

# A model with a precipitous ideal but without normal one.

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## Abstract

Starting with a measurable cardinal  $\kappa$  of the Mitchell order  $\kappa^{++}$  we construct a model with a precipitous ideal on  $\aleph_1$  but without normal precipitous ideals. This answers a question by T. Jech and K. Prikry. In the constructed model there are no  $Q$ -point precipitous filters on  $\aleph_1$ , i.e. those isomorphic to extensions of  $Cub_{\aleph_1}$ .

## 1 Introduction and Basic ideas

Precipitous ideals were introduced by T. Jech and K. Prikry [10]. A  $\kappa$ -complete ideal  $I$  on  $\kappa$  is *precipitous* if the generic ultrapower  $V \cap {}^\kappa V/G$  is well-founded for every generic ultrafilter  $G \subseteq I^+$ . Precipitousness can be viewed as a weakening of measurability which is compatible with small cardinals. Given a  $\kappa$ -complete ultrafilter  $U$  over a measurable  $\kappa$  there always exists a normal ultrafilter  $U^*$  over  $\kappa$  as well. Just take a function  $f : \kappa \rightarrow \kappa$  which represents  $\kappa$  in the ultrapower by  $U$ , i.e.  $[f]_U = \kappa$ , and project  $U$  using  $f$ .  $U^* := \{X \mid f^{-1} \upharpoonright X \in U\}$  will be a normal ultrafilter over  $\kappa$ . There are two obstacles that prevent implementation of the same approach to a precipitous filter  $F$ . The first is that it does not necessarily exist a *single* function that represents  $\kappa$  in a generic ultrapower (the choice of such function may depend on particular condition, i.e. a set in  $F^+$ . In [5] an example of a precipitous filter without a normal filter below it in the Rudin-Keisler order was given). It is easy to fix this by simply restricting  $F$  to its positive set that decides a function  $f$  which represents  $\kappa$  in the ultrapower. The second much more serious obstacle is that the projection of  $F$  (or its restriction that decides  $f$ ) by  $f$  need not in general be precipitous. The first example of this type was given by R. Laver [11] using a supercompact cardinal. Later in [5] an example

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using only a measurable cardinal was produced.

Let us briefly explain the idea used in this construction since it will be relevant for the present one. We started with a GCH model with a measurable  $\kappa$  and a normal ultrafilter  $U$  over it. Using the Backward Easton iteration (in order to preserve the measurability of  $\kappa$ )  $\kappa^+$ -many Cohen functions  $\langle f_\beta \mid \beta < \kappa^+ \rangle$  from  $\kappa$  to  $\kappa$  were added. A precipitous filter  $F$  was defined over  $\kappa^2$  and its generic embedding extended  $i_{02}$  the second iterated ultrapower of  $U$ , i.e.  $i_{01} : V \rightarrow M_1 \simeq {}^\kappa V/U, \kappa_1 = i_{01}(\kappa), i_{12} : M_1 \rightarrow M_2 \simeq {}^{\kappa_1} M_1/i_{01}(U)$ , and  $i_{02} = i_{12} \circ i_{01} : V \rightarrow M_2$ . The projection  $F^*$  of  $F$  to a normal filter was not precipitous due to the following reason. Each of the Cohen functions  $f_\beta(\beta < \kappa^+)$  was a potential candidate to represent  $\kappa_1$  in a generic ultrapower, but no particular order was specified between them. This eventually insured a non-precipitousness of  $F^*$ , since having  $\kappa^+$ -many of them allowed always to produce a decreasing sequence consisting of  $f_\beta$ 's in generic ultrapowers by  $F^*$ . It is not by chance that this  $f_\beta$ 's were candidates for the function that represents  $\kappa_1$ . Further results starting with Section 4 of [5], then 2.4 of [6] and [7] suggest that the only ordinals have a chance to produce an ill-foundness must be of the form  $\kappa_\alpha$  (i.e. critical points of iterated ultrapowers).

On the other hand if the number critical points is too small (i.e. the length of the iteration is too short), say at most  $\kappa^+$ , then (at least under GCH-type assumptions and in absence of too large cardinals) there will be always normal precipitous filters by [6], [7].

So it is natural to try the following:

start with a normal ultrafilter  $U$  over  $\kappa$  and iterate it  $\kappa^{++}$ -many times. This will create critical points  $\langle \kappa_\alpha \mid \alpha < \kappa^{++} \rangle$ . Next add  $\kappa^{++}$ -blocks each consisting  $\kappa^+$ -many Cohen function from  $\kappa$  to  $\kappa$ . Arrange this (say by adding clubs) so that the functions of  $\alpha$ -th block are the candidates to represent  $\kappa_\alpha$ . Note that by J.-P. Levinski [12] no assumptions beyond the measurability are needed in order to blow up the power and to preserve precipitousness. A problematic point is that his arguments and their extensions in [4] produce large (size  $\kappa^{++}$ ) antichains which allow using method of [6] to construct normal precipitous ideals.

A way around this obstacle will be to collapse in advance a measurable  $\kappa$  to  $\aleph_1$  and to rely on  $Col(\omega, < \kappa^{++})$  (which satisfies  $\kappa^{++}$ -c.c.) in order to extend generically relevant embeddings (namely  $i_{0\kappa^{++}}$ ).

An additional problem with this approach is that the models of the iteration (iterated ultrapowers)  $M_\alpha, \alpha \leq \kappa^{++}$  are very unclosed. Thus already starting with  $M_\omega$  we loose a closure even under  $\omega$ -sequences. For example  $\langle \kappa_n \mid n < \omega \rangle \notin M_\alpha$  for every  $\alpha, \omega \leq \alpha \leq \kappa^{++}$ . This turns out to be bad once we try to change values of Cohen functions in order to insure the

right representations. Remember that  $\alpha$ -th block should provide potential candidates for functions representing  $\kappa_\alpha$ . Which makes values changing a crucial issue.

The way to gain the missing closure will be to switch from dealing with a single normal ultrafilter and its iterated ultrapowers to  $\kappa^{++}$ -many ultrafilters. We will use an initial model the model from [3] which have a Rudin-Keisler increasing sequence of ultrafilters of the length  $\kappa^{++}$ .

The actual construction will be as follows. We start with a model of [3] (assuming that there is a measurable  $\kappa$  of the Mitchell order  $\kappa^{++}$ ). Collapse  $\kappa$  to  $\aleph_1$  and add  $\kappa^{++}$ -blocks of Cohen functions. Organize suitable generics using  $Col(\omega, < \kappa^{++})$ . Define an  $\infty$ -semi precipitous filter over  $\aleph_1$ . Add clubs in order to turn it into  $Cub_{\aleph_1}$  restricted to a certain set. Next argue that there is no normal precipitous filters on  $\aleph_1$  (and, hence once started with the core model for  $o(\kappa) = \kappa^{++}$ , no normal precipitous at all). Finally a precipitous filter on  $\aleph_1$  will be constructed using methods of [6].

