A variant of Namba Forcing

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Abstract

Ronald Jensen asked whether it is possible to change cofinality of every regular cardinal between $\aleph_1$ and an inaccessible λ to ω preserving λ (and of course $\aleph_1$). We show that it is possible assuming large cardinals.

1 Introduction.

Let λ be a cardinal above $\aleph_1$. It is not hard to change cofinality of every regular cardinal in the interval $(\aleph_1, \lambda)$ to ω and to preserve $\aleph_1$ using Namba forcing (see [1] or [4]). For example we can use subtrees of $P_{\aleph_2}(\lambda)$ with splitting unbounded in $P_{\aleph_2}(\lambda)$. If λ was a regular then it will change its cofinality to ω and by Shelah [3], 4.9 p.304 λ^+ will not be a cardinal in the extension (actually it will change its cofinality to $\aleph_1$.
Ronald Jensen asked the following question:

Is it possible to change cofinality of every regular cardinal between $\aleph_1$ and an inaccessible λ to ω preserving λ (and of course $\aleph_1$).

Our purpose will be to prove the following:

Theorem 1.1 [1] *Assume GCH. Suppose that κ is a λ-supercompact cardinal for a Mahlo cardinal λ. Then in $V^{Col(\omega_1, <\kappa)}$ there is a forcing that does not add reals, changes cofinality of every regular cardinal $\delta, \aleph_1 < \delta < \lambda$ to ω and preserves λ.*

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[1]Menachem Magidor found a simpler proof using a weaker assumption. He pointed out that Woodin's Stationary Tower Forcing can be used to give the result.

Notation. For finite sequences t, t' $\quad t \trianglelefteq t'$ means that t is an initial (probably not proper) segment of t'. Trees here will be subtrees of ${}^{\omega >}(\lambda \times \lambda)$. If $t \in T$, for a tree T, then T_t denotes the set $\{t' \in T \mid t \trianglelefteq t'\}$ and $Suc_T(t)$ denotes the set of all immediate successors of t in T. Denote by $Lev_n(T)$ the set of all the points of T from the level n, i.e. $Lev_n(T) = \{t \in T \mid |t| = n\}$. Let $lim(T)$ denotes the set of all maximal branches of T.

2 The main construction.

Let κ is a λ-supercompact cardinal for a Mahlo cardinal λ. Work in $V^{Col(\omega_1, <\kappa)}$. For each regular $\delta, \aleph_1 < \delta < \lambda$ let U_δ be a uniform $\aleph_2$-complete filter over δ such that U_δ^+ has a dense σ-closed subset.

Definition 2.1 A subtree T of ${}^{\omega >}(\lambda \times \lambda)$ is called a good tree iff

1. if $t^\frown \langle \nu_1, \nu_2 \rangle \in T$, then ν_1 is a regular cardinal in the interval $(\aleph_1, \lambda)$ and ν_2 is an ordinal below ν_1.
   Intuition here is that ν_2 will be an element of an ω-sequence for ν_1.
2. For each $t \in T$, if $\langle \nu_1, \mu_1 \rangle, \langle \nu_2, \mu_2 \rangle \in Suc_T(t)$, then $\nu_1 = \nu_2$.
3. If $t \in T$, $\langle \nu, \mu_1 \rangle, \langle \nu, \mu_2 \rangle$ appear in t and $\langle \nu, \mu_1 \rangle$ appears in t below $\langle \nu, \mu_2 \rangle$ then $\mu_1 < \mu_2$.
   If $t \in T$ and there is a regular cardinal $\nu, \aleph_1 < \nu < \lambda$, such that the set
   $$\{\mu \mid \langle \nu, \mu \rangle \in Suc_T(t)\} \in U_\nu^+,$$
   then let us call t a *ν-splitting point.*
4. If for some ν, μ and t we have $t^\frown \langle \nu, \mu \rangle \in T$, then for each $f \in lim(T_{t^\frown \langle \nu, \mu \rangle})$ the set
   $$\{n < \omega \mid f(n) \text{ is a } \nu\text{-splitting point of } T_{t^\frown \langle \nu, \mu \rangle}\}$$
   is infinite.
   Note that if $f \in lim(T)$ and for some n we have $f(n) = \langle \nu, \mu \rangle$, then ν appears infinitely many times in rng(f) as a splitting point- just take $t = f \upharpoonright n + 1$.

