Another method for constructing models of not approachability and not SCH.

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Abstract

We present a new method of constructing a model of ¬SCH+¬AP.

At least to the best of our knowledge, the only method to get a singular strong limit cardinal κ such that $\neg\mathrm{AP}_\kappa$ and $2^\kappa > \kappa^+$ was the one introduced in [6]. Here a different approach will be suggested.

We will use the method for blowing up the power of a singular cardinal of [4] in order to get models of not approachability and not SCH. The advantage of the present technique is that no cardinal is collapsed or changes its cofinality.

1 A model in which both AP and SCH fail at a singular cardinal.

We will combine the forcing of [4] with the approach of Section 3 of [5].[1]

Let κ be a supercompact cardinal. Fix a regular cardinal $\eta < \kappa$. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals. Denote $\bigcup_{\alpha<\eta} \kappa_\alpha$ by $\bar{\kappa}_\eta$. Let $\langle E_\alpha \mid \alpha < \eta \rangle$ be a sequence of extenders such that for every $\alpha < \eta$

1. $\kappa < \kappa_0$,

2. $E(\alpha)$ is a $(\kappa_\alpha, \bar{\kappa}_\eta^{++})-$extender,

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[1]Section 3 of [5] contains an essential flow, which is due solely to the first author, but it turns out that with the forcing of [4], it is possible to make the idea work.

3. $E(\alpha) \lhd E(\alpha+1)$.

Let $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq, \leq^* \rangle$ be the forcing of Section 2 of [4].
For every limit $\alpha \leq \eta$ denote $\bar{\kappa}_\alpha = \bigcup_{\alpha'<\alpha} \kappa_{\alpha'}$.
By [4], Section 2, it has the following properties:

1. $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq, \leq^* \rangle$ is a Prikry type forcing,
2. the forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$:
   (a) blows up the power of $\bar{\kappa}_\eta$ to $\bar{\kappa}_\eta^{++}$,
   (b) blows up the power of $\bar{\kappa}_\alpha$ above $\bar{\kappa}_\alpha^{+}$, for every limit $\alpha < \eta$,
   (c) preserves cardinals and cofinalities,
   (d) preserves strong limitness of each of κ_α's, for every $\alpha \leq \eta$, and $\bar{\kappa}_\alpha$'s, for every limit $\alpha \leq \eta$
   (e) does not add new subsets to κ_0.
3. For every $p \in \mathcal{P}$ and every $\mathcal{P}-$name $\underset{\sim}{\zeta}$ of an ordinal, there is $p^* \geq^* p$ such that the number of possible decisions of $\underset{\sim}{\zeta}$ above p^* is at most λ.
   I.e. $|\{\xi \mid \exists q \geq p^*(q \Vdash_{\langle \mathcal{P},\leq\rangle} \underset{\sim}{\zeta} = \xi)\}| \leq \bar{\kappa}_\eta$. [2]
4. The forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq^* \rangle$ is equivalent to the product of Cohen forcings $Cohen(\kappa_\alpha^+, \bar{\kappa}_\eta^{++})$.[3]
   Namely, we just remove or ignore sets of measure one A^p_α in each coordinate $p(\alpha) = \langle f^p_\alpha, A^p_\alpha\rangle$ of a condition $p = \langle p(\alpha) \mid \alpha < \eta\rangle \in \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$. More precisely, if $p = \langle p(\alpha) \mid \alpha < \eta\rangle$ and $q = \langle q(\alpha) \mid \alpha < \eta\rangle$ are in $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$, then set $p \sim q$ iff for every $\alpha < \eta$
   (a) $p(\alpha)$ is non-pure iff $q(\alpha)$ is non-pure. Require then that $p(\alpha) = q(\alpha)$.
   (b) If $p(\alpha) = \langle f^p_\alpha, A^p_\alpha\rangle$, i.e. is pure, then $q(\alpha) = \langle g^p_\alpha, B^p_\alpha\rangle$ is pure as well, and require that $f^p_\alpha = g^p_\alpha$.

[2]This condition basically says that one entree given dense open set by taking a direct extension and then specifying finitely many coordinates. Usually, this property has the same proof, as the Prikry condition and is used to show that λ^+ is preserved in $V^{\langle \mathcal{P},\leq\rangle}$.

