On number of generators of normal ultrafilters mod the closed unbounded filter

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Abstract

Starting with $o(\kappa) = \kappa^{+3} + 1$, a model in which κ carries a normal ultrafilter $\mathcal{U}$ with number of generators mod Cub_κ less than 2^κ is constructed.

1 Introduction

Let $\mathcal{U}$ be a normal ultrafilter over a measurable cardinal κ.
How many sets are needed in order to generate $\mathcal{U}$ mod the closed unbounded filter over κ, i.e. What is the least cardinal λ for which there exists a set $\mathcal{A} \subseteq U$ such that for every $X \in U$ there is $A \in \mathcal{A}$, $A \subseteq X$ mod Cub_κ. This is a basic question that we will deal with.
Clearly, that if $2^\kappa = \kappa^+$, then $\lambda = \kappa^+$. By T. Carlson, H. Woodin, see also [1], it is possible to have $2^\kappa > \kappa^+$ and the number of generators of $\mathcal{U}$, even mod bounded, is κ^+. Supercompacts and the Mathias forcing were used in this constructions. Here we will use a different methods and reduce greatly the initial assumptions.

2 Basic settings and ideas

Let us present the construction first under a stronger assumption.

Assume GCH. Let E' be a $(\kappa, \kappa^{+3})-$extender.
Pick some $\rho, \kappa^{++} < \rho < \kappa^{+3}$ of cofinality κ^+ which is a limit of generators of E'. We will use $E = E' \restriction \rho$. Note that $\rho = (\kappa^{+3})^{M_E}$ and M_E is a direct limit of

$$\langle M_{E\restriction\xi} \mid \xi < \rho, \xi \text{ is a generator of } E\rangle.$$

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Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa\rangle$ be the Easton support iteration of Cohen forcings $Cohen(\nu, \nu^{+3})$, for inaccessible $\nu < \kappa$ and let Q_κ be $Cohen(\kappa, \rho)$.
Take a generic $G \subseteq P_{\kappa+1}$. Denote by $\langle f_{\nu\xi} \mid \xi < \nu^{+3}\rangle$ the Cohen functions added by G to $\nu < \kappa$ and $\langle f_{\kappa\xi} \mid \xi < \rho = (\kappa^{+3})^{M_E}\rangle$ to κ.
The elementary embedding $j = j_E$ extends in $V[G]$ to $j^* : V[G] \to M_E[G^*]$.
Define the corresponding normal ultrafilter $\mathcal{U}$ by setting

$$X \in \mathcal{U} \text{ iff } \kappa \in j^*(X).$$

For every $A \in \mathcal{U}$, there are $r, s, t \in M_E$ such that $r \in P_{\kappa+1} \cap G^*, s \in P_{(\kappa+1, j(\kappa))} \cap G^*, t \in Q_{j(\kappa)} \cap G^*$ such that

$$r^\frown s^\frown t \Vdash \kappa \in j(\underset{\sim}{A}).$$

We can assume that s, t are in the range of the canonical embedding $k : M_{E_\kappa} \to M_E$, i.e. $s = k(s')$ and $t = k(t')$. Choose functions f_s and f_t on κ such that $s = j(f_s)(\kappa)$ and $t = j(f_t)(\kappa)$.
In addition, for some generator (or a pair of generators) $\eta < \rho$, $r = j(f_r)(\eta)$.
Reflect down and set

$$B_s = \{\nu < \kappa \mid f_s(\nu) \in G \cap P_{(\nu+1,\kappa)}\},$$

$$B_t = \{\nu < \kappa \mid f_t(\nu) \in G \cap Q_\kappa\},$$

and

$$B_r = \{\nu < \kappa \mid f_r(f_{\kappa\eta}(\nu)) \in G \cap P_{\nu+1}\},$$

where $f_{\kappa\eta}$ is the $\eta-$th Cohen function for κ and it will represent η.
Set

$$X = \{\rho < \kappa \mid f_r(\rho)^\frown f_s(\nu)^\frown f_t(\nu) \Vdash \nu \in \underset{\sim}{A}\},$$

where $\nu = \pi_{\eta\kappa}(\rho)$ and $\pi_{\eta\kappa}$ is the canonical projection from E_η to E_κ.
Then X is in V and belongs to E_η.
Note that the number of possibilities for s's is κ^+, but for t's it is already κ^{++}.
Suppose now, that there is a family $\mathcal{Y}$ of cardinality κ^+,[1] such that:

[1]Further we will take $\mathcal{Y}$ to consist of the sets $\{\nu < \kappa \mid f_{\kappa\rho_{\xi+2}}(\nu^+) = 0\}, \xi < \kappa^+$ and their intersections.