Definition 2.2 Let T be a good tree and $\nu, \aleph_1 < \nu < \lambda$ be a regular cardinal. We call T a *ν-splitting tree* iff for every $f \in lim(T)$ the set

$$\{n < \omega \mid f(n) \text{ is a } \nu\text{-splitting point of } T\}$$

is infinite.

Note that for a good tree T the set

$$\{\nu \mid T \text{ is } \nu\text{-splitting tree}\}$$

is at most countable.

The next lemma allows us to shrink positive sets in a good tree.

Lemma 2.3 *Let T be a good tree and T' is a subtree of T such that for every $t \in T'$ if t was a ν-splitting point of T, for some ν then t remains a ν-splitting point of T' (i.e. $Suc_{T'}(t) \in U_\nu^+$, but it may be smaller than $Suc_T(t)$). Then T' is a good tree.*

Proof. We need to check only the last condition of Definition 2.1. So let $t^\frown\langle\nu,\mu\rangle \in T'$ and $f \in lim(T'_{t^\frown\langle\nu,\mu\rangle})$. Then, clearly, $f \in lim(T_{t^\frown\langle\nu,\mu\rangle})$. But T is a good tree, hence the set

$$S = \{n < \omega \mid f(n) \text{ is a } \nu\text{-splitting point of } T_{t^\frown\langle\nu,\mu\rangle}\}$$

is infinite. Now, for each $n \in S$, $f(n) \in T'$ and so it remains a ν-splitting point in T'.
□

We will use few partial orders over the set of good trees. The first one below is just the usual Namba forcing order.

Definition 2.4 Let T, T' be good trees. Define $T \leq_1 T'$ iff T' is a subtree of T.

The next lemma is obvious.

Lemma 2.5 *Let T be a good tree. Then there is $T' \geq_1 T$ such that for every $t \in T'$ either $|Suc_{T'}(t)| = 1$ or t is a ν-splitting point in T' for some regular cardinal $\nu, \aleph_1 < \nu < \lambda$.*

Lemma 2.6 *Let T be a good tree. Then there is $T' \geq_1 T$ such that for every $n < \omega$ and $t_1, t_2 \in Lev_n(T')$ $\quad t_1$ is a splitting point of T' iff t_2 is a splitting point of T'.*

Proof. By Lemma 2.5, we can assume that for every $t \in T$ either $|Suc_T(t)| = 1$ or t is a ν-splitting point in T for some regular cardinal $\nu, \aleph_1 < \nu < \lambda$.
Let t_0 be the trunk of T. Then t_0 is a ν_0-splitting point of T. Denote the set $\{\mu < \nu_0 \mid \langle\nu_0,\mu\rangle \in Suc_T(t)\}$ by A_0. Consider a function

$$F : A_0 \to \omega, F(\mu) = \min(|s| \mid s \trianglerighteq t \text{ and } s \text{ is a splitting point })).$$

Shrink A_0 to a positive set A_{00} on which F has a constant value n_0. Let T_{00} be the tree obtained from T by the shrinking the first splitting level to A_{00}. Clearly T_{00} is a good tree.

Now each $t \in Lev_{n_0}(T_{00})$ is a splitting point of T_{00}. We repeat the process with every such t. Thus let

$$A_t = \{\mu < \nu_t \mid \langle \nu_t, \mu \rangle \in Suc_{T_{00}}(t)\}.$$

Define $F_t : A_t \to \omega$ by setting $F_t(\mu) = \min(|s| \mid s \trianglerighteq t$ and s is a splitting point). Shrink A_t to a positive set A_{t0} on which F_t has a constant value n_t.

Return to A_{00}. Shrink it in order to stabilize n_t's. Let A_1 be the result and n_1 be the stabilized value of n_t. Shrink T_{00} to the tree T_1 by shrinking the first splitting level to A_1 and for each t at the second splitting level n_0 shrink the set A_t to A_{t0}. By Lemma 2.3 T_1 is a good tree.

Continue in a similar fashion and define T_2,n_2,....,T_k, n_k,....$(k < \omega)$.

Set $T' = \bigcap_{k<\omega} T_k$.

By the σ-completeness of U^+_α's and Lemma 2.3, the tree T' is a good tree.