[3]This is the crucial difference from the long extenders Prikry forcing $\langle \mathcal{P}, \leq, \leq^* \rangle$ of Sec. 2 of [3]. The conditions in $\mathcal{P}$ consist basically of two parts one of cardinality $< \kappa_n, (n < \omega)$ (assignment functions) and another of cardinality κ_ω (Cohen functions). As a result, $\langle \mathcal{P}, \leq^* \rangle$ collapses κ_ω^+ and, as Asaf Sharon pointed out, $\langle \mathcal{P}, \leq, \leq^* \rangle$ adds $\square^*_{\kappa_\omega}$.
In the present setting both parts are put into one of cardinality κ_n.

Then $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}/\sim, \leq^* \rangle$ is the product of Cohen forcings.

Let us assume that the supercompact cardinal κ was made indestructible under $\kappa-$directed closed forcings using the Laver forcing. Denote by G a generic subset of the Laver forcing.
Then force with $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$. Denote further $\mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}$ by $\mathcal{P}$. We claim that the resulting generic extension is as desired, i.e. it satisfies $\neg \mathrm{AP}_{\bar{\kappa}_\eta}$ and $2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$.
$2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$ follows by (2(a)) above. Let deal with the approachability. Denote $\bar{\kappa}_\eta$ by λ.

Theorem 1.1 $\neg \mathrm{AP}_\lambda$ *holds in* $V[G]^{\langle \mathcal{P}, \leq \rangle}$.

Proof.
The argument that follows closely Section 3 of [5].
We will use the following result of S. Shelah [8], (Fact 2.6, 4),(ii):

Fact 1 Suppose that λ is a strong limit cardinal singular cardinal of cofinality η, $\langle \lambda_i \mid i < \eta \rangle$ is an ascending sequence of regular cardinals with limit λ and
$d : \lambda^+ \times \lambda^+ \to \eta$ is such that for every $\alpha, \beta, \gamma < \lambda^+$:

1. $d(\alpha, \beta) = c(\beta, \alpha)$,
2. if $\alpha < \beta < \gamma$, then $d(\alpha, \gamma) \leq \max(d(\alpha, \beta), d(\beta, \gamma))$.
3. for all $i < \eta$, $|\{\beta < \alpha \mid d(\alpha, \beta) = i\}| \leq \lambda_i$.

Then

$$S(d) =_{def} \{\delta < \lambda^+ \mid \text{ there is an unbounded subset } A \subseteq \delta, \text{ such that for every } \gamma < \delta,$$
$$d'' A \cap \gamma \times A \cap \gamma \text{ is bounded in } \eta\} = \{\delta < \lambda^+ \mid \text{ if } \operatorname{cof}(\delta) > \eta, \text{ then for every unbounded}$$
$$\text{subset } A \subseteq \delta, \text{ there is an unbounded subset } A' \text{ such that } d'' A' \times A' \text{ is bounded in } \eta\}.$$

Moreover, $S(d)$ is a maximal (mod non-stationary) set in the ideal $I[\lambda^+]$.

Also, by S. Shelah [8] (Lemma 4.1), a function $d : \lambda^+ \times \lambda^+ \to \eta$ as above, always exists.
Fix such $d \in V$.
Suppose that AP_λ holds in $V[G]^{\langle \mathcal{P}, \leq \rangle}$.
Then, in $V[G]^{\langle \mathcal{P}, \leq \rangle}$ there is a club $C, C \subseteq S(d)$.
Suppose for simplicity that

$$0_{\mathcal{P}} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{C} \subseteq \lambda^+ \text{ is a club and } \underset{\sim}{C} \subseteq \underset{\sim}{S}(d)).$$

Let G_κ be a generic subset of $\langle \mathcal{P}, \leq^* \rangle$. Then, κ remains a supercompact in $V[G, G_\kappa]$. Using supercompactness, we pick $N^* \prec \langle (H(\lambda^{++}))^{V[G,G_\kappa]}, \in \rangle$ such that

1. $N^* = N[G, G_\kappa]$, for some $N \prec \langle (H(\lambda^{++}))^V, \in \rangle$, $N^* \cap V = N$,
2. $N \cap \kappa \in \kappa$,
3. $|N| < \kappa$,
4. $\underset{\sim}{C}, d \in N^*$,
5. for every $A \subseteq N^* \cap \lambda^+$ there is $B \in N^*$ such that $B \cap N^* = A$.