## 2 Construction of the model

Start with a measurable  $\kappa$ ,  $o(\kappa) = \kappa^{++}$  and preform the forcing of [3] to produce a GCH model with a Rudin - Keisler increasing commutative sequence of  $Q$ -points ultrafilters  $\langle U_\alpha \mid \alpha < \kappa^{++} \rangle$  over  $\kappa$ . For every  $\alpha < \kappa^{++}$  let  $M_{\alpha+1}$  be the transitive collapse of  $V^\kappa/U_\alpha$  and  $i_{0\alpha+1}$  the corresponding elementary embedding. Set  $M_0 = V$ ,  $i_{00} = id$ . For every limit  $\alpha \leq \kappa^{++}$  let  $M_\alpha$  be the direct limit of  $M_\beta, \beta < \alpha$  and  $\langle i_{\beta\alpha} \mid \beta < \alpha \rangle$  the corresponding elementary embeddings.

Denote  $i_{0\alpha}(\kappa)$  by  $\kappa_\alpha$ . Then  $\kappa^+ < \kappa_\alpha < \kappa^{++}$ , for every  $\alpha < \kappa^{++}$ , and  $\kappa_{\kappa^{++}} = \kappa^{++}$ .

We have by [3],  ${}^\kappa M_{\kappa^{++}} \subseteq M_{\kappa^{++}}$ .

Assume that the core model exists. It will be used with a single exception (Lemma 3.1) only to compute the Mitchell order of cardinals. It is possible to avoid this assumption (for example if one likes to do the construction below a supercompact) by using the coherent sequence of measures of [3] directly and computing the Mitchell rank of cardinals according to this sequence. The final conclusion will be that there are no precipitous filters which extend  $Cub_{\aleph_1} + \{\nu < \kappa \mid o(\nu) = 0\}$ .

Denote by  $\mathcal{K}_\alpha$  the core model of  $M_\alpha$ .

We collapse  $\kappa$  to  $\aleph_1$  by  $Col(\omega, < \kappa)$ . Then let us add  $\kappa^{++}$  blocks of functions from  $\kappa$  to  $\kappa$  as follows: in  $V^{Col(\omega, < \kappa)}$  set

$$Cohen(\kappa, \kappa^{++} \times \kappa^+) = \{f \mid |f| < \kappa, f \text{ is a partial function from } \kappa \times (\kappa^{++} \times \kappa^+) \text{ to } \kappa\}.$$

Let  $G_1 \subseteq Col(\omega, < \kappa)$  be generic over  $V$  and  $G_2 \subseteq Cohen(\kappa, \kappa^{++} \times \kappa^+)$  be a generic over  $V[G_1]$ . Set  $\bar{G}_2 = \bigcup G_2$  and for every  $\alpha < \kappa^{++}, \beta < \kappa^+, \nu < \kappa$  let  $f_{\alpha\beta}(\nu) = \bar{G}_2(\nu, \alpha, \beta)$ .

Denote by

$$F_\alpha = \{f_{\alpha\beta} \mid \beta < \kappa^+\}.$$

It will be our  $\alpha$ -th block of functions.

Let show that the elementary embedding  $i_{0\kappa^{++}} : V \rightarrow M_{\kappa^{++}}$  extends (generically). It is possible to use [4], but we need to do this in a different fashion in order to preserve  $\kappa^{++}$ . One of the issues will be to generate  $M_{\kappa^{++}}^{Col(\omega, < \kappa^{++})}$ -generic subset of  $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$  using  $Col(\omega, < \kappa^{++})$ .

## 2.1 Constructing generics

For each  $\alpha < \kappa^{++}$  let us add only  $(o(\kappa_\alpha))^{\mathcal{K}_{\kappa^{++}}} < \kappa^{++}$  blocks of Cohen functions over  $M_{\alpha+1}^{Col(\omega, < i_{0\alpha}(\kappa))}$ . More generally  $M_{\alpha+1}^{Col(\omega, < i_{0\alpha}(\kappa))}$ -generic subsets of iterations of length  $(o(\kappa_\alpha))^{\mathcal{K}_{\kappa^{++}}}$  need to be constructed, since we will add also certain clubs further. Dealing with them is very similar, so let us concentrate on blocks of Cohen functions.

Note that  $(o(\kappa_\alpha))^{\mathcal{K}_{\kappa^{++}}} = (o(\kappa_\alpha))^{\mathcal{K}_{\kappa_{\alpha+1}}} = i_{0\alpha}(\alpha)$ . Also  $i_{0\kappa^{++}}(\kappa^{++}) = \bigcup_{\alpha < \kappa^{++}} i_{0\kappa^{++}}(\alpha)$  and  $i_{0\kappa^{++}}(\alpha) = i_{\alpha\kappa^{++}}(i_{0\alpha}(\alpha))$ .

Let  $P = i'_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$  or the image of a  $\kappa$ -support iteration of forcings of cardinality  $\kappa$  with closure properties, where  $i'_{0\kappa^{++}} : V^{Col(\omega, < \kappa)} \rightarrow M_{\kappa^{++}}^{Col(\omega, < (\kappa^{++})^V)}$  is the obvious extension extension of  $i_{0\kappa^{++}}$ . Note that in  $V^{Col(\omega, < \kappa)}$  we have  $\kappa = \aleph_1$  and  $\kappa^{++} = \aleph_3$ . In order to simplify the notation, let us use  $i_{0\kappa^{++}}$  to denote also  $i'_{0\kappa^{++}}$  and by  $\kappa^{++}$  we will mean  $(\kappa^{++})^V$ .

We would like to construct an  $M_{\kappa^{++}}^{Col(\omega, < \kappa^{++})}$ -generic subset of  $i_{0\kappa^{++}}(Cohen(\kappa, \kappa^{++} \times \kappa^+))$  in  $V^{Col(\omega, < \kappa) * Cohen(\kappa, \kappa^{++} \times \kappa^+) * Col(\omega, [\kappa, \kappa^{++})}$ . Note that  $G \subseteq P$  will be  $M_{\kappa^{++}}^{Col(\omega, < \kappa^{++})}$ -generic iff for every subforcing  $Q \in M_{\kappa^{++}}^{Col(\omega, < \kappa^{++})}$  of  $P$  of cardinality at most  $\kappa^{++}$  we have  $G \cap Q$  is  $M_{\kappa^{++}}^{Col(\omega, < \kappa^{++})}$ -generic subset of  $Q$ . This follows since each antichain has size at most  $\kappa^{++}$ .

