1 The Preparation Forcing

We assume GCH.

A condition in the preparation forcing $P'$, which we define below, will consist basically of an elementary chain of models of cardinality $\kappa^{++}$ and a directed system elementary submodels of cardinality $\kappa^+$. Inside this directed system a crucial role will be played by a certain elementary chain which will be called central line. Let us give first a definition of both elementary chains.

**Definition 1.1** The set $P''$ consists of elements of the form

$$\langle B^{1\kappa^+}, A^{1\kappa^{++}} \rangle$$

so that the following hold:

1. $A^{1\kappa^{++}}$ is a continuous closed chain of length less than $\kappa^3$ of elementary submodels of $\langle H(\kappa^3), \in, <, \subseteq, \kappa \rangle$ each of cardinality $\kappa^{++}$.

2. For each $X \in A^{1\kappa^{++}}$, we have $X \cap \kappa^3 \in On$. So, $X \supseteq \kappa^+$. Further we shall frequently identify such model $X$ with the ordinal $X \cap \kappa^3$ and also view $A^{1\kappa^{++}}$ as a closed set of ordinals.

3. If $X$ is a non-limit element of the chain $A^{1\kappa^{++}}$ then

   (a) $A^{1\kappa^{++}} \upharpoonright X := \{ Y \mid Y \subseteq X, Y \in A^{1\kappa^{++}} \} \in X$,
(b) $\kappa^+ X \subseteq X$.

4. $B^{1\kappa^+}$ is a continuous closed chain of length less than $\kappa^{++}$ of elementary submodels of $\langle H(\kappa^{+3}), \in, <, \subseteq, \kappa \rangle$, each of cardinality $\kappa^+$. $B^{1\kappa^+}$ has the last element which we denote by $\text{max}(B^{1\kappa^+})$.

5. For each $X \in B^{1\kappa^+}$, we have $X \cap \kappa^{++} \in On$. Hence $X \supseteq \kappa^+$.

6. If $X$ is a non-limit element of the chain $B^{1\kappa^+}$ then

   (a) $B^{1\kappa^+} \upharpoonright X := \{ Y \mid Y \subseteq X, Y \in B^{1\kappa^+} \} \in X$,

   (b) $\kappa X \subseteq X$,

   (c) If $\delta < \text{sup}(X)$ for some $\delta \in A^{1\kappa^{++}}$ (we identify here an element of $A^{1\kappa^{++}}$ with an ordinal), then $\text{min}(X \setminus \delta) \in A^{1\kappa^{++}}$.

The following technical notion will be needed in order to define $\mathcal{P}'$ (and will be used further as well).

**Definition 1.2** Suppose that $\langle B^{1\kappa^+}, A^{1\kappa^{++}} \rangle \in \mathcal{P}'$, $F \in B^{1\kappa^+}$ and $F_0, F_1 \in F$. We say that the triple $F_0, F_1, F$ is of $\Delta$-system type iff

1. $F_0$ is the immediate predecessor of $F$ in the chain $B^{1\kappa^+}$,

2. $F_1 \prec F$,

3. ?(it looks like it is possible also without this) $\kappa F_1 \subseteq F_1$,

4. $A^{1\kappa^{++}} \upharpoonright \text{sup}(F_1) \in F_1$. Replacement:

5. If $\delta < \text{sup}(F_1 \cap On)$ for some $\delta \in A^{1\kappa^{++}}$, then $\text{min}((F_1 \cap On) \setminus \delta) \in A^{1\kappa^{++}}$.

6. There are $\alpha_0, \alpha_1 \in A^{1\kappa^{++}}$ such that

   (a) $\text{cof}(\alpha_0) = \text{cof}(\alpha_1) = \kappa^{++}$,

   (b) $\alpha_0 \in F_0$ and $\alpha_1 \in F_1$, 

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(c) \( F_0 \cap F_1 \cap On = F_0 \cap \alpha_0 = F_1 \cap \alpha_1 \),

(d) either \( \alpha_0 > \sup(F_1 \cap On) \) or \( \alpha_1 > \sup(F_0 \cap On) \).

Intuitively, this means that \( F_0, F_1 \) behave as in a \( \Delta \)-system with the common part below \( \min \alpha_0, \alpha_1 \).

Further let us call \( \alpha_0, \alpha_1 \) the witnessing ordinals for \( F_0, F_1, F \).

The next condition will require more similarity:

7. (isomorphism condition)

the structures

\[
\langle F_0, \in, <, \subseteq, \kappa, A^{1\kappa++} \cap F_0, f_{F_0} \rangle
\]

and

\[
\langle F_1, \in, <, \subseteq, \kappa, A^{1\kappa++} \cap F_1, f_{F_1} \rangle
\]

are isomorphic over \( F_0 \cap F_1 \), i.e. the isomorphism \( \pi_{F_0,F_1} \) between them is the identity on \( F_0 \cap F_1 \), where ? (it seems unnecessary for gap 3 to have this \( f_{F_1} \)) \( f_{F_0} : \kappa^+ \longleftrightarrow F_0, f_{F_1} : \kappa^+ \longleftrightarrow F_1 \) are some fixed in advance bijections.

Note that, in particular, we will have that \( \otp(F_0) = \otp(F_1) \) and \( F_0 \cap \kappa^{++} = F_1 \cap \kappa^{++} \).

**Definition 1.3** The set \( \mathcal{P}' \) consists of elements of the form

\[
\langle \langle A^{0\kappa}, A^{1\kappa+}, C^{\kappa^+} \rangle, A^{1\kappa++} \rangle
\]

so that the following hold:

1. \( A^{0\kappa+} \in A^{1\kappa+} \),

2. every \( X \in A^{1\kappa+} \) is either equal to \( A^{0\kappa+} \) or belongs to it,

3. \( C^{\kappa^+} : A^{1\kappa+} \to P(A^{1\kappa+}) \),

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4. for every $X \in A^{1\kappa^+}$, \( \langle \kappa^+(X), A^{1\kappa^+} \rangle \in P'' \) and $X$ is the maximal model of $\kappa^+(X)$. In particular, each $\kappa^+(X)$ is an increasing continuous chain of models of cardinality $\kappa^+$.

5. (Coherence) If $X, Y \in A^{1\kappa^+}$ and $X \in \kappa^+(Y)$, then $\kappa^+(X)$ is an initial segment of $\kappa^+(Y)$ with $X$ being the largest element of it.

We call $\kappa^+(A^{0\kappa^+})$ central line of $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^\kappa \rangle, A^{1\kappa^+} \rangle$. The following conditions describe a special way in which $A^{1\kappa^+}$ is generated from the central line.

6. Let $B \in A^{1\kappa^+}$. Then $B \in C^\kappa(A^{0\kappa^+})$ (i.e., it is on the central line) or there are $n < \omega$ and sequences $\langle A_1, ..., A_n \rangle$, $\langle B_1, ..., B_n \rangle$ of elements of $A^{1\kappa^+}$ such that

(a) $A_1 \in C^\kappa(A^{0\kappa^+})$ is the least model of the central line $C^\kappa(A^{0\kappa^+})$ that contains $B$.

(b) $A_1$ is a successor model in $C^\kappa(A^{0\kappa^+})$. Let $A_{-1}^1$ denotes its immediate predecessor in $C^\kappa(A^{0\kappa^+})$.

(c) The triple $A_{-1}^1, B_1, A_1$ is of a $\Delta$-system type with respect to $A^{1\kappa^+}$.

(d) For each $m, 1 < m \leq n$,

i. $A_m \in C^\kappa(B_{m-1})$ (i.e. it is on the central line of $B_{m-1}$) is the least model in $C^\kappa(B_{m-1})$ that contains $B$.

ii. $A_m$ is a successor model in $C^\kappa(B_{m-1})$. Let $A_{-m}^{-1}$ denotes its immediate predecessor in $C^\kappa(B_{m-1})$.

iii. The triple $A_{-m}^{-1}, B_m, A_m$ is of a $\Delta$-system type with respect to $A^{1\kappa^+}$.

(e) $B \in C^\kappa(B_n)$.

We refer to the sequence $\langle A_1, A_1^{-1}, B_1, ..., A_{n-1}^{-1}, B_{n-1}, A_n, A_n^{-1}, B_n \rangle$ as the walk from $A^{0\kappa^+}$ (or from the central line) to $B$. Denote it by $wk(A^{0\kappa^+}, B)$. 

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Let us call \( n \) distance of \( B \) from the central line, denote it by \( \text{dcl}(B) \). If it is on the central line, then set \( \text{dcl}(B) = 0 \).

The next condition strengthens a bit the isomorphism condition (7) of Definition 1.2.

7. (isomorphism condition) Let \( F_0, F_1, F \in A^{1 \kappa^+} \) be of a \( \Delta \)-system type and \( X \in A^{1 \kappa^+} \). Then \( X \in F_0 \) iff \( \pi_{F_0 \cap F_1} [X] \in F_1 \cap A^{1 \kappa^+} \). This means that the structures of 1.2(7) remain isomorphic even if we add \( F_0 \cap A^{1 \kappa^+} \) to the first and \( F_1 \cap A^{1 \kappa^+} \) to the second.

8. (uniqueness) Let \( F_0, F_1, F' \in A^{1 \kappa^+} \). If both triples \( F_0, F_1, F \) and \( F_0, F'_1, F \) are of a \( \Delta \)-system type, then \( F_1 = F'_1 \).

Note that both conditions 7, 8 can be stated equivalently only in the case when \( F \) is on the central line.

Let us define also a walk to an ordinal.

**Definition 1.4** Let \( \langle \langle A^{0 \kappa^+}, A^{1 \kappa^+}, C^{\kappa^+} \rangle, A^{1 \kappa^{++}} \rangle \in P' \) and \( \alpha \in A^{1 \kappa^{++}} \cap A^{0 \kappa^+} \).

The sequence \( \langle A_1, A_{-1}, B_1, ..., A_{n-1}, A_n, B_{n-1}, A_n, A_{n-1}, A_{n-2}, ..., A_1 \rangle \) of elements of \( A^{1 \kappa^+} \) is called a walk from \( A^{0 \kappa^+} \) to \( \alpha \) iff

1. \( A_1 \in C^{\kappa^+}(A^{0 \kappa^+}) \) is the least model of \( C^{\kappa^+}(A^{0 \kappa^+}) \) with \( \alpha \in A_1 \),

2. either

   - \( A_1 \) is the least model of \( C^{\kappa^+}(A^{0 \kappa^+}) \) and then \( A_n^- = A_1 \), i.e. the walk consists of \( A_1 \) alone,
   
   or

   - \( A_1^- \) exists, it is the immediate predecessor of \( A_1 \) on \( C^{\kappa^+}(A^{0 \kappa^+}) \). If \( A_1^- \) is the unique immediate predecessor of \( A_1 \), or there is no other one but \( \alpha \) does belong to it, then the walk consists of \( \langle A_1, A_1^- \rangle \).

   Otherwise, \( A_1, B_1, A_1 \) are of \( \Delta \)-system type, \( \alpha \in B_1 \) and the walk continues.
3. For each $m, 1 < m \leq n$,

(a) $A_m \in C_{\kappa^+}^{-}(B_{m-1})$ (i.e. it is on the central line of $B_{m-1}$) is the least model in $C_{\kappa^+}^{-}(B_{m-1})$ with $\alpha \in A_m$, either

- $A_m$ is the least model of $C_{\kappa^+}^{-}(B_{m-1})$ and then $B_{n-} = A_m$,

or

- $A_m^-$ exists, it is the immediate predecessor of $A_m$ on $C_{\kappa^+}^{-}(B_{m-1})$. If $A_m^-$ is the unique immediate predecessor of $A_m$, or there is an other one but $\alpha$ does belong to it, then $A_{n-} = A_m^-$. Otherwise, $A_m, B_m, A_m$ are of $\Delta$-system type, $\alpha \in B_m$ and the walk continues.

4. $\alpha \in A_n$ and either

- $A_n$ is the least model of $C_{\kappa^+}^{-}(B_{n-1})$ and then $A_{n-} = A_{n-1} = A_n$, i.e. the walk terminates at $A_n$;

or

- there exists the immediate predecessor of $A_n$ in $C_{\kappa^+}^{-}(B_{n-1})$. Then $A_{n-}$ is this immediate predecessor of $A_n$ and there is no $Z \in A_{1\kappa^+}^{-}$ such that $A_{n-}, Z, A_n$ is of a $\Delta$-system type. In this case $A_{n-1} = A_{n-}$ and the walk terminates at $A_{n-}$;

or

- there exists the immediate predecessor of $A_n$ in $C_{\kappa^+}^{-}(B_{n-1})$. Then $A_{n-}$ is this immediate predecessor of $A_n$ and there is $Z \in A_{1\kappa^+}^{-}$ such that $A_{n-}, Z, A_n$ is of a $\Delta$-system type, witnessed by $\xi_0 \in A_{n-} \cap A_{1\kappa^+}^{-}, \xi_1 \in Z \cap A_{1\kappa^+}^{-}$. Then $\alpha \not\in Z$. If $\alpha \not\in [\xi_1, \sup(Z)]$, then $A_{n-1} = A_{n-}$ and the walk to $\alpha$ terminates at $A_{n-}$. If $\alpha \in [\xi_1, \sup(Z)]$, then $A_{n-1} = Z$.

Note that walks to ordinals terminate by the last model $A_n$ to which the ordinal belongs followed by its immediate predecessor in $C_{\kappa^+}^{-}(A_n)$, whenever such predecessor exists.
Definition 1.5 (Complexity of walks)
Let $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in \mathcal{P}'$.

- Suppose that $A, B \in A^{1\kappa^+}$. We say that the walk from $A^{0\kappa^+}$ to $A$ is \textit{simpler} than the walk from $A^{0\kappa^+}$ to $B$ iff
  1. $A \subset B$, or
  2. $A \not\subset B, B \not\subset A, A \neq B$ and if $F \in A^{1\kappa^+}$ is the last common point of both walks, then $A \subseteq F_0$, where $F_0$ is the immediate predecessor of $F$ in $C^{\kappa^+}(F)$. Note that necessarily, there is $F_1 \in A^{1\kappa^+}$ such that $F_0, F_1, F$ is a triple of a $\Delta$-system type and $B \subseteq F_1$.

- Suppose that $A \in A^{1\kappa^+}$ and $\alpha \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. We say that the walk from $A^{0\kappa^+}$ to $A$ is \textit{simpler} than the walk from $A^{0\kappa^+}$ to $\alpha$ iff
  1. $A$ is one of the models of the walk to $\alpha$,
  or
  2. if $F$ is the last common model of the walks, then $A \subseteq C^{\kappa^+}(F)$, or $A \not\subseteq C^{\kappa^+}(F)$ and $A \subseteq F_0$, where $F_0$ is the immediate predecessor of $F$ in $C^{\kappa^+}(F)$. Note, if the second possibility occurs, then, necessarily, there is $F_1 \in A^{1\kappa^+}$ such that $F_0, F_1, F$ is a triple of a $\Delta$-system type and $\alpha \in F_1$.