By [7](Claim 27), $\delta =_{def} \sup(N^* \cap \lambda^+) = \sup(N \cap \lambda^+) \notin (S(d))^{V[G,G_\kappa]}$.

The forcing $\langle \mathcal{P}, \leq^* \rangle$ does not add new subsets to κ or even new $\kappa-$sequences. Hence, by Fact 1,

$$(S(d))^{V[G]} \cap \{\nu < \lambda^+ \mid \mathrm{cof}(\nu) \leq \kappa\} = (S(d))^{V[G,G_\kappa]} \cap \{\nu < \lambda^+ \mid \mathrm{cof}(\nu) \leq \kappa\}.$$

Note that $\mathrm{cof}(\delta) < \kappa$, and hence $\delta \notin (S(d))^{V[G]}$.

Now, the forcing $\langle \mathcal{P}, \leq \rangle$ (over $V[G]$) does not add new subsets to κ. So, by Fact 1,

$$(S(d))^{V[G]} \cap \{\nu < \lambda^+ \mid \mathrm{cof}(\nu) \leq \kappa\} = (S(d))^{V[G]^{\langle \mathcal{P}, \leq \rangle}} \cap \{\nu < \lambda^+ \mid \mathrm{cof}(\nu) \leq \kappa\},$$

and hence, $\delta \notin (S(d))^{V[G]^{\langle \mathcal{P}, \leq \rangle}}$.

Set $M = N[G]$. Then $M \in V[G]$ and $M \preceq (H(\lambda^{++}))^{V[G]}$. Also, $N^* = M[G_\kappa]$. Let $\bar{M}$ be the transitive collapse of M, and let $\pi : \bar{M} \to M$ be the collapsing map. Set $\bar{N}^*$ to be the transitive collapse of N^*, and let $\pi^* : \bar{N}^* \to N^* \preceq (H(\lambda^{++}))^{V[G,G_\kappa]}$ be the collapsing map. Then $\bar{N}^* = \bar{M}[\pi^{*-1}(G_\kappa)]$ and $\pi^* \restriction \bar{M} = \pi$. Denote $\kappa \cap N = \kappa \cap M = \kappa \cap N^*$ by $\kappa(N)$. Then, $\bar{G} =_{def} \pi^{-1}(G)$ will be just the restriction of G to $\kappa(N)$.
Note that $\bar{N}^* \in V[G]$, since it is a transitive set of cardinality $< \kappa$ and the forcing $\langle \mathcal{P}, \leq^* \rangle$ does not add new subsets to κ.

Set $\bar{G}_\kappa = \pi^{*-1}(G_\kappa)$. Then $\pi^{*\prime\prime}\bar{G}_\kappa$ can be easily turned into a condition in $\langle \mathcal{P}, \leq^* \rangle$. Denote it by $p(M)$. Note that $\pi^{*\prime\prime}\bar{G}_\kappa = \pi^{\prime\prime}\bar{G}_\kappa$, and $\pi^{\prime\prime}\bar{G}_\kappa \in V[G]$. Hence, $p(M) \in V[G]$.

Turn now $p(M)$ into a condition $p^*(M) \in \mathcal{P}$ as follows:
for every coordinate $\alpha < \eta$, we replace $p(M)(\alpha)$ (a non-pure condition in $\mathcal{P}_{E(\alpha)}$) by a pure

one $\langle p(M)(\alpha), \bigcap\{A \mid A \in E(\alpha)(\mathrm{dom}(p(M)(\alpha))) \cap M\}\rangle$.
Note that $|M| < \kappa < \kappa_\alpha$, so $\bigcap\{A \mid A \in E(\alpha)(\mathrm{dom}(p(M)(\alpha))) \cap M\} \in E(\alpha)(\mathrm{dom}(p(M)(\alpha)))$.

We shall show, in order to derive a contradiction, that

$$p^*(M) \Vdash_{\langle \mathcal{P}, \leq \rangle} \delta \in \underset{\sim}{C}.$$

Claim 1 For every $\alpha < \lambda^+$ the set

$$D_\alpha = \{p \in \mathcal{P} \mid p \geq^* 0_{\mathcal{P}} \text{ and } |\{\xi \mid \exists q \geq p (q \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{\zeta}_\alpha = \xi)\}| \leq \lambda\}$$

is dense in $\langle \mathcal{P}, \leq^* \rangle$, where $\underset{\sim}{\zeta}_\alpha$ is a canonical name of the first element of C above α.

The claim follows from the condition (3) on $\langle \mathcal{P}, \leq, \leq^* \rangle$ above.

Claim 2 For every $\alpha \in N$, $p^*(M) \Vdash_{\langle \mathcal{P}, \leq \rangle}$ the first element of $\underset{\sim}{C}$ above α is below δ.

Proof. We have $D_\alpha \in M$. Then the set

$$D_\alpha^\sim = \{[p]_\sim \in \mathcal{P}/\sim \mid p \in D_\alpha\}$$

is in M and is dense in $\langle \mathcal{P}/\sim, \leq^* \rangle$.
Now, $p(M)$ is $\langle \mathcal{P}/\sim, \leq^* \rangle$−generic over M, so there is $p^*_\sim \leq^* p(M), p^*_\sim \in D^\sim_\alpha \cap M$. By elementarity, then there is $p^* \in N \cap D_\alpha$ in the equivalence class of $p^*_\sim$. The definition of $p^*(M)$ implies then that $p^* \leq^* p^*(M)$.
Also, $M \preceq (H(\lambda^+))^{V[G]}$. By the previous claim, the number of possible decisions made by conditions stronger (in the order $\leq$) than p^* of the first element of C above α is bounded below λ^+. By elementarity, there is such bound inside $M \cap \lambda^+$.
$\square$ of the claim.

By the previous claim,

$$p^*(M) \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{C} \text{ is unbounded in } \delta.$$

So,

$$p^*(M) \Vdash_{\langle \mathcal{P}, \leq \rangle} \delta \in \underset{\sim}{C}.$$

This provides the desired contradiction.
$\square$

As corollary, we obtain the following:

Theorem 1.2 *Let κ be a supercompact cardinal, $\eta < \kappa$ be a regular cardinal and $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of strong cardinals above κ. Let $\bar{\kappa}_\alpha = \bigcup_{\alpha' < \alpha} \kappa_{\alpha'}$, for every limit $\alpha \leq \eta$.*
Then there is cofinalities preserving extension which satisfies the following, for every $\alpha \leq \eta$ limit:

1. *each $\bar{\kappa}_\alpha$ remains strong limit cardinal,*
2. $2^{\bar{\kappa}_\alpha} > \bar{\kappa}_\alpha^+$,
3. $\neg \mathrm{AP}_{\bar{\kappa}_\eta}$.

The initial assumptions of the theorem are stronger than those made in [6], for countable cofinality, and in D.Sinapova [9] for uncountable one. However, cardinals were collapsed in the previous approach and are preserved here.

2 A model in which both AP and SCH fail on a proper class club of singular cardinals.

In [2], O. Ben-Neria, C. Lambie-Hanson, S. Unger use the supercompact Radin forcing to constract a model in which both AP and SCH fail on a proper class club of singular cardinals. Here we would like to use [4] instead, in order to obtain the same result. Again, the initial assumption will be stronger, however no cardinals will change their cofinality.

Theorem 2.1 *Suppose that θ is the least inaccessible cardinal which is a limit of supercompact cardinals.*
Then there is a cofinalities preserving extension such that

- *θ remains inaccessible,*
- *there is a club in θ consisting of singular cardinals for which both AP and SCH fail.*

Proof.
Let $\langle \delta_\alpha \mid \alpha < \theta \rangle$ be an increasing sequence of supercompact cardinals. Set $\kappa_\alpha = \delta_{\alpha+1}$, for every $\alpha < \theta$. Clearly, each κ_α is strong.
We follow the lines of the previous section with obvious adjustments.
Let $\langle E_\alpha \mid \alpha < \theta \rangle$ such that for every $\alpha < \theta$

1. κ_α is a limit of supercompact cardinals,

2. $E(\alpha)$ is a $(\kappa_\alpha, \theta)-$extender,

3. $E(\alpha) \lhd E(\alpha+1)$.