Fix such subforcing  $Q$ . Pick the least  $\alpha < \kappa^{++}$  such that  $Q = i_{\alpha+1, \kappa^{++}}(Q^*)$ , for some subforcing  $Q^*$  of  $P_{\alpha+1}$ , where  $P_{\alpha+1}$  is a corresponding forcing over  $M_{\alpha+1}^{Col(\omega, < \kappa_{\alpha+1})}$ . Now we use  $Col(\omega, \{((\kappa_{\alpha+1})^+)^{M_{\alpha+1}}\})$  to produce  $M_{\alpha+1}^{Col(\omega, < \kappa_{\alpha+1})}$ -generic for  $Q^*$ . Actually we will split  $Col(\omega, \{((\kappa_{\alpha+1})^+)^{M_{\alpha+1}}\})$  into  $\kappa^+$  many collapses– simplest way will just to use instead of  $Col(\omega, \{((\kappa_{\alpha+1})^+)^{M_{\alpha+1}}\})$  the  $Col(\omega, [((\kappa_{\alpha+1})^+)^{M_{\alpha+1}}, ((\kappa_{\alpha+1})^+)^{M_{\alpha+1}} + \kappa^+])$ . Let us replace its part below  $\kappa_\alpha$  as follows. Consider  $i_{\alpha\alpha+1}'' o^{\mathcal{K}_{\alpha+1}}(\kappa_\alpha)$  intersected with the coordinates of  $Q^*$ . There will be a set  $A \subseteq o^{\mathcal{K}_{\alpha+1}}(\kappa_\alpha)$  in  $M_\alpha$  of cardinality at most  $\kappa_\alpha$  such that  $i_{\alpha\alpha+1}'' A$  gives this intersection. Now  $i_{\alpha\alpha+1}'' A \in M_{\alpha+1}$  since  $M_{\alpha+1}$  is an ultrapower of  $M_\alpha$ . We replace

the functions of the generic object on coordinates in  $i_{\alpha\alpha+1}''A$  restricted to  $\kappa_\alpha$  by the one constructed for  $M_\alpha$ . Replace each  $i_{\alpha\alpha+1}(\nu)$  by  $\nu$ , for coordinates  $\nu$  in  $A$ .

Let us describe the process more carefully. Note that at limit stages  $\alpha < \kappa^{++}$  we will not have  $M_\alpha^{Col(\omega, < \kappa_\alpha)}$ -generics due to the fact that lengths of the iterations are increasing. Basically we would like to take the union of things created below  $\alpha$ . It will not be possible to make this literally, instead we will do this for pieces of size at most  $\kappa_\alpha$ . So the main work will be done at successor stages.

Let  $\alpha < \kappa^{++}$ . Consider in  $M_{\alpha+1}^{Col(\omega, < \kappa_{\alpha+1})}$  the forcing

$$i_{0\alpha+1}(Cohen(\kappa, \alpha \times \kappa^+)) = (Cohen(\kappa_{\alpha+1}, i_{0\alpha+1}(\alpha) \times \kappa_{\alpha+1}^+))^{M_{\alpha+1}^{Col(\omega, < \kappa_{\alpha+1})}}.$$

It is  $\kappa_{\alpha+1}^+$ -c.c. forcing of size  $\kappa_{\alpha+1}^+$ . Use  $Col(\omega, \{(\kappa_{\alpha+1}^+)^{M_{\alpha+1}}\})$  to build an  $M_{\alpha+1}^{Col(\omega, < \kappa_{\alpha+1})}$ -generic subset  $G'_{\alpha+1}$ . For each  $\beta < \kappa^+$ ,  $G'_{\alpha+1}$  produces a Cohen function

$$f'_{i_{0\alpha+1}(\alpha), i_{0\alpha+1}(\beta)} : \kappa_{\alpha+1} \rightarrow \kappa_{\alpha+1}.$$

Let us change  $f'_{i_{0\alpha+1}(\alpha), i_{0\alpha+1}(\beta)} \upharpoonright \kappa_\alpha + 1$ . Fix  $\beta < \kappa^+$ . First we use  $Col(\omega, \{(\kappa_\alpha^+)^{M_{\alpha+1}} + \beta\})$  to change  $f'_{i_{0\alpha+1}(\alpha), i_{0\alpha+1}(\beta)} \upharpoonright \kappa$  to  $f_{\alpha\beta}$ , i.e. actually we factor the forcing  $Col(\omega, \{(\kappa_\alpha^+)^{M_{\alpha+1}} + \beta\})$  by  $f_{\alpha\beta}$ . Next use  $Col(\omega, \{(\kappa_\alpha^+)^{M_{\alpha+1}} + \kappa^+ + \beta\})$  to produce an  $M_\alpha^{Col(\omega, < \kappa_\alpha)}$ -generic Cohen function  $h_\beta : \kappa_\alpha + 1 \rightarrow \kappa_\alpha + 1$  which extends  $f_{\alpha\beta}$ . Finally replace  $f'_{i_{0\alpha+1}(\alpha), i_{0\alpha+1}(\beta)} \upharpoonright \kappa_\alpha + 1$  in  $f'_{i_{0\alpha+1}(\alpha), i_{0\alpha+1}(\beta)}$  by  $h_\beta$ .

Note that this way the order between the Cohen function of the  $\alpha$ -th block  $F_\alpha$  is defined generically. Namely values  $i_{0\alpha+1}(f_{\alpha\beta})(\kappa)$  depend on collapses  $Col(\omega, \{(\kappa_\alpha^+)^{M_{\alpha+1}} + \kappa^+ + \beta\})$ . This will be crucial for the argument showing non existence of a normal precipitous filter. Make this replacement for every  $\beta < \kappa^+$ . Let  $G_{\alpha+1}$  be the result. Note that  $G_{\alpha+1}$  is still an  $M_{\alpha+1}^{Col(\omega, < \kappa_{\alpha+1})}$ -generic, since for every  $X \in M_{\alpha+1}$  of size at most  $\kappa_{\alpha+1}$  there, we have  $|i_{0\alpha+1}''\kappa^+ \cap X| \leq \kappa$ .

Let us explain what will be done at limit stages. Consider the case  $\alpha = \omega$ . In a general case only the notation are bit more complicated.

Then for each  $n < \omega$  we have  $G_n$  which is  $M_n^{Col(\omega, < \kappa_n)}$ -generic for  $Cohen(\kappa_n, n + 1 \times \eta_n)$  obtained from  $Col(\omega, \{(\kappa_n^+)^{M_n}\})$ , where  $\eta_0 < \kappa^+$  is some fixed ordinal,  $\eta_n = i_{0n}(\eta_0)$  (remember that we deal with at most  $\kappa_n$  many things at once, since over  $\kappa_\omega$  we would like to combine together only pieces of sizes  $\leq \kappa_\omega$ ). Suppose for simplicity that  $\eta_0 = \kappa$  otherwise we will need to fix an enumeration of  $\eta_0$  in  $V$  and deal with its images.

We assume by induction that each  $G_{n+1}$  was constructed such that  $G_{n+1} \upharpoonright \kappa_n \times n \times \kappa_n = G_n$ . Namely, first  $M_1^{Col(\omega, < \kappa_1)}$ -generic subset  $G'_1$  of  $Cohen(\kappa_1, 2 \times \kappa_1)$  obtained from

$Col(\omega, \{(\kappa_1^+)^{M_1}\})$  with the second coordinate restricted to  $\kappa$  being generic over  $V^{Col(\omega, <\kappa^++1)}$ . This puts restrictions on collapses, but we will force actually with collapse over Cohen functions, so from a point of view of  $M_1^{Col(\omega, <\kappa_1)}$  only the collapse matters.

Also require that  $G'_1$  will be read from  $Col(\omega, \{(\kappa_1^+)^{M_1}\})$  as part of a bigger generic object—namely those for  $Cohen(\kappa_1, \kappa^+ \times \kappa_1)$ . This is needed since eventually we would like to put together all the pieces (over  $\kappa^{++}$ ). So the compatibility of them is necessary.

Then we replace in  $G'_1$  the first coordinate restricted to  $\kappa$  by  $G_0$  ( where  $G_0$  was obtained from  $Col(\omega, \{\kappa^+\})$ ). The result will be the desired  $G_1$ . Continue similar for each  $n > 1$ .