□

Lemma 2.7 *Let T' be good tree. Then there is $T \geq_1 T'$ such that for every $t \in T$ either there is ν such that for every $\langle \eta, \xi \rangle \in Suc_T(t)$ there is μ with $\langle \nu, \mu \rangle \in Suc_T(t^\frown\langle \eta, \xi \rangle)$*

or

for some $\eta^*, \aleph_2 \leq \eta^* \leq \eta$ there are an increasing sequence of regular cardinals $\langle \rho_i \mid i < \eta^* \rangle$ and a partition $\langle A_i \mid i < \eta^* \rangle$ of the set

$$\{\zeta < \eta \mid \langle \eta, \zeta \rangle \in Suc_T(t)\}$$

such that

1. *for every $j < \eta^*$, the set $\bigcup_{i<j} A_i$ is in the dual to U_η ideal,*
2. *for every $i < \eta^*, \zeta \in A_i$, the immediate successors of $t^\frown\langle \eta, \zeta \rangle$ are of the form $\langle \rho_i, \mu \rangle$, for some $\mu < \rho_i$.*

Further let us denote η by $sp(t)$ and $\bigcup_{i<\eta^*} \rho_i$ by $sp(t, 2)$.

Proof. Easy.

□

Lemma 2.8 *Let $\tilde{T}$ be a tree as in the conclusion of the previous lemma. Then there is $T \geq_1 \tilde{T}$ such that for every $\langle \eta, \xi \rangle, \langle \eta, \xi' \rangle \in Suc_T(t)$, either $sp(t^\frown\langle \eta, \xi \rangle, 2) = sp(t^\frown\langle \eta, \xi' \rangle, 2)$*

or

for some $\eta^*, \aleph_2 \leq \eta^* \leq \eta$ there are an increasing sequence of regular cardinals $\langle \rho_i \mid i < \eta^* \rangle$ and a partition $\langle A_i \mid i < \eta^* \rangle$ of the set

$$\{\zeta < \eta \mid \langle \eta, \zeta \rangle \in Suc_T(t)\}$$

such that

1. *for every $j < \eta^*$, the set $\bigcup_{i<j} A_i$ is in the dual to U_η ideal,*

2. *for every $i < \eta^*, \zeta \in A_i$, $\quad sp(t^\frown \langle \eta, \zeta \rangle, 2) = \rho_i$.*

Further denote $\bigcup_{i<\eta^*} \rho_i$ by $sp(t, 3)$.

Similar we can define $sp(t, n)$ for each $n > 3$. Now using σ-completeness of U^+_α, we can conclude the following:

Lemma 2.9 *Let $\tilde{T}$ be a tree as in the conclusion of the previous lemma. Then there is $T \geq_1 \tilde{T}$ such that for every $n, 1 < n < \omega$, $\quad sp(\langle\rangle, n)$ is defined.*

Definition 2.10 A good tree T is called a *very good tree* iff it satisfies the conclusions of 2.5-2.9.

The next lemma follows from 2.5-2.9.

Lemma 2.11 *Let $\tilde{T}$ be a good tree. Then there is a very good tree $T \geq_1 \tilde{T}$.*

Our next tusk will be to provide a way of adding new cardinals to trees.

Definition 2.12 Let A be a set of cardinals and $\langle \langle \eta_1, \xi_1 \rangle, ..., \langle \eta_n, \xi_n \rangle \rangle \in {}^\omega(\lambda \times \lambda)$. Set

$$\text{proj}_A(\langle \langle \eta_1, \xi_1 \rangle, ..., \langle \eta_n, \xi_n \rangle \rangle) = \{\langle \eta_i, \xi_i \rangle \mid 1 \leq i \leq n, \eta_i \in A\}.$$

Definition 2.13 Let A be a set of cardinals and T be a good tree. Set

$$\text{proj}_A(T) = \{\text{proj}_A(t) \mid t \in T\}.$$

Let us denote for a good tree T by *supp*(T) the set

$$\{\eta < \lambda \mid \exists t \in T, \xi < \eta \quad \langle \eta, \xi \rangle \in t\}.$$

Definition 2.14 Let T_1, T_2 be good trees. Set $T_1 \equiv T_2$ iff

1. T_1, T_2 are the same above their trunks,

2. the trunks of T_1 and T_2 have the same length,

3. the trunk of T_1 is obtained by a permutation of the trunk of T_2.

Definition 2.15 Let T_1, T_2 be good trees. Set $T_1 \leq T_2$ iff there is a good trees T_1' such that

1. $T_1' \geq_1 T_1$,

2. $T_1' \equiv \text{proj}_{supp(T_1')}(T_2)$.

   In particular $\text{proj}_{supp(T_1')}(T_2)$ is a good tree.

3. For every $f \in lim(T_2)$, $\quad \text{proj}_{supp(T_1')}[f] \in lim(T_1')$.

   This condition insures that if T_1 was a ν-splitting tree then a stronger tree T_2 will be ν-splitting as well.

Let us check the transitivity of the relation $\leq$ defined above. Split the proof into few lemmas.

Lemma 2.16 *Suppose that $T_1 \leq T_2 \leq_1 T_3$. Then $T_1 \leq T_3$.*

Proof. Let $T_1' \geq_1 T_1$ be as in Definition 2.15. Consider $\text{proj}_{supp(T_1')}(T_3)$. By using a permutation of its trunk if necessary we obtain an equivalent condition T_1'' such that $T_1' \leq_1 T_1''$. Then T_1'' witnesses $T_1 \leq T_3$.
□

Lemma 2.17 *Suppose that $T_1 \leq T_2 \equiv T_3$. Then $T_1 \leq T_3$.*

Proof. Let $T_1' \geq_1 T_1$ be as in Definition 2.15. Note that $\text{proj}_{supp(T_1')}(T_2) \equiv \text{proj}_{supp(T_1')}(T_3)$. Hence T_1' witnesses $T_1 \leq T_3$.
□

Lemma 2.18 *Suppose that $T_1 \leq T_2 \leq T_3$. Then $T_1 \leq T_3$.*

Proof. Let $T_2' \geq_1 T_2$ witnesses $T_2 \leq T_3$. Then T_2' is equivalent to $\text{proj}_{supp(T_2')}(T_3)$. By previous lemmas then $T_1 \leq \text{proj}_{supp(T_2')}(T_3)$. Let $T_1' \geq_1 T_1$ witnesses this. Then

$$T_1' \equiv \text{proj}_{supp(T_1')}(\text{proj}_{supp(T_2')}(T_3)).$$

But

$$\mathrm{proj}_{supp(T_1')}(\mathrm{proj}_{supp(T_2')}(T_3)) = \mathrm{proj}_{supp(T_1')}(T_3),$$

since $supp(T_1') \subseteq supp(T_2')$. Hence T_1' witnesses $T_1 \leq T_3$.

□

Let $\mathcal{P} = \langle$ the set of all good trees $, \leq \rangle$. By Lemma 2.11, the set of very good trees is dense in $\mathcal{P}$.

Lemma 2.19 *Let T be a good tree and $\underset{\sim}{f}$ a name of a function from ω to V. Then there is $T^* \geq T$ such that for every $n < \omega$ and $t \in T^*$ which passes through n many splitting in T^*, T^*_t decides $\underset{\sim}{f} \restriction n$.*

Proof. The proof is standard, but only note passing to a stronger tree (in sense of Definition **??**) may turn splitting points into non-splitting ones. So after ω stages we may loose a goodness. In order to prevent this let us make some bookkeeping and if a tree T in the process if it is a ν-splitting (see Definition 2.2) for some ν, then unboundedly many times we preserve splitting into ν. Note that for each T there are at most countably many such ν's, so this can be done.

□

Lemma 2.20 *The forcing $\mathcal{P}$ does not add reals.*

Proof. Let T be a good tree and $\underset{\sim}{f} : \omega \to \omega$ a name of a real. Pick $T^* \geq T$ as Lemma 2.19. Shrink the first splitting level of T^* in order to decide $\underset{\sim}{f}(0)$. Note that there only $\aleph_0$ possible values and all the filters involved are $\aleph_2$-complete, hence this is possible. Now above each point t of the first splitting level consider the next splitting level. Again shrink and decide a value of $\underset{\sim}{f}(1)$. Shrink the first splitting level once more in order to obtain the same decision. Continue further in the same fashion. Finally, σ-completeness of positive sets is used to intersect all the trees that were constructed.

□

Now we can to conclude the following:

Theorem 2.21 *The forcing with good trees does not add reals (and so preserves $\aleph_1$). Each regular cardinal $\delta, \aleph_1 < \delta < \lambda$ changes its cofinality to ω.*