1. there is a set $A_s \in \mathcal{Y} \cup E_\kappa$ such that $A_s \subseteq B_s \mod Cub_\kappa \restriction Inaccesibles$,

2. there is a set $A_t \in \mathcal{Y} \cup E_\kappa$ such that $A_t \subseteq B_t \mod Cub_\kappa \restriction Inaccesibles$,

3. there is a set $A_r \in \mathcal{Y} \cup E_\kappa$ such that $A_r \subseteq B_r \mod Cub_\kappa \restriction Inaccesibles$,

4. there is $Y \in \mathcal{Y} \cup E_\kappa$ such that $f_{\kappa\eta}{}''Y \subseteq X \mod Cub_\kappa \restriction Inaccesibles$.

Then

$$A \supseteq B_r \cap B_s \cap B_t \cap Y,$$

since if $\nu \in B_r \cap B_s \cap B_t \cap Y$, then

$$f_r(f_{\kappa\eta}(\nu))^\frown f_s(\nu)^\frown f_t(\nu) \Vdash \nu \in \underset{\sim}{A}$$

and $f_r(f_{\kappa\eta}(\nu))^\frown f_s(\nu)^\frown f_t(\nu) \in G$.

This means that $\mathcal{U}$ is generated by $\mathcal{Y} \cup E_\kappa \mod Cub_\kappa \restriction Inaccesibles$, assuming that $\mathcal{Y} \cup E_\kappa$ is closed under intersections.

3 An order on Cohen functions and old functins in ${}^\kappa\kappa$

We force clubs such that

(a) $\{\nu < \kappa \mid f_{\kappa\xi}(\nu) < f_{\kappa\zeta}(\nu)\} \supseteq club \cap Inc$,

for every $\xi < \zeta < \kappa^{++}$.

We need to ensure that

$$f_{j(\kappa)j(\eta)} \restriction \kappa = f_{\kappa\eta},$$

for every $\eta < \kappa^{++}$.

Consider

$$X_\eta = \{\nu < \kappa \mid f_{\kappa\eta} \restriction \nu = f_{\nu f_{\kappa\eta}(\nu)}\}.$$

Force a club C_η such that $C_\eta \cap Inc \subseteq X_\eta$.

4 Generating sets

We would like to add clubs intersected with inaccessibles into many sets of $\mathcal{U}$. The main difficulty is that such forcing should be defined inside M_E in order to be able to extend the embedding j_E. However, $E \notin M_E$, and so, the actual forcing will be wider - more sets and more clubs will be added than actually needed for a small generating family. We will first add a Cohen function over M_E for its κ^{+3}, then change some of its values and use it as a guide for which sets to add clubs.

Consider the Cohen forcing for adding a function for κ^{+3}, $Cohen(\kappa^{+3}, 2)$ in M_E.
In V, we can construct M_E-generic subset of it, but let us do this via constructing generics for $M_{E\restriction\xi}, \xi$ is a generator less than ρ. Note that such ξ's are κ^{+3}'s of $M_{E\restriction\xi}$'s. Hence, given $\xi < \xi'$, we can start building $Cohen(\kappa^{+3}, 2)$ generic over $M_{E\restriction\xi'}$ with $k_{E\restriction\xi E\restriction\xi'}{}''G_\xi = G_\xi$, since $crit(k_{E\restriction\xi E\restriction\xi'}) = (\kappa^{+3})^{M_\xi} = \xi$, where $k_{E\restriction\xi E\restriction\xi'} : M_{E\restriction\xi} \to M_{E\restriction\xi'}$ is the canonical embedding.
For a limit ξ^*, let $G_{\xi^*} = \bigcup_{\xi<\xi^*} G_\xi$ and finally, $G = G_\rho = \bigcup_{\xi<\rho} G_\xi$.