Note that the sequence  $\langle G_n \mid n < \omega \rangle$  is defined from  $\langle \kappa_n \mid n < \omega \rangle$ , and so it is in  $M_{\omega+1}^{Col(\omega, <\kappa_{\omega+1})}$ .

It is crucial that  $\langle \kappa_n \mid n < \omega \rangle$  is in  $M_{\omega+1}$ . Actually  $M_{\omega+1}$  thinks of it as Prikry generic sequence for  $\kappa_\omega$ .

Set  $G_\omega = \bigcup_{n < \omega} G_n$ . Such  $G_\omega$  will not be  $M_\omega^{Col(\omega, <\kappa_\omega)}$ , but its restriction to first  $n$ -coordinates will generic over  $M_\omega^{Col(\omega, <\kappa_\omega)}$  separately for every  $n < \omega$ .

Now we generate  $M_{\omega+1}^{Col(\omega, <\kappa_{\omega+1})}$ -generic subset  $G'_{\omega+1}$  of  $Cohen(\kappa_{\omega+1}, \omega + 1 \times \kappa_{\omega+1})$  from  $Col(\omega, \{(\kappa_{\omega+1}^+)^{M_{\omega+1}}\})$ . Replace  $G'_{\omega+1} \upharpoonright \kappa_\omega \times \omega \times \kappa_\omega$  by  $G_\omega$ . This will be our  $G_{\omega+1}$ .

Note that the way generics are generated from the collapses reflects down to  $\kappa$ . We will need, in order to keep  $\kappa^{++}$ -c.c. of the final thing, to add clubs into following sets

$$\{\nu < \kappa \mid f_{00} \upharpoonright \nu \text{ is build from the generic for } Col(\omega, \{(\nu^+)^V\})$$

in the same fashion as it was done above },

$$\{\nu < \kappa \mid \exists \xi < \kappa^+ f_{10} \upharpoonright \nu \text{ is build from the generic for } Col(\omega, \{(f_{0\xi}(\nu)^+)^V\})$$

in the same fashion as it was done above }, etc.

## 2.2 $\infty$ -semi precipitous filter

Recall that  $G_1 \subseteq Col(\omega, <\kappa)$  is generic over  $V$  and  $G_2 \subseteq Cohen(\kappa, \kappa^{++} \times \kappa^+)$  is generic over  $V[G_1]$ . For every  $\alpha < \kappa^{++}$  we denote the  $\alpha$ -th block of Cohen functions by  $F_\alpha = \{f_{\alpha\beta} \mid \beta < \kappa^+\}$ .

Next we would like to make functions of  $F_\alpha$  to be those that have a chance to represent  $\kappa_\alpha$  in a generic ultrapower. For this purpose let us add clubs by forcing over  $V[G_1, G_2]$ .

Force with  $< -\kappa$  support iteration a club into

$$\{\nu < \kappa \mid f_{\alpha\beta}(\nu) < f_{\alpha'\beta'}(\nu)\},$$

for every  $\alpha < \alpha' < \kappa^{++}$  and  $\beta, \beta' < \kappa^+$ .

In addition for each  $f \in {}^\kappa\kappa \cap V$  and  $\beta < \kappa^+$  force a club into

$$\{\nu < \kappa \mid f(\nu) < f_{0\beta}(\nu)\}.$$

Also, for each  $n < \omega$ ,  $f \in {}^{[\kappa]^n}\kappa \cap V$ ,  $\alpha_1 < \dots < \alpha_n < \alpha < \kappa^{++}$ ,  $\beta_1, \dots, \beta_n, \beta < \kappa^+$  we force a club into

$$\{\nu < \kappa \mid f(f_{\alpha_1\beta_1}(\nu), \dots, f_{\alpha_n\beta_n}(\nu)) < f_{\alpha\beta}(\nu)\}.$$

It is insured this way that in any normal filter the block  $F_\alpha$  of functions will strictly above each of the blocks  $F_{\alpha'}$  with  $\alpha' < \alpha$ .

If a generic ultrapower with such a filter is well-founded, then functions from  $F_\alpha$  will represent ordinals inside the interval  $[\kappa_\alpha, \kappa_{\alpha+1})$ . Note that each ordinal there is of a form  $i_{0,\alpha+1}(f)(\kappa_{\alpha_1}, \dots, \kappa_{\alpha_n})$  for some  $f \in {}^{[\kappa]^n}\kappa \cap V$ ,  $\alpha_1 < \dots < \alpha_n \leq \alpha$ .

Let  $G_3$  be a corresponding generic object. It easy to reorganize both adding blocks of Cohen functions and forcings this clubs into one iteration of the length  $\kappa^{++}$ .

Let us define a filter  $F$  over  $\kappa$  in  $V[G_1, G_2, G_3]$  as follows:

$$F = \{X \subseteq \kappa \mid 0_{Col(\omega, < i_{0\kappa^{++}}(\kappa))/G_1 * G_2 * G_3} \Vdash \kappa \in i_{0\kappa^{++}}(\tilde{X})\}.$$

Then

$$F^+ = \{X \subseteq \kappa \mid \exists p \in Col(\omega, < i_{0\kappa^{++}}(\kappa))/G_1 * G_2 * G_3 \quad p \Vdash \kappa \in i_{0\kappa^{++}}(\tilde{X})\}.$$

The next lemma is immediate.

**Lemma 2.1**  *$F$  is  $\infty$ -semi precipitous<sup>1</sup> filter with a witnessing forcing  $Col(\omega, < \kappa^{++})/G_1 * G_2 * G_3$  and with a generic embedding which extends  $i_{0\kappa^{++}}$ .*

## 2.3 Adding clubs

Next we would like to add clubs to sets in  $F$ , and then to extensions of  $F$  as it was done in Jech-Magidor-Mitchell-Prikry [9], but picking generics over  $M_{\kappa^{++}}$  using the procedure above. It should be done a bit more careful in order to keep a resulting generic embedding to extend  $i_{0\kappa^{++}}$ . Just note that the set  $\{\kappa_\alpha \mid \alpha < \kappa^{++}\}$  is a club, so we cannot force a club for example into the set  $\{\nu < \kappa \mid o^\mathcal{K}(\nu) = 0\} \in F$  and still to extend  $i_{0\kappa^{++}}$ , since for every  $\alpha, 0 < \alpha < \kappa^{++}$ ,  $o^{\mathcal{K}_{\kappa^{++}}}(\kappa_\alpha) = \alpha > 0$ .

<sup>1</sup>We refer to [1] and [2] for this notion. The meaning is that after forcing with  $Col(\omega, < \kappa^{++})/G_1 * G_2 * G_3$  it is possible to extend  $i_{0\kappa^{++}}$  to an elementary embedding of  $V[G, G_2, G_3]$  into a transitive model.

So let us add clubs only to subsets  $X$  of  $\kappa$  in  $F$  such that for a final segment of  $\alpha$ 's below  $\kappa^{++}$ ,

$$0_{Col(\omega, < i_{0\kappa^{++}}(\kappa))/G_1 * G_2 * G_3} \Vdash \kappa_\alpha \in i_{0\kappa^{++}}(\mathcal{X}).$$

Then, in particular, the set  $\{\nu < \kappa \mid \nu \text{ is an accessible ordinal in } V\}$  will be nonstationary.

