A variant of Namba Forcing

Moti Gitik *
School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Science
Tel Aviv University
Ramat Aviv 69978, Israel

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Abstract
Ronald Jensen asked whether it is possible to change cofinality of every regular cardinal between $\aleph_1$ and an inaccessible $\lambda$ to $\omega$ preserving $\lambda$ (and of course $\aleph_1$). We show that it is possible assuming large cardinals.

1 Introduction.

Let $\lambda$ be a cardinal above $\aleph_1$. It is not hard to change cofinality of every regular cardinal in the interval $(\aleph_1, \lambda)$ to $\omega$ and to preserve $\aleph_1$ using Namba forcing (see [1] or [4]). For example we can use subtrees of $P_{\aleph_2}(\lambda)$ with splitting unbounded in $P_{\aleph_2}(\lambda)$. If $\lambda$ was a regular then it will change its cofinality to $\omega$ and by Shelah [3], 4.9 p.304 $\lambda^+$ will not be a cardinal in the extension (actually it will change its cofinality to $\aleph_1$).

Ronald Jensen asked the following question:

Is it possible to change cofinality of every regular cardinal between $\aleph_1$ and an inaccessible $\lambda$ to $\omega$ preserving $\lambda$ (and of course $\aleph_1$).

Our purpose will be to prove the following:

Theorem 1.1 1 Assume GCH. Suppose that $\kappa$ is a $\lambda$-supercompact cardinal for a Mahlo cardinal $\lambda$. Then in $V^{Col(\omega_1, <\kappa)}$ there is a forcing that does not add reals, changes cofinality of every regular cardinal $\delta, \aleph_1 < \delta < \lambda$ to $\omega$ and preserves $\lambda$.

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1Menachem Magidor found a simpler proof using a weaker assumption. He pointed out that Woodin’s Stationary Tower Forcing can be used to give the result.
Notation. For finite sequences \( t, t' \), \( t \leq t' \) means that \( t \) is an initial (probably not proper) segment of \( t' \). Trees here will be subtrees of \( ^{\rightarrow>(\lambda \times \lambda)} \). If \( t \in T \), for a tree \( T \), then \( T_t \) denotes the set \( \{ t' \in T \mid t \leq t' \} \) and \( \text{Suc}_T(t) \) denotes the set of all immediate successors of \( t \) in \( T \). Denote by \( \text{Lev}_n(T) \) the set of all the points of \( T \) from the level \( n \), i.e. \( \text{Lev}_n(T) = \{ t \in T \mid |t| = n \} \). Let \( \text{lim}(T) \) denotes the set of all maximal branches of \( T \).

2 The main construction.

Let \( \kappa \) is a \( \lambda \)-supercompact cardinal for a Mahlo cardinal \( \lambda \). Work in \( V^{\text{Col}(\omega_1, <\kappa)} \). For each regular \( \delta, \aleph_1 < \delta < \lambda \) let \( U_\delta \) be a uniform \( \aleph_2 \)-complete filter over \( \delta \) such that \( U_\delta^+ \) has a dense \( \sigma \)-closed subset.

Definition 2.1 A subtree \( T \) of \( ^{\rightarrow>(\lambda \times \lambda)} \) is called a good tree iff

1. if \( t^\frown<\nu_1, \nu_2> \in T \), then \( \nu_1 \) is a regular cardinal in the interval \( (\aleph_1, \lambda) \) and \( \nu_2 \) is an ordinal below \( \nu_1 \).
   Intuition here is that \( \nu_2 \) will be an element of an \( \omega \)-sequence for \( \nu_1 \).

2. For each \( t \in T \), if \( <\nu_1, \mu_1>, <\nu_2, \mu_2> \in \text{Suc}_T(t) \), then \( \nu_1 = \nu_2 \).

3. If \( t \in T \), \( <\nu, \mu_1>, <\nu, \mu_2> \) appear in \( t \) and \( <\nu, \mu_1> \) appears in \( t \) below \( <\nu, \mu_2> \) then \( \mu_1 < \mu_2 \).
   If \( t \in T \) and there is a regular cardinal \( \nu, \aleph_1 < \nu < \lambda \), such that the set
   \[ \{ \mu \mid <\nu, \mu> \in \text{Suc}_T(t) \} \in U_\nu^+ \],
   then let us call \( t \) a \( \nu \)-splitting point.