Let us deal with the construction at successor stages.

Lemma 4.1 *Let $\xi, \kappa^{++} < \xi < \rho$ be a successor generator of E, then* $\mathrm{cof}(\xi) = \kappa^+$.

Proof. For every $\zeta < \kappa^{++}$, consider $E_\zeta = \{X \subseteq \kappa \mid \zeta \in j_E(X)\}$. It is a $\kappa-$complete ultrafilter over κ. There is a natural embedding $k_\zeta : M_{E_\zeta} \to M_E$ defined by setting $k_\zeta([f]_{E_\zeta}) = j_E(f)(\zeta)$, for every $f : \kappa \to V$.
We have, in M_{E_ζ}, $\kappa^{++} < [id]_{E_\zeta} < \kappa^{+3}$. The critical point of k_ζ is $(\kappa^{++})^{M_{E_\zeta}}$ which is moved to the real κ^{++}. Also, $k_\zeta([id]_{E_\zeta}) = \zeta$ and $k_\zeta((\kappa^{+3})^{M_{E_\zeta}}) = \rho$.

Let ξ' be the last generator below ξ or just κ^{++} if ξ was the first generator above κ^{++}. Then, in $M_{E\restriction\xi}$, $\xi = \kappa^{+3}$.
Let $\alpha < \xi$. Then there are $m < \omega$, $f : [\kappa]^m \to \kappa$, for every $\nu_1, ..., \nu_m < \kappa, f(\nu_1, ..., \nu_m) < \nu_1^{+3}$, and $\xi_2, ..., \xi_m \leq \xi'$ such that $j_{E\restriction\xi}(f)(\kappa, \xi_2, ..., \xi_m) = \alpha$.
Define $g : \kappa \to \kappa$ by setting $g(\nu) = \sup_{\nu_1,...,\nu_n \leq \nu} f(\nu_1, ..., \nu_n)$. Then, still $g(\nu) < \nu_1^{+3}$, and hence, $\alpha \leq j_{E\restriction\xi}(g)(\xi') < \xi$.

This shows that if $k_{\xi'\xi} : M_{E_{\xi'}} \to M_{E\restriction\xi}$ is the canonical embedding, then $k_{\xi'\xi}{}''(\kappa^{+3})^{M_{\xi'}}$ is unbounded in ξ. Clearly, $(\kappa^{+3})^{M_{\xi'}}$ is an ordinal of cardinality and of cofinality κ^+. So, we are done.
□

Lemma 4.2 *Let $\xi, \kappa^{++} < \xi < \rho$ be a successor generator of E, and let ξ' be the last generator below ξ or just κ^{++}, if ξ was the first generator above κ^{++}.*

Then

1. *there is* $G_{\xi'}$ *which is* $M_{E_{\xi'}}-$*generic for* $(Cohen(\kappa^{+3},2))^{M_{E_{\xi'}}}$*;*

2. $k_{\xi'\xi}{}''G_{\xi'}$ *generates a* $M_{E\restriction\xi}-$*generic for* $(Cohen(\kappa^{+3},2))^{M_{E\restriction\xi}}$*;*

Proof. (1) is clear, since there are only κ^+-many dense subsets to meet and $M_{E_{\xi'}}$ is closed under its $\kappa-$sequences.
Let us deal with (2). Let D be a dense open subset of $Cohen(\kappa^{+3},2)$ in $M_{E\restriction\xi}$.
Then there are $m<\omega$, $f:[\kappa]^m\to V_\kappa$, such that for every $\nu_1,...,\nu_m<\kappa, f(\nu_1,...,\nu_m)$ is a dense open subset of $Cohen(\nu_1^{+3},2)$, and $\xi_2,...,\xi_m\leq\xi'$ such that $j_{E\restriction\xi}(f)(\kappa,\xi_2,...,\xi_m)=D$.
Define $g:\kappa\to V_\kappa$ by setting $g(\nu)=\bigcap_{\nu_1,...,\nu_n\leq\nu}f(\nu_1,...,\nu_n)$. Then, still $g(\nu)$ is a dense open subset of $Cohen((\nu_1^{+3},2)$, and hence, $j_{E\restriction\xi}(g)(\xi')\subseteq D$. So, we are done.
□

Let $\langle\rho_i\mid i<\kappa^+\rangle$ be an increasing sequence of all generators of E above κ^{++}.