We would like to arrange a situation where each filter which extends  $Cub_{\aleph_1}$  concentrates on the set  $\{\nu < \kappa \mid o(\nu) = 0\}$ . The simplest way to guarantee this is to shoot a club into  $\{\nu < \kappa \mid o(\nu) = 0\}$ . But doing it will destroy  $i_{0\kappa^{++}}$  completely, since  $\{\kappa_\alpha \mid 0 < \alpha < \kappa^{++}\}$  is a club in  $\kappa^{++}$  and it is disjoint to the image of a club in  $\{\nu < \kappa \mid o(\nu) = 0\}$ . So adding such a club will cause a changing cofinality of  $\kappa^{++}$  to  $\omega$  and eventually will produce a normal precipitous filter. An other way is to shoot clubs disjoint from  $\{\nu < \kappa \mid o(\nu) = 1\}$ ,  $\{\nu < \kappa \mid o(\nu) = 2\}$  etc., and this way prevent the ground model ultrafilters  $U_2, U_3$ , etc. to have a normal extensions. This works nicely, but unfortunately not for all ground model ultrafilters. Remember that we have a sequence of  $\kappa^{++}$ -many of them. So up-repeat points must be on the sequence, i.e. for some  $\alpha < \kappa^{++}$ , for every  $X \in U_\alpha$  there will be  $\beta > \alpha$  (even  $\kappa^{++}$  many of them) with  $X \in U_\beta$ . Shooting clubs for them will not work.

The actual approach will be as follows. We add together with blocks  $F_\alpha$ 's of Cohen functions an additional sequence  $\langle g_\alpha \mid \alpha < \kappa^{++} \rangle$  of Cohen functions from  $\kappa$  to  $\kappa$  (it is possible just to use the first function of each block instead). Require that for each  $\nu < \kappa$  with  $o(\nu) > 0$ ,  $g_\alpha(\nu) < \nu^{++}$ .

Now, as in 2.1, by changing values of generics, we insure that

1.  $\langle i_{0\kappa^{++}}(g_\alpha)(\kappa) \mid \alpha < \kappa^{++} \rangle$  is an increasing sequence,
2. for every  $\alpha < \kappa^{++}$ ,  $\langle i_{0\kappa^{++}}(g_\gamma)(\kappa_\alpha) \mid \gamma < \alpha \rangle$  is an increasing sequence of ordinals below  $o(\kappa_\alpha)$ .

Now, for every  $\gamma < \alpha < \kappa^{++}$ , we force clubs into the set

$$\{\nu < \kappa \mid g_\gamma(\nu) < g_\alpha(\nu)\}$$

and into the compliment of the set

$$\{\nu < \kappa \mid o(\nu) > 0, o(\nu) \leq g_\alpha(\nu)\}.$$

Note  $\kappa$  and final segments of  $\kappa_\xi$ 's are in the images under  $i_{0\kappa^{++}}$  of this sets.

We will show further in Lemma 3.1 that this does the job.

Clubs will be added to any set of the form  $A \cup \{\nu < \kappa \mid o(\nu) > 0\}$  with  $A \in U_0$  etc. In addition clubs would be forced into the sets that describe the way of generating Cohen functions from collapses (at the end of 2.1).

In addition, we will have that for any  $\omega$ -sequences  $\langle \alpha_n \mid n < \omega \rangle, \langle \beta_n \mid n < \omega \rangle$  consisting of different elements of  $\kappa^{++}$  and of  $\kappa^+$  respectively, the following set contains a club

$$\bigcup_{n < \omega} \{\nu < \kappa \mid f_{\alpha_n \beta_n}(\nu) > f_{\alpha_n \beta_{n+1}}(\nu)\}.$$

Thus, by genericity of the collapse there always will be  $p$  in a generic object such that for some  $n < \omega$

$$p \Vdash f_{i_{0\kappa^{++}}(\alpha_n) i_{0\kappa^{++}}(\beta_n)}(\kappa) > f_{i_{0\kappa^{++}}(\alpha_n) i_{0\kappa^{++}}(\beta_{n+1})}(\kappa).$$

In general, suppose that we have a sequence  $\langle A_\eta \mid \eta < \kappa^{++} \rangle$  of  $F$ -positive sets. Let  $\langle p_\eta \mid \eta < \kappa^{++} \rangle$  be a sequence of conditions in  $Col(\omega, < \kappa)$  such that for every  $\eta < \kappa^{++}$ ,  $p_\eta \Vdash \kappa \in \check{i}^*(A_\eta)$ . Shrink if necessary the sequence  $\langle p_\eta \mid \eta < \kappa^{++} \rangle$  in order to form a  $\Delta$ -system. If the kernel of it is empty, then for any sequence  $\langle \eta_n \mid n < \omega \rangle$  of different ordinals below  $\kappa^{++}$  the set  $\bigcup_{n < \omega} A_{\eta_n}$  contains a club.

Similar, if  $p$  is a kernel and for some  $A \subseteq \kappa$  we have  $p = \|\kappa \in \check{i}^*(A)\|^{Col(\omega, < \kappa^{++})}$ , then the set  $(\kappa \setminus A) \cup \bigcup_{n < \omega} A_{\eta_n}$  contains a club.

Denote by  $\tilde{F}$  such extension of  $F$ .

Let  $\tilde{V}$  be a generic extension obtained by this forcing. Note that a bit nicer way to organize the iteration used here will be an inductive definition of an iteration of the length  $\kappa^{++}$ . Thus suppose that at a stage  $\alpha < \kappa^{++}$  we have a forcing  $P_{< \alpha}$  defined with a generic subset  $G_{< \alpha}$ . Force the  $\alpha$ -th block of Cohen functions  $F_\alpha$ . Now over  $V[G_1, G_{< \alpha}, F_\alpha]$  we add clubs relevant for the blocks of Cohen functions  $\langle F_\gamma \mid \gamma \leq \alpha \rangle$ . This will be  $Q_\alpha$ . Its length is below  $\kappa^{++}$ . Set  $P_{< \alpha+1} = P_{< \alpha} * Q_\alpha$ .

### 3 No normal precipitous ideals

We will prove a bit more general statement– in  $\tilde{V}$  there is no precipitous filter on  $\aleph_1$  which contains  $Cub_{\aleph_1}$ , i.e. which is a  $Q$ -point filter. If the initial ground model had no large cardinals above  $\kappa$  (say  $o(\kappa) = \kappa^{++}$  but nothing more), then there will be no normal precipitous filters at all.

Suppose otherwise. Let  $H$  be a precipitous filter over  $\kappa = \aleph_1^{\tilde{V}}$  which includes  $Cub_{\aleph_1}$ .

**Lemma 3.1** *Assume that there is no inner model with a strong cardinal. Then  $H \supseteq U_0$ .*

**Remark.** Here is actually the only place where the core model is used in an essential way. If we restrict ourself initially only to filters which extend  $Cub_{\aleph_1} + \{\nu < \kappa \mid o(\nu) = 0\}$ , then no  $\mathcal{K}$  is needed.

