r_1}(\mu) \in G \cap P_\kappa \}$$

and

$$E_{r_2} = \{ \nu < \kappa \mid h_{r_2}(\nu) \in G \cap P_{(\nu,\kappa)} \}.$$ 

Now, there is a set $C'_A \in U \times U$ such that

if $(\mu, \nu) \in C'_A \cap B_{i(\alpha)} \cap E_{r_1}, \mu \in E_{r_2}$, then $\nu \in A$.

Note that $U$ is a normal ultrafilter in $V$, so $C'_A$ can be picked to be of the form $[C_A]^2$, for some $C_A \in U$.

Shrink $B_{i(\alpha)}$ to the following set in $\tilde{W}$:

$$B'_{i(\alpha)} = \{ \nu < \kappa \mid (f_\kappa(\nu), \nu) \in B_{i(\alpha)} \}.$$ 

Set

$$F_A = \{ \nu < \kappa \mid f_\kappa(\nu) \in C_A \cap E_{r_1} \}.$$ 

Clearly, both $B'_{i(\alpha)}$ and $F_A$ are in $\tilde{W}$.

Note that $U \subseteq \tilde{W}$ and $E_{r_2} \in \tilde{W}$. Hence, $C_A \cap B'_{i(\alpha)} \cap F_A \cap E_{r_2} \in \tilde{W}$. So,

if $\nu \in C_A \cap B'_{i(\alpha)} \cap F_A \cap E_{r_2}$, then $\nu \in A$.

We specified sets $C_A, E_{r_1}, E_{r_2}, B_{i(\alpha)}$ for every $A \in \tilde{W}$.

Note that $C_A, E_{r_1}, E_{r_2} \in U^* \cap \tilde{W}$, and so, by Lemma 2.4, $\kappa, \kappa_1 \in j_{\tilde{W}}(C_A)$ and $\kappa, \kappa_1 \in j_{\tilde{W}}(E_{r_1})$.

Denote $E_{r_1}$ by $E_{A1}$, $E_{r_2}$ by $E_{A2}$ and $B_{i(\alpha)}$ by $B_A$.

Now, we are ready to show the Galvin property of $\tilde{W}$.

Let $\{ A_\gamma \mid \gamma < \kappa^+ \} \subseteq \tilde{W}$. For every $\gamma < \kappa^+$, we pick $C_{A_\gamma}, E_{A_{1,\gamma}}, E_{A_{2,\gamma}}, B_{A_\gamma}$, as above. Apply Lemmas 1.5, 1.6 to the families $\{ C_{A_\gamma} \mid \gamma < \kappa^+ \}, \{ E_{A_{1,\gamma}} \mid \gamma < \kappa^+ \}, \{ E_{A_{2,\gamma}} \mid \gamma < \kappa^+ \}$ and $\{ B_{A_\gamma} \mid \gamma < \kappa^+ \}$.

Then there will be $I \subseteq \kappa^+, |I| = \kappa$ such that

1. $\bigcap_{\gamma \in I} C_{A_\gamma} \in U \cap W$,
2. $\bigcap_{\gamma \in I} E_{A_{1,\gamma}} \in U \cap W$. 

3. $\bigcap_{\gamma \in I} E_{A, 2} \in U \cap W$.
4. $\bigcap_{\gamma \in I} B_{A, \gamma} \in W$.

Set

$$F = \{ \nu < \kappa \mid f_{\kappa}(\nu) \in \bigcap_{\gamma \in I} C_{A, \gamma} \cap \bigcap_{\gamma \in I} E_{A, 1} \}.$$ 

Then for every $\alpha \in I$,

if $\nu \in \bigcap_{\gamma \in I} C_{A, \gamma} \cap \bigcap_{\gamma \in I} B_{A, \gamma} \cap F \cap \bigcap_{\gamma \in I} E_{A, 2}$, then $\nu \in A_\alpha$.

We have $\bigcap_{\gamma \in I} C_{A, \gamma} \cap \bigcap_{\gamma \in I} B_{A, \gamma} \cap F \cap \bigcap_{\gamma \in I} E_{A, 2} \in \hat{W}$, so this completes the proof. □

**Remark 2.8** The idea used in the construction above works for variety of other forcing notions. The crucial point was a domination of functions $h(x, y)$ by functions $g(y)$ of the second variable.

### 3 Additional examples of non-Galvin ultrafilters

We show here that basic forcings over a measurable $\kappa$ which preserve measurability, add non-Galvin ultrafilters extending Cub$_\kappa$.

Assume GCH and let $\kappa$ be a measurable cardinal. Let $U$ be a normal ultrafilter over $\kappa$. We will deal with $j_U : V \rightarrow M_U, j_{U \times U} : V \rightarrow M_{U \times U}, j_{j_U(U)} : M_U \rightarrow M_{j_U(U)} = M_{U \times U}$. Denote $j_U$ by $j_1$, $M_U$ by $M_1$, $j_U(\kappa)$ by $\kappa_1$, $j_{U \times U}$ by $j_2$, $M_{U \times U}$ by $M_2$, $j_{U \times U}(\kappa)$ by $\kappa_2$ and $j_{j_U(U)}$ by $k$.

Let

$$\langle P_\alpha, Q_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle$$

be an Easton support iteration of Cohen forcings Cohen($\beta$) which add a Cohen function $g_\beta : \beta \rightarrow 2$ to every regular $\beta \leq \kappa$. Let $G$ be a generic subset of $P_{\kappa + 1}$.