4. If for some \( \nu, \mu \) and \( t \) we have \( t^\frown<\nu, \mu> \in T \), then for each \( f \in \text{lim}(T_t^\frown<\nu, \mu>) \) the set
   \[ \{ n < \omega \mid f(n) \text{ is a } \nu \text{-splitting point of } T_t^\frown<\nu, \mu> \} \]
   is infinite.
   Note that if \( f \in \text{lim}(T) \) and for some \( n \) we have \( f(n) = <\nu, \mu> \), then \( \nu \) appears infinitely many times in \( \text{rng}(f) \) as a splitting point- just take \( t = f \upharpoonright n + 1 \).

Definition 2.2 Let \( T \) be a good tree and \( \nu, \aleph_1 < \nu < \lambda \) be a regular cardinal. We call \( T \) a \( \nu \)-splitting tree iff for every \( f \in \text{lim}(T) \) the set
   \[ \{ n < \omega \mid f(n) \text{ is a } \nu \text{-splitting point of } T \} \]
   is infinite.
Note that for a good tree $T$ the set 

$$\{ \nu \mid T \text{ is } \nu\text{-splitting tree} \}$$

is at most countable.

The next lemma allows us to shrink positive sets in a good tree.

**Lemma 2.3** Let $T$ be a good tree and $T'$ is a subtree of $T$ such that for every $t \in T'$ if $t$ was a $\nu$-splitting point of $T$, for some $\nu$ then $t$ remains a $\nu$-splitting point of $T'$ (i.e. $\text{Suc}_{T'}(t) \in U^+_\nu$, but it may be smaller than $\text{Suc}_T(t)$). Then $T'$ is a good tree.

**Proof.** We need to check only the last condition of Definition 2.1. So let $t^\frown\langle \nu, \mu \rangle \in T'$ and $f \in \text{lim}(T'_{t^\frown\langle \nu, \mu \rangle})$. Then, clearly, $f \in \text{lim}(T_{t^\frown\langle \nu, \mu \rangle})$. But $T$ is a good tree, hence the set

$$S = \{ n < \omega \mid f(n) \text{ is a } \nu\text{-splitting point of } T_{t^\frown\langle \nu, \mu \rangle} \}$$

is infinite. Now, for each $n \in S$, $f(n) \in T'$ and so it remains a $\nu$-splitting point in $T'$.

$\square$

We will use few partial orders over the set of good trees. The first one below is just the usual Namba forcing order.

**Definition 2.4** Let $T, T'$ be good trees. Define $T \leq_1 T'$ iff $T'$ is a subtree of $T$.

The next lemma is obvious.

**Lemma 2.5** Let $T$ be a good tree. Then there is $T' \geq_1 T$ such that for every $t \in T'$ either $|\text{Suc}_{T'}(t)| = 1$ or $t$ is a $\nu$-splitting point in $T'$ for some regular cardinal $\nu, \aleph_1 < \nu < \lambda$.

**Lemma 2.6** Let $T$ be a good tree. Then there is $T' \geq_1 T$ such that for every $n < \omega$ and $t_1, t_2 \in \text{Lev}_n(T')$ $t_1$ is a splitting point of $T'$ iff $t_2$ is a splitting point of $T'$.

**Proof.** By Lemma 2.5, we can assume that for every $t \in T$ either $|\text{Suc}_T(t)| = 1$ or $t$ is a $\nu$-splitting point in $T$ for some regular cardinal $\nu, \aleph_1 < \nu < \lambda$.

Let $t_0$ be the trunk of $T$. Then $t_0$ is a $\nu_0$-splitting point of $T$. Denote the set $\{ \mu < \nu_0 \mid \langle \nu_0, \mu \rangle \in \text{Suc}_T(t) \}$ by $A_0$. Consider a function

$$F : A_0 \rightarrow \omega, F(\mu) = \min(|s| \mid s \supseteq t \text{ and } s \text{ is a splitting point}) .$$

Shrink $A_0$ to a positive set $A_{00}$ on which $F$ has a constant value $n_0$. Let $T_{00}$ be the tree obtained from $T$ by the shrinking the first splitting level to $A_{00}$. Clearly $T_{00}$ is a good tree.
Now each \( t \in \text{Lev}_{n_0}(T_{00}) \) is a splitting point of \( T_{00} \). We repeat the process with every such \( t \). Thus let
\[
A_t = \{ \mu < \nu_t \mid \langle \nu_t, \mu \rangle \in \text{Suc}_{T_{00}}(t) \}.
\]
Define \( F_t : A_t \to \omega \) by setting \( F_t(\mu) = \min(|s| \mid s \supseteq t \text{ and } s \text{ is a splitting point}) \). Shrink \( A_t \) to a positive set \( A_{t0} \) on which \( F_t \) has a constant value \( n_t \).