*Proof.* Let  $G \subseteq H^+$  be a generic ultrafilter and  $j : \tilde{V} \rightarrow N$  be the corresponding generic elementary embedding. Now  $j \upharpoonright \mathcal{K}$  is an iterated ultrapower of  $\mathcal{K}$ . Let  $E_\alpha$  be the extender (actually a measure) used to move  $\kappa$  in this iteration. If  $\alpha = 0$ , then we are done. Suppose otherwise. Consider  $\delta = (o(\kappa))^{\mathcal{K}^N}$ . Then  $\delta = (o(\kappa))^{\mathcal{K}_\alpha} = \alpha$ , where  $\mathcal{K}_\alpha = Ult(\mathcal{K}, E_\alpha)$ . Now  $\alpha < (\kappa^{++})^{\mathcal{K}_\alpha} < (\kappa^{++})^{\mathcal{K}} = \kappa^{++\tilde{V}}$ , since a club was forced into  $\{\nu < \kappa \mid o(\nu) < (\nu^{++})^{\mathcal{K}}\}$ . Consider now the sequence  $\langle j(g_\xi)(\kappa) \mid \xi < (\kappa^{++})^{\tilde{V}} \rangle$ . It is an increasing sequence of ordinals of the order type  $(\kappa^{++})^{\tilde{V}}$ . But  $\delta < (\kappa^{++})^{\tilde{V}}$ , hence there is  $\eta < (\kappa^{++})^{\tilde{V}}$  with  $\delta \leq j(g_\eta)(\kappa)$ . By elementarity, then  $\{\nu < \kappa \mid o(\nu) > 0, o(\nu) \leq g_\eta(\nu)\} \in H^+$ . This is impossible since we added a club into its compliment and  $H \supseteq Cub_{\aleph_1}$ .

□

Note that if  $\delta$  (the Mitchell order of  $\kappa$  as computed in the ground model of  $N$ ) is less than  $(\kappa^{++})^{\tilde{V}}$ , then the argument above still provides the desired conclusion.

By the lemma we have in particular that  $\{\nu < \kappa \mid o(\nu) = 0\}$  is in  $H$ . Hence  $H \supseteq Cub_{\aleph_1} + \{\nu < \kappa \mid o(\nu) = 0\}$ , since  $\tilde{F} + \{\nu < \kappa \mid o(\nu) = 0\}$  is  $Cub_{\aleph_1} + \{\nu < \kappa \mid o(\nu) = 0\}$ . Let us further assume that every  $A \in H^+$  under the consideration is automatically a subset of  $\{\nu < \kappa \mid o(\nu) = 0\}$ .

For each  $\alpha < \kappa^{++}$ , pick a set  $A_\alpha \in H^+$  deciding (in the forcing with  $H$ -positive sets) which of the functions in  $F_\alpha = \{f_{\alpha\beta} \mid \beta < (\kappa^+)^V\}$  represents a smallest ordinal in a generic ultrapower.

Denote by  $\beta_\alpha$  the index of this function, i.e.

$$A_\alpha \Vdash_{H^+} \forall \beta < (\kappa^+)^V \quad [f_{\alpha\beta_\alpha}]_{\mathcal{G}(H^+)} \leq [f_{\alpha\beta}]_{\mathcal{G}(H^+)}.$$

It is tempting to look at  $\beta < (\kappa^+)^V$  such that  $f_{\alpha\beta}$  represents  $\kappa_\alpha$  in the generic ultrapower, but it need not exist in general since the functions of previous blocks can spread and represent  $\kappa_\gamma$ 's of higher blocks. So  $f_{\alpha\beta}$  may represent  $\kappa_\gamma$  for  $\gamma > \alpha$ . Still it is possible to deal with the least such  $\gamma$  and continue under the lines below.

Note that all but finitely many of  $A_\alpha$ 's are not in  $H$ , since otherwise, by a countable completeness of  $H$ , we will have in  $\tilde{V}$  a countable sequence  $\langle \alpha_n \mid n < \omega \rangle$  with  $f_{\alpha_n\beta_{\alpha_n}}$  being the least function of the block  $F_{\alpha_n}$ . By countable completeness of  $H$  then the set

$$\{\nu < \kappa \mid \forall n < \omega \quad f_{\alpha_n\beta_{\alpha_n}}(\nu) < f_{\alpha_n\beta_{\alpha_n+1}}(\nu)\}$$

is in  $H$ . But its complement contains a club. Contradiction.

We can assume that for every  $\alpha < \kappa^{++}$  the set  $A_\alpha$  is not in  $H$ . Actually the argument below will not be effected even if some of  $A_\alpha$ 's are in  $H$ .

We will use now the fact that the iteration  $P_{<\kappa^{++}}$  over  $V[G_1]$  (adding blocks of Cohen functions and clubs) satisfies  $\kappa^+$ -c.c. So each of  $A_\alpha$ 's depends only on at most  $\kappa$ -many Cohen functions and clubs.

Let  $\alpha < \kappa^{++}$ . Consider the characteristic function  $\chi_\alpha : \kappa \rightarrow 2$  of  $A_\alpha$ .

There are Cohen functions  $\{f_{\eta,\xi} \mid \langle \eta, \xi \rangle \in a_\alpha\}$ , clubs  $\{c_\eta \mid \eta \in \text{dom}(a_\alpha)\}$  and a continuous function  $t_\alpha \in V[G_1]$ , such that  $|a_\alpha| \leq \kappa$  and  $\chi_\alpha = t_\alpha(\langle f_{\eta,\xi} \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \{c_\eta \mid \eta \in \text{dom}(a_\alpha)\} \rangle)$ .

We can assume, by shrinking if necessary, that for some  $t$  each  $t_\alpha = t$ , and that  $\langle \text{dom}(a_\alpha) \mid \alpha < \kappa^{++} \rangle$  form a  $\Delta$ -system.

Now, for every  $\alpha < \kappa^{++}$ , pick some  $\beta_\alpha^* \in (\kappa^+)^V \setminus \text{ran}(a_\alpha) \cup \{\beta_\alpha\}$ . Consider the set

$$B_\alpha = \{\nu < \kappa \mid o(\nu) > 0 \text{ or } (o(\nu) = 0 \text{ and } f_{\alpha\beta_\alpha^*}(\nu) < f_{\alpha\beta_\alpha}(\nu))\}.$$

Then  $B_\alpha$  and even  $A_\alpha \cap B_\alpha$  are  $\tilde{F}$ -positive, by the choice of  $\beta_\alpha^*$ . Recall that we have

$$H \supseteq \tilde{F} + \{\nu < \kappa \mid o(\nu) = 0\} = \text{Cub}_{\aleph_1} + \{\nu < \kappa \mid o(\nu) = 0\}.$$

So each of  $A_\alpha$ 's is  $\tilde{F} + \{\nu < \kappa \mid o(\nu) = 0\}$ -positive.

On the other hand the set  $A_\alpha \cap B_\alpha$  is in the ideal dual to  $H$ , since  $A_\alpha$  forces in the forcing with  $H^+$  that  $f_{\alpha\beta}$  is the least function of the block  $F_\alpha$ .