We assume that for every $i<\kappa^+$, $E\restriction\rho_i\lhd E\restriction\rho_{i+1}$, say, for example, that the ground model is $\mathcal{K}$.
Then, the construction of a $M_{E\restriction\rho_i}-$generic for $(Cohen(\kappa^{+3},2))^{M_{E\restriction\rho_i}}$ of Lemma 4.2, can be preformed inside $M_{E\restriction\rho_{i+1}}$.

Instead of a single Cohen function from κ^{+3} to κ^{+3}, we will use the following variation:

$$Q=\{t\mid t:[\kappa^{+3}]^2\to\kappa^{+3},|t|\leq\kappa^{++},(\alpha,\beta)\in\mathrm{dom}(t)\to t(\alpha,\beta)>\alpha\}.$$

The construction of generics will be preformed inside $M_{E\restriction\rho_{i+1}}$, as it was described above. At limit steps we will take the union.
Let $i<\kappa^+$ be a successor and describe a small change of a generic in the construction inside $M_{E\restriction\rho_{i+1}}$.
Denote the generic function constructed for $M_{E\restriction\rho_i}$ by F_i. Then, F_i is in $M_{E\restriction\rho_{i+1}}$, since the construction was in $M_{E\restriction\rho_{i+1}}$, and so, it is a condition there. Start with it. Proceed to ρ_i. It is the largest generator of $E\restriction\rho_{i+1}$. The relevant α's are $<\rho_i$, i.e., if $\alpha\geq\rho_i$, then $t(\alpha,\rho_\xi)$ is undefined for every $\xi\leq i$. Hence we can put all (α,ρ_i)'s ($\alpha<\rho_i$) into a single condition.
Arrange the right values for such pairs, i.e. $F_{i+1}(\langle\alpha,h\rangle,\rho_i)=\gamma$ iff $j_{E\restriction\rho_i}(h)(\alpha)=\gamma$, for any $h:\kappa\to\kappa$.
The assumption $E\restriction\rho_i\lhd E\restriction\rho_{i+1}$ is used for this. Just $E\restriction\rho_i\in M_{E\restriction\rho_{i+1}}$, and so, $j_{E\restriction\rho_i}$ is definable there.

Then extend the resulting condition to $M_{E\restriction\rho_{i+1}}-$generic or just replace in the previously constructed generic such values.

We will use $\{f_{\kappa\rho_{i+2}} \mid i < \kappa^+\}$ in order to form a generating family. Thus, let

$$A_i = \{\nu < \kappa \mid f_{\kappa\rho_{i+2}}(\nu^+) = 0\},$$

for every $i < \kappa^+$.[2]
Let $B_\beta = \{\nu < \kappa \mid f_{\kappa\beta}(\nu^+) = 0\}$, for every $\beta < \rho$.
By changing generics in the ultrapower, we can assume that

$$f_{j(\kappa)j(\beta)}(\kappa^+) = 0 \text{ iff } \beta = \rho_i, \text{ for some } i < \kappa^+.$$

Then B_β will be in the ultrafilter iff $\beta = \rho_i$, for some $i < \kappa^+$, i.e. $B_\beta = A_i$.

Let us explain now how the family $\{A_i \mid i < \kappa^+\}$ will be turn into generating (mod Cub_κ).

For every $\alpha_0 < \rho$, $h : \kappa \to \kappa$ in V and $\alpha < \rho$ consider the following set:

$$Y^h_{\alpha_0\alpha} = \{\nu < \kappa \mid h(f_{\kappa\alpha_0}(\nu)) = f_{\kappa\alpha}(\nu)\}.$$

Let $F : \rho^2 \to \rho$ be a final generic with values $F(\langle\alpha_0, h\rangle, \rho_i)$ altered in the right way.[3]
Namely, if $F(\langle\alpha_0, h\rangle, \rho_i) = \gamma$, then $j(h)(j(f_{\kappa\alpha_0})(\kappa)) = j(f_{\kappa\gamma})(\kappa)$.
At the next stage we will add clubs $\cap Inc$ which avoid $B_\alpha \setminus Y^h_{\alpha_0\gamma}$, whenever $F(\langle\alpha_0, h\rangle, \alpha) = \gamma$.
In particular such clubs will be forced into $A_i \setminus Y^h_{\alpha_0\gamma}$, whenever $F(\langle\alpha_0, h\rangle, \rho_i) = \gamma$.