Lemma 2.22 *The forcing with good trees satisfies λ-c.c.*

Proof. Let $\langle T_\alpha \mid \alpha < \lambda, \alpha \text{ inaccessible } \rangle$ be a sequence of very good trees.
For each α, let A_α be at most countable set of cardinals in $(\aleph_1, \lambda)$ which are splitting cardinals of T_α or are as in Lemma 2.9, i.e. $sp_{T_\alpha}(\langle\rangle, n)$, for $n < \omega$.
Form a Δ-system of $\langle A_\alpha \mid \alpha < \lambda, \alpha \text{ inaccessible } \rangle$. Let $\langle A_\alpha \mid \alpha \in S\rangle$ be a Δ-system with S stationary subset of λ and a kernel $A \subseteq \min(S)$.
Denote the least inaccessible $\geq \max(A)$ by η. Clearly we can assume that $\min(S) > \eta$.
By shrinking more if necessary, we can assume that for each $\alpha < \beta$ in S all the ordinals that appear in T_α are below β. Denote by α^* the sup of all the ordinals that appear in T_α. Then $\alpha^* < \beta$.
Let $\alpha \in S$. Pick M_α to be an elementary submodel of cardinality η so that $M_\alpha \supseteq \eta$ and $T_\alpha \in M_\alpha$. Let $\bar{M_\alpha}$ be the transitive collapse of M_α and $\pi_\alpha : M_\alpha \to \bar{M}$ the collapsing map. Set $\bar{T_\alpha} = \pi_\alpha(T_\alpha)$.
By shrinking if necessary, we can assume that for some $\langle \bar{M}, \bar{T}\rangle$ for every $\alpha \in S \quad \langle \bar{M_\alpha}, \bar{T_\alpha}\rangle = \langle \bar{M}, \bar{T}\rangle$.

We claim now that for any $\alpha, \beta \in S$ the trees T_α and T_β are compatible. Let $\alpha < \beta$ be in S. We will combine them (actually their subtrees) together into one good tree.
Suppose for simplicity the trees have empty trunks. Otherwise we just put one above or equivalently inside an other.

Consider $\eta_1 := sp_{T_\beta}(\langle\rangle)$. If it is below η, then do nothing. Otherwise, we remove the set $\{\langle \eta_1, \mu\rangle \mid \mu < \alpha^*\}$ from $Suc_{T_\beta}(\langle\rangle)$. Note that all the filters are uniform, hence the set in ideal dual to U_{η_1} is removed.
Repeat this splitting $t \in T_\beta$ instead of just $\langle\rangle$.

Proceed now to the next splitting level of T_β, it exists and does not depend on particular points, since T_β is a very good tree. Denote this level by n_2. Let $\eta_2 := sp_{T_\beta}(\langle\rangle, 2)$. One of the two possibilities may occur: every $t \in T_\beta$ of the length $n_2 - 1$ is η_2-splitting point or for some $\eta_2^*, \aleph_2 \leq \eta_2^* \leq \eta_2$, there are an increasing sequence of regular cardinals $\langle \rho_i \mid i < \eta_2^*\rangle$ with $\eta_2 = \bigcup_{i<\eta_2^*} \rho_i$ and a partition $\langle A_i \mid i < \eta_2^*\rangle$ of the set

$$\{\zeta < \eta_1 \mid \langle \eta_1, \zeta\rangle \in Suc_{T_\beta}(\langle\rangle)\}$$

such that

1. for every $j < \eta_2^*$, the set $\bigcup_{i<j} A_i$ is in the dual to U_{η_1} ideal,
2. for every $i < \eta, \zeta \in A_i$, the immediate successors of $t^\frown\langle \eta_1, \zeta\rangle$ are of the form $\langle \rho_i, \mu\rangle$, for some $\mu < \rho_i$.

If the first possibility occurs then we do nothing (just all the ordinals below α^* were already removed at the previous stage). Assume that the second possibility occurs. In this case we remove from $Suc_{T_\beta}(\langle\rangle)$ the set $\bigcup_{i<j} A_i$ where j is the least with $\rho_j > \alpha^*$.

Continue further in the same fashion. Finally we use σ-completeness of positive sets to take intersections. Denote the final tree still by T_β. Let T_α be a corresponding shrink of the original T_α.

Now we put T_α and T_β together into a good tree T.

Suppose that the first level of T_α splits into U_ζ-positive set for some $\zeta > \eta$. Then $\zeta \geq \alpha$ and the first level of T_β splits into U_ρ-positive for some $\rho \geq \beta$.
Let $\langle\zeta, \mu\rangle \in Suc_{T_\alpha}(\langle\rangle)$ and $\Sigma_{\langle\zeta,\mu\rangle}$ be the type of this point over $\bar{M}$, i.e. we pick first a pair in M_α which realizes the type as those of $\langle\zeta, \mu\rangle$ and then take its image under π_α.
Now we put as an immediate successor of $\langle\zeta, \mu\rangle$ every point of $Suc_{T^*_\beta}(\langle\rangle)$ which realizes $\Sigma_{\langle\zeta,\mu\rangle}$. Note that this may turn the first level of T_β into a non-splitting level of T. So we will need to compensate it further up by just switching and putting elements of T_α above those of T_β which form a ρ-splitting.