**Case 1.** *The kernel of the  $\Delta$ -system is empty.*

For each  $\alpha < \kappa^{++}$  pick a condition  $p_\alpha$  in the collapse  $\text{Col}(\omega, < \kappa^{++})$  of the smallest size which forces “ $\kappa \in i_{0\kappa^{++}}(\underline{A}_\alpha \cap \underline{B}_\alpha)$ ” which means more explicitly:

$$i_{0\kappa^{++}}(t)(\langle \dot{\imath}(f_{\eta,\xi}) \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \dot{\imath}(c_\eta) \mid \eta \in \text{dom}(a_\alpha) \rangle)(\kappa) = 1 \text{ and } \dot{\imath}(f_{\alpha\beta_\alpha^*})(\kappa) < \dot{\imath}(f_{\alpha\beta_\alpha})(\kappa).$$

The value of  $i_{0\kappa^{++}}(t)(\langle \dot{\imath}(f_{\eta,\xi}) \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \dot{\imath}(c_\eta) \mid \eta \in \text{dom}(a_\alpha) \rangle)$  on  $\kappa$  depends only on an initial segment of  $\langle \dot{\imath}(f_{\eta,\xi}) \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \dot{\imath}(c_\eta) \mid \eta \in \text{dom}(a_\alpha) \rangle$ . Assume that  $p_\alpha$  already decides it, i.e. there is  $\xi_\alpha < \kappa^{++}$  such that  $p_\alpha$  forces “ $i_{0\kappa^{++}}(t)(\langle \dot{\imath}(f_{\eta,\xi}) \upharpoonright \xi_\alpha \mid \langle \eta, \xi \rangle \in a_\alpha \rangle, \langle \dot{\imath}(c_\eta) \upharpoonright \xi_\alpha \mid \eta \in \text{dom}(a_\alpha) \rangle)(\kappa) = 1$ ”. Let  $S \subseteq \kappa^{++}$  such that for every  $\alpha < \alpha'$  in  $S$  we have  $\xi_\alpha < \alpha'$ .

Now, if  $\alpha < \alpha'$  are in  $S$ , then  $\min(\text{dom}(p_{\alpha'}))$  is above  $\max(\text{dom}(p_\alpha))$ . This follows from the way of constructing generics from the collapse in 2.1.

Let  $\langle \alpha_n \mid n < \omega \rangle$  be an increasing sequence of elements of  $S$  (in  $\tilde{V}$ ). Then one of  $\langle p_{\alpha_n} \mid n < \omega \rangle$  always will in any generic subset of the collapse. But this means that  $\bigcup_{n < \omega} A_{\alpha_n} \cap B_{\alpha_n} \in$

$\tilde{F} + \{\nu < \kappa \mid o(\nu) = 0\} = \text{Cub}_{\aleph_1} + \{\nu < \kappa \mid o(\nu) = 0\} = H$ . But remember that  $A_\alpha \cap B_\alpha$  is in the dual to  $H$  ideal, for every  $\alpha < \kappa^{++}$ . This contradicts  $\sigma$ -completeness of  $H$ .

**Case 2.** *The kernel of the  $\Delta$ -system is not empty.*

We may assume that all  $\text{ran}(a_\alpha)$ ,  $\alpha < \kappa^{++}$  are the same, since there are only  $\kappa^+$  many possibilities for them. Pick  $\eta < \kappa^+$  to be an ordinal which includes all the ranges. Assume for simplicity that they are  $\eta$ .

Let  $a$  be the kernel of  $\langle \text{dom}(a_\alpha) \mid \alpha < \kappa^{++} \rangle$ . Then  $|a| \leq \kappa$ . Suppose that  $|a| = \kappa$ . The case  $|a| < \kappa$  is similar. Let  $a = \{\rho_\tau \mid \tau < \kappa\}$  be its enumeration in  $\tilde{V}$ . Denote  $\{\rho_\tau \mid \tau < \nu\}$  by  $a \upharpoonright \nu$ , for every  $\nu < \kappa$ .

Now we have

$$A_\alpha = \{\nu < \kappa \mid t(\langle f_{\alpha\beta} \mid \alpha \in a, \beta < \eta \rangle, \langle c_{\alpha\beta} \mid \alpha \in a, \beta < \eta \rangle, \langle f_{\alpha\beta} \mid \alpha \in \text{dom}(a_\alpha) \setminus a, \beta < \eta \rangle, \langle c_{\alpha\beta} \mid \alpha \in \text{dom}(a_\alpha) \setminus a, \beta < \eta \rangle)(\nu) = 1, o(\nu) = 0\}.$$

Consider also the following set

$$Z = \{\nu < \kappa \mid o(\nu) > 0 \text{ or } o(\nu) = 0 \text{ and there is no Cohens * Clubs generic } \vec{f}, \vec{c} \text{ such that } t(\langle f_{\alpha\beta} \mid \alpha \in a, \beta < \eta \rangle, \langle c_{\alpha\beta} \mid \alpha \in a, \beta < \eta \rangle, \vec{f}, \vec{c})(\nu) = 1\}.$$

Clearly  $A_\alpha \cap Z = \emptyset$ , for every  $\alpha < \kappa^{++}$ .

Let  $B_\alpha$ 's be as above (even we take  $\beta_\alpha^*$  always to be  $\eta$ ). The choice of  $\eta$  insures that  $A_\alpha \cap B_\alpha \in F^+$ , but clearly not in  $H^+$ , since on  $\nu$ 's in  $A_\alpha \cap B_\alpha$  we have  $f_{\alpha\eta}(\nu) < f_{\alpha\beta_\alpha}(\nu)$ .

Using  $\kappa^{++}$ -c.c. of  $\text{Col}(\omega, < \kappa^{++})$  find  $\xi_\alpha < \kappa^{++}$  such that the weakest condition forces that

$$\kappa \in i_{0\kappa^{++}}(Z) \text{ or } i_{0\kappa^{++}}(t(\langle i_{0\kappa^{++}}(f_{\alpha\beta}) \upharpoonright \xi_\alpha \mid \alpha \in a, \beta < \eta \rangle, \langle i_{0\kappa^{++}}(c_{\alpha\beta}) \upharpoonright \xi_\alpha \mid \alpha \in a, \beta < \eta \rangle, \langle i_{0\kappa^{++}}(f_{\alpha\beta}) \upharpoonright \xi_\alpha \mid \alpha \in \text{dom}(a_\alpha) \setminus a, \beta < \eta \rangle, \langle i_{0\kappa^{++}}(c_{\alpha\beta}) \upharpoonright \xi_\alpha \mid \alpha \in \text{dom}(a_\alpha) \setminus a, \beta < \eta \rangle))(\kappa) = 1.$$

Find  $S \subseteq \kappa^{++}$  such that for every  $\alpha < \alpha'$  in  $S$  we have  $\xi_\alpha < \alpha'$ . Let  $\langle \alpha_n \mid n < \omega \rangle$  be an increasing sequence of elements of  $S$  (in  $\tilde{V}$ ). The construction of generics for blocks of Cohen functions in 2.1 implies then that the set  $Z \cup (\bigcup_{n < \omega} (A_{\alpha_n} \cap B_{\alpha_n}))$  contains a club. This is impossible, since  $A_\alpha$ 's are  $H$ -positive, disjoint to  $Z$ , each  $A_{\alpha_n} \cap B_{\alpha_n}$  is in the ideal dual to  $H$  and  $H$  is countably complete.

## 4 A construction of a precipitous ideal.

Let show now how to construct a precipitous filter in  $\tilde{V}$ .