Lemma 4.3 *For every generator ζ of E, E_ζ is generated by $f_{\kappa\zeta}$ applied to the closure of A_i's under finite intersection (mod clubs $\cap$ Inc).*

Proof. Let $B \in E_\zeta$. There are $h_1, h_2 \in V$ functions from κ to κ such that

$$B = \{\tau < \kappa \mid h_1(\tau) = h_2(\tau)\}.$$

Take any $\alpha, \zeta < \alpha < \rho$ and consider

$$Y^{h_1}_{\zeta\alpha} \cap Y^{h_2}_{\zeta\alpha} = \{\nu < \kappa \mid h_1(f_{\kappa\zeta}(\nu)) = f_{\kappa\alpha}(\nu) = h_2(f_{\kappa\zeta}(\nu))\}.$$

Then,

$$f_{\kappa\zeta}{}'' Y^{h_1}_{\zeta\alpha} \cap Y^{h_2}_{\zeta\alpha} \subseteq B.$$

[2]It is possible to reserve κ^+-many sets like this - one for every $h : \kappa \to \kappa$ in V. Also, it is possible to use other similar families of sets as generating families.

[3]$\rho^2 = \{(\alpha, \beta) \mid \alpha < \beta < \rho\}$.

□

Let us deal now with a general case, i.e. take an arbitrary set in $\mathcal{U}$. Consider the requirements (1)-(3) from the first section.
The treatment of all three is basically the same, so, let show how to arrange (1).

We need to deal with $h : \kappa \to P_{\kappa+1}$, such that for every $\nu < \kappa$, $h(\nu) \in P_{(\nu+1,\kappa+1)}$.
Let $\langle h_\gamma \mid \gamma < \rho_i \rangle$ be an enumeration in $M_{E \restriction \rho_i}$ of all such h's and if $i < i'$, then $\langle h_\gamma \mid \gamma < \rho_{i'} \rangle$ extends $\langle h_\gamma \mid \gamma < \rho_i \rangle$. We use here that $\rho_i = (\kappa^{+3})^{M_{E \restriction \rho_i}}$ is the critical point of the canonical embedding $k_{ii'} : M_{E \restriction \rho_i} \to M_{E \restriction \rho_{i'}}$.
Also, $E \restriction \rho_i \lhd E \restriction \rho_{i'}$.
Set

$$B_\gamma = \{\nu < \kappa \mid h_\gamma(\nu) \in G \cap P_{(\nu+1,\kappa+1)}\}.$$

Use, in M_E, the forcing Q defined above. Let F_1 be M_E-generic subset of Q constructed as above, only now we change the values in order to have the following:

$$\text{If } F_1(\alpha, \rho_i) = \gamma, \text{ then } j(h_\gamma)(\kappa) \in G^* \cap P_{(\kappa+1,j(\kappa)+1)}.$$

Again, $E \restriction \rho_i \lhd E \restriction \rho_{i'} \lhd E$ is used for this.

A club$\cap Inc$ which avoids $A_i \setminus B_\gamma$ will be forced, in order to arrange $A_i \subseteq B_\gamma$ mod $Cub_\kappa \kappa$.

We will need to repeat the above process κ^+-many times in order "to catch the tail". Note that the forcings for adding clubs$\cap Inc$ are very close to Cohen forcings. Actually, after adding a single club which avoids inaccessibles (clearly, we are not going to force it) they turn to be equivalent to Cohens. This allows to repeat the construction above.

5 Preparation forcing

We will now go back in order to define a preparation forcing. It will be Easton support iteration of the forcings described in the previous sections. The main point is that in M_E, the needed forcing will appear over κ, by elementarity. So, we are able to reflect down and to use the same definition (with different ordinal parameters) at stages of preparation.

6 Some remarks related to the consistency strength

6.1 Basic assumptions

Suppose that E is a $(\kappa, \kappa^{++})-$extender in $\mathcal{K}$, in V, $2^\kappa = \kappa^{++}$ and there is a normal ultrafilter U such that:

1. $U \cap \mathcal{K} = E_\kappa$,
2. $j_U \restriction \mathcal{K} = j_E$,
3. U is generated mod Cub_κ by κ^+-many sets.

Note that if $\mathcal{A}$ is a generating (mod Cub_κ) family of U, then $\mathcal{A} \notin M_U$, since $U \notin M_U$. In particular, E_κ cannot generate U, since $E_\kappa \in M_E \subseteq M_U$.

6.2 Constructing in M_U functions different mod U

Let $\langle f_{\kappa\xi} \mid \xi < \kappa^{++}\rangle \in M_U$ be a list of pairwise different functions from κ to κ.
Fix $\nu \mapsto \langle f_{\nu\xi} \mid \xi < \nu^{++}\rangle$ which represents $\langle f_{\kappa\xi} \mid \xi < \kappa^{++}\rangle$ in M_U. Note that $\langle\langle \nu, \langle f_{\nu\xi} \mid \xi < \nu^{++}\rangle\rangle \mid \nu < \kappa\rangle \in V_{\kappa+1} \subseteq M_U$.
Let $\eta < \kappa^{++}$ and $\sigma : \kappa \to \kappa$.

Set

$$X_{\eta\sigma} = \{\nu < \kappa \mid f_{\kappa\eta} \restriction \nu = f_{\nu\sigma(\nu)}\}.$$

The next lemma follows from the elementarity of j_U:

Lemma 6.1 *$X_{\eta\sigma} \in U$ iff $[\sigma]_U = \eta$.*

Proof. ($\Rightarrow$) Suppose that $X_{\eta\sigma} \in U$. Then, in M_U, $\kappa \in j_U(X_{\eta\sigma})$. Hence,

$$\kappa \in j_U(X_{\eta\sigma}) = \{\nu < j_U(\kappa) \mid j_U(f_{\kappa\eta}) \restriction \nu = f_{\nu\sigma(\nu)}\}.$$

Then, $j_U(f_{\kappa\eta}) = f_{j_U(\kappa)j_U(\eta)}$ and $f_{j_U(\kappa)j_U(\eta)} \restriction \kappa = f_{\kappa j_U(\sigma)(\kappa)}$.
Note that for every $\xi < \kappa^{++}$, $j_U(f_{\kappa\xi}) \restriction \kappa = f_{\kappa\xi}$, since $\kappa = \text{crit}(j_U)$.
Hence, $[\sigma]_U = j_U(\sigma)(\kappa) = \eta$, since the functions in the list $\langle f_{\kappa\xi} \mid \xi < \kappa^{++}\rangle$ are different.

($\Leftarrow$) Suppose that $[\sigma]_U = \eta$.
As above, we have $j_U(f_{\kappa\eta}) \restriction \kappa = f_{\kappa\eta}$, since $\kappa = \text{crit}(j_U)$. So, $f_{j_U(\kappa) j_U(\eta)} \restriction \kappa = f_{\kappa j_U(\sigma)(\kappa)}$. Hence,

$$\kappa \in \{\nu < j_U(\kappa) \mid j_U(f_{\kappa\eta}) \restriction \nu = f_{\nu\sigma(\nu)}\} = j_U(X_{\eta\sigma}).$$

So, $X_{\eta\sigma} \in U$.
□

Now, for every $\eta < \kappa^{++}$, let $\sigma_\eta : \kappa \to \kappa$ be such that $X_{\eta\sigma_\eta} \in U$. By the lemma, $[\sigma_\eta]_U = \eta$.
For every $\eta < \kappa^{++}$, pick a generator A_η of U such that $A_\eta \subseteq X_{\eta\sigma_\eta}$ mod Cub_κ.
We have only κ^+ such generators, hence there are $S \subseteq \kappa^{++}, |S| = \kappa^{++}$ and $A \in U$ such that $A \subseteq X_{\eta_i\sigma_{\eta_i}}$ mod Cub_κ, for every $i \in S$.
We have $X_{\eta\sigma}$'s and A in M_U, so inside M_U it is possible to construct by induction a set $S^* \subseteq \kappa^{++}, |S| = \kappa^{++}$ such that $A \subseteq X_{\eta_i\sigma_{\eta_i}}$ mod Cub_κ, for every $i \in S^*$.
Then $\langle \sigma_{\eta_i} \mid i \in S^* \rangle$ and $\langle \eta_i \mid i \in S^* \rangle$ are in M_U and for every $i < i' \in S^*$,

$$[\sigma_{\eta_i}]_U = \eta_i < \eta_{i'} = [\sigma_{\eta_{i'}}]_U.$$

So, we have a sequence of $\kappa^{++}-$many functions from κ to κ in M_U which represent increasing, mod U, sequence of ordinals $< \kappa^{++}$, which also belongs to M_U.

6.3 Constructing generators in M_U

Let $\langle f_\alpha \mid \alpha < \kappa^{++} \rangle$ be a list in M_U of functions κ to κ in M_U which represent an increasing, mod U, sequence of ordinals $< \kappa^{++}$.

For every $\alpha_0, \alpha < \kappa^{++}$ and $h : \kappa \to \kappa, h \in \mathcal{K}$, consider

$$Y^h_{\alpha_0\alpha} = \{\nu < \kappa \mid \nu^{++} > h(f_{\alpha_0}(\nu)) \geq f_\alpha(\nu)\}.$$

Note that for every $A \in U, \alpha_0 < \kappa^{++}$

$$|\{\alpha < \kappa^{++} \mid \exists h \in {}^\kappa\kappa \cap \mathcal{K} \quad A \subseteq Y^h_{\alpha_0\alpha} \text{ mod } Cub_\kappa\}| \leq \kappa^+.$$

Just otherwise, we will have $[f_\alpha]_U = [f_\beta]_U$, for some $\alpha \neq \beta$, and actually, for $\kappa^{++}-$many of them.

Assume that there is a list $\langle B_\xi \mid \xi < \kappa^{++} \rangle$ in M_U of subsets of κ such that a generating family (mod Cub_κ) of U is listed below some $i^* < \kappa^{++}$.

This holds, for example if $2^\kappa = \kappa^{++} = (2^\kappa)^{M_U}$.

Now, in M_U let $\mathbb{B}$ be a set consisting of all $B \subseteq \kappa$ such that

1. B is stationary,

2. $\forall \alpha_0 < \kappa^{++}, |\{\alpha < \kappa^{++} \mid \exists h \in {}^\kappa\kappa \cap \mathcal{K} \quad B \subseteq Y^h_{\alpha_0 \alpha} \bmod Cub_\kappa\}| \leq \kappa^+$,

3. the index of B is below i^*.

Note that $\mathbb{B}$ covers the generating family of U.
For every $\gamma < \kappa^{++}$, let γ^* be the least such that

$$\forall \delta \geq \gamma^* \forall h \in {}^\kappa\kappa \cap \mathcal{K} \forall B \in \mathbb{B} \quad B \not\subseteq Y^h_{\gamma\delta} \bmod Cub_\kappa\}.$$

It exists, since otherwise, for every $\gamma^* < \kappa^{++}$ there will be $\delta_{\gamma^*} \geq \gamma^*, h_{\gamma^*}, B_{\gamma^*}$ such that $B_{\gamma^*} \subseteq Y^{h_{\gamma^*}}_{\gamma\delta_{\gamma^*}} \bmod Cub_\kappa\}$. But $|\mathbb{B}| \leq \kappa^+$, so we can freeze $h_{\gamma^*}, B_{\gamma^*}$ and this will lead to a contradiction due to the second condition of the definition of $\mathbb{B}$.

Now, we define inside M_U an increasing sequence $\langle \alpha_\xi \mid \xi < \kappa^{++} \rangle$ of ordinals below κ^{++} such that $\alpha_\xi > \alpha^*_\zeta$, for every $\zeta < \xi$. Then $\langle f_{\alpha_\xi} \mid \xi < \kappa^{++}$ is a limit ordinal $\rangle$ will represent (mod U) an unbounded set of generators of E.
If, initially, we have

$$\langle \langle \alpha, [f_\alpha]_U \rangle \mid \alpha < \kappa^{++} \rangle \in M_U,$$

then

$$\langle \langle \xi, [f_{\alpha_\xi}]_U \rangle \mid \xi < \kappa^{++} \rangle \in M_U,$$

since $\langle \langle \xi, \alpha_\xi \rangle \mid \xi < \kappa^{++} \rangle \in M_U$. So, an unbounded set of generators of E belongs to M_U.
Note that such a set cannot be in M_E since M_E is a direct limit of $M_{E \restriction \alpha+1}, \alpha < \kappa^{++}$ and it cannot have a pre-image in non of $M_{E \restriction \alpha+1}$'s.