Return to \( A_{00} \). Shrink it in order to stabilize \( n_t \)'s. Let \( A_1 \) be the result and \( n_1 \) be the stabilized value of \( n_t \). Shrink \( T_{00} \) to the tree \( T_1 \) by shrinking the first splitting level to \( A_1 \) and for each \( t \) at the second splitting level \( n_0 \) shrink the set \( A_t \) to \( A_{t0} \). By Lemma 2.3 \( T_1 \) is a good tree.

Continue in a similar fashion and define \( T_2, n_2, \ldots, T_k, n_k, \ldots \) \((k < \omega)\).

Set \( T' = \bigcap_{k<\omega} T_k \).

By the \( \sigma \)-completeness of \( U_\alpha^+ \)'s and Lemma 2.3, the tree \( T' \) is a good tree.

\( \square \)

**Lemma 2.7** Let \( T' \) be good tree. Then there is \( T \geq_1 T' \) such that for every \( t \in T \) either there is \( \nu \) such that for every \( \langle \eta, \xi \rangle \in \text{Suc}_T(t) \) there is \( \mu \) with \( \langle \nu, \mu \rangle \in \text{Suc}_T(t \triangleleft \langle \eta, \xi \rangle) \) or for some \( \eta^*, N_2 \leq \eta^* \leq \eta \) there are an increasing sequence of regular cardinals \( \langle \rho_i \mid i < \eta^* \rangle \) and a partition \( \langle A_i \mid i < \eta^* \rangle \) of the set
\[
\{ \zeta < \eta \mid \langle \eta, \zeta \rangle \in \text{Suc}_T(t) \}
\]
such that

1. for every \( j < \eta^* \), the set \( \bigcup_{i<j} A_i \) is in the dual to \( U_\eta \) ideal,
2. for every \( \eta^* \), \( \zeta \in A_i \), the immediate successors of \( t \triangleleft \langle \eta, \zeta \rangle \) are of the form \( \langle \rho_i, \mu \rangle \), for some \( \mu < \rho_i \).

Further let us denote \( \eta \) by \( sp(t) \) and \( \bigcup_{\eta \leq j} \rho_i \) by \( sp(t, 2) \).

**Proof.** Easy.

\( \square \)

**Lemma 2.8** Let \( \tilde{T} \) be a tree as in the conclusion of the previous lemma. Then there is \( T \geq_1 \tilde{T} \) such that for every \( \langle \eta, \xi \rangle, \langle \eta, \xi' \rangle \in \text{Suc}_T(t) \), either \( sp(t \triangleleft \langle \eta, \xi \rangle, 2) = sp(t \triangleleft \langle \eta, \xi' \rangle, 2) \) or
for some $\eta^*, \aleph_2 \leq \eta^* \leq \eta$ there are an increasing sequence of regular cardinals $\langle \rho_i \mid i < \eta^* \rangle$ and a partition $\langle A_i \mid i < \eta^* \rangle$ of the set

$$\{ \zeta < \eta \mid \langle \eta, \zeta \rangle \in \text{Suc}_T(t) \}$$

such that

1. for every $j < \eta^*$, the set $\bigcup_{i<j} A_i$ is in the dual to $U_\eta$ ideal,

2. for every $i < \eta^*, \zeta \in A_i, \ sp(t^\sim \langle \eta, \zeta \rangle, 2) = \rho_i$.

Further denote $\bigcup_{i < \eta^*} \rho_i$ by $sp(t, 3)$.

Similar we can define $sp(t, n)$ for each $n > 3$. Now using $\sigma$-completeness of $U^+_\alpha$, we can conclude the following:

**Lemma 2.9** Let $\tilde{T}$ be a tree as in the conclusion of the previous lemma. Then there is $T \geq_1 \tilde{T}$ such that for every $n, 1 < n < \omega, \ sp(\langle \rangle, n)$ is defined.

**Definition 2.10** A good tree $T$ is called a **very good tree** iff it satisfies the conclusions of 2.5-2.9.

The next lemma follows from 2.5-2.9.

**Lemma 2.11** Let $\tilde{T}$ be a good tree. Then there is a very good tree $T \geq_1 \tilde{T}$.

Our next tusk will be to provide a way of adding new cardinals to trees.

**Definition 2.12** Let $A$ be a set of cardinals and $\langle \langle \eta_1, \xi_1 \rangle, ..., \langle \eta_n, \xi_n \rangle \rangle \in \omega(\lambda \times \lambda)$. Set

$\text{proj}_A(\langle \langle \eta_1, \xi_1 \rangle, ..., \langle \eta_n, \xi_n \rangle \rangle) = \{ \langle \eta_i, \xi_i \rangle \mid 1 \leq i \leq n, \eta_i \in A \}$. 

**Definition 2.13** Let $A$ be a set of cardinals and $T$ be a good tree. Set

$\text{proj}_A(T) = \{ \text{proj}_A(t) \mid t \in T \}$. 

Let us denote for a good tree $T$ by $\text{supp}(T)$ the set

$$\{ \eta < \lambda \mid \exists t \in T, \xi < \eta \ \langle \eta, \xi \rangle \in t \}$$

**Definition 2.14** Let $T_1, T_2$ be good trees. Set $T_1 \equiv T_2$ iff

1. $T_1, T_2$ are the same above their trunks,
2. the trunks of $T_1$ and $T_2$ have the same length,
3. the trunk of $T_1$ is obtained by a permutation of the trunk of $T_2$.

**Definition 2.15** Let $T_1, T_2$ be good trees. Set $T_1 \leq T_2$ iff there is a good trees $T'_1$ such that

1. $T'_1 \geq_1 T_1$,
2. $T'_1 \equiv \text{proj}_{\text{supp}(T'_1)}(T_2)$.

In particular $\text{proj}_{\text{supp}(T'_1)}(T_2)$ is a good tree.
3. For every $f \in \text{lim}(T_2)$, $\text{proj}_{\text{supp}(T'_1)}[f] \in \text{lim}(T'_1)$.

This condition insures that if $T_1$ was a $\nu$-splitting tree then a stronger tree $T_2$ will be $\nu$-splitting as well.

Let us check the transitivity of the relation $\leq$ defined above. Split the proof into few lemmas.

**Lemma 2.16** Suppose that $T_1 \leq T_2 \leq_1 T_3$. Then $T_1 \leq T_3$.

**Proof.** Let $T'_1 \geq_1 T_1$ be as in Definition 2.15. Consider $\text{proj}_{\text{supp}(T'_1)}(T_3)$. By using a permutation of its trunk if necessary we obtain an equivalent condition $T''_1$ such that $T'_1 \leq_1 T''_1$. Then $T''_1$ witnesses $T_1 \leq T_3$.

□

**Lemma 2.17** Suppose that $T_1 \leq T_2 \equiv T_3$. Then $T_1 \leq T_3$.

**Proof.** Let $T'_1 \geq_1 T_1$ be as in Definition 2.15. Note that $\text{proj}_{\text{supp}(T'_1)}(T_2) \equiv \text{proj}_{\text{supp}(T'_1)}(T_3)$. Hence $T'_1$ witnesses $T_1 \leq T_3$.

□

**Lemma 2.18** Suppose that $T_1 \leq T_2 \leq T_3$. Then $T_1 \leq T_3$.

**Proof.** Let $T'_2 \geq_1 T_2$ witnesses $T_2 \leq T_3$. Then $T'_2$ is equivalent to $\text{proj}_{\text{supp}(T'_2)}(T_3)$. By previous lemmas then $T_1 \leq \text{proj}_{\text{supp}(T'_2)}(T_3)$. Let $T'_1 \geq_1 T_1$ witnesses this. Then

$$T'_1 \equiv \text{proj}_{\text{supp}(T'_1)}(\text{proj}_{\text{supp}(T'_2)}(T_3)).$$
But
\[ \text{proj}_{\text{supp}(T'_1)}(\text{proj}_{\text{supp}(T'_2)}(T_3)) = \text{proj}_{\text{supp}(T'_1)}(T_3), \]

since \( \text{supp}(T'_1) \subseteq \text{supp}(T'_2) \). Hence \( T'_1 \) witnesses \( T_1 \leq T_3 \).