Let $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq, \leq^* \rangle$ be the forcing like those of Section 2 of [4], but of inaccessible length θ.[4]

For every limit $\alpha < \theta$ denote $\bar{\kappa}_\alpha = \bigcup_{\alpha'<\alpha} \kappa_{\alpha'}$. The arguments of Section 2 of [4] show the following:

1. $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq, \leq^* \rangle$ is a Prikry type forcing,

2. the forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq \rangle$:

   (a) blows up the power of $\bar{\kappa}_\alpha$ above $\bar{\kappa}_\alpha^+$, for every limit $\alpha < \theta$,

   (b) preserves cardinals and cofinalities,

   (c) preserves strong limitness of each of κ_α's, for every $\alpha \leq \theta$, and $\bar{\kappa}_\alpha$'s, for every limit $\alpha \leq \eta$.

   (d) If for some $\alpha < \theta$, a non-direct extension was made over κ_α, then the forcing $\mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}$ can be split into $\mathcal{P}_{\langle E(\alpha')\restriction\rho_\alpha|\alpha'<\alpha\rangle}$ and $\mathcal{P}_{\langle E(\alpha')|\alpha\leq\alpha'<\theta\rangle}$, where $\rho_\alpha < \kappa_\alpha$ is the reflection of θ below κ_α. Such splitting behave nicely, namely:

      i. $\mathcal{P}_{\langle E(\alpha')\restriction\rho_\alpha|\alpha'<\alpha\rangle}$ has size $\rho_\alpha < \kappa_\alpha$,

      ii. $\mathcal{P}_{\langle E(\alpha')|\alpha\leq\alpha'<\theta\rangle}$ does not add new subsets to κ_α.

3. The forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq^* \rangle$ is equivalent to the product of Cohen forcings $Cohen(\kappa_\alpha^+, \theta)$.

We force with the Laver preparation forcings to ensure indestructibility of supercompactness of each $\delta_\alpha, \alpha < \theta$, even under $\delta_\alpha-$directed closed forcings which preserve cardinals, as it is done in Apter [1].

Let G be a corresponding generic set.

Note that it is easy to extend the extender $E(\alpha)$ and its elementary embedding in $V[G]$. Let us abuse the notation a bit and still denote the extension of $E(\alpha)$ in $V[G]$ by $E(\alpha)$.

Force with $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq \rangle$ over $V[G]$. Let us argue that this generic extension is as desired.

[4]Either the Magidor or Easton support can be used for this.

The only thing to check is that for every limit $\alpha < \theta$, $\mathrm{AP}_{\bar{\kappa}_\alpha}$ breaks down.

Fix a limit ordinal $\alpha^* < \theta$.

By the assumption on minimality of θ, $\mathrm{cof}(\bar{\kappa}_{\alpha^*}) < \bar{\kappa}_{\alpha^*}$. Pick some $\beta^* < \alpha^*$ such that $\bar{\kappa}_{\beta^*} > \mathrm{cof}(\bar{\kappa}_{\alpha^*})$.

Now we split the forcing $\mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}$ into $\mathcal{P}_{\langle E(\alpha')\restriction\rho_{\beta^*}|\alpha'<\beta^*\rangle}$ and $\mathcal{P}_{\langle E(\alpha')|\beta^*\leq\alpha'<\theta\rangle}$, where $\rho_{\beta^*} < \kappa_{\beta^*}$ is the reflection of θ below κ_{β^*}.

Pick now a supercompact cardinal κ such that $\max(\rho_{\beta^*}, \mathrm{cof}(\bar{\kappa}_{\alpha^*})) < \kappa < \kappa_{\beta^*}$.

Now we deal with the upper part $\mathcal{P}_{\langle E(\alpha')|\beta^*\leq\alpha'<\theta\rangle}$.

The conditions (1)-(3) above insure that the argument of the previous suction applies and so, $\neg\mathrm{AP}_{\bar{\kappa}_{\alpha^*}}$ holds in a generic extension by $\langle \mathcal{P}_{\langle E(\alpha')|\beta^*\leq\alpha'<\theta\rangle}, \leq \rangle$.

The remaining forcing $\mathcal{P}_{\langle E(\alpha')\restriction\rho_{\beta^*}|\alpha'<\beta^*\rangle}$ has small cardinality relatively to κ, by (2(d)i) above, and so, by Shelah [7], $\neg\mathrm{AP}_{\bar{\kappa}_{\alpha^*}}$ will still hold in such further extension. Hence, $\neg\mathrm{AP}_{\bar{\kappa}_{\alpha^*}}$ holds in a generic extension by $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\theta\rangle}, \leq \rangle$.

□

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