It is not hard to construct a model such that a set of functions which represent the generators of E is inside, however no unbounded set of generators of E belongs to M_U. Just add $\kappa^{++}-$many Cohen functions $\langle f_{\kappa\beta} \mid \beta < \kappa^{++} \rangle$ and change the value of $j(f_{\kappa\beta})(\kappa)$ to the $\beta-$th generator of E, for every $\beta < \kappa^{++}$. Then $\langle f_{\kappa\beta} \mid \beta < \kappa^{++} \rangle$ will be in M_U and each $f_{\kappa\beta}$ will represent (mod U) the $\beta-$th generator of E.

Fix $\eta < \kappa^{++}$. Note that there are $\kappa^{++}-$many $\mathcal{K}$ which generator is α_η.
Enumerate all extenders F of $\mathcal{K}$ which generator is α_η and $j_F(\alpha_\eta) < \alpha^*_\eta$. Let $\langle F^\eta_i \mid i < \kappa^+ \rangle$ be the least (in $\mathcal{K}$) such enumeration.
Each of this extenders is a $-$point, so there are $Z^\eta_i \subseteq \kappa$ such that $Z^\eta_i \in F^\eta_i$ and for every $i < i'$, $Z^\eta_i \notin F^\eta_{i'}$.

For every $\xi < \kappa^{++}$, let $i_\xi < \kappa^+$ be such that $E \restriction \alpha_\xi + 1 = F^\xi_{i_\xi}$.
Now, find a set A_ξ from a small generating family of U such that $f_{\alpha_\xi}{}'' A_\xi \subseteq Z^\xi_{i_\xi}$ (mod Cub_κ).

Note that i_ξ is the last one for which such set A_ξ exists, due to the separation property of the sequence $\langle F_i^\xi \mid i < \kappa^+ \rangle$.
There will be a single A which works for every ξ in a set S of cardinality κ^{++}.
If such S can be found in M_U, then this will lead to a contradiction, since then E will be reconstructible in M_U from $\langle F_{i_\xi}^\xi \mid \xi \in S \rangle$.

7 Additional results

Starting with two extenders E_1, E_2 such that $E_1(\kappa) \neq E_2(\kappa)$, instead of a single E in the above construction, it is possible to obtain two normal ultrafilters $\mathcal{U}_1, \mathcal{U}_2$ such that $\mathcal{U}_1$ is generated by κ^+-many sets mod CUB_κ and every family which generates $\mathcal{U}_2$ mod CUB_κ has cardinality κ^{++}.
In order to achieve this, we pick some $S \in E_1(\kappa) \setminus E_2(\kappa)$ and preform the above construction only over S leaving $\kappa \setminus S$ untouched.

In a similar fashion it is possible to obtain a model with $2^\kappa = \kappa^{+3}$ and three normal ultrafilters $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ which are generated exactly by κ^+, κ^{++} and $\kappa^{+3}-$many sets respectevely mod CUB_κ.

Note that, at least to our best knowledge, such possibilities mod bounded are open even assuming supercompact cardinals.

8 Concluding remarks

The following remains open:

Let U be a normal ultrafilter over κ which is generated by less than 2^κ sets mod Cub_κ.

Question 1. *Is it possible that $\mathcal{K}$ exists, no cardinal of $\mathcal{K}$ changes its cofinality in V and cardinals of $M_U, \mathcal{K}$ which are less or equal to 2^κ are the same.*

Question 3. *What about generating families mod bounded?*

For example, suppose that U is generated by less than 2^κ sets mod bounded. Does this imply an inner model with a strong cardinal?

References

[1] M. Gitik and S. Shelah, On density of box products,