Then the embeddings $j_1, j_2, k$ extend to $j_1^* : V[G] \rightarrow M_1[G_1], j_2^* : V[G] \rightarrow M_2[G_2], k^* : M_1[G_1] \rightarrow M_2[G_2]$.

Fix, in $V$, an increasing cofinal in $\kappa_1$ sequence $\langle \eta_\alpha \mid \alpha < \kappa^+ \rangle$ and a sequence of functions $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ from $\kappa$ to $\kappa$ such that $[f_\alpha]_U = \eta_\alpha$, for every $\alpha < \kappa^+$.

Now, in $V[G]$, for every $\alpha < \kappa^+$, define

$$A_\alpha = \{ \nu < \kappa \mid g_{\kappa}(f_\alpha(\nu)) = 1 \}.$$ 

Now we would like to define a $\kappa$–complete ultrafilter $W$ over $\kappa$ which extends $U, \text{Cub}_\kappa$ and such that the sets $\{ A_\alpha \mid \alpha < \kappa^+ \}$ witness that $W$ is not Galvin.

The argument will be very similar to those of 2.6 of [2].
First we change $g_{\kappa_1}$ by setting the values on each $\eta_\alpha$ to 1. Let $g'_{\kappa_1}$ be the resulting function. Then the choice of $\eta_\alpha$’s insure that $g'_{\kappa_1}$ is still generic over $M_1[G_1 \cap P_{\kappa_1}]$. Denote $G'_1 = G_1 \cap P_{\kappa_1} * g'_{\kappa_1}$ and let $j'_1 : V[G] \to M_1[G'_1]$ be the corresponding embedding. We will have as a result that $\kappa \in j'_1(A_\alpha)$, for every $\alpha < \kappa^+$.

Apply now $k$ and move to $M_2$. $k$ extends naturally to $k' : M_1[G'_1] \to M_2[G'_2]$.

Let us change same values of $g'_{\kappa_2}$.

Let

$$\langle \eta_\gamma^1 \mid \gamma < j_1(\kappa^+) \rangle = j_1(\langle \eta_\gamma \mid \gamma < \kappa^+ \rangle).$$

Then by elementarity, $\langle \eta_\gamma^1 \mid \gamma < j_1(\kappa^+) \rangle$ will be a cofinal sequence in $\kappa_2$ in $M_1$.

Let

$$\langle f^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle = j_1(\langle f_\gamma \mid \gamma < \kappa^+ \rangle).$$

Then, $f^1_\gamma$ will represent, mod $j_1(U)$, $\eta_\gamma^1$ in $M_1$.

Set

$$\langle f^2_\gamma \mid \gamma < j_2(\kappa^+) \rangle = j_2(\langle f_\gamma \mid \gamma < \kappa^+ \rangle) = k(\langle f^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle).$$

Then, whenever $\gamma < \delta < j_1(\kappa^+)$,

$$f^2_{k(\gamma)}(\kappa_1) = k(\langle f^1_\gamma \rangle)(\kappa_1) = \eta_\gamma^1 < \eta_\delta^1 = k(\langle f^1_\delta \rangle)(\kappa_1) = f^2_{k(\delta)}(\kappa_1).$$

We change the value of $g^2_{\kappa_2}(f^2_{j_2(\alpha)}(\kappa_1))$ to 1 , for every $\alpha < \kappa^+$. In addition, change $g^2_{\kappa_2}(f^2_{k(\gamma)}(\kappa_1))$ to 0 , for every $\gamma \in j_1(\kappa^+) \setminus j''_1\kappa^+$.

Let $g^*_{\kappa_2}$ denotes the resulting function. As in 2.6 of [2], $g^*_{\kappa_1}$ is still generic over $M_2[G'_2 \cap P_{\kappa_2}]$. Denote $G'_2 = G_2 \cap P_{\kappa_2} * g^*_{\kappa_2}$ and let $j^*_2 : V[G] \to M_2[G'_2], k^* : M_1[G'_1] \to M_2[G'_2]$ be the corresponding embeddings.

We will have as a result that $\kappa_1 \in j^*_2(A_\alpha)$, for every $\alpha < \kappa^+$ and $\kappa_1 \notin k^*(A^1_\gamma)$, for every $\gamma \in j_1(\kappa^+) \setminus j''_1\kappa^+$, where $\langle A^1_\gamma \mid \gamma < j_1(\kappa^+) \rangle = j'_1(\langle A_\alpha \mid \alpha < \kappa^+ \rangle)$.

Thus hold, since by elementarity,

$$j^*_2(A_\alpha) = \{ \nu < \kappa_2 \mid g^*_{\kappa_2}(f^2_{j_2(\alpha)}(\nu)) = 1 \},$$

for every $\alpha < \kappa^+$ and

$$k^*(A^1_\gamma) = \{ \nu < \kappa_2 \mid g^*_{\kappa_2}(f^2_{k(\gamma)}(\nu)) = 1 \},$$

for every $\gamma \in j_1(\kappa^+)$. Set
\[ W = \{ X \subseteq \kappa \mid \kappa_1 \in j_2^*(X) \} \]

and

\[ U^* = \{ X \subseteq \kappa \mid \kappa \in j_2^*(X) \}. \]

Then \( W >_{R-K} U^* \) both extend \( U \), \( W \supseteq Cub_\kappa \) and \( U^* \) is normal. Moreover, \( W \) is non-Galvin witnessed by \( \{ A_\alpha \mid \alpha < \kappa^+ \} \subseteq U^* \).

Similar constructions can be used with iterations of other forcing notions. What is needed is possibilities to extend the elementary embeddings \( j_1, j_2, k \) and \( \beta \)-closure of iterants \( Q_\beta \).
References