The basic idea will be to use  $\kappa^{++}$  as an additional generator. We continue the iteration from  $M_{\kappa^{++}}$  to  $M_{\kappa^{++}+1}$  using  $i_{0\kappa^{++}}(U_0)$ .

Deal now with a two dimensional analog  $F_2$  of  $F$ :

$$F_2 = \{X \subseteq \kappa^2 \mid 0_{Col(\omega, < i_{0\kappa^{++}+1}(\kappa)) / G_1 * G_2 * G_3} \Vdash \langle \kappa, \kappa^{++} \rangle \in \underset{\sim}{i}(X)\}.$$

The crucial difference between  $F$  and  $F_2$  is that  $F_2$  has anti-chains of a length  $\kappa^{++}$ . Thus we have here  $Col(\omega, \{\kappa^{++}\})$ . Let  $\underset{\sim}{H}$  be an  $F_2^+$  name of a generic function from  $\omega$  onto  $\kappa^{++}$ . Fix a maximal antichain of elements  $\langle A_\xi \mid \xi < \kappa^{++} \rangle$  of  $F_2^+$  which decide  $\underset{\sim}{H}(0)$ .

Now we turn to an inductive process of extending  $F_2$  similar to those used in [6]. Let  $\langle X_\alpha \mid \alpha < \kappa^{++} \rangle$  be an enumeration of all  $F_2$ -positive subsets of  $\kappa$  (in  $V[G_0, G_1]$ ).

Start with  $n = 0$ . Define a sequences of ordinals  $\langle \xi_{\langle \alpha \rangle} \mid \alpha < \kappa^{++} \rangle$  and filters  $\langle F_{\langle \alpha \rangle} \mid \alpha < \kappa^{++} \rangle$  by induction as follows. Let  $\alpha < \kappa^{++}$ . In case  $\alpha < \omega$ , we shift everything by 2.

If there is  $\xi < \kappa^{++}$  such that  $\xi \neq \xi_{\langle \beta \rangle}$ , for each  $\beta < \alpha$  and  $X_\alpha \cap A_\xi \in F_2^+$ , then let  $\xi_{\langle \alpha \rangle}$  be the least such  $\xi$ . Extend  $F_2$  to  $F_2 + X_\alpha \cap A_{\xi_{\langle \alpha \rangle}}$ . Then pick  $\beta_\alpha$  to be the least  $\beta < \kappa^+$  such that for each  $k < \omega$ ,  $\beta_1, \dots, \beta_k \in \kappa^+ \setminus \{\beta\}$  and  $t \in {}^{\kappa^k} \kappa \cap V$  the set

$$\{\langle \nu_0, \nu_1 \rangle \mid f_{\xi_{\langle \alpha \rangle} \beta}(\nu_0) < t(f_{\xi_{\langle \alpha \rangle} \beta_1}(\nu_0), \dots, f_{\xi_{\langle \alpha \rangle} \beta_k}(\nu_0))\} \in F_2 + X_\alpha \cap A_{\xi_{\langle \alpha \rangle}}.$$

Note that always there is such  $\beta_\alpha$ , since  $F_2 + X_\alpha \cap A_{\xi_{\langle \alpha \rangle}}$  cannot decide which of the functions  $f_{\xi_{\langle \alpha \rangle} \beta'}$ , for  $\beta' < \kappa^+$ , is the least one. Now let  $F_{\langle \alpha \rangle}$  be the  $\aleph_1$ -complete filter generated by  $F_2 + X_\alpha \cap A_{\xi_{\langle \alpha \rangle}}$  together with all the sets  $\{\langle \nu_0, \nu_1 \rangle \mid f_{\xi_{\langle \alpha \rangle} \beta_\alpha}(\nu_0) < t(f_{\xi_{\langle \alpha \rangle} \beta_1}(\nu_0), \dots, f_{\xi_{\langle \alpha \rangle} \beta_k}(\nu_0))\}$ . Intuitively,  $f_{\xi_{\langle \alpha \rangle} \beta_\alpha}$  is the function corresponding to  $\kappa_{\xi_{\langle \alpha \rangle}}$  below  $F_{\langle \alpha \rangle}$ . If  $\xi_{\langle \alpha \rangle} < \omega$ , then actually  $f_{\xi_{\langle \alpha \rangle} \beta_\alpha}$  is the function corresponding to  $\kappa_{\xi_{\langle \alpha \rangle} + 2}$ , rather than to  $\kappa_{\xi_{\langle \alpha \rangle}}$ . Note also that it is impossible to find some  $f_{\xi_{\langle \alpha \rangle} \beta}$  that will correspond to  $\kappa_1$ . Actually the only functions that come into account are  $f_{0\beta}$ 's. Suppose for a moment that  $f_{0\beta^*}$  is such a function. Use the function  $g$  of the lemma above. Then for  $\kappa^{++}$  many  $\alpha$ 's we will have the set  $\{\langle \nu_0, \nu_1 \rangle \in \kappa^2 \mid f_{0\beta}(\nu_1) \geq f_{\alpha 0}(\nu_0)\}$  in  $F_2^+$ , but now  $\{\langle \nu_0, \nu_1 \rangle \in \kappa^2 \mid f_{0\beta}(\nu_1) \geq f_{\alpha 0}(\nu_0)\} = \{\langle \nu_0, \nu_1 \rangle \in \kappa^2 \mid f_{0\beta}(f_{0\beta^*}(\nu_0)) \geq f_{\alpha 0}(\nu_0)\}$  and the compliment of the projection of the last set contains a club for any  $\alpha \geq 1$ .

If there is no  $\xi$  as above then we leave  $\xi_{\langle \alpha \rangle}$  and  $F_{\langle \alpha \rangle}$  undefined.

Note that for each  $\mu < \kappa^{++}$ ,  $A_\mu$  appears in the list  $\langle X_\alpha \mid \alpha < \kappa^{++} \rangle$ . Hence,  $\{\xi_{\langle \alpha \rangle} \mid \alpha < \kappa^{++}, \xi_{\langle \alpha \rangle} \text{ is defined}\} = \kappa^{++}$ . In particular each  $\kappa_\mu$  ( $\mu > 1$ ) has a chance to get a corresponding function.

Set  $F(0) = \bigcap \{F_{\langle \alpha \rangle} \mid F_{\langle \alpha \rangle} \text{ is defined}\}$ . Denote the corresponding dual ideals by  $I_{\langle \alpha \rangle}$  and  $I(0)$ .

The following lemma follows from the construction (or see [6]):

**Lemma 4.1** For each  $X \in F_2^+$ , we have  $X \in F_{\langle \alpha \rangle}$ , for some  $\alpha < \kappa^{++}$ , or the set

$$\{\xi < \kappa^{++} \mid X \cap A_\xi \in F_2^+\}$$

has cardinality at most  $\kappa^+$ .

Next we deal with  $n = 1$ .

Let  $\alpha < \kappa^{++}$  and  $F_{\langle \alpha \rangle}$  be defined. We split  $(\text{mod}(F_{\langle \alpha \rangle}) X_\alpha \cap A_{\xi_\alpha})$  into  $\kappa^{++}$ -many sets which decide  $\widetilde{H}(n_\alpha)$ , where  $n_\alpha$  is the least possible that allows  $\kappa^{++}$ -many possible values. