An other method for constructing models of not approachability and not SCH.

Moti Gitik

May 28, 2018

At least to the best of our knowledge, the only method to get a singular strong limit cardinal κ such that $\neg \mathrm{AP}_{\kappa^+}$ and $2^\kappa > \kappa^+$ was the one introduced in [5]. Here a different approach will be suggested.

We will use the method for blowing up the power of a singular cardinal of [3] in order to get models of not approachability and not SCH. The advantage of the present technique is that no cardinal is collapsed or changes its cofinality.

1 A model in which both AP and SCH fail at a singular cardinal.

We will combine the forcing of [3] with the approach of Section 3 of [4].[1]

Let κ be a supercompact cardinal. Fix a regular cardinal $\eta < \kappa$. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals and let $\langle E_\alpha \mid \alpha < \eta \rangle$ be a sequence of extenders such that for every $\alpha < \eta$

1. $\kappa < \kappa_0$,
2. $E(\alpha)$ is a $(\kappa_\alpha, \bar{\kappa}_\eta^{++})-$extender, where $\bar{\kappa}_\eta = \bigcup_{\alpha<\eta} \kappa_\alpha$,
3. $E(\alpha) \lhd E(\alpha+1)$.

Let $\langle \mathcal{P}_{\langle E(\alpha) \mid \alpha < \eta \rangle}, \leq, \leq^* \rangle$ be the forcing of Section 2 of [3].
For every limit $\alpha \leq \eta$ denote $\bar{\kappa}_\alpha = \bigcup_{\alpha' < \alpha} \kappa_{\alpha'}$.
By [3], Section 2, it has the following properties:

[1]Section 3 of [4] contains an essential flow, which is due solely to the first author, but it turns out that with the forcing of [3], it is possible to make the idea work.

1. $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq, \leq^* \rangle$ is a Prikry type forcing,

2. the forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$:

   (a) blows up the power of $\bar{\kappa}_\eta$ to $\bar{\kappa}_\eta^{++}$,

   (b) blows up the power of $\bar{\kappa}_\alpha$ above $\bar{\kappa}_\alpha^+$, for every limit $\alpha < \eta$,

   (c) preserves cardinals and cofinalities,

   (d) preserves strong limitness of each of κ_α's, for every $\alpha \leq \eta$, and $\bar{\kappa}_\alpha$'s, for every limit $\alpha \leq \eta$

   (e) does not add new subsets to κ_0.

3. For every $p \in \mathcal{P}$ and every $\mathcal{P}-$name $\underset{\sim}{\zeta}$ of an ordinal, there is $p^* \geq^* p$ such that the number of possible decisions of $\underset{\sim}{\zeta}$ above p^* is at most λ.
   I.e. $|\{\xi \mid \exists q \geq p^*(q \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{\zeta} = \xi)\}| \leq \lambda$. [2]

4. The forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq^* \rangle$ is equivalent to the product of Cohen forcings $Cohen(\kappa_\alpha^+, \bar{\kappa}_\eta^{++})$.
   Namely, we just remove or ignore sets of measure one A_α^p in each coordinate $p(\alpha) = \langle f_\alpha^p, A_\alpha^p \rangle$ of a condition $p = \langle p(\alpha) \mid \alpha < \eta \rangle \in \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$. More precisely, if $p = \langle p(\alpha) \mid \alpha < \eta \rangle$ and $q = \langle q(\alpha) \mid \alpha < \eta \rangle$ are in $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$, then set $p \sim q$ iff for every $\alpha < \eta$

   (a) $p(\alpha)$ is non-pure iff $q(\alpha)$ is non-pure. Require then that $p(\alpha) = q(\alpha)$.

   (b) If $p(\alpha) = \langle f_\alpha^p, A_\alpha^p \rangle$, i.e. is pure, then $q(\alpha) = \langle g_\alpha^p, B_\alpha^p \rangle$ is pure as well, and require that $f_\alpha^p = g_\alpha^p$.