Continue further up in the same fashion and switching all the time between putting elements of T_β as immediate successors to those of T_α and those of T_α as immediate successors of T_β. This process constructs a good tree T which is stronger than both T_α and T_β.
$\square$

3 Preserving successors of singulars of uncountable cofinality

Similar, but a simpler argument may be used to show the following:

Theorem 3.1 [2] *Assume GCH. Suppose that κ is a $< \kappa^{+\omega_1}$-supercompact cardinal for a Mahlo cardinal λ. Then in $V^{Col(\omega_1,<\kappa)}$ there is a forcing that does not add reals, changes cofinality of every regular cardinal $\delta, \aleph_1 < \delta < \aleph_{\omega_1}$ to ω and preserves $\aleph_{\omega_1+1}$.*

Proof. We would like to limit the number of cardinals that appear in condition T to $\aleph_0$. The only obstacle to this in the previous construction is the argument of Lemma 1.14. It may require adding many new cardinals to a initial tree T. The crucial observation here is that

[2] Jensen showed that no large cardinals are needed for this result using a different construction. Magidor pointed out that it is possible to modify a bit the Namba forcing (again without any large cardinals assumptions) in order to obtain the result.

the total number of cardinals is now ω_1 (our target is $\aleph_{\omega_1}$ and not an inaccessible λ), so we have enough completeness (each U_δ is $\aleph_2$-complete) to shrink to one.
It is possible to use a simpler trees from the beginning here. Just for a fixed at most countable set $supp(T)$ of regular cardinals in the interval $[\aleph_2, \aleph_{\omega_1})$ we require that T splits always above its trunk and at each level accordingly to a cardinal from $supp(T)$. The forcing will satisfy the Prikry condition.
$\square$

If one likes to replace $\aleph_{\omega_1}$ by say $\aleph_{\omega_2}$, then it is already problematic. Thus, ω_2 changes its cofinality to ω and so $(\aleph_{\omega_2})^V$ will have cofinality ω in the extension which is different from its cardinality there which is $\aleph_1$. Under mild assumptions, by Shelah [3], 4.9 p.304 , this implies that $(\aleph_{\omega_2+1})^V$ cannot be a cardinal in the extension.

If we allow to preserve $\aleph_2$ or to change its cofinality to ω_1, then it is possible to change cofinalities of all regular cardinals in the interval $[\aleph_3, \aleph_{\omega_2})$ to ω and to preserve $\aleph_{\omega_2+1}$. What is needed then is just a bit more completeness-namely $\aleph_2$-completeness. So instead of $Col(\omega_1, < \kappa)$ we force with $Col(\omega_2, < \kappa)$. The rest of the argument is the same.

Let us conclude with the following related questions:

Question 1. Are large cardinals really needed? Is it possible to force over L and change cofinality of every regular cardinal between $\aleph_1$ and an inaccessible λ to ω preserving λ and $\aleph_1$?

Question 2.[3] What if we replace $\aleph_1$ by $\aleph_2$, i.e. is it possible to change cofinality of every regular cardinal between $\aleph_2$ and an inaccessible λ to ω preserving λ, $\aleph_2$ and $\aleph_1$?

For this large cardinals are needed by the Jensen Covering Lemma and its generalizations. An attempt to generalize the construction above to higher cardinals breaks down due to the luck of completeness in the argument of Lemma 2.19.
An other approach which seems natural is to try to use variations of Supercompact Extender Based Prikry forcing of Merimovich [2]. The problem with this is that a supercompact cardinal itself is collapsed (say is an ordinal of cardinality $\aleph_1$ or $\aleph_2$) too many Prikry sequences reflect down below κ and so new reals are added.
An additional way may be to arrange at each regular $\delta \in (\aleph_1, \lambda)$ a δ-complete filter U_δ over δ such that U_δ^+ has $\aleph_2$-closed dense subset. Unfortunately we do not know how to do this over the successor of a singular. The following probably is the simplest case:

Question 3. Let δ be the successor of a singular cardinal of cofinality ω. Is it possible to have a δ-complete filter U_δ over δ such that U_δ^+ has $\aleph_2$-closed dense subset?

[3]Magidor's argument with Woodin's Stationary Tower Forcing provides an affirmative answer

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