□

Let \( \mathcal{P} = \langle \text{the set of all good trees }, \leq \rangle \). By Lemma 2.11, the set of very good trees is dense in \( \mathcal{P} \).

Lemma 2.19 Let \( T \) be a good tree and \( f \) a name of a function from \( \omega \) to \( V \). Then there is \( T^* \geq T \) such that for every \( n < \omega \) and \( t \in T^* \) which passes through \( n \) many splitting in \( T^* \), \( T^*_t \) decides \( f \restriction n \).

Proof. The proof is standard, but only note passing to a stronger tree (in sense of Definition ??) may turn splitting points into non-splitting ones. So after \( \omega \) stages we may loose a goodness. In order to prevent this let us make some bookkeeping and if a tree \( T \) in the process if it is a \( \nu \)-splitting (see Definition 2.2) for some \( \nu \), then unboundedly many times we preserve splitting into \( \nu \). Note that for each \( T \) there are at most countably many such \( \nu \)'s, so this can be done.

□

Lemma 2.20 The forcing \( \mathcal{P} \) does not add reals.

Proof. Let \( T \) be a good tree and \( f : \omega \to \omega \) a name of a real. Pick \( T^* \geq T \) as Lemma 2.19. Shrink the first splitting level of \( T^* \) in order to decide \( f(0) \). Note that there only \( \aleph_0 \) possible values and all the filters involved are \( \aleph_2 \)-complete, hence this is possible. Now above each point \( t \) of the first splitting level consider the next splitting level. Again shrink and decide a value of \( f(1) \). Shrink the first splitting level once more in order to obtain the same decision. Continue further in the same fashion. Finally, \( \sigma \)-completeness of positive sets is used to intersect all the trees that were constructed.

□

Now we can to conclude the following:

Theorem 2.21 The forcing with good trees does not add reals (and so preserves \( \aleph_1 \)). Each regular cardinal \( \delta, \aleph_1 < \delta < \lambda \) changes its cofinality to \( \omega \).

Lemma 2.22 The forcing with good trees satisfies \( \lambda \)-c.c.
Proof. Let $\langle T_\alpha \mid \alpha < \lambda, \alpha$ inaccessible $\rangle$ be a sequence of very good trees.

For each $\alpha$, let $A_\alpha$ be at most countable set of cardinals in $(\aleph_1, \lambda)$ which are splitting cardinals of $T_\alpha$ or are as in Lemma 2.9, i.e. $sp_{T_\alpha}(\langle \rangle, n)$, for $n < \omega$.

Form a $\Delta$-system of $\langle A_\alpha \mid \alpha < \lambda, \alpha$ inaccessible $\rangle$. Let $\langle A_\alpha \mid \alpha \in S \rangle$ be a $\Delta$-system with $S$ stationary subset of $\lambda$ and a kernel $A \subseteq \min(S)$.

Denote the least inaccessible $\geq \max(A)$ by $\eta$. Clearly we can assume that $\min(S) > \eta$.

By shrinking more if necessary, we can assume that for each $\alpha < \beta$ in $S$ all the ordinals that appear in $T_\alpha$ are below $\beta$. Denote by $\alpha^*$ the sup of all the ordinals that appear in $T_\alpha$. Then $\alpha^* < \beta$.

Let $\alpha \in S$. Pick $M_\alpha$ to be an elementary submodel of cardinality $\eta$ so that $M_\alpha \supseteq \eta$ and $T_\alpha \in M_\alpha$. Let $\tilde{M}_\alpha$ be the transitive collapse of $M_\alpha$ and $\pi_\alpha : M_\alpha \rightarrow \tilde{M}$ the collapsing map.

Set $\tilde{T}_\alpha = \pi_\alpha(T_\alpha)$.

By shrinking if necessary, we can assume that for some $\langle \tilde{M}, \tilde{T} \rangle$ for every $\alpha \in S$ $\langle \tilde{M}_\alpha, \tilde{T}_\alpha \rangle = \langle \tilde{M}, \tilde{T} \rangle$.

We claim now that for any $\alpha, \beta \in S$ the trees $T_\alpha$ and $T_\beta$ are compatible. Let $\alpha < \beta$ be in $S$. We will combine them (actually their subtrees) together into one good tree.

Suppose for simplicity the trees have empty trunks. Otherwise we just put one above or equivalently inside an other.

Consider $\eta_1 := sp_{T_\beta}(\langle \rangle)$. If it is below $\eta$, then do nothing. Otherwise, we remove the set $\{\langle \eta_1, \mu \rangle \mid \mu < \alpha^* \}$ from $Suc_{T_\beta}(\langle \rangle)$. Note that all the filters are uniform, hence the set in ideal dual to $U_{\eta_1}$ is removed.

Repeat this splitting $t \in T_\beta$ instead of just $\langle \rangle$.

Proceed now to the next splitting level of $T_\beta$, it exists and does not depend on particular points, since $T_\beta$ is a very good tree. Denote this level by $n_2$. Let $\eta_2 := sp_{T_\beta}(\langle \rangle, 2)$. One of the two possibilities may occur: every $t \in T_\beta$ of the length $n_2 - 1$ is $\eta_2$-splitting point or for some $\eta_2^*, N_2 \leq \eta_2^* \leq \eta_2$, there are an increasing sequence of regular cardinals $\langle \rho_i \mid i < \eta_2^* \rangle$ with $\eta_2 = \bigcup_{i < \eta_2^*} \rho_i$ and a partition $\langle A_i \mid i < \eta_2^* \rangle$ of the set

$$\{\zeta < \eta_1 \mid \langle \eta_1, \zeta \rangle \in Suc_{T_\beta}(\langle \rangle)\}$$

such that

1. for every $j < \eta_2^*$, the set $\bigcup_{i < j} A_i$ is in the dual to $U_{\eta_1}$ ideal,

2. for every $i < \eta_2^*, \zeta \in A_i$, the immediate successors of $t^\zeta\langle \eta_1, \zeta \rangle$ are of the form $\langle \rho_i, \mu \rangle$, for some $\mu < \rho_i$.
If the first possibility occurs then we do nothing (just all the ordinals below $\alpha^*$ were already removed at the previous stage). Assume that the second possibility occurs. In this case we remove from $Suc_{T_\beta}(\langle \rangle)$ the set $\bigcup_{i<j} A_i$ where $j$ is the least with $\rho_j > \alpha^*$.

Continue further in the same fashion. Finally we use $\sigma$-completeness of positive sets to take intersections. Denote the final tree still by $T_\beta$. Let $T_\alpha$ be a corresponding shrink of the original $T_\alpha$.

Now we put $T_\alpha$ and $T_\beta$ together into a good tree $T$.

Suppose that the first level of $T_\alpha$ splits into $U_\zeta$-positive set for some $\zeta > \eta$. Then $\zeta \geq \alpha$ and the first level of $T_\beta$ splits into $U_\rho$-positive for some $\rho \geq \beta$.

Let $\langle \zeta, \mu \rangle \in Suc_{T_\alpha}(\langle \rangle)$ and $\Sigma_{\langle \zeta, \mu \rangle}$ be the type of this point over $\bar{M}$, i.e. we pick first a pair in $M_\alpha$ which realizes the type as those of $\langle \zeta, \mu \rangle$ and then take its image under $\pi_\alpha$.

Now we put as an immediate successor of $\langle \zeta, \mu \rangle$ every point of $Suc_{T_\beta}(\langle \rangle)$ which realizes $\Sigma_{\langle \zeta, \mu \rangle}$. Note that this may turn the first level of $T_\beta$ into a non-splitting level of $T$. So we will need to compensate it further up by just switching and putting elements of $T_\alpha$ above those of $T_\beta$ which form a $\rho$-splitting.

Continue further up in the same fashion and switching all the time between putting elements of $T_\beta$ as immediate successors to those of $T_\alpha$ and those of $T_\alpha$ as immediate successors of $T_\beta$. This process constructs a good tree $T$ which is stronger than both $T_\alpha$ and $T_\beta$.