   Then $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle} / \sim, \leq^* \rangle$ is the product of Cohen forcings.

Let us assume (or make) the supercompact cardinal κ was made indestructible under $\kappa-$directed closed forcings using the Laver forcing. Denote by G a generic subset of the Laver forcing.
Then force with $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$. Denote further $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$ by $\mathcal{P}$. We claim that the resulting generic extension is as desired, i.e. it satisfies $\neg \mathrm{AP}_{\bar{\kappa}_\eta^+}$ and $2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$.
$2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$ follows by (2(a)) above. Let deal with the approachability. Denote $\bar{\kappa}_\eta$ by λ.

[2]This condition basically says that one entree given dense open set by taking a direct extension and then specifying finitely many coordinates. Usually, this property has the same proof, as the Prikry condition and is used to show that λ^+ is preserved in $V^{\langle \mathcal{P}, \leq \rangle}$.

Theorem 1.1 *$\neg \mathrm{AP}_{\lambda^+}$ holds in $V[G]^{\langle \mathcal{P}, \leq \rangle}$.*

Proof.

The argument that follows closely Section 3 of [4].

Let us recall the following basic definitions of S. Shelah [6]:

Definition 1.2 1. A function $d : [\lambda^+]^2 \to \eta$ is called

   (a) *normal*, if for every $\nu < \eta, \beta < \lambda^+$ the set $\{\alpha < \beta \mid d(\alpha, \beta) \leq \nu\}$ has cardinality $< \lambda$;

   (b) *subadditive*, if for every $\alpha < \beta < \gamma < \lambda^+$,$d(\alpha, \gamma) \leq \max(d(\alpha, \beta), d(\beta, \gamma))$.

2. $S_0(d) = \{\xi < \lambda^+ \mid \exists A, B$ unbounded in ξ such that $\forall \beta \in B \exists \nu_\beta < \eta \forall \alpha \in A \cap \beta \quad d(\alpha, \beta) \leq \nu_\beta\}$.

Fact 1 (S. Shelah [6]) Suppose that $d, d' : [\lambda^+]^2 \to \omega$ are two normal subadditive functions. Then $S_0(d) \equiv S_0(d')$ (mod the closed unbounded filter).

Fact 2 (S. Shelah [6]) The statement AP_λ is equivalent to the existence of a normal subadditive function $d : [\lambda^+]^2 \to \eta$ such that $S_0(d)$ contains a club.

Suppose that AP_{λ^+} holds in $V[G]^{\langle \mathcal{P}, \leq \rangle}$.
Let $d : [\lambda^+]^2 \to \eta$ be a normal subadditive function in V.
Then, in $V[G]^{\langle \mathcal{P}, \leq \rangle}$ there is a club $C, C \subseteq S_0(d)$.
Suppose for simplicity that

$$0_{\mathcal{P}} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{C} \subseteq \lambda^+ \text{ is a club and } \underset{\sim}{C} \subseteq S_0(d)).$$

Now, in $V[G]$, using the indestructibility of supercompactness of κ under the forcing $\langle \mathcal{P}/\sim, \leq^* \rangle$, let us pick $N \prec \langle H(\lambda^{++}), \in \rangle$ such that

1. $N \cap \kappa \in \kappa$,
2. $|N| < \kappa$,
3. $\underset{\sim}{C}, d \in N$,
4. for every $A \subseteq N \cap \lambda^+$ there is $B \in N$ such that $B \cap N = A$.

By [6], $\sup(N \cap \lambda^+) \notin S_0(d)$.

Let $\bar{N}$ be the transitive collapse of N, and let $\pi : \bar{N} \to N \prec H(\lambda^{++})$. Denote $\kappa \cap N$ by $\kappa(N)$.

Now the assumption that κ was forced to be indestructible applied to the forcing $\langle \mathcal{P}, \leq^* \rangle$, provides a $\bar{N}-$generic set. Its image under π can be easily turned into a condition in $\langle \mathcal{P}/\sim, \leq^* \rangle$. Let $p(N)$ be such a condition. Then for every G_κ generic for $\langle \mathcal{P}, \leq^* \rangle$ with $p(N) \in G_\kappa$, $N[p(N)]$ will be a generic extension of N and an elementary submodel of $(H(\lambda^+))[G_\kappa]$, satisfying the same properties as N.

So, $\delta = \sup(N[p(N)] \cap \lambda^+) = \sup(N \cap \lambda^+) \notin S_0(d)$. It is crucial here that λ, λ^+ remain cardinals in $V[G]^{\langle \mathcal{P}, \leq^* \rangle}$, and so, $S_0(d)$ makes sense in $V[G]^{\langle \mathcal{P}/\sim, \leq^* \rangle}$.

Turn now $p(N)$ into a condition $p^*(N) \in \mathcal{P}$ as follows:
for every coordinate $\alpha < \eta$, we replace $p(N)(\alpha)$ (a non-pure condition in $\mathcal{P}_{E(\alpha)}$) by a pure one $\langle p(N)(\alpha), \bigcap\{A \mid A \in E(\alpha)(\mathrm{dom}(p(N)(\alpha))) \cap N\} \rangle$.
Note that $|N| < \kappa < \kappa_\alpha$, so $\bigcap\{A \mid A \in E(\alpha)(\mathrm{dom}(p(N)(\alpha))) \cap N\} \in E(\alpha)(\mathrm{dom}(p(N)(\alpha)))$.

We shall show, in order to derive a contradiction, that

$$p^*(N) \Vdash_{\langle \mathcal{P}, \leq \rangle} \delta \in \underset{\sim}{C}.$$

Claim 1 For every $\alpha < \lambda^+$ the set

$$D_\alpha = \{p \in \mathcal{P} \mid p \geq^* 0_{\mathcal{P}} \text{ and } |\{\xi \mid \exists q \geq p (q \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{\zeta}_\alpha = \xi)\}| \leq \lambda\}$$

is dense in $\langle \mathcal{P}, \leq^* \rangle$, where $\underset{\sim}{\zeta}_\alpha$ is a canonical name of the first element of C above α.

The claim follows from the condition (3) on $\langle \mathcal{P}, \leq, \leq^* \rangle$ above.

Claim 2 For every $\alpha \in N$, $p^*(N) \Vdash_{\langle \mathcal{P}, \leq \rangle}$ the first element of $\underset{\sim}{C}$ above α is below δ.

Proof. We have $D_\alpha \in N$. Then the set

$$D_\alpha^\sim = \{[p]_\sim \in \mathcal{P}/\sim \mid p \in D_\alpha\}$$

is in N and is dense in $\langle \mathcal{P}/\sim, \leq^* \rangle$.
Now, $p(N)$ is $\langle \mathcal{P}/\sim, \leq^* \rangle-$generic over N, so there is $p^*_\sim \leq^* p(N), p^*_\sim \in D^\sim_\alpha \cap N$. By elementarity, then there is $p^* \in N \cap D_\alpha$ in the equivalence class of $p^*_\sim$. The definition of $p^*(N)$ implies then that $p^* \leq^* p^*(N)$.
Also, $N \prec H(\lambda^+)$. By the previous claim, the number of possible decisions made by conditions stronger than p^* of the first element of C above α is bounded below λ^+. By elementarity,

there is such bound inside $N \cap \lambda^+$.
$\square$ of the claim.

By the previous claim,

$$p^*(N) \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{C} \text{ is unbounded in } \delta.$$

So,

$$p^*(N) \Vdash_{\langle \mathcal{P}, \leq \rangle} \delta \in \underset{\sim}{C}.$$

This provides the desired contradiction.
$\square$

As corollary, we obtain the following:

Theorem 1.3 *Let κ be a supercompact cardinal, $\eta < \kappa$ be a regular cardinal and $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of strong cardinals above κ. Let $\bar{\kappa}_\alpha = \bigcup_{\alpha' < \alpha} \kappa_{\alpha'}$, for every limit $\alpha \leq \eta$.*
Then there is cofinalities preserving extension which satisfies the following, for every $\alpha \leq \eta$ limit:

1. *each $\bar{\kappa}_\alpha$ remains strong limit cardinal,*
2. $2^{\bar{\kappa}_\alpha} > \bar{\kappa}_\alpha^+$,
3. $\neg \mathrm{AP}_{\bar{\kappa}_\eta^+}$.

The initial assumptions of the theorem are stronger than those made in [5], for countable cofinality, and in D.Sinapova [7] for uncountable one. However, cardinals were collapsed in the previous approach and are preserved here.

2 A model in which both AP and SCH fail on a proper class club of singular cardinals.

In [2], O. Ben-Neria, C. Lambie-Hanson, S. Unger use the supercompact Radin forcing to constract a model in which both AP and SCH fail on a proper class club of singular cardinals. Here we would like to use [3] instead, in order to obtain the same result. Again, the initial assumption will be stronger, however no cardinals will change their cofinality.

Theorem 2.1 *Suppose that θ is the least inaccessible cardinal which is a limit of supercompact cardinals.*
Then there is a cofinalities preserving extension such that

- *θ remaining inaccessible,*
- *there is a club in θ consisting of singular cardinals for which both AP and SCH fail.*

Proof.
Let $\langle \delta_\alpha \mid \alpha < \theta \rangle$ be an increasing sequence of supercompact cardinals. Set $\kappa_\alpha = \delta_{\alpha+1}$, for every $\alpha < \theta$. Clearly, each κ_α is strong.
We follow the lines of the previous section with obvious adjustments.
Let $\langle E_\alpha \mid \alpha < \theta \rangle$ such that for every $\alpha < \theta$

1. κ_α is a limit of supercompact cardinals,
2. $E(\alpha)$ is a $(\kappa_\alpha, \theta)-$extender,
3. $E(\alpha) \lhd E(\alpha + 1)$.

Let $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq, \leq^* \rangle$ be the forcing like those of Section 2 of [3], but of inaccessible length θ.[3]
For every limit $\alpha < \theta$ denote $\bar{\kappa}_\alpha = \bigcup_{\alpha'<\alpha} \kappa_{\alpha'}$. The arguments of Section 2 of [3] show the following:

1. $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq, \leq^* \rangle$ is a Prikry type forcing,
2. the forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq \rangle$:
   (a) blows up the power of $\bar{\kappa}_\alpha$ above $\bar{\kappa}_\alpha^+$, for every limit $\alpha < \theta$,
   (b) preserves cardinals and cofinalities,
   (c) preserves strong limitness of each of κ_α's, for every $\alpha \leq \theta$, and $\bar{\kappa}_\alpha$'s, for every limit $\alpha \leq \eta$.
   (d) If for some $\alpha < \theta$, a non-direct extension was made over κ_α, then the forcing $\mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}$ can be split into $\mathcal{P}_{\langle E(\alpha')\restriction\rho_\alpha|\alpha'<\alpha\rangle}$ and $\mathcal{P}_{\langle E(\alpha')|\alpha\leq\alpha'<\theta\rangle}$, where $\rho_\alpha < \kappa_\alpha$ is the reflection of θ below κ_α. Such splitting behave nicely, namely:
      i. $\mathcal{P}_{\langle E(\alpha')\restriction\rho_\alpha|\alpha'<\alpha\rangle}$ has size $\rho_\alpha < \kappa_\alpha$,
      ii. $\mathcal{P}_{\langle E(\alpha')|\alpha\leq\alpha'<\theta\rangle}$ does not add new subsets to κ_α.
3. The forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq^* \rangle$ is equivalent to the product of
   Cohen forcings $Cohen(\kappa_\alpha^+, \theta)$.

[3]Either the Magidor or Easton support can be used for this.

We force with the Laver preparation forcings to ensure indestructibility of supercompactness of each $\delta_\alpha \mid \alpha < \theta$ even under $\delta_\alpha-$directed closed forcings which preserve cardinals, as it is done in Apter [1].
Let G be a corresponding generic set.
Note that it is easy to extend the extender $E(\alpha)$ and its elementary embedding in $V[G]$. Let us abuse the notation a bit and still denote the extension of $E(\alpha)$ in $V[G]$ by $E(\alpha)$.

Force with $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq \rangle$ over $V[G]$. Let us argue that this generic extension is as desired.

The only thing to check is that for every limit $\alpha < \theta$, $\mathrm{AP}_{\bar{\kappa}_\alpha^+}$ breaks down.

Fix a limit ordinal $\alpha^* < \theta$.
By the assumption on minimality of θ, $\mathrm{cof}(\bar{\kappa}_{\alpha^*}) < \bar{\kappa}_{\alpha^*}$. Pick some $\beta^* < \alpha^*$ such that $\bar{\kappa}_{\beta^*} > \mathrm{cof}(\bar{\kappa}_{\alpha^*})$.
Now we split the forcing $\mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}$ into $\mathcal{P}_{\langle E(\alpha')\restriction\rho_{\beta^*}|\alpha'<\beta^*\rangle}$ and $\mathcal{P}_{\langle E(\alpha')|\beta^*\leq\alpha'<\theta\rangle}$, where $\rho_{\beta^*} < \kappa_{\beta^*}$ is the reflection of θ below κ_{β^*}.
Pick now a supercompact cardinal κ such that $\max(\rho_{\beta^*}, \mathrm{cof}(\bar{\kappa}_{\alpha^*})) < \kappa < \kappa_{\beta^*}$.
Now we deal with the upper part $\mathcal{P}_{\langle E(\alpha')|\beta^*\leq\alpha'<\theta\rangle}$.
The conditions (1)-(3) above insure that the argument of the previous suction applies and so, $\neg\mathrm{AP}_{\bar{\kappa}_{\alpha^*}^+}$ holds in a generic extension by $\langle \mathcal{P}_{\langle E(\alpha')|\beta^*\leq\alpha'<\theta\rangle}, \leq \rangle$.
The remaining forcing $\mathcal{P}_{\langle E(\alpha')\restriction\rho_{\beta^*}|\alpha'<\beta^*\rangle}$ has small cardinality relatively to κ, by (2(d)i) above, and so, by Shelah [6], $\neg\mathrm{AP}_{\bar{\kappa}_{\alpha^*}^+}$ will still hold in such further extension. Hence, $\neg\mathrm{AP}_{\bar{\kappa}_{\alpha^*}^+}$ holds in a generic extension by $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq \rangle$.
$\square$

References

[1] A. Apter, Laver Indestructibility and the Class of Compact Cardinals, Journal of Symbolic Logic 63, 1998, 149-157.

[2] O. Ben-Neria, C. Lambie-Hanson, S. Unger, Diagonal supercompact Radin forcing,

[3] M. Gitik, Blowing up the power of a singular cardinal of uncountable cofinality,

[4] M. Gitik and M. Magidor, Extender based forcings. J.of Symbolic Logic 59:2 (1994), 445-460.

[5] M. Gitik and A. Sharon, On SCH and approachability property, Proc. AMS, 136(1), 2008, 311-320.

[6] S. Shelah, On successors of singular cardinals, Logic Colloquim, 78, (M. Boffa, D. van Dallen, K. McAlloon, editors) North-Holland, Amsterdam, 357-380.

[7] D. Sinapova, A model for a very good scale and a bad scale, J. of Symbolic Logic, 73:4,2008, 1361-1372.