- Suppose that $\alpha, \beta \in A^{1\kappa^{++}} \cap A^{0\kappa^+}$. We say that the walk from $A^{0\kappa^+}$ to $\alpha$ is \textit{simpler} than the walk from $A^{0\kappa^+}$ to $\beta$ iff $\alpha \neq \beta$, there is $F \in A^{1\kappa^+}$ which is the last common point of both walks and
  1. there are $D, E \in C^{\kappa^+}(F)$ such that $\alpha \in D \in E$ and $\beta \in E \setminus D$,
  or
  2. there are $F_0, F_1 \in A^{1\kappa^+}$ such that $F_0, F_1, F$ are of a $\Delta$-system type, $F_0 \in C^{\kappa^+}(F), \alpha \in F_0$ and $\beta \in F_1$,
3. there are $F_0, F_1 \in A^{1\kappa^+}$ such that $F_0, F_1, F$ are of a $\Delta$-system type, $F_0 \in C^{\kappa^+}(F), \xi_0, \xi_1$ the witnessing ordinals, and $\beta \in F \setminus (F_0 \cup F_1)$, $\xi_1 \leq \beta \leq \sup(F_1)$ and $\alpha \in F_1$, or

4. there are $F_0, F_1 \in A^{1\kappa^+}$ such that $F_0, F_1, F$ are of a $\Delta$-system type, $F_0 \in C^{\kappa^+}(F), \xi_0, \xi_1$ the witnessing ordinals, and $\alpha \in F \setminus (F_0 \cup F_1)$, $\beta \in F_1$ and $\alpha < \xi_1$ or $\alpha > \sup(F_1)$.

The above defines a well-founded relation. We will use further the walks complexity in inductive arguments.

**Lemma 1.6** Let $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^++} \rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Then

1. $\langle \langle B, A^{1\kappa^+} \cap (B \cup \{B\}), C^{\kappa^+} \upharpoonright A^{1\kappa^+} \cap (B \cup \{B\}) \rangle, A^{1\kappa^++} \rangle \in \mathcal{P}'$.

2. If $B' \in A^{1\kappa^+}$ and $B' \not\subseteq B$, then $B' \in B$.

**Proof.** We prove both statements simultaneously by an induction on $dcl(B)$ -the distance from the central line. If $B$ is on the central line, then it is clear. Suppose that $B$ is not on the central line. Consider the walk $\langle A_1, A_1^-, B_1, ..., A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n \rangle$ from $A^{0\kappa^+}$ to $B$. We have

$\langle \langle A_1^-, A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\}), C^{\kappa^+} \upharpoonright A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\}) \rangle, A^{1\kappa^++} \rangle \in \mathcal{P}'$.

Recall that $A_1^-, B_1, A_1$ are of the $\Delta$-system type. Hence we have the isomorphism $\pi_{A_1^-,B_1}$ between $A_1^-$ and $B_1$ which preserves all the relevant structure. In particular, it will move the walk from $A_1^-$ to a model in $A^{1\kappa^+} \cap (A_1^- \cup \{A_1^-\})$ to the walk from $B_1$ to the corresponding under $\pi_{A_1^-,B_1}$ model of $A^{1\kappa^+} \cap (B_1 \cup \{B_1\})$. This easily implies that

$\langle \langle B_1, A^{1\kappa^+} \cap (B_1 \cup \{B_1\}), C^{\kappa^+} \upharpoonright A^{1\kappa^+} \cap (B_1 \cup \{B_1\}) \rangle, A^{1\kappa^++} \rangle \in \mathcal{P}'$.

Suppose now that we have some $B' \in A^{1\kappa^+}, B' \not\subseteq B_1$. If $B' \not\subseteq A_1^-$, then the walk from $A^{0\kappa^+}$ to $B'$ goes via $B_1$, and hence $B' \in B_1$. Suppose that
$B' \subseteq A_1^-$. It is impossible to have $B' = A_1^-$, since then

$$A_1^- \cap B_1 \supseteq B' = A_1^-,$$

which is clearly not the case. So, $B' \not\subseteq A_1^-$. Then the walk from $A^{0\kappa^+}$ to $B'$ goes via $A_1^-$, and hence $B' \in A_1^-$. Then $\pi_{A_1^-,B_1}(B') \in B_1$, but

$$\pi_{A_1^-,B_1}(B') = \pi_{A_1^-,B_1} " B' = B'.$$

So we are done.

Hence, $A_1^{1\kappa^+} \cap (B_1 \cup \{B_1\}) = A_1^{1\kappa^+} \cap \mathcal{P}(B_1)$.

Now we deal with $B$ and $\langle\langle B_1, A^{1\kappa^+} \cap (B_1 \cup \{B_1\}), C^{\kappa^+} \upharpoonright A^{1\kappa^+} \cap (B_1 \cup \{B_1\}), A^{1\kappa^{++}} \rangle, \mathcal{P}'\rangle$. The walk distance from $B_1$ to $B$ is shorter than those from $A^{0\kappa^+}$ to $B$. So the induction hypothesis applies.

□

The next lemma is trivial.

**Lemma 1.7** Let $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}, A^{1\kappa^{++}} \rangle, A^{1\kappa^{++}} \rangle \in \mathcal{P}'$ and $Z \in A^{1\kappa^{++}}$ is so that $Z \cap \kappa^{+3} \geq \sup(A^{0\kappa^+})$. Then $\langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}, \{Y \in A^{1\kappa^{++}} \mid Y \subseteq Z\} \rangle \in \mathcal{P}'$.

Let us introduce the following notation:

**Definition 1.8** Let $p = \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}, A^{1\kappa^{++}} \rangle, A^{1\kappa^{++}} \rangle \in \mathcal{P}'$ and $B \in A^{1\kappa^+}$. Then set

$$p \upharpoonright B := \langle\langle B, A^{1\kappa^+} \cap (B \cup \{B\}), C^{\kappa^+} \upharpoonright A^{1\kappa^+} \cap (B \cup \{B\}), A^{1\kappa^{++}} \rangle, A^{1\kappa^{++}} \rangle.$$

We call $p \upharpoonright B$ the restriction of $p$ to $B$.

Similar, if $Z \in A^{1\kappa^{++}}$, then set

$$p \upharpoonright Z := \langle\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+}, \{Y \in A^{1\kappa^{++}} \mid Y \subseteq Z\} \rangle.$$

Also, let $p \upharpoonright (B, Z) := (p \upharpoonright B) \upharpoonright Z$.

By the previous lemmas, $p \upharpoonright (B, Z) \in \mathcal{P}'$.

The next lemma follows easily from the definitions.
Lemma 1.9 Let \( \langle \langle A^0+, A^{1\kappa+}, C^{\kappa+} \rangle, A^{1\kappa++} \rangle \in \mathcal{P}' \), \( A \in A^{1\kappa+} \) and \( \delta \in A^{1\kappa++} \). If \( \delta < \sup(A) \), then \( \min(A \setminus \delta) \in A^{1\kappa++} \).

Proof. By 1.3(4), \( \langle C^{\kappa+}(A), A^{1\kappa++} \rangle \in \mathcal{P}'' \). So, it satisfies 1.1(6(d)), (or ???) and we are done, if \( A \) is a successor model of \( C^{\kappa+}(A) \). Suppose \( A \) is a limit model of \( C^{\kappa+}(A) \). Let \( \langle A_i \mid i < \eta \rangle \) be an increasing sequence of successor models of \( C^{\kappa+}(A) \) with \( \bigcup_{i<\eta} A_i = A \). Now, \( \delta < \sup(A) \), so starting with some \( i^* < \eta \), we have \( \delta < \sup(A_i) \). Just note that \( i < j \) implies \( A_i \subseteq A_j \), hence \( \langle \sup(A_i) \mid i < \eta \rangle \) is an increasing sequence of ordinals with limit \( \sup(A) \). Set \( \alpha_i = \min(A_i \setminus \delta) \), for each \( i, i^* \leq i < \eta \). By 1.1(6(d)), \( \alpha_i \in A^{1\kappa++} \). Clearly, \( i \geq j \) implies \( \alpha_i \leq \alpha_j \). Hence, the sequence \( \langle \alpha_i \mid i^* \leq i < \eta \rangle \) is eventually constant. Let \( \alpha^* \) be this constant value. Then \( \min(A \setminus \delta) = \alpha^* \) and we are done.

\( \square \)

Definition 1.10 Let \( \langle \langle A^0+, A^{1\kappa+}, C^{\kappa+} \rangle, A^{1\kappa++} \rangle \in \mathcal{P}' \) and \( A, B \in A^{1\kappa+} \). We say that \( A \) satisfies the intersection property with respect to \( B \) or shortly \( ip(A, B) \) iff either

1. \( A \supseteq B \), or
2. \( B \supseteq A \), or
3. \( A \not\supseteq B \), \( B \not\supseteq A \), and then there are \( A' \in A^{1\kappa+} \cap (A \cup \{A \}) \) and \( \eta \in A^{1\kappa++} \cap A' \) such that
\[
A \cap B = A' \cap \eta,
\]
or just
\[
A \cap B = A'.
\]

Let \( ipb(A, B) \) denotes that both \( ip(A, B) \) and \( ip(B, A) \) hold.

Lemma 1.11 (The intersection lemma) Let \( \langle \langle A^{0\kappa+}, A^{1\kappa+}, C^{\kappa+} \rangle, A^{1\kappa++} \rangle \in \mathcal{P} \) and \( X, Y \in A^{1\kappa+} \). Then \( ipb(X, Y) \).
Proof. Assume that $X \not\supseteq Y, Y \not\supseteq X$.

Consider the walks $\langle A_1, A_1^-, B_1, \ldots, A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n \rangle$ from $A^{0\kappa^+}$ to $X$ and $\langle D_1, D_1^-, E_1, \ldots, D_{m-1}, D_{m-1}^-, E_{m-1}, D_m, D_m^-, E_m \rangle$ from $A^{0\kappa^+}$ to $Y$.

Let $B_k = E_k$ be the last place up to which the walks coincide. Then we have both $A_{k+1}, D_{k+1}$ in $C^{\kappa^+}(B_k)$ but at different places.

Suppose first that $A_{k+1}$ is above $D_{k+1}$. Then $A_{k+1}^- = D_{k+1}$ or $A_{k+1} \supset D_{k+1}$, and then $D_{k+1} \in A_{k+1}^-$. Now, $A_{k+1}^-, B_{k+1}, A_{k+1}$ are of a $\Delta$-system type. Hence by Definition 1.2(6), there are ordinals $\alpha_0, \alpha_1 \in A^{1\kappa^+} \cap A_{k+1}, \alpha_0 \in A_{k+1}^-$ and $\alpha_1 \in B_{k+1}$ such that

$$A_{k+1}^- \cap B_{k+1} = A_{k+1}^- \cap \alpha_0 = B_{k+1} \cap \alpha_1.$$ 

Recall that $X \subseteq B_{k+1}$ and $Y \subseteq A_{k+1}^-$. Hence,

$$X \cap Y = (X \cap B_{k+1}) \cap (Y \cap A_{k+1}^-) = (X \cap \alpha_1) \cap (Y \cap \alpha_0).$$

Let us use (7) of 1.3. Then

$$X' = \pi_{B_{k+1}, A_{k+1}^-} [X] \in A_{k+1} \cap A^{1\kappa^+}.$$ 

Also,

$$X \cap \alpha_1 = X' \cap \alpha_0,$$

since the isomorphism $\pi_{B_{k+1}, A_{k+1}^-}$ is the identity over $B_{k+1} \cap A_{k+1}^-$. Hence,

$$X \cap Y = X \cap \alpha_1 \cap Y = X' \cap \alpha_0 \cap Y.$$ 

Consider

$$p := \langle \langle A_{k+1}^-, A^{1\kappa^+} \cap (A_{k+1}^- \cup \{A_{k+1}^-\}), C^{\kappa^+} \upharpoonright A^{1\kappa^+} \cap (A_{k+1}^- \cup \{A_{k+1}^-\}), A^{1\kappa^+} \rangle.$$ 

By Lemma 1.6, it is in $\mathcal{P}'$. We can apply the inductive hypothesis to $p, X'$ and $Y$, since the walk from $A_{k+1}^-$ to $X'$ shorter than those from $A^{0\kappa^+}$ to $X$ (it is just a copy under $\pi_{B_{k+1}, A_{k+1}^-}$ of the final segment $\langle B_{k+1}, \ldots, A_n, A_n^-, B_n \rangle$. 

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of the original walk to $X$ from $A^{0\kappa^+}$). Hence there are $Y' \in A^{1\kappa^+} \cap (Y \cup \{Y\})$ and $\eta \in A^{1\kappa^+} \cap A$ such that

$$X' \cap Y = Y' \cap \eta.$$  

Then

$$X \cap Y = Y \cap Y' \cap \eta \cap \alpha_0.$$  

If $\alpha_0 \in Y$, then we are done. Suppose otherwise. If $\alpha_0 \geq \sup(Y)$, then we can just remove it from the intersection above. If $\alpha_0 < \sup(Y)$, then replace it by $\min(Y \cap \alpha_0)$, which is in $A^{1\kappa^+}$ by Lemma 1.9.

This shows $ip(Y, X)$. Finally, using $\pi_{A_{k+1}, B_{k+1}}$ and moving $Y$ to $B_{k+1}$, the same argument shows $ip(X, Y)$. 

□

It is easy to deduce the following generalization using an induction:

**Lemma 1.12** Let $\langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle \in P$ and $A_1, \ldots, A_n \in A^{1\kappa^+}$, for some $n < \omega$. Then there are $A' \in A^{1\kappa^+} \cap (A_1 \cup \{A_1\})$ and $\eta \in A^{1\kappa^+} \cap A'$ such that $A_1 \cap \ldots \cap A_n = A' \cap \eta$ or just $A_1 \cap \ldots \cap A_n = A'$.

We need to allow a possibility to change the component $C^{\kappa^+}$ in elements of $P'$ and replace one central line by another. It is essential for the definition of an order on $P'$ given below.

**Definition 1.13** Let $p = \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle \in P'$ and $B \in A^{1\kappa^+}$. Define $swt(p, B)$ (here $swt$ stands for switch) to be

$$\langle A^{0\kappa^+}, A^{1\kappa^+}, D^{\kappa^+}, A^{1\kappa^+} \rangle,$$

where $D^{\kappa^+}$ is obtained from $C^{\kappa^+}$ as follows:

$D^{\kappa^+} = C^{\kappa^+}$ unless $B$ has exactly two immediate predecessors in $A^{1\kappa^+}$. If $B_0 \neq B_1$ are such predecessors of $B$ and, say $B_0 \in C^{\kappa^+}(B)$, then we set $D^{\kappa^+}(B) = C^{\kappa^+}(B_1) \setminus B$. Extend $D^{\kappa^+}$ on the rest in the obvious fashion just replacing $C^{\kappa^+}(B_0)$ by $C^{\kappa^+}(B_1)$ for models including $B$ and then moving over
isomorphic models.  

Intuitively, we switched here from $B_0$ to $B_1$.

Note that $\text{swt}(\text{swt}(p, B), B) = p$.

Define $q = \text{swt}(p, B_1, \ldots, B_n)$ by applying the operation $\text{swt}$ $n$-times:

\[ p_{i+1} = \text{swt}(p_i, B_i), \] for each $1 \leq i \leq n$, where $p_1 = p$ and $q = p_{n+1}$.

The following simple observation will be useful further.

**Lemma 1.14** Let $p = \langle \langle A_0^{\kappa^+}(p), A_1^{\kappa^+}(p), C^{\kappa^+}(p) \rangle, A_1^{\kappa^+}(p) \rangle \in P'$ and $B \in A_1^{\kappa^+}(p)$. Then there are $E_1, \ldots, E_m \in A_1^{\kappa^+}(p)$ such that $B \in C^{\kappa^+}(q)(A_0^{\kappa^+}(p))$, where

\[ q = \langle \langle A_0^{\kappa^+}(p), A_1^{\kappa^+}(p), C^{\kappa^+}(q) \rangle, A_1^{\kappa^+}(p) \rangle = \text{swt}(p, E_1, \ldots, E_m). \]

**Proof.** If $B \in C^{\kappa^+}(p)(A_0^{\kappa^+}(p))$, then let $q = p$. Otherwise, Consider the walk $(A_1, A_1^-, B_1, \ldots, A_{n-1}, A_{n-1}^-, B_{n-1}, A_n, A_n^-, B_n)$ from $A_0^{\kappa^+}$ to $B$. Then $q = \langle \langle A_0^{\kappa^+}(p), A_1^{\kappa^+}(p), C^{\kappa^+}(q) \rangle, A_1^{\kappa^+}(p) \rangle = \text{swt}(p, A_1^-, B_1, A_2^-, B_2, \ldots, A_n^-, B_n)$ will be as desired. \[ \square \]

**Definition 1.15** Let $r, q \in P'$. Then $r \geq q$ ($r$ is stronger than $q$) iff there is $p = \text{swt}(r, B_1, \ldots, B_n)$ for some $B_1, \ldots, B_n$ appearing in $r$ so that the following hold, where

\[ p = \langle \langle A_0^{\kappa^+}, A_1^{\kappa^+}, C^{\kappa^+} \rangle, A_1^{\kappa^+} \rangle \]
\[ q = \langle \langle B_0^{\kappa^+}, B_1^{\kappa^+}, D^{\kappa^+} \rangle, B_1^{\kappa^+} \rangle \]

(1) $A_1^{\kappa^+} \cap (\text{max}(B_1^{\kappa^+}) + 1) = B_1^{\kappa^+}$

(2) $B_0^{\kappa^+} \in C^{\kappa^+}(A_0^{\kappa^+})$ and $D^{\kappa^+}(B_0^{\kappa^+})$ is an initial segment of $C^{\kappa^+}(A_0^{\kappa^+})$

(3) $q = p \upharpoonright (B_0^{\kappa^+}, B_0^{\kappa^+})$ (as it was defined in 1.8).
Remarks (1) Note that if \( t = swt(p, B_0, \ldots, B_n) \) is defined, then \( t \geq p \) and \( p = swt(swt(p, B_0, \ldots, B_n), B_n, B_{n-1}, \ldots, B_0) = swt(t, B_n, \ldots, B_0) \geq t \). Hence the switching produces equivalent conditions.

(2) We need to allow \( swt(p, B_0) \) for the \( \Delta \)-system argument. Since in this argument two conditions are combined into one and so \( C^{\kappa^+} \) should pick one of them only. Also it is needed for proving a strategic closure of the forcing.

(3) The use of finite sequences \( B_0, \ldots, B_n \) is needed in order to insure transitivity of the order \( \leq \) on \( P' \).

Let \( p = \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle \in P' \). Set \( p\backslash \kappa^+ = A^{1\kappa^+} \). Define \( P'_{\geq \kappa^+} \) to be the set of all \( p\backslash \kappa^+ \) for \( p \in P' \).

The next lemma is obvious.

**Lemma 1.16** \( \langle P'_{\geq \kappa^+}, \leq \rangle \) is \( \kappa^{+3} \)-closed.

Set \( p \upharpoonright \kappa^+ = \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle \) where \( p = \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle \in P' \).

Let \( G(P'_{\geq \kappa^+}) \) be a generic subset of \( P'_{\geq \kappa^+} \). Define \( P'_{\leq \kappa^+} \) to be the set of all \( p \upharpoonright \kappa^+ \) for \( p \in P' \) with \( p\backslash \kappa^+ \in G(P'_{\geq \kappa^+}). \)

Let \( p \in P' \) and \( q \in P'_{\geq \kappa^+} \). Then \( q\upharpoonright p \) denotes the set obtained from \( p \) by adding \( q \) to the last component of \( p \), i.e. to \( A^{1\kappa^+} \).

The following lemma is trivial.

**Lemma 1.17** Let \( p \in P' \), \( q \in P'_{\geq \kappa^+} \) and \( q \geq P'_{\geq \kappa^+}, p\backslash \kappa^+ \). Then \( q\upharpoonright p \in P' \) and \( q\upharpoonright p \geq p \).

It follows now that \( P' \) projects to \( P'_{\leq \kappa^+} \).

Let us turn to the chain condition.

**Lemma 1.18** The forcing \( P'_{\leq \kappa^+} \) satisfies \( \kappa^{+3} \)-c.c. in \( V^{P'_{\geq \kappa^+}} \).

**Proof.** Suppose otherwise. Let us assume that

\[
\emptyset \models_{P'_{\geq \kappa^+}} (\langle p_\alpha = \langle A^{0\alpha^+}_\alpha, A^{1\alpha^+}_\alpha, C^{\alpha^+}_\alpha \rangle | \alpha < \kappa^{+3} \rangle \) is an antichain in \( P'_{\leq \kappa^+} \))
\]
Without loss of generality we can assume that each $A_{\alpha}^{0_{\kappa^+}}$ is forced to be a successor model, otherwise just extend conditions by adding one additional model on the top. Define by induction, using Lemma 1.16, an increasing sequence $\langle q_\alpha \mid \alpha < \kappa^+ \rangle$ of elements of $P'_{\geq \kappa^+}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$, $p_\alpha = \langle A_{\alpha}^{0_{\kappa^+}}, A_{\alpha}^{1_{\kappa^+}}, C_{\alpha}^{\kappa^+} \rangle$ so that for every $\alpha < \kappa^+$

$$q_\alpha \models_{P_{\geq \kappa^+}} \langle A_{\alpha}^{0_{\kappa^+}}, A_{\alpha}^{1_{\kappa^+}}, C_{\alpha}^{\kappa^+} \rangle = \bar{p}_\alpha.$$  

For a limit $\alpha < \kappa^+$ let

$$q_\alpha = \bigcup_{\beta < \alpha} q_\beta \cup \{ \sup_{\beta < \alpha} q_\beta \}$$

and $q_\alpha$ be its extension deciding $p_\alpha$. Also assume that $\max q_\alpha \geq \sup (A_{\alpha}^{0_{\kappa^+}} \cap \kappa^+)$. 

We form a $\Delta$-system. By shrinking if necessary assume that for some stationary $S \subseteq \kappa^+$ and $\delta < \kappa^+$ we have the following for every $\alpha < \beta$ in $S$:

(a) $A_{\alpha}^{0_{\kappa^+}} \cap \alpha = A_{\beta}^{0_{\kappa^+}} \cap \beta \subseteq \delta$

(b) $A_{\alpha}^{0_{\kappa^+}} \setminus \alpha \neq \emptyset$

(c) $\sup A_{\alpha}^{0_{\kappa^+}} < \beta$

(d) $\sup q_\alpha = \alpha + 1$

(e) $\langle A_{\alpha}^{0_{\kappa^+}}, \leq, \subseteq, \kappa, C_{\alpha}^{\kappa^+}, f_{A_{\alpha}^{0_{\kappa^+}}}, A_{\alpha}^{1_{\kappa^+}}, q_\alpha \cap A_{\alpha}^{0_{\kappa^+}} \rangle$

$$\langle A_{\beta}^{0_{\kappa^+}}, \leq, \subseteq, \kappa, C_{\beta}^{\kappa^+}, f_{A_{\beta}^{0_{\kappa^+}}}, A_{\beta}^{1_{\kappa^+}}, q_\beta \cap A_{\beta}^{0_{\kappa^+}} \rangle$$

are isomorphic over $\delta$, i.e. by isomorphism fixing every ordinal below $\delta$, where

$$f_{A_{\alpha}^{0_{\kappa^+}}} : \kappa^+ \longleftrightarrow A_{\alpha}^{0_{\kappa^+}}$$

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and

\[ f_{A_0^{\kappa^+}} : \kappa^+ \leftrightarrow A_0^{\kappa^+} \]

are the fixed enumerations.

We claim that for \( \alpha < \beta \) in \( S \) it is possible to extend \( q_\beta \) to a condition forcing compatibility of \( p_\alpha \) and \( p_\beta \). Proceed as follows. Pick \( A \) to be an elementary submodel of cardinality \( \kappa^+ \) so that

(i) \( A_\alpha^{1\kappa^+}, A_\beta^{1\kappa^+} \in A \)

(ii) \( C_\alpha^{\kappa^+}, C_\beta^{\kappa^+} \in A \)

(iii) \( q_\beta \in A \).

Extend \( q_\beta \) to \( q = q_\beta \cup \sup(A \cap \kappa^3) \). Set \( p = \langle A, A_1^{1\kappa^+}, C_\kappa^{\kappa^+} \rangle \), where \( A_1^{1\kappa^+} := A_\alpha^{1\kappa^+} \cup A_\beta^{1\kappa^+} \cup \{ A \} \), \( C_\kappa^{\kappa^+} := C_\alpha^{\kappa^+} \cup C_\beta^{\kappa^+} \cup \langle A, C_\beta^{(A_0^\kappa)^-} \rangle \).

Clearly, \( \langle C_\kappa^{\kappa^+}(A), q \rangle \in \mathcal{P}' \).

The triple \( A_\beta^{0\kappa^+}, A_\alpha^{0\kappa^+}, A \) is of a \( \Delta \)-system type relatively to \( q \), by (e) above. It follows that \( \langle p, q \rangle \in \mathcal{P}' \). Thus the condition (6) of Definition 1.3 holds since each of \( \langle p_\alpha, q \rangle, \langle p_\beta, q \rangle \) satisfies it. The condition (7) of Definition 1.3 follows from (e) above and since both \( \langle p_\alpha, q \rangle, \langle p_\beta, q \rangle \) satisfy it.

\[ \square \]

**Lemma 1.19** \( \mathcal{P}' \) is \( \kappa^{++} \)-strategically closed.

**Proof.** We define a winning strategy for the player playing at even stages. Thus suppose \( \langle p_j \mid j < i \rangle, p_j = \langle \langle A_j^{0\kappa^+}, A_j^{1\kappa^+}, C_j^{\kappa^+} \rangle, A_j^{1\kappa^+} \rangle \) is a play according to this strategy up to an even stage \( i < \kappa^{++} \). Set first

\[
B_i^{0\kappa^+} = \bigcup_{j < i} A_j^{0\kappa^+}, B_i^{1\kappa^+} = \bigcup_{j < i} A_j^{1\kappa^+} \cup \{ B_i^{0\kappa^+} \},
\]

\[
D_i^{\kappa^+} = \bigcup_{j < i} C_j^{\kappa^+} \cup \{ B_i^{0\kappa^+}, \{ B_i^{0\kappa^+} \} \cup \{ C_j^{\kappa^+}(A_0^{0\kappa^+}) \mid j \text{ is even} \} \}.
\]
\[ B_i^{1\kappa^+} = \bigcup_{j<i} B_j^{1\kappa^+} \cup \{ \sup \bigcup_{j<i} B_j^{1\kappa^+} \}. \]

Then pick \( A_0^{0\kappa^+} \) to be a model of cardinality \( \kappa^+ \) such that

(a) \( A_0^{0\kappa^+} \subseteq A_i^{0\kappa^+} \)

(b) \( B_i^{0\kappa^+}, B_i^{1\kappa^+}, D_i^{\kappa^+}, B_i^{1\kappa^+} \in A_i^{0\kappa^+} \).

Set \( A_i^{1\kappa^+} = B_i^{1\kappa^+} \cup \{ A_0^{0\kappa^+} \}, C_i^{\kappa^+} = D_i^{\kappa^+} \cup \{ \langle A_0^{0\kappa^+}, D_i^{\kappa^+} (B_i^{0\kappa^+}) \cup \{ A_0^{0\kappa^+} \} \rangle \} \) and \( A_i^{1\kappa^+} = B_i^{1\kappa^+} \cup \{ \sup (A_0^{0\kappa^+} \cap \kappa^+3) \} \). As an inductive assumption we assume that at each even stage \( j < i \), \( p_j \) was defined in the same fashion. Then \( p_i = \langle \langle A_i^{0\kappa^+}, A_i^{1\kappa^+}, C_i^{\kappa^+} \rangle, A_i^{1\kappa^+} \rangle \) will be a condition in \( P' \) stronger than each \( p_j \) for \( j < i \). The switching may be required here once moving from an odd stage to its immediate successor even stage.

\[ \square \]

2 Suitable structures and assignment functions

In the gap 2 case assignment functions \( a_n \) (those connecting the level \( \kappa \) with level \( \kappa_n, n < \omega \) were order preserving. In other words \( a_n \) is an isomorphism between structures in the language containing only the predicate for the order relation. Here, in the gap 3 case (and beyond ), \( a_n \)'s will be isomorphisms between structures in more complicated languages.

Let us start with two definitions which will specify relevant structures.

**Definition 2.1** A three sorted structure \( \langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle \) is called suitable structure iff

1. \( X \) has a maximal under inclusion element. Denote it by \( \max(X) \).
2. \( Y \subseteq \max(X) \),
3. $C$ is a binary relation $X$,

4. $\langle \langle \max(X), X, C \rangle, Y \rangle \in \mathcal{P}'$, where for every $A \in X$ we identify $C(A)$ with the set $\{B \in X \mid \langle A, B \rangle \in C\}$.

5. $Z = \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in X \cup Y\}$.

Note that by Lemma 1.11, an intersection $t_1 \cap ... \cap t_n$ above is really simple, thus it is equal to an element of $X$ or of $Y$ or to $s \cap \alpha$, where $s \in X$ and $\alpha \in Y$.

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.

**Definition 2.2** A suitable structure $\langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle$ is called suitable generic structure iff there is $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle \in G(\mathcal{P}')$ such that

1. $\langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle$ is a substructure (not necessarily elementary) of
   
   $\langle \langle A^{1\kappa^+}, A^{1\kappa^+}, \{t_1 \cap ... \cap t_n \mid n < \omega, t_1, ..., t_n \in A^{1\kappa^+} \cup A^{1\kappa^+}\} \rangle, C^{\kappa^+}, \in, \subseteq \rangle$,

2. $\max(X) \in C^{\kappa^+}(A^{0\kappa^+})$,

3. $\langle \langle \max(X), X, C \rangle, Y \rangle$ and $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle$ agree about the walks to members of $X$ and to ordinals in $\max(X) \cap Y$. In other words we require that all the elements of walks in $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle$ to elements of $X$ and to ordinals in $\max(X) \cap Y$ are in $X$.

Note that, as a condition in $\mathcal{P}'$, $\langle \langle \max(X), X, C \rangle, Y \rangle$ need not be weaker than $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^+} \rangle$, and hence it need not be in $G(\mathcal{P}')$. Thus, for example, $A^{1\kappa^+}$ need not be an end extension of $Y$.

Note also, that any stronger condition $\langle \langle B^{0\kappa^+}, B^{1\kappa^+}, D^{\kappa^+} \rangle, B^{1\kappa^+} \rangle \in G(\mathcal{P}')$ with $C^{\kappa^+}(A^{0\kappa^+})$ being an initial segment of $D^{\kappa^+}(B^{0\kappa^+})$ will witness that $\langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle$ is a suitable generic structure.

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Lemma 2.3 Let \( \langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle \) be a suitable generic structure as witnessed by \( \langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{0\kappa^+}, A^{1\kappa^+} \rangle \in G(\mathcal{P}) \). Suppose that \( F_0, F_1, F \in A^{1\kappa^+}, F_0, F \in C^{\kappa^+}(A^{0\kappa^+}) \) is a triple of a \( \Delta \)-system type with \( \alpha_0, \alpha_1 \) as in Definition 1.2, and \( \alpha_1 \in Y \). Then \( F_0, F_1, F \in X \cap \max(X), F \in X, \alpha_0 \in \max(X) \cap Y \).

Proof. The walk to \( \alpha_1 \) from \( \max(X) \) (or the same from \( A^{0\kappa^+} \)) passes through \( F \) and turns to \( F_1 \). Hence, by 2.2(3), \( F_0, F_1, F \in X \). Recall that by 2.1(3) we have \( \langle \langle \max(X), X, C \rangle, Y \rangle \in \mathcal{P}'. \) Hence \( F_0, F_1, F \) are of a \( \Delta \)-system type in \( \langle \langle \max(X), X, C \rangle, Y \rangle \). Then there are \( \alpha'_0 \in F_0 \cap Y, \alpha'_1 \in F_1 \cap Y \) such that

\[
F_0 \cap F_1 = F_0 \cap \alpha'_0 = F_1 \cap \alpha'_1.
\]

But, also

\[
F_0 \cap F_1 = F_1 \cap \alpha_1
\]

and \( \alpha_1, \alpha'_1 \in Y \subseteq A^{1\kappa^+} \).

Hence, \( \alpha_1 = \alpha'_1 \). Finally, \( \alpha'_0 = \pi_{F_1,F_0}(\alpha_1) = \alpha_0 \). Hence, \( \alpha_0 \in \max(X) \cap Y \).

□

Lemma 2.4 Let \( p = \langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle \) and \( p' = \langle \langle X', Y', Z' \rangle, C', \in, \subseteq \rangle \) be isomorphic suitable structures (even over different cardinals) and \( a \) an isomorphism between them. Suppose that \( F_0, F_1, F \) is a triple in \( X \) of a \( \Delta \)-system type and \( \alpha_0 \in F_0 \cap Y, \alpha_1 \in F_1 \cap Y \) are witnessing this ordinals. Then \( a(F_0), a(F_1), a(F) \) is a triple in \( X' \) of a \( \Delta \)-system type witnessed by \( a(\alpha_0) \) and \( a(\alpha_1) \).

Proof. Obviously, \( \alpha_0 \) and \( \alpha_1 \) are uniquely determined by \( F_0 \) and \( F_1 \).
Denote \( a(F_0) \) by \( F'_0, a(F_1) \) by \( F'_1, a(F) \) by \( F' \), \( a(\alpha_0) \) by \( \alpha'_0 \) and \( a(\alpha_1) \) by \( \alpha'_1 \).
Now, \( F'_0, F'_1 \in F' \), moreover \( F'_0 \) is the immediate predecessor of \( F' \) in \( C(F') \) and \( F'_1 \) is an additional predecessor of \( F' \) under the inclusion relation, since \( a \) is an isomorphism between \( p \) and \( p' \). Note that by 2.1(3) this implies that
Lemma 2.5 Let $p = \langle\langle X, Y, Z \rangle, C, \in, \subseteq \rangle$ and $p' = \langle\langle X', Y', Z' \rangle, C', \in, \subseteq \rangle$ be isomorphic suitable structures (even over different cardinals) and $a$ an isomorphism between them. Then $a$ respects walks, i.e. for every $A \in X$ and $B \in (X \cup Y) \cap A$, $a$ maps the walk between $A$ and $B$ in $p$ onto the walk between $a(A)$ and $a(B)$.

Proof. Induction on walks length. Thus, if $B$ in $C(A)$ or if $B \in Y$ and the walk to it from $A$ involves only $C(A)$, then the isomorphism $a$ guarantees the same for the images. Suppose that the walk proceeds with splitting. Let $F_0, F_1, F$ be the first split on the way to $B$, i.e. $F \in C(A)$, the triple $F_0, F_1, F$ is of a $\Delta$-system type, $B \notin F_0$ (or, if $B \in Y$, $B \notin F_0$) and $B \subseteq F_1$ (or $B \in F_1 \cup \{F_1\}$). By the previous lemma (Lemma 2.4), $a(F_0), a(F_1), a(F)$ is a triple in $X'$ of a $\Delta$-system type. $a$ is isomorphism, hence $a(F) \in C(a(A)), a(F_0) \in C(a(F_0)), a(B) \notin a(F_0)$ (or, if $B \in Y$, $a(B) \notin a(F_0)$) and $a(B) \subseteq a(F_1)$ (or $a(B) \in a(F_1) \cup \{a(F_1)\}$).

But this means that the walk from $a(A)$ to $a(B)$ goes via $a(F_1)$. Now we can apply induction to the walk from $F_1$ to $B$, since it is shorter than the original one from $A$ to $B$.

□

Lemma 2.6 Let $p = \langle\langle X, Y, Z \rangle, C, \in, \subseteq \rangle$ and $p' = \langle\langle X', Y', Z' \rangle, C', \in, \subseteq \rangle$ be isomorphic suitable structures (even over different cardinals), $a$ an isomorphism between them and $F_0, F_1, F \in X$ a triple of a $\Delta$-system type. Then $F_0, F_1, F'$ is a $\Delta$-system type triple in $p'$.

Let $\alpha_0' \in F_0' \cap a(Y)$ and $\alpha_1' \in F_1' \cap a(Y)$ be such that

$$F_0' \cap F_1' = F_0' \cap \alpha_0'' = F_1' \cap \alpha_1''.$$
a respects \( \pi_{F_0,F_1} \), i.e. for every \( A \in F_0 \cap (X \cup Y) \) we have \( a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A)) \).

Proof. Let \( F_0, F_1, F \in X \) be a triple of a \( \Delta \)-system type and \( A \in F_0 \cap (X \cup Y) \). We prove the lemma by induction on the length of the walk from \( F_0 \) to \( A \).

Suppose first that \( A \in C(F_0) \) (or in case \( A \in Y \) the walk to \( A \) involves only \( C(F_0) \)). The isomorphism \( a \) moves \( C(F_0) \) to \( C(a(F_0)) \) and \( C(F_1) \) to \( C(a(F_1)) \). By Lemma 2.4, the triple \( a(F_0), a(F_1), a(F) \) is of a \( \Delta \)-system type. So, \( \pi_{a(F_0),a(F_1)} \) moves \( C(a(F_0)) \) onto \( C(a(F_1)) \) respecting the inclusion relation. Then \( \pi_{a(F_0),a(F_1)}(a(A)) \) should an element of \( C(a(F_1)) \) at the same place as \( a(A) \) in \( C(a(F_0)) \), which, in turn is at the same place as \( A \) in \( C(F_0) \) and \( \pi_{F_0,F_1}(A) \) in \( C(F_1) \). Hence

\[
a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A)).
\]

Suppose no that \( A \not\in C(F_0) \). Let \( H_0, H_1, H \) be the first splitting on the way to \( A \) from \( F_0 \). The induction applies to \( H_1, A \). Hence

\[
a(\pi_{H_1,H_0}(A)) = \pi_{a(H_1),a(H_0)}(a(A)).
\]

Let \( A' = \pi_{H_1,H_0}(A) \). Apply the induction to \( F_0, A' \). Then

\[
a(\pi_{F_0,F_1}(A')) = \pi_{a(F_0),a(F_1)}(a(A')).
\]

Again, apply induction to \( F_0, H_0 \) and \( F_0, H_1 \). So,

\[
a(\pi_{F_0,F_1}(H_0)) = \pi_{a(F_0),a(F_1)}(a(H_0))
\]

and

\[
a(\pi_{F_0,F_1}(H_1)) = \pi_{a(F_0),a(F_1)}(a(H_1)).
\]

Finally,

\[\pi_{F_0,F_1}(A) = \pi_{\pi_{F_0,F_1}(H_0),\pi_{F_0,F_1}(H_1)}(\pi_{F_0,F_1}(\pi_{H_1,H_0}(A')))\].

Applying \( a \), we obtain

\[
a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A)).
\]
Note that the proofs of Lemmas 2.5, 2.6 rely only on Lemma 2.4 and 
(also this lemma does not use $Z$) do not use the component of suitable 
structures consisting of intersections. Let us isolate a weaker notion that 
still will capture all the essential parts.

**Definition 2.7** A two sorted structure $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ is called **weak suitable structure** iff

1. $X$ has a maximal under inclusion element. Denote it $\text{max}(X)$,
2. $Y \subseteq \text{max}(X)$,
3. $C$ is a binary relation $X$,
4. $\langle \langle \text{max}(X), X, C \rangle, Y \rangle \in P'$, where for every $A \in X$ we identify $C(A)$ with the set \{ $B \in X$ | $\langle A, B \rangle \in C$ \}.

The following analogs of Lemmas 2.5, 2.6 were actually proved above:

**Lemma 2.8** Let $p = \langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ and $p' = \langle \langle X', Y' \rangle, C', \in, \subseteq \rangle$ be isomorphic weak suitable structures (even over different cardinals) and $a$ an isomorphism between them. Then $a$ respects walks, i.e. for every $A \in X$ and $B \in (X \cup Y) \cap A$, $a$ maps the walk between $A$ and $B$ in $p$ onto the walk between $a(A)$ and $a(B)$.

**Lemma 2.9** Let $p = \langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ and $p' = \langle \langle X', Y' \rangle, C', \in, \subseteq \rangle$ be isomorphic weak suitable structures (even over different cardinals), $a$ an isomorphism between them and $F_0, F_1, F \in X$ a triple of a $\Delta$-system type. Then $a$ respects $\pi_{F_0,F_1}$, i.e. for every $A \in F_0 \cap (X \cup Y)$ we have $a(\pi_{F_0,F_1}(A)) = \pi_{a(F_0),a(F_1)}(a(A))$. 
Let \( p = \langle \langle X, Y \rangle, C, \in, \subseteq \rangle \) be a weak suitable structure. Consider \( Z = \{ t_1 \cap \ldots \cap t_n \mid n < \omega, t_1, \ldots, t_n \in X \cup Y \} \). Then \( \langle X, Y, Z \rangle, C, \in, \subseteq \rangle \) is a suitable structure. Let us call it \textit{expansion} of \( p \) to a suitable structure.

**Lemma 2.10** Suppose that \( p = \langle \langle X, Y \rangle, C, \in, \subseteq \rangle \) and \( p' = \langle \langle X', Y' \rangle, C', \in, \subseteq \rangle \) are isomorphic weak suitable structures (even over different cardinals). Then their expansions are isomorphic as well.

**Proof.** Let \( a \) be the isomorphism between \( p \) and \( p' \). We show that it extends to an isomorphism between the expansions. Let \( Z = \{ t_1 \cap \ldots \cap t_n \mid n < \omega, t_1, \ldots, t_n \in X \cup Y \} \) and \( Z' = \{ t_1 \cap \ldots \cap t_n \mid n < \omega, t_1, \ldots, t_n \in X' \cup Y' \} \). Extend \( a \) to a function \( b \) in the obvious fashion:

\[
b \upharpoonright \text{dom}(a) = a \quad \text{and} \quad b(t_1 \cap \ldots \cap t_n) = a(t_1) \cap \ldots \cap a(t_n), \quad \text{for any} \quad t_1, \ldots, t_n \in X \cup Y.
\]

We need to check that such defined \( b \) is a function and an isomorphism.

Note first that for every \( A, B \in X, A' \in (A \cup \{A\}) \cap X \) and \( \alpha \in Y \cap A' \) such that \( A \cap B = A' \cap \alpha \) we have \( a(A) \cap a(B) = a(A') \cap a(\alpha) \). Use induction on the walks complexity from \( \max(X) \) to \( A, B \) as in Lemma 1.11. The inductive step follows since \( a \) preserves \( \Delta \)-system triples. Also, by Lemmas 2.8,2.9, \( a \) respects walks and images under \( \Delta \)-system triples isomorphisms.

Similar, if instead of two sets we have finitely many \( A_1, \ldots, A_n \in X, A' \in (A_1 \cup \{A_1\}) \cap X \) and \( \alpha \in Y \cap A' \) such that \( A_1 \cap \ldots \cap A_n = A' \cap \alpha \), then \( a(A_1) \cap \ldots \cap a(A_n) = a(A') \cap a(\alpha) \). Also, the same holds if some (or actually one) of \( A_i \)’s is in \( Y \), i.e. is an ordinal.

Now, by Lemma 1.12, for every \( A_1, \ldots, A_n \in X \) there are \( A' \in (A_1 \cup \{A_1\}) \cap X \) and \( \eta \in Y \cap A' \) such that \( A_1 \cap \ldots \cap A_n = A' \cap \eta \), or just \( A_1 \cap \ldots \cap A_n = A' \).

An alternative proof that works for higher gaps as well proceeds as follows. Suppose that

\[
A_1 \cap \ldots \cap A_n = B_1 \cap \ldots \cap B_n,
\]

for some \( A_1, \ldots, A_n, B_1, \ldots, B_n \in X \cup Y \). We need to show that then

\[
a(A_1) \cap \ldots \cap a(A_n) = a(B_1) \cap \ldots \cap a(B_n).
\]
The proof is by induction on complexity of the walks to components of the intersections. Thus, suppose that $A_1$ has a maximal walk complexity among the components of the intersection. Consider the walks from $\max(X)$ to $A_1$ and to $A_2$. Go to the last point until which the walks coincide. Then, as in the proof of Lemma 1.11, we replace $A_1$ by $A_1' \in X$ and $\alpha_1 \in Y$ which are simpler than $A_1$ in the walk sense and such that

$$A_1 \cap A_2 = A_1' \cap \alpha_1 \cap A_2.$$  

Now the induction applies.

□

Fix $n < \omega$. We define an analog $P'_n$ of $P'$ on the level $n$ just replacing $\kappa$ by $\kappa_n^{+n}$. An assignment function $a_n$ will be an isomorphism between a suitable generic structure of cardinality less than $\kappa_n$ over $\kappa$ and a suitable structure over $\kappa_n^{+n}$.

Define $Q_{n0}$.

**Definition 2.11** Let $Q_{n0}$ be the set of the triples $\langle a, A, f \rangle$ so that:

1. $f$ is partial function from $\kappa^{+3}$ to $\kappa_n$ of cardinality at most $\kappa$

2. $a$ is an isomorphism between a suitable generic structure $\langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle$ of cardinality less than $\kappa_n$ and a suitable structure $\langle \langle X', Y', Z' \rangle, C', \in, \subseteq \rangle$ in $P'_n$ so that

   (a) $\max(X')$ is above every $t \in X' \cup Y'$ in the order $\leq_{E_n}$ of the extender $E_n$, (or actually, the ordinal which codes $\max(X')$ in the fixed in advance nice coding of $[\kappa^{+n+3}]^{<\kappa_n}$.

   We need that each element of $[\kappa^{+n+3}]^{<\kappa_n}$ is coded by a stationary many ordinals below $\kappa_n^{+n+3}$.

(b) if $t \in X' \cup Y'$ then for some $k, 2 < k < \omega$, $\exists t < H(\chi^k)$, with $\chi$ big enough fixed in advance. (Alternatively, may be to work with subsets of $\kappa_n^{+n+3}$ only and further require it
is a restriction of such model to $\kappa^{+n+3}$.) We deal with elementary submodels of $H(\chi^k)$, instead of those of $H(\kappa^{+n+3})$.

Further passing from $Q_{n0}$ to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many $n$’s the $n$-th image of a model $t \in X \cup Y$ will be an elementary submodel of $H(\chi^k)$.

The way to compare such models $t_1 \prec H(\chi^{k_1})$, $t_2 \prec H(\chi^{k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:

move to $H(\chi^{k_1})$, i.e. compare $t_1$ with $t_2 \cap H(\chi^{k_1})$.

3. $A \in E_{n, \text{max}(X')}$,

4. for every ordinals $\alpha, \beta, \gamma$ which are elements of $Y'$ or the ordinals coding models in $X'$ we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \quad \text{implies}$$

$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho))$$

for every $\rho \in \pi^{\text{max}(X'), \alpha}_n(A)$.

Define a partial order on $Q_{n0}$ as follows.

**Definition 2.12** Let $\langle a, A, f \rangle$ and $\langle b, B, g \rangle$ be in $Q_{n0}$. Set $\langle a, A, f \rangle \geq_{n0} \langle b, B, g \rangle$ iff

1. $a \supseteq b$,
2. $f \supseteq g$,
3. $\pi_{\text{max}(\text{rng}(a))}, \text{max}(\text{rng}(b)) A \subseteq B$,
4. $\text{dom}(f) \cap Y^b = \text{dom}(g) \cap Y^b$, where $Y^b$ is the second component (i.e. the set of ordinals) of the suitable structure on which $b$ is defined.

Note that here we do not require disjointness of the domain of $g$ and of $Y^b$, but as it will follow from the further definition of non-direct extension, the value given by $g$ will be those that eventually counts.
Definition 2.13 $Q_{n1}$ consists of all partial functions $f: \kappa^{+3} \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

Definition 2.14 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq^*_n = \leq_{n0} \cup \leq_{n1}$.

Let $p = \langle a, A, f \rangle \in Q_{n0}$ and $\nu \in A$. Set $p \triangleright \nu = f \cup \{ \langle \alpha, \pi_{\max(\text{rng}(a)), a(\alpha)}(\nu) \mid \alpha \in \text{dom}(a) \setminus \text{dom}(f) \}$. Note that here $a$ contributes only the values for $\alpha$'s in $\text{dom}(a) \setminus \text{dom}(f)$ and the values on common $\alpha$'s come from $f$. Also only the ordinals in $\text{dom}(a)$ are used to produce non direct extensions, models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_{n} q$ iff either $p \geq^*_n q$ or $p \in Q_{n1}, q = \langle b, B, g \rangle \in Q_{n0}$ and for some $\nu \in B, p \geq_{n1} q \triangleright \nu$.

Definition 2.15 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) for every $n < \omega$, $p_n \in Q_n$,

(2) there is $\ell(p) < \omega$ such that

(i) for every $n < \ell(p)$, $p_n \in Q_{n1}$,

(ii) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$,

(iii) there is $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \in G(\mathcal{P}')$ which witnesses that $\text{dom}(a_n(p))$ is a suitable generic structure (i.e. $\text{dom}(a_n(p))$ and $\langle \langle A^{0\kappa^+}, A^{1\kappa^+}, C^{\kappa^+} \rangle, A^{1\kappa^{++}} \rangle \text{ satisfy 2.2} \rangle$, simultaneously for every $n, \ell(p) \leq n < \omega$.

(3) for every $n \geq m \geq \ell(p)$, $\text{dom}(a_m) \subseteq \text{dom}(a_n)$,

(4) for every $n$, $\ell(p) \leq n < \omega$, and $X \in \text{dom}(a_n)$ we have that for each $k < \omega$ the set $\{ m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k})) \}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \kappa_m^{+m+3}$ but there is $\tilde{X} \prec H(\chi^{+k})$ such that $a_m(X) = \tilde{X} \cap \kappa_m^{+m+3}$. It is possible to define being $k$-good this way as well).
(5) For every $n \geq \ell(p)$ and $\alpha \in \text{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \text{dom}(a_m) \setminus \text{dom}(f_m)$.

Next lemma deals with extensions of elements of $\mathcal{P}$. The analogs for the gap 2 are trivial.

**Lemma 2.16** Let $p \in \mathcal{P}$ and $\langle \langle B^{0^k+}, B^{1^k+}, D^{k^+} \rangle, B^{1k++} \rangle \in G(\mathcal{P}')$. Then

1. for every $\alpha \in B^{1k++}$ there is $q \geq^* p$ such that $\alpha \in \text{dom}(a_n(q))$ for all but finitely many $n$’s;

2. for every $A \in B^{1^k+}$ there is $q \geq^* p$ such that $A \in \text{dom}(a_n(q))$ for all but finitely many $n$’s. Moreover, if $\langle \langle A^{0^k+}, A^{1^k+}, C^{k^+} \rangle, A^{1k++} \rangle \geq \langle \langle B^{0^k+}, B^{1^k+}, D^{k^+} \rangle, B^{1k++} \rangle$ witnesses a generic suitability of $p$ and $A \in C^{k^+}(A^{0^k+})$, then the addition of $A$ does not require adding of ordinals and the only models that probably will be added together with $A$ are its images under $\Delta$-system type isomorphisms for triples in $p$.

**Proof.** Pick some $\langle \langle A^{0^k+}, A^{1^k+}, C^{k^+} \rangle, A^{1k++} \rangle \in G(\mathcal{P}')$ stronger than $\langle \langle B^{0^k+}, B^{1^k+}, D^{k^+} \rangle, B^{1k++} \rangle$ such that

1. $\alpha \in A^{1k++}$,

2. $A \in A^{1^k+}$,

3. $\langle \langle A^{0^k+}, A^{1^k+}, C^{k^+} \rangle, A^{1k++} \rangle$ witnesses that $\text{dom}(a_n(p))$ is a suitable generic structure (i.e. $\text{dom}(a_n(p))$) and $\langle \langle A^{0^k+}, A^{1^k+}, C^{k^+} \rangle, A^{1k++} \rangle$ satisfy 2.2), for every $n, l(p) \leq n < \omega$.

Note first that it is easy to add to $p$ any $A \in C^{k^+}(A^{0^k+})$ such that the maximal models of $p_n$’s belong to $A$. Just at each level $n \geq l(p)$ pick an elementary submodel of $H(\chi^{+\omega})$ of cardinality $\kappa_{n+1}^{+n+1}$ which includes $\text{rng}(a_n)$ as an element. Map $A$ to such a model.

Hence it is enough to deal with $\alpha, A$ which are the members of the maximal model of $p$, just, if not, then we can add first $A^{0^k+}$.
We prove the lemma simultaneously for $\alpha$ and $A$ by induction on the walk distance or complexity.

Fix $n \geq l(p)$. Let $\text{dom}(a_n(p)) = \langle \langle X, Y, Z \rangle, C, \in, \subseteq \rangle$.

Suppose that the walk to $\alpha$ involves only the central line. The general case is treated similar.

Let $A_1 \in C^{\kappa+}(\max(X))$ be the least model of $C^{\kappa+}(\max(X))$ with $\alpha \in A_1$. We assume that $A_1 \in X$. Just otherwise use the induction to add it. This is possible, since the walk to $A_1$ is simpler than those to $\alpha$.

**Case 1.** $A_1$ is the least model of $C^{\kappa+}(\max(X))$.

The walk to $\alpha$ from $\max(X)$ (or from $A_0^{\kappa+}$) consists of $A_1$ alone. So, in order to add $\alpha$ we do not have to add models or other ordinals first.

Consider $\beta_1 = \min(A_1 \cap Y) \setminus \alpha$ and $\gamma_1 = \max(A_1 \cap Y \cap \alpha)$ whenever defined. Suppose that both $\beta_1$ and $\gamma_1$ are defined. If one of them or both are undefined then the argument below will be only simpler.

Let us denote $a_n(\beta_1)$ by $\beta_1^*$, $a_n(\gamma_1)$ by $\gamma_1^*$, $a_n(X)$ by $X^*$ and $a_n(A_1)$ by $A_1^*$. Let $C^*$ be the function that corresponds to $C$ in $\text{rng}(a_n)$. Then $A_1^* \in C^*(X^*)$.

Also, $\beta_1^*, \gamma_1^* \in A_1^* \cap a_n"Y$ and $\gamma_1^* < \beta_1^*$.

Assume that $A_1^*$ and $\beta_1^*$ are $k$-good, for some $k >> 2$. Pick now $M \in A_1^*$ such that

1. $M \in \beta_1^*$,
2. $|M| = \kappa_n^{+n+2}$,
3. $M$ is $k - 1$-good,
4. $\gamma_1^* \in M$.

Now, extend $a_n$ by mapping $\alpha$ to $M$ and all the images of it under $\Delta$-system types triples isomorphisms to those of $M$.

**Case 2.** $A_1$ is not the least element of $C^{\kappa+}(\max(X))$.

Then we will need to add also the immediate predecessor $A_1^-$ of $A_1$ in $C^{\kappa+}(\max(X))$. Do this using the induction.
Split the argument into three cases.

**Case 2.1.** $\alpha > \sup(A_1^-)$.

Then we proceed exactly as in Case 1 above only require in addition that $a_n(A_1^-) \in M$.

**Case 2.2.** $\alpha = \sup(A_1^-)$.

Set $B = a_n(A_1)$. Then, its immediate predecessor $B^- = a_n(A_1^-)$. Pick $k < \omega$ such that $B^- \prec H(\chi^{+k+1})$ and $B \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$. Then $H(\chi^+) \in B^-$. Hence

$$B^- \models \forall \nu < \kappa_n^{+n+3} \forall t \in [H(\chi^+)]<\kappa^{+n+3} \exists M \prec H(\chi^+) \quad (M \supseteq \nu \cup t \text{ and } |M| < \kappa_n^{+n+3}).$$

Let $\delta = \sup(B^- \cap \kappa_n^{+n+3})$. Set $M$ to be the Skolem hull of $\delta \cup (B^- \cap H(\chi^+))$ in $H(\chi^+)$. Then $M \cap \kappa_n^{+n+3} = \delta$. Also, $M \in B$.

Now, extend $a_n$ by mapping $\alpha$ to $M$ and all the images of it under $\Delta$-system types triples isomorphisms to those of $M$.

**Case 2.3.** $\alpha < \sup(A_1^-)$.

Consider $\alpha_1 = \min(A_1^- \setminus \alpha)$. We need to add $\alpha_1$ before $\alpha$ and this can be done using the induction, since the walk to $\alpha_1$ is simpler than those to $\alpha$. So assume that $\alpha_1$ is already in $Y$. Note that $\text{cof}(\alpha_1) = \kappa^{++}$, since $A_1^- \supseteq \kappa^+$ and it is an elementary submodel of $H(\kappa^{+3})$.

We split the proof now into two cases.

**Case 2.3.1.** $\alpha = \sup(\alpha_1 \cap A_1^-)$.

This case is similar to Case 2.2 above. Set $B = a_n(A_1)$. Then, its immediate predecessor $B^- = a_n(A_1^-)$. Let $E = a_n(\alpha_1)$.

Pick $k < \omega$ such that $E \prec H(\chi^{+k+1})$, $B^- \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$ and $B \cap H(\chi^{+k+1}) \prec H(\chi^{+k+1})$. Then $H(\chi^+) \in E \cap B^-$. Hence

$$E \cap B^- \models \forall \nu < \kappa_n^{+n+3} \forall t \in [H(\chi^+)]<\kappa^{+n+3} \exists M \prec H(\chi^+) \quad (M \supseteq \nu \cup t \text{ and } |M| < \kappa_n^{+n+3}).$$

Let $\delta = \sup(E \cap B^- \cap \kappa_n^{+n+3})$. Set $M$ to be the Skolem hull of $\delta \cup (E \cap B^- \cap H(\chi^+))$ in $H(\chi^+)$. Then $M \cap \kappa_n^{+n+3} = \delta$. Also, $M \in B$. 

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Now, extend $a_n$ by mapping $\alpha$ to $M$ and all the images of it under $\Delta$-system types triples isomorphisms to those of $M$.

**Case 2.3.2.** $\alpha > \sup(\alpha_1 \cap A_1^\gamma)$.

Consider $\beta_1 = \min((A_1 \cap Y) \setminus \alpha)$ and $\gamma_1 = \max(A_1 \cap Y \cap \alpha)$ whenever defined. Suppose that both $\beta_1$ and $\gamma_1$ are defined. If one of them or both are undefined then the argument below will be only simpler.

Let us denote $a_n(\beta_1)$ by $\beta_1^*$, $a_n(\gamma_1)$ by $\gamma_1^*$, $a_n(X)$ by $X^*$ and $a_n(A_1)$ by $A_1^*$. Let $C^*$ be the function that corresponds to $C$ in $\text{rng}(a_n)$. Then $A_1^* \in C^*(X^*)$ and $a_n(A_1^\gamma)$ is the immediate predecessor of $A_1^*$ in $C^*(A_1^*)$. Also, $\beta^*, \gamma^* \in A_1^* \cap a_n''Y$ and $\gamma^* < \beta^*$.

Assume that $A_1^*$ and $\beta_1^*$ are $k$-good, for some $k >> 2$. Pick now $M \in A_1^*$ such that

1. $M \in \beta_1^*$,
2. $|M| = \kappa_n^{\omega n+2}$,
3. $M$ is $k - 1$-good,
4. $\gamma_1^*, a_n(A_1^\gamma) \cap a_n(\alpha_1) \in M$.

Now, extend $a_n$ by mapping $\alpha$ to $M$ and all the images of it under $\Delta$-system types triples isomorphisms to those of $M$.

Set

$$Y_1 = Y \cup \{ \alpha' \mid \alpha' \text{ is the image of } \alpha \text{ under } \Delta - \text{system types triples (of X) isomorphisms} \}.$$ 

**Claim 2.16.1** $Y_1$ is a closed set.

**Proof.** We just prove that every limit point of $Y_1$ is a limit point of $Y$, and hence, is in $Y$. It is enough to deal limits of $\omega$-sequences, since if every limit of an $\omega$-sequence from $Y_1$ is in $Y$, then any limit will be in $Y$, because $Y$ is closed.

Such images are generated as follows. Pick the smallest triple $F_0^1, F_1^1, F^1 \in X$
of a \(\Delta\)-system type with \(F_0^1, F_1^1 \in C(\max(X))\) and \(F_0^1 \subseteq A\). We add \(\alpha^1 = \pi_{F_0^1, F_1^1}(\alpha)\) to \(Y\). Note that it is possible to have \(\alpha = \alpha^1\). Let \(\xi_0^1 \in F_0^1 \cap Y, \xi_1^1 \in F_1^1 \cap Y\) be as in Definition 1.2(6d). Then \(\alpha^1 > \alpha\) implies \(\xi_0^1 \leq \alpha < \xi_1^1 \leq \alpha^1\).

Then pick the smallest triple \(F_0^2, F_1^2, F_2^2 \in X\) of a \(\Delta\)-system type with \(F_0^2, F_2^2 \in C(\max(X))\) and \(F_0^2 \subseteq F_1^2\). We add \(\alpha_{20} = \pi_{F_0^2, F_1^2}(\alpha)\) and \(\alpha_{21} = \pi_{F_0^2, F_1^2}(\alpha^1)\) to \(Y\). Again it is possible to have \(\alpha_{2i} \in \{\alpha, \alpha^1\}\), where \(i < 2\).

Let \(\xi_0^2 \in F_0^2 \cap Y, \xi_1^2 \in F_1^2 \cap Y\) be as in Definition 1.2(6d). Again, if one of the new \(\alpha_{2i}'s\) is above its pre-image, then the corresponding \(\xi_i^2\) will be above \(\sup(F_0^2)\), and so, above both \(\alpha, \alpha^1\).

Continue further all the way up to \(\max(X)\). This way all the images of \(\alpha\) are generated. Note that we move up over the central line of \(X\).

At each stage \(j\) in the process the same effect observed above will take place— if one of \(\alpha_{ji}'s\) is above its pre-image, then the corresponding \(\xi_j^i\) will be above \(\sup(F_j^i)\), and so, above all the images \(\alpha_{j'i'}\) of \(\alpha\) generated at stages \(j' < j\). But all such \(\xi_j^i\) are in \(Y\). Hence, their limit, which is the same as those of increasing sequence of \(\alpha_{ji}'s\), is in \(Y\) as well.

\(\square\) of the claim.

Turn now to the adding of a model.

Assume first that a model \(A\) is on the central line. Let us observe that no collision with ordinals in \(Y\) can occur. Thus if some \(\alpha \in Y, \alpha \not\in A\) and \(\sup(A) > \alpha\) (if \(\alpha = \sup(A)\), then by the walk closure we must have \(A \in X\), then the same should hold with images, i.e. the image \(A^*\) of \(A\) must have supremum above \(\alpha^* := a_n(\alpha)\) and \(\alpha^* \not\in A^*\). There may be infinitely many such \(\alpha\)’s and then, in general, it will be impossible to find \(A^*\). In present situation, we have the advantage - \(X\) is closed under walks to ordinals of \(Y\). This means, in particular, that there is \(B_\alpha \in C(\max(X))\) such that \(\alpha \in B_\alpha\) and \(B_\alpha\) is the least model of \(C(\max(X))\), or \(B_\alpha\) has the immediate predecessor \(B_\alpha^-\) in \(C(\max(X))\) and \(\alpha \not\in B_\alpha^-\). In our case the first possibility is just impossible. Thus, we assumed that \(A \in C^{a^+}(A^{0c^+})\), \(\alpha \in B_\alpha \setminus A\). So,
$B_\alpha$ is not the least element of $C^{\kappa^+}(A_0^{0\kappa^+})$, which by 2.2(3) implies that $B_\alpha$ is not the least element of $C(\max(X))$ as well.

Hence, $B_\alpha^-$ exists and $A \subseteq B_\alpha^-$. Consider now a set

$$T = \{B_\alpha^- \mid \alpha \in Y, \alpha \notin A, \sup(A) > \alpha\}.$$  

$T$ is a subset of the closed chain $C(\max(X))$. Let $E$ be the least element of $T$ under the inclusion. Then $A \subseteq E$, since $T \subseteq C^{\kappa^+}(A_0^{0\kappa^+})$ and so, both $E$ and $A$ are inside the chain $C^{\kappa^+}(A_0^{0\kappa^+})$, but $E$ is of the form $B_\alpha^-$, for some $\alpha \in B_\alpha \setminus A$, and $B_\alpha^- \in X, A \notin X$.

Now it is easy to add $A$ in a fashion similar to adding an ordinal above.

First we pick the least $D \in C(E)$ which contains $A$. Let $F$ be the last model of $C(E)$ inside $D$. Note that $D$ can be a limit model of $C^{\kappa^+}(A_0^{0\kappa^+})$ and so $D^-$ may not exist. Even if $D^-$ exists, still it cannot be in $X$, since otherwise $A = D^-$ will be in $X$.

Set $\beta = \min((D \cap Y) \setminus \sup(A))$ whenever defined. Suppose that $\beta$ is defined. If it is undefined then the argument below will be only simpler. Note that necessarily $\beta > \sup(A)$. Otherwise, $\sup(A) = \beta$ and it is in $Y$. Then the largest model $W$ of $C^{\kappa^+}(A_0^{0\kappa^+})$ with $\sup(A) \notin W$ must be in $X$ (walks closure to ordinals). But then $W = A$, since $W \neq A$ will imply $W \in A$ or $A \in W$, both possibilities are clearly impossible.

Note that every $\gamma \in D \cap Y \cap \beta$ is in $F$. Otherwise, let some $\gamma \in D \cap Y \cap \beta$ be not in $F$. The walk to $\gamma$ goes via $D$ but does continue further on $C^{\kappa^+}(D)$. Hence, $D$ must be a successor model of $C^{\kappa^+}(A_0^{0\kappa^+})$ and $D^-$ must be in $X$, which is impossible, as was observed above.

Let us denote $a_n(\beta)$ by $\beta^*$, $a_n(D)$ by $D^*$, $a_n(X)$ by $X^*$ and $a_n(F)$ by $F^*$. Let $C^*$ be the function that corresponds to $C$ in $\text{rng}(a_n)$. Then $D^*, F^* \in C^*(X^*)$ and $\beta^* \in D^* \cap a_n" Y$.

Assume that $D^*$ and $\beta^*$ are $k$-good, for some $k >> 2$. Pick now $M \in D^*$ such that

1. $M \in \beta^*$,
2. $|M| = \kappa_n^{+n+1}$,

3. $M$ is $k-1$-good,

4. $F^* \in M$.

Now, extend $a_n$ by mapping $A$ to $M$ and all the images of it under $\Delta$-system types triples isomorphisms to those of $M$.

Note that no new ordinals were added in the process and only models that are images of $A$ under $\Delta$-system types isomorphisms for triples in $X$ were added.

Suppose that $A$ is not on the central line. In this case we are supposed to add to $p$ the whole walk from $A^{0\kappa^+}$ to $A$. We can concentrate, using the induction, only on the case of a $\Delta$-system triple. Namely given $F_0, F_1, F \in A^{1\kappa^+}$ of a $\Delta$-system type with $F_0$ being the immediate predecessor of $F$ in $C^{\kappa^+}(A^{0\kappa^+})$. We need to add $F_1$ (and probably also $F_0, F$ if they are not inside) to $p$. $F_0, F$ are on the central line, hence we may assume that they are in $p$. Let $\alpha_0, \alpha_1 \in F \cap A^{1\kappa^+}$ be so that $\alpha_0 \in F_0, \alpha_1 \in F_1, F_0 \cap F_1 = \alpha_0 \cap F_0 = \alpha_1 \cap F_1$ and either $\alpha_0 > \sup(F_1)$ or $\alpha_1 > \sup(F_0)$. By the argument above, we can assume that $\alpha_0$ is already in $p$.

Note that $F_1 \notin p$ implies that $\alpha_1 \notin p$, since otherwise the walk to $\alpha_1$ must be in $p$, by the definition of a suitable structure, but $F_1$ which is a part of this walk (actually the final model of it) is not in $p$. This provides a freedom to define the image of $\alpha_1$ which will be crucial further in choosing the image of $F_1$.

Fix $n \geq l(p)$. We need to add $F_1$ to $\text{dom}(a_n(p))$. Let $\text{dom}(a_n(p)) = \langle\langle X, Y, Z\rangle, C, \in, \subseteq\rangle$. We assume that $F_0, F \in C(\text{max}(X))$ and $\alpha_0 \in Y$.

Note that $Y \cap [\alpha_1, \sup(F_1)] = \emptyset$, since if some $\xi \in Y \cap [\alpha_1, \sup(F_1)]$, then all models of the walk to $\xi$ are in $X$, but $F_1$ is one of them.

Split into two cases.

**Case 1.** $\alpha_0 > \alpha_1$. 

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Then sup($F_1$) < $\alpha_0$. Consider the images $F_0^*$, $F^*$ and $M_0$ of $F_0$, $F$ and $\alpha_0$ under $a_n$.

Let us deal first with a little bit simplified situation, but which still contains the main elements of the construction.

**Subcase 1.A.** No elements of $Y \cap (\text{sup}(F_0 \cap \alpha_0), \alpha_0)$ are in $\text{dom}(a_n) \cap F$.

By Definition 1.2, we have $\text{cof}(\alpha_0) = \kappa^+$. Hence $\text{cof}(M_0 \cap \kappa_n^{+n+3}) = \kappa_n^{+n+2}$. So, $\kappa_n^{+n+1} > M_0 \subseteq M_0$. In particular, $M_0 \cap F_0^* \in M_0$. Clearly, $M_0 \cap F_0^* \in F^*$, as well. We assume that $M_0$ is $k$-good for $k$ big enough. Hence there is a $k - 1$-good $M_1 \in M_0$ realizing the same $k - 1$ type over $M_0 \cap F_0^*$ as $M_0$ does. By elementarity, we can find such $M_1$ inside $F^*$. Finally, pick $F_1^*$ to be an element of $F^* \cap M_0$ which realizes over $\langle M_0 \cap F_0^*, M_1 \rangle$ the same $k - 1$ type as $F_0^*$ realizes over $\langle M_0 \cap F_0^*, M_0 \rangle$.

Extend $a_n$ by mapping $F_1$ to $F_1^*$ and all the images of it under $\Delta$-system types triples isomorphisms. In particular, $M_1$ is added as the image of $M_0$ under $\pi_{F_0^*, F_1^*}$.

Turn now to a general case.

**Subcase 1.B.** There are elements of $Y \cap (\text{sup}(F_0 \cap \alpha_0), \alpha_0)$ in $\text{dom}(a_n) \cap F$.

Let $\gamma$ denotes the last such element below $\alpha_1$ and $\beta$ the first such element above $\alpha_1$. If one of them does not exists, then the argument below applies with obvious simplifications. Note that, as was observed above, there is no elements of $Y$ in the interval $[\alpha_1, \text{sup}(F_1)]$.

Denote $a_n(\beta)$ by $N$ and $a_n(\gamma)$ by $\gamma^*$. We assume that $M_0$ and $N$ are $k$-good for $k$ big enough. $\text{sup}(F_0^* \cap M_0) \cap \kappa_n^{+n+3} < N \cap \kappa_n^{+n+3}$, hence $F_0^* \cap M_0 \cap \kappa_n^{+n+3} \in N$ (as a set of ordinals of small cardinality). There is a $k - 1$-good $M_1 \in N$ realizing the same $k - 1$ type over $F_0^* \cap M_0 \cap \kappa_n^{+n+3}$ as $M_0$ does and with $\gamma^* \in M_1$. By elementarity, we can find such $M_1$ inside $F^*$. Finally, pick $F_1^*$ to be an element of $F^* \cap N$ which realizes over $\langle F_0^* \cap M_0 \cap \kappa_n^{+n+3}, M_1 \rangle$ the same $k - 1$ type as $F_0^*$ realizes over $\langle M_0 \cap F_0^*, M_0 \rangle$.

Extend $a_n$ by mapping $F_1$ to $F_1^*$ and all the images of it under $\Delta$-system types triples isomorphisms. In particular, $M_1$ is added as the image of $M_0$ under $\pi_{F_0^*, F_1^*}$.
under $\pi_{E_0^*,F_1^*}$.

**Case 1.** $\alpha_0 < \alpha_1$.

The construction is similar. The only change is that we pick $M_1$ above $M_0$.

This completes the inductive construction, and hence the proof of the lemma.

\[\square\]

The ordering $\leq^*$ on $\mathcal{P}$ and $\leq_n$ on $Q_{n0}$ seems to be not closed in the present situation. Thus it is possible to find an increasing sequence of $\aleph_0$ conditions $\biglangle \langle a_i, A_i, f_i \rangle \mid i < \omega \bigrangle$ in $Q_{n0}$ with no simple upper bound. The reason is that the union of maximal models of these conditions, i.e. $\bigcup_{i<\omega} \text{max}(\text{dom } a_i)$ need not be in $A^{1_{\kappa^+}}$ for any $A^{1_{\kappa^+}}$ in $G(\mathcal{P'})$. The next lemma shows that still $\leq_n$ and so also $\leq^*$ share a kind of strategic closure.

**Lemma 2.17** Let $n < \omega$. Then $\langle Q_{n0}, \leq_n \rangle$ does not add new sequences of ordinals of the length $< \kappa_n$, i.e. it is $(\kappa_n, \infty)$-distributive.

**Proof.** Let $\delta < \kappa_n$ and $\dot{h}$ be a $Q_{n0}$-name of a function from $\delta$ to ordinals. Without loss of generality assume that $\delta$ is a regular cardinal.

Using genericity of $G(\mathcal{P'})$ (or stationarity of the set $\{ A^{0_{\kappa^+}} \mid A^{0_{\kappa^+}}$ appears in an element of $G(\mathcal{P'}) \}$) it is not hard to find elementary submodel $M$ of some $H(\nu)$ for $\nu$ big enough so that

(a) $Q_{n0}, \dot{h}, \mathcal{P'} \in M$,

(b) $|M| = \kappa^+$,

(c) there is $\biglangle \langle A^{0_{\kappa^+}}, A^{1_{\kappa^+}}, C^{1_{\kappa^{++}}} \rangle \bigrangle \in G(\mathcal{P'})$ such that $A^{0_{\kappa^+}} = M \cap H(\kappa^{+3})$ and $\max(A^{1_{\kappa^{++}}} \cap \kappa^{+3}) = \sup(M \cap \kappa^{+3})$.

(d) $cf(M^* \cap \kappa^{++}) = \delta$,

(e) $\delta > M \subseteq M$. 

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Note that for such $M$, $M^* = M \cap H(\kappa^{+3})$ must be a limit model, since by Definition 1.1(3) successor models are closed under $\kappa$ sequences, but $M^*$ is not by (d) above.

We have $C^{\kappa^+}(M^*) \setminus \{M^*\} \subseteq M^*$. Let $B \in C^{\kappa^+}(M^*) \setminus \{M^*\}$. We claim that then $C^{\kappa^+}(B) \in M$. Thus, by elementarity there are $B^{1\kappa^+}, D^{\kappa^+}, B^{1\kappa^{++}} \in M$ such that

$$\langle\langle B, B^{1\kappa^+}, D^{\kappa^+}\rangle, B^{1\kappa^{++}}\rangle \in G(\mathcal{P}^') \cap M.$$ 

Note that $C^{\kappa^+} \upharpoonright B^{1\kappa^+}$ may be different from $D^{\kappa^+}$, but by the definition of order on $\mathcal{P}^'$ (1.15) and since $B \in C^{\kappa^+}(M^*)$, there are $E_1, ..., E_n \in B^{1\kappa^+}$ such that the switch with $E_1, ..., E_n$ turns $D^{\kappa^+}$ into $C^{\kappa^+} \upharpoonright B^{1\kappa^+}$. But $B^{1\kappa^+} \in M$ and $|B^{1\kappa^+}| \leq \kappa^+$. Hence $B^{1\kappa^+} \subseteq M$. So $E_1, ..., E_n \in M$, and then the corresponding switch is in $M$ as well. This implies that its result $C^{\kappa^+} \upharpoonright B^{1\kappa^+}$ is in $M$.

The cofinality of $C^{\kappa^+}(M^*) \setminus \{M^*\}$ under the inclusion must be $\delta$, since it is an $\in$-increasing continuous sequence of elements of $M^*$ with limit $M^*$ and by (d) above $\text{cf}(M^* \cap \kappa^{++}) = \delta$. Fix an increasing continuous sequence $\langle A_i \mid i < \delta \rangle$ of elements of $C^{\kappa^+}(M^*) \setminus \{M^*\}$ such that $\bigcup_{i<\delta} A_i = M^*$, $A_0$ is a successor model and for each limit model $A_i$ in the sequence $A_{i+1}$ is its immediate successor in $C^{\kappa^+}(M^*)$. By (e), each initial segment of it will be in $M$. Now we decide inside $M$ one by one values of $h_\gamma$ and put models from $\langle A_i \mid i < \delta \rangle$ to be maximal models of conditions used. This way we insure that unions of such conditions is a condition.

We define by induction an increasing sequence of conditions

$$\langle\langle a(i), A(i), f(i)\rangle\mid i \leq \delta \rangle.$$ 

and an increasing continuous subsequence

$$\langle A_{k_i} \mid i < \delta \rangle$$

of $\langle A_i \mid i < \delta \rangle$ such that for each $i < \delta$

$$(1) \quad \langle a(i), A(i), f(i)\rangle \in M,$$
\[(a(i + 1), A(i + 1), f(i + 1)) \text{ decides } h(i),\]

(3) \(A_{k_i}, A_{k_{i+1}} \in \text{dom}(a(i))\), \(A_{k_{i+1}}\) is the maximal model of \(\text{dom}(a(i))\) and \((\langle A_{k_{i+1}}, T, C^{\kappa^+} \upharpoonright T \rangle, R) \in G(P') \cap M\) witnesses a generic suitability of \(\text{dom}(a(i))\), for some \(T, R\), with \(R \subseteq A_{k_{i+1}} \cup \sup(A_{k_{i+1}})\).

There is no problem with \(A(i)\)'s and \(f(i)\)'s in this construction. Thus we have enough completeness to take intersections of \(A(i)\)'s and unions of \(f(i)\)'s.

The only problematic part is \(a(i)\). So let us concentrate only on building of \(a(i)\)’s.

\(i=0\)

Then let us pick some \(Z_0 \prec Z_1 \prec H(\chi^{+\omega}) \cap M\) of cardinality \(\kappa_n^{+n+1}\), closed under \(\kappa_n^{+n}\) - sequences of its elements and \(Z_0 \in Z_1\). Set \(a(0) = \langle \langle A_0, Z_0 \rangle, \langle A_1, Z_1 \rangle \rangle\).

\(i+1\)

Then we first extend \((a(i), A(i), f(i))\) to a condition \((a(i)', A(i)', f(i)') \in M\) which decides \(h(i)\). Then perform \textit{swt} (see 1.13) to turn \((a(i)', A(i)', f(i)')\) into an equivalent condition \((a(i)'', A(i)'', f(i)'')\) with \(A_{k_i} \in C^{\kappa^+} (\text{max}(\text{dom}(a(i)''))).\)

Pick a successor model \(A_j\) (from the cofinal sequence \(A_i \mid i < \delta\)) including \(\text{max}(\text{dom}(a(i)''))\). Set \(k_{i+1} = j\) and add it to \(\text{dom}(a(i)'')\), using \textit{swt} inside \(A_j\) if necessary. Finally we add \(A_{j+1}\).

\(i\) is a limit ordinal

Then we need to turn \(a = \bigcup_{j<i} a(j)\) into condition. For this we will need to add to \(\text{dom}(a)\) models and ordinals which are limits of elements of \(\text{dom}(a)\).

First we extend \(a\) by adding to it \(\langle A_{k_i}, \bigcup_{j<i} a(A_k) \rangle\), where \(k_i = \bigcup_{j<i} k_j\). Then for each non decreasing sequence \(\langle \alpha_j \mid j < i \rangle\) of ordinals in \(\text{dom}(a)\) we add the pair \(\langle \bigcup_{j<i} \alpha_j, \bigcup_{j<i} (a(\alpha_j) \cap H(\chi^{+\ell})) \rangle\), if it is not already in the \(\text{dom}(a)\), where \(\ell \leq \omega\) the maximal such that for unboundedly many \(j\)'s in \(i\) \(a(\alpha_j) \prec H(\chi^{+\ell})\), if the maximum exists or \(\ell >> n\) otherwise. Finally, for each model \(B \in \text{dom}(a)\) if there is a nondecreasing sequence \(\langle B_j \mid j < i \rangle\) of elements of \(C^{\kappa^+}(B)\) in \(\text{dom}(a)\) and \(B\) is the least possible (under inclusion or with least sup)
including the sequence, then we add the pair \( \langle \bigcup_{j<i} B_j, \bigcup_{j<i} (a(B_j) \cap H(\chi^+\ell)) \rangle \), if it is not already in the \( \text{dom}(a) \), where \( \ell \leq \omega \) is the minimum between the least \( k \) such that \( a(B) \subseteq H(\chi^k) \) and the maximal \( \ell' \) such that for unboundedly many \( j \)'s in \( i \) \( a(B_j) \prec H(\chi^{+\ell'}) \), if the maximum exists or

it is \( k \), if the maximum does not exist and \( k < \omega \),
or

\( \ell >> n \), if the maximum does not exist and \( k = \omega \).

We will need to extend a bit more if the following hold:

1. \( B \in \text{dom}(a) \),
2. \( \langle B_j \mid j < i \rangle \) is a nondecreasing sequence of elements of \( C^{\kappa^+}(B) \) in \( \text{dom}(a) \),
3. \( B \) is the least element of \( \text{dom}(a) \) such that \( \bigcup_{j<i} B_j \in B \),
4. \( \langle \alpha_j \mid j < i \rangle \) is a sequence of ordinals such that
   1. \( \alpha_j \in B_j \),
   2. \( \alpha_j \in \text{dom}(a) \),
   3. \( \bigcup_{j<i} \alpha_j \notin \text{dom}(a) \).

Set \( \alpha = \bigcup_{j<i} \alpha_j \). Then \( \alpha \in B \).

Let us consider two cases.

**Case 1.** \( \alpha \notin \bigcup_{j<i} B_j \).

If \( B \) is the real immediate successor of \( \bigcup_{j<i} B_j \), i.e. the one in \( C^{\kappa^+}(A^{0\kappa^+}) \) of \( G(\mathcal{P}') \), then the extension made above suffices. Otherwise, we need to add the real successor of \( \bigcup_{j<i} B_j \) in order to insure walks to ordinals closure. Denote such successor by \( E \). We map it to a model \( E^* \) such that \( \bigcup_{j<i} (a(B_j) \cap H(\chi^+\ell)) \prec E^* \prec a(B) \cap H(\chi^{+\ell}) \) and \( E^* \) is good enough, where \( \ell \) is as above. Note that each \( \gamma \in B \cap \text{dom}(a) \) is already in \( B_j \), for some \( j < i \), by walks to ordinals closure of \( \text{dom}(a) \). Finally we map \( \alpha \) to \( \bigcup_{j<i} (a(B_j) \cap a(\alpha_j)) \).
Case 2. \( \alpha \in \bigcup_{j<i} B_j \).

Let \( E \) be the smallest model in \( C^\kappa^+(B) \) with \( \alpha \in E \).

Subcase 2.1. \( E \) is the least (under the inclusion) element of \( C^\kappa^+(B) \).

If for some \( j < i \) we have \( \alpha_j \in E \), then by the walk closure of \( \text{dom}(a) \), the model \( E \) is in \( \text{dom}(a) \). It is easy now to extend \( a \) by adding only \( \alpha \) which is mapped to an appropriate element of \( a(E) \).

Suppose that for each \( j < i \), \( \alpha_j \not\in E \). Consider \( \alpha_0 \). Let \( D_0 \) be the largest model in \( C^\kappa^+(B) \) with \( \alpha_0 \not\in D_0 \). By the walk closure of \( \text{dom}(a) \), we have \( D_0 \in \text{dom}(a) \). Assume that \( D_0 \neq E \), otherwise proceed as above. Clearly \( D_0 \supset E \), and hence \( \alpha_0 < \alpha < \sup(D_0) \). Then \( \alpha_0 := \min(D_0 \cap \alpha_0) \in \text{dom}(a) \). So, \( \alpha_0 < \alpha_0_1 < \alpha \). Let \( D_{01} \) be the largest model in \( C^\kappa^+(B) \) with \( \alpha_{01} \notin D_0 \).

By the walk closure of \( \text{dom}(a) \), we have \( D_{01} \in \text{dom}(a) \). Again, we assume that \( D_{01} \neq E \). Clearly \( D_0 \supset D_{01} \supset E \), and hence \( \alpha_{01} < \alpha < \sup(D_{01}) \). Then \( \alpha_{02} := \min(D_{01} \cap \alpha_{01}) \in \text{dom}(a) \). So, \( \alpha_0 < \alpha_{01} < \alpha_{02} < \alpha \). We continue and define \( D_{02} \) etc. The sequence of such \( D_{0k} \) will be \( \epsilon \)-decreasing, and hence at certain stage \( D_{0k} = E \).

Subcase 2.2. \( E \) is not the least (under the inclusion) element of \( C^\kappa^+(B) \).

Then \( E \) has the immediate predecessor \( E^- \) in \( C^\kappa^+(B) \). Suppose first that \( \alpha \) is a limit point of \( E^- \). Note that then necessarily \( E^- \) is a limit model, as successor ones are closed under \( < \kappa^+ \)-sequences.

Claim 2.17.1 There is an increasing sequence \( \langle \alpha_j' \mid j < i \rangle \) in \( E^- \cap \text{dom}(a) \) with limit \( \alpha \).

Proof. Let \( j < i \). If \( \alpha_j \in E^- \), then we take it. Suppose that \( \alpha_j \not\in E^- \). Pick \( D_j \) to be the largest model in \( C^\kappa^+(B) \) with \( \alpha_j \not\in D_j \). Then, \( D_j \in \text{dom}(a) \), and clearly, \( D_j \supset E^- \). Also, \( \alpha_j < \alpha \) and \( \alpha \) is a limit point of \( E^- \). Hence \( \alpha_j < \sup(D_j) \). Then \( \alpha_{j1} := \min(D_j \setminus \alpha_j) \in D_j \cap \text{dom}(a) \). If \( \alpha_{j1} \in E^- \), then we pick it. Otherwise, continue and consider \( D_{j1} \) the largest model in \( C^\kappa^+(B) \) with \( \alpha_{j1} \not\in D_{j1} \). Then, \( D_{j1} \in \text{dom}(a) \), and clearly, \( D_{j1} \supset E^- \). Also, \( \alpha_{j1} < \alpha \) and \( \alpha \) is a limit point of \( E^- \). Hence \( \alpha_{j1} < \sup(D_{j1}) \). Then \( \alpha_{j2} := \min(D_{j1} \setminus \alpha_{j1}) \in D_{j1} \cap \text{dom}(a) \). If \( \alpha_{j2} \in E^- \), then we pick it. Otherwise,
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Proof. Let us check the condition (6(6c)) of Definition 1.1. Thus let

Claim 2.17.2 dom(b) is a suitable generic structure.

Proof. Let us check the condition (6(6c)) of Definition 1.1. Thus let

A, α

∈

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dom(b), \( A \in C(\max(\text{dom}(b))) \) a non-limit model and sup(\( A \)) > \( \alpha \). We need to show that min(\( A \setminus \alpha \)) \( \in \text{dom}(b) \).

**Case 1.** \( A \in \text{dom}(a(l)) \) for some \( l < i \).

If \( \alpha \in \text{dom}(a) \), then for some \( j < i \) big enough we will have \( A, \alpha \in \text{dom}(a_j) \), and then min(\( A \setminus \alpha \)) \( \in \text{dom}(a_j) \). Note that if \( \alpha \) is a non-limit element of \( \text{dom}(b) \), then \( \alpha \in \text{dom}(a) \).

Suppose that \( \alpha \) is a limit point of \( \text{dom}(b) \) and \( \alpha \not\in \text{dom}(a) \). Let \( \langle \alpha_j | j < i \rangle \) be a nondecreasing sequence from \( \text{dom}(a) \) converging to \( \alpha \). By (6(6c)) of Definition 1.1, \( \gamma_j = \min(A \setminus \alpha_j) \in \text{dom}(a) \). If \( \langle \gamma_j | j < i \rangle \) is eventually constant, then the constant value will be as desired. Suppose otherwise. Then \( \langle \gamma_j | j < i \rangle \) will be also a converging to \( \alpha \) sequence. But remember that \( A \) is non-limit, hence \( \kappa A \subseteq A \), and so \( \alpha \subseteq A \). Then min(\( A \setminus \alpha \)) = \( \alpha \in \text{dom}(b) \) and we are done.

**Case 2.** \( A \not\in \text{dom}(a) \).

Assume that \( \alpha \not\in A \), just otherwise min(\( A \setminus \alpha \)) = \( \alpha \) and we are done. Denote \( \text{min}(A \setminus \alpha) \) by \( \alpha^* \).

**Subcase 2.1.** \( \alpha \in \text{dom}(a) \).

Consider then the smallest model \( E_\alpha \) in \( C(\max(\text{dom}(b))) \) with \( \alpha \) inside. Let \( E_\alpha^- \) be its immediate predecessor in \( C(\max(\text{dom}(b))) \). Then \( A \subseteq E_\alpha^- \), since \( \alpha \not\in A \), and \( A \neq E_\alpha^\top \), since \( E_\alpha^- \in \text{dom}(a) \) and \( A \not\in \text{dom}(a) \). Then sup(\( E_\alpha^- \)) > \( \alpha \), hence \( \alpha_1 := \min(E_\alpha^- \setminus \alpha) > \alpha \) and \( \alpha_1 \in \text{dom}(a) \). \( E_\alpha^- \supseteq A \) implies that \( \alpha_1 \leq \alpha^* \). If \( \alpha_1 = \alpha^* \), then \( \alpha^* \in \text{dom}(a) \) and we are done. Suppose otherwise. Then \( \alpha_1 < \alpha^* \). Consider then the smallest model \( E_{\alpha_1} \) in \( C(\max(\text{dom}(b))) \) with \( \alpha_1 \) inside. Let \( E_{\alpha_1}^- \) be its immediate predecessor in \( C(\max(\text{dom}(b))) \). Then \( A \subseteq E_{\alpha_1}^- \), since \( \alpha_1 \not\in A \), and \( A \neq E_{\alpha_1}^- \), since \( E_{\alpha_1}^- \in \text{dom}(a) \) and \( A \not\in \text{dom}(a) \). Then sup(\( E_{\alpha_1}^- \)) > \( \alpha_1 \), since \( \alpha^* \in E_{\alpha_1}^- \) and \( \alpha^* > \alpha_1 \). Hence \( \alpha_2 := \min(E_{\alpha_1}^- \setminus \alpha_1) > \alpha_1 \) and \( \alpha_2 \in \text{dom}(a) \). If \( \alpha_2 = \alpha^* \), then \( \alpha^* \in \text{dom}(a) \) and we are done. Otherwise, \( \alpha_2 < \alpha^* \). We continue and consider \( E_{\alpha_2}, E_{\alpha_2}^- \) etc. Note that the sequence of models \( E_{\alpha_m} \) constructed this way is decreasing. So the process stops after finitely many steps. Which means that \( \alpha^* \in \text{dom}(a) \).
**Subcase 2.2.** $\alpha \notin \text{dom}(a)$.

Then $\alpha$ is a limit of an increasing sequence $\langle \alpha_j \mid j < i \rangle$ of elements of $\text{dom}(a)$. If an unbounded subsequence of the sequence $\langle \alpha_j \mid j < i \rangle$ is in $A$, then $\alpha$ will be in $A$ as well, since $A$ is a non-limit model and so is closed under $\delta$ sequences of its elements. Hence there is $j^* < i$ such that for every $j, j^* \leq j < i, \alpha_j \notin A$.

Let $j^* \leq j < i$. We have $\sup(A) > \alpha > \alpha_j$. Set $\alpha^*_j = \min(A \setminus \alpha_j)$. By Subcase 2.1, $\alpha^*_j \in \text{dom}(a)$. If $\alpha^*_j > \alpha$, then $\alpha^*_j = \alpha^*$ and we are done. Assume, hence that $\alpha^*_j < \alpha$, for every $j < i$. But the sequence $\langle \alpha^*_j \mid j < i \rangle$ is a sequence of elements of $A$ which converges to $\alpha$. So, $\alpha \in A$. Contradiction. □ of the claim.

The next claim is similar.

**Claim 2.17.3** $\text{rng}(b)$ is a suitable structure over $\kappa_n$.

We need to check that $b$ is an isomorphism between the suitable structures $\text{dom}(b)$ and $\text{rng}(b)$. By Lemma 2.10, it is enough to show that the restriction of $b$ is an isomorphism between between the corresponding weak suitable structures. But this is obvious, since no $\Delta$-system type triples are added at limit stages. □

It is possible to work in $V$ rather than in $V[G(P')]$ or $M$. Combining arguments of 1.19 and the previous lemma it is not hard to show the following:

**Lemma 2.18** $P' * Q_{n0}$ is $< \kappa_n$-strategically closed.

**Lemma 2.19** $\langle P, \leq' \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.

*Proof.* Repeat the argument of Lemma 2.17 with $P$ replacing $Q_{n0}$. □

The argument of Lemma 2.17 can be used in a standard fashion to show the Prikry condition (i.e. the standard argument runs inside elementary submodel $M$ with $\delta$ replaced by $\kappa^+$).
Lemma 2.20 \(\langle \mathcal{P}, \leq^* \rangle\) satisfies the Prikry condition.

Finally we define \(\rightarrow\) on \(\mathcal{P}\) similar to those of [1] or [2].

Lemma 2.21 \(\langle \mathcal{P}, \rightarrow \rangle\) satisfies \(\kappa^{++}\)-c.c.

Proof. Suppose otherwise. Work in \(V\). Let \(\langle p_\alpha \mid \alpha < \kappa^{++} \rangle\) be a name of an antichain of the length \(\kappa^{++}\). Using 1.19 we find an increasing sequence \(\langle \langle A^{0\kappa^+}_\alpha, A^{1\kappa^+}_\alpha, C^{\kappa^+}_\alpha \rangle, A^{1\kappa^{++}}_\alpha \rangle \mid \alpha < \kappa^{++} \rangle\) of elements of \(\mathcal{P}'\) and a sequence \(\langle p_\alpha \mid \alpha < \kappa^{++} \rangle\) so that for every \(\alpha < \kappa^{++}\) the following hold:

(a) \(\langle \langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1} \rangle, A^{1\kappa^{++}}_{\alpha+1} \rangle \Vdash p_\alpha \leq \bar{p}_\alpha,\)

(b) \(\bigcup_{\beta < \alpha} A^{0\kappa^+}_\beta = A^{0\kappa^+}_\alpha\), if \(\alpha\) is a limit ordinal,

(c) \(\kappa A^{0\kappa^+}_{\alpha+1} \subseteq A^{0\kappa^+}_{\alpha+1}\),

(d) \(A^{0\kappa^+}_{\alpha+1}\) is a successor model,

(e) \(\langle A^{1\kappa^+}_\beta \mid \beta < \alpha \rangle \in A^{0\kappa^+}_{\alpha+1}\),

(f) for every \(\alpha \leq \beta < \kappa^{++}\) we have

\[ C^{\kappa^+}_\alpha (A^{0\kappa^+}_\alpha) \text{ is an initial segment of } C^{\kappa^+}_\beta (A^{0\kappa^+}_\beta), \]

(g) \(p_\alpha = \langle p_{\alpha n} \mid n < \omega \rangle,\)

(h) for every \(n \geq l(p_\alpha)\), \(A^{0\kappa^+}_{\alpha+1}\) is the maximal model of \(\text{dom}(a_{\alpha n})\) and \(A^{0\kappa^+}_\alpha \in \text{dom}(a_{\alpha n})\), where \(p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n} \rangle\).

Actually this condition is the reason for not requiring the equality in (a) above.

Let \(p_{\alpha n} = \langle a_{\alpha n}, A_{\alpha n}, f_{\alpha n} \rangle\) for every \(\alpha < \kappa^{++}\) and \(n \geq l(p_\alpha)\).

Let \(\alpha < \kappa^{++}\). Fix some

\[ \langle \langle B^{0\kappa^+}_{\alpha+1}, B^{1\kappa^+}_{\alpha+1}, D^{\kappa^+}_{\alpha+1} \rangle, B^{1\kappa^{++}}_{\alpha+1} \rangle \leq \mathcal{P}' \langle \langle A^{0\kappa^+}_{\alpha+1}, A^{1\kappa^+}_{\alpha+1}, C^{\kappa^+}_{\alpha+1} \rangle, A^{1\kappa^{++}}_{\alpha+1} \rangle \]
which witnesses a generic suitability of structure \( \text{dom}(a_{\alpha n}) \) for each \( n, l(p_\alpha) \leq n < \omega \), as in Definition 2.2. Note that \( B_{\alpha+1}^{0\kappa^+} \) need not be in \( C_{\alpha+1}^{\kappa^+}(A_{\alpha+1}^{0\kappa^+}) \) and even if it does, then \( D_{\alpha+1}(B_{\alpha+1}^{0\kappa^+}) \) need not be an initial segment of \( C_{\alpha+1}(A_{\alpha+1}^{0\kappa^+}) \). By the definition of the order \( \leq_\mathcal{P} \) (Definition 1.15) there are \( m < \omega \) and \( E_1, ..., E_m \in A_{\alpha+1}^{1\kappa^+} \) such that

\[
\text{swt}(\langle A_{\alpha+1}^{0\kappa^+}, A_{\alpha+1}^{1\kappa^+}, C_{\alpha+1}^{\kappa^+}, A_{\alpha+1}^{1\kappa^+} \rangle, E_1, ..., E_m) \text{ and } \langle B_{\alpha+1}^{0\kappa^+}, B_{\alpha+1}^{1\kappa^+}, D_{\alpha+1}^{\kappa^+}, B_{\alpha+1}^{1\kappa^+} \rangle
\]

satisfy (1)-(3) of Definition 1.15.

By Lemma 2.16 it is possible to add all \( E_i (i = 1, ..., m) \) to \( \text{dom}(a_{\alpha n}) \), for a final segment of \( n \)'s. By adding and taking non-direct extension if necessary, we can assume that \( E_i \)'s are already in \( \text{dom}(a_{\alpha n}) \), for every \( n \geq l(p_\alpha) \).

Now we can apply the opposite switch (i.e. the one starting with \( E_m \), then \( E_m - 1 \), ..., and finally \( E_1 \)) to \( \text{dom}(a_{\alpha n}) \) (and the corresponding to it under \( a_{\alpha n} \) to \( \text{rng}(a_{\alpha n}) \)). Denote the result still by \( a_{\alpha n} \).

Finally, \( \langle A_{\alpha+1}^{0\kappa^+}, A_{\alpha+1}^{1\kappa^+}, C_{\alpha+1}^{\kappa^+}, A_{\alpha+1}^{1\kappa^+} \rangle \) will witness a generic suitability of structure \( \text{dom}(a_{\alpha n}) \) for each \( n, l(p_\alpha) \leq n < \omega \).

In particular, we have now that the central line of \( \text{dom}(a_{\alpha n}) \) is a part of \( C_{\alpha+1}(A_{\alpha+1}^{0\kappa^+}) \) and \( A_{\alpha}^{0\kappa^+} \) is on it, for every \( n, l(p_\alpha) \leq n < \omega \).

Shrinking if necessary, we assume that for all \( \alpha, \beta < \kappa^+ \) the following holds:

1. \( \ell = \ell(p_\alpha) = \ell(p_\beta) \),
2. for every \( n < \ell \) \( p_{\alpha n} \) and \( p_{\beta n} \) are compatible in \( Q_{n1} \) i.e. \( p_{\alpha n} \cup p_{\beta n} \) is a function,
3. for every \( n, \ell \leq n < \omega \), \( \langle \text{dom}(f_{\alpha n}) \mid \alpha < \kappa^{++} \rangle \) form a \( \Delta \)-system with the kernel contained in \( A_0^{\kappa^+} \),
4. for every \( n, \omega > n \geq \ell \), \( \text{rng}(a_{\alpha n}) = \text{rng}(a_{\beta n}) \).

Shrink now to the set \( S \) consisting of all the ordinals below \( \kappa^{++} \) of cofinality \( \kappa^+ \). Let \( \alpha \) be in \( S \). For each \( n, \ell \leq n < \omega \), there will be \( \beta(\alpha, n) < \alpha \)
such that
\[ \text{dom}(a_{\alpha n}) \cap A_\alpha^{0\kappa^+} \subseteq A_\beta^{0\kappa^+}. \]

Just recall that \(|a_{\alpha n}| < \kappa_n\). Shrink \(S\) to a stationary subset \(S^*\) so that for some \(\alpha^* < \min S^*\) of cofinality \(\kappa^+\) we will have \(\beta(\alpha, n) < \alpha^*\), whenever \(\alpha \in S^*, \ell \leq n < \omega\). Now, the cardinality of \(A_{\alpha^*}^{0\kappa^+}\) is \(\kappa^+\). Hence, shrinking \(S^*\) if necessary, we can assume that for each \(\alpha, \beta \in S^*, \ell \leq n < \omega\)
\[ \text{dom}(a_{\alpha n}) \cap A_\alpha^{0\kappa^+} = \text{dom}(a_{\beta n}) \cap A_\beta^{0\kappa^+}. \]

Let us add \(A_{\alpha^*}^{0\kappa^+}\) to each \(p_\alpha\) with \(\alpha \in S^*\).

By 2.16(2), we can add it without adding ordinals and the only other models that probably were added are the images of \(A_{\alpha^*}^{0\kappa^+}\) under \(\Delta\)-system type isomorphisms. Denote the result for simplicity by \(p_\alpha\) as well.

Let now \(\beta < \alpha\) be ordinals in \(S^*\). We claim that \(p_\beta\) and \(p_\alpha\) are compatible in \(\langle \mathcal{P}, \rightarrow \rangle\).

First extend \(p_\alpha\) by adding \(A_{\beta}^{0\kappa^+}\). This will not add other additional models or ordinals except the images of \(A_{\beta}^{0\kappa^+}\) under isomorphisms to \(p_\alpha\), as was remarked above.

Let \(p\) be the resulting extension. Denote \(p_\beta\) by \(q\). Assume that \(\ell(q) = \ell(p)\). Otherwise just extend \(q\) in an appropriate manner to achieve this.

Let \(n \geq \ell(p)\) and \(p_n = \langle a_n, A_n, f_n \rangle\). Let \(q_n = \langle b_n, B_n, g_n \rangle\). Without loss of generality we may assume that \(a_n(A_{\beta+2}^{0\kappa^+})\) is an elementary submodel of \(\mathfrak{A}_{n,k_n}\) with \(k_n \geq 5\). Just increase \(n\) if necessary. Now, we can realize the \(k_n - 1\)-type of \(\text{rng}(b_n)\) inside \(a_n(A_{\beta+2}^{0\kappa^+})\) over the common parts \(\text{dom}(b_n)\) and \(\text{dom}(a_n)\). This will produce \(q'_n = \langle b'_n, B_n, g_n \rangle\) which is \(k_n - 1\)-equivalent to \(q_n\) and with \(\text{rng}(b'_n) \subseteq a_n(A_{\beta+2}^{0\kappa^+})\). Doing the above for all \(n \geq \ell(p)\) we will obtain \(q' = \langle q'_n \mid n < \omega \rangle\) equivalent to \(q\) (i.e. \(q' \iff q\)).

Extend \(q'\) to \(q''\) by adding to it \(\langle A_{\beta+2}^{0\kappa^+}, a_n(A_{\beta+2}^{0\kappa^+}) \rangle\) as the maximal set for every \(n \geq \ell(p)\). Recall that \(A_{\beta+1}^{0\kappa^+}\) was its maximal model. So we add a top model. Hence no additional models or ordinals are added at all. Let \(q''_n = \langle b''_n, B_n, g_n \rangle\), for every \(n \geq \ell(p)\).
Combine now $p$ and $q''$ together. Thus for each $n \geq \ell(p)$ we add $b''_n$ to $a_n$ as well as all of its isomorphic images under $\Delta$-system type isomorphisms of triples in $a_n$. The rest of the parts are combined in the obvious fashion (we put together the functions and intersect sets of measure one moving first to the same measure). Add if necessary $A^{0\kappa^+}_{\alpha + 3}$ as a new top model in order to insure 2.11(2(2a)). Let $r = \langle r_n \mid n < \omega \rangle$ be the result, where $r_n = \langle c_n, C_n, h_n \rangle$, for $n \geq \ell(p)$.

**Claim 2.21.1** For each $\gamma, \alpha + 3 < \gamma < \kappa^{++}$,

$$\langle \langle A^{0\kappa^+}_\gamma, A^{1\kappa^+}_\gamma, C^{\kappa^+}_\gamma \rangle, A^{1\kappa^{++}}_\gamma \rangle \models r \in \mathcal{P}.$$  

**Proof.** Let $\gamma \in (\alpha + 3, \kappa^{++})$ and $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$ with $\langle \langle A^{0\kappa^+}_\gamma, A^{1\kappa^+}_\gamma, C^{\kappa^+}_\gamma \rangle, A^{1\kappa^{++}}_\gamma \rangle \in G(\mathcal{P}')$.

Fix $n \geq \ell(p)$. The main points here are that $b''_n$ and $a_n$ agree on the common part and adding of $b''_n$ to $a_n$ does not required other additions of models or of ordinals except the images of $b''_n$ under $\Delta$-system type isomorphisms for triples in $a_n$.

We need to check that dom$(c_n)$ is a suitable generic structure and rng$(a_n)$ is a suitable structure. Let us deal with dom$(c_n)$. The range is similar. By Lemma 2.10 it is enough to deal with a weak suitable structures. Let $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ be the corresponding redact of dom$(c_n)$.

Clearly, $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ is a submodel of $\langle \langle A^{1\kappa^+}_\gamma, A^{1\kappa^{++}}_\gamma, C^{\kappa^+}_\gamma, \in, \subseteq \rangle$.

Let us check that the structures $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ and $\langle \langle A^{1\kappa^+}_\gamma, A^{1\kappa^{++}}_\gamma, C^{\kappa^+}_\gamma, \in, \subseteq \rangle$ agree about walks to members of $X$ and to ordinals in $Y$. This will show, in particular that $\langle \langle X, Y \rangle, C, \in, \subseteq \rangle$ is walks closed and, hence $\langle \langle \max(X), X, C, Y \rangle \in \mathcal{P}' \rangle$.

Fix $t \in X \cup Y$ (a model or an ordinal). Note that, by the choice of the top model $\max(X)$ of $X$ we have $\max(X) \in C^{\kappa^+}_\gamma (A^{0\kappa^+}_\gamma)$. Hence, the walk from $A^{0\kappa^+}_\gamma$ to $t$ will go via $\max(X)$. If $t$ appears in dom$(a_n)$, then the continuation of the walk will be inside dom$(a_n)$, since $\max(a_n) = A^{0\kappa^+}_{\alpha + 1} \in C(\max(X))$.  

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It will co-inside with the walk from $A_{\alpha+1}^{0\kappa+}$ to $t$, since $\text{dom}(a_n)$ is a suitable structure. Hence all the members of the walk are in $X \cup Y$.

Note that if $t$ is in the common part, i.e. if $t$ appears in both $\text{dom}(a_n)$ and $\text{dom}(b_n)$, then $t \in A_{\alpha}^{0\kappa+}$. So the walk to $t$ passes through $A_{\alpha}^{0\kappa+}$, since $A_{\alpha}^{0\kappa+} \in C^{\kappa+}(A_7^{0\kappa+})$.

If $t$ appears in $\text{dom}(b_n'') = \text{dom}(b_n) \cup \{A_{\beta+2}^{0\kappa+}\}$, then the walk to $t$ will proceed via $A_{\beta+2}^{0\kappa+}$, since $t \in A_{\beta+2}^{0\kappa+}$ and $A_{\beta+2}^{0\kappa+} \in C(\max(X))$. Now, it will co-inside with the walk from $A_{\beta+1}^{0\kappa+}$ to $t$, since $\text{dom}(b_n)$ is a suitable structure and $A_{\beta+1}^{0\kappa+} \in C(A_{\beta+2}^{0\kappa+})$.

The agreement between the walks follows.

□ of the claim.

Now we have $r \geq p, q''$. Hence, $p \rightarrow r$ and $q \rightarrow r$. Contradiction.

□

References