$\square$

3 Preserving successors of singulars of uncountable cofinality

Similar, but a simpler argument may be used to show the following:

**Theorem 3.1** Suppose GCH. Suppose that $\kappa$ is a $\kappa^{+\omega_1}$-supercompact cardinal for a Mahlo cardinal $\lambda$. Then in $V^{Col(\omega_1, < \kappa)}$ there is a forcing that does not add reals, changes cofinality of every regular cardinal $\delta, \aleph_1 < \delta < \aleph_{\omega_1}$ to $\omega$ and preserves $\aleph_{\omega_1+1}$.

**Proof.** We would like to limit the number of cardinals that appear in condition $T$ to $\aleph_0$. The only obstacle to this in the previous construction is the argument of Lemma 1.14. It may require adding many new cardinals to a initial tree $T$. The crucial observation here is that

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2Jensen showed that no large cardinals are needed for this result using a different construction. Magidor pointed out that it is possible to modify a bit the Namba forcing (again without any large cardinals assumptions) in order to obtain the result.
the total number of cardinals is now $\omega_1$ (our target is $\aleph_{\omega_1}$ and not an inaccessible $\lambda$), so we have enough completeness (each $U_\delta$ is $\aleph_2$-complete) to shrink to one. It is possible to use a simpler trees from the beginning here. Just for a fixed at most countable set $supp(T)$ of regular cardinals in the interval $[\aleph_2, \aleph_{\omega_1}]$ we require that $T$ splits always above its trunk and at each level accordingly to a cardinal from $supp(T)$. The forcing will satisfy the Prikry condition. □

If one likes to replace $\aleph_{\omega_1}$ by say $\aleph_{\omega_2}$, then it is already problematic. Thus, $\omega_2$ changes its cofinality to $\omega$ and so $(\aleph_{\omega_2})^V$ will have cofinality $\omega$ in the extension which is different from its cardinality there which is $\aleph_1$. Under mild assumptions, by Shelah [3], 4.9 p.304, this implies that $(\aleph_{\omega_2+1})^V$ cannot be a cardinal in the extension.

If we allow to preserve $\aleph_2$ or to change its cofinality to $\omega_1$, then it is possible to change cofinalities of all regular cardinals in the interval $[\aleph_3, \aleph_{\omega_2}]$ to $\omega$ and to preserve $\aleph_{\omega_2+1}$. What is needed then is just a bit more completeness-namely $\aleph_2$-completeness. So instead of $Col(\omega_1, < \kappa)$ we force with $Col(\omega_2, < \kappa)$. The rest of the argument is the same.

Let us conclude with the following related questions:

**Question 1.** Are large cardinals really needed? Is it possible to force over $L$ and change cofinality of every regular cardinal between $\aleph_1$ and an inaccessible $\lambda$ to $\omega$ preserving $\lambda$ and $\aleph_1$?

**Question 2.** What if we replace $\aleph_1$ by $\aleph_2$, i.e. is it possible to change cofinality of every regular cardinal between $\aleph_2$ and an inaccessible $\lambda$ to $\omega$ preserving $\lambda$, $\aleph_2$ and $\aleph_1$?

For this large cardinals are needed by the Jensen Covering Lemma and its generalizations. An attempt to generalize the construction above to higher cardinals breaks down due to the luck of completeness in the argument of Lemma 2.19. An other approach which seems natural is to try to use variations of Supercompact Extender Based Prikry forcing of Merimovich [2]. The problem with this is that a supercompact cardinal itself is collapsed (say is an ordinal of cardinality $\aleph_1$ or $\aleph_2$) too many Prikry sequences reflect down below $\kappa$ and so new reals are added. An additional way may be to arrange at each regular $\delta \in (\aleph_1, \lambda)$ a $\delta$-complete filter $U_\delta$ over $\delta$ such that $U_\delta^+$ has $\aleph_2$-closed dense subset. Unfortunately we do not know how to do this over the successor of a singular. The following probably is the simplest case:

**Question 3.** Let $\delta$ be the successor of a singular cardinal of cofinality $\omega$. Is it possible to have a $\delta$-complete filter $U_\delta$ over $\delta$ such that $U_\delta^+$ has $\aleph_2$-closed dense subset?

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3Magidor’s argument with Woodin’s Stationary Tower Forcing provides an affirmative answer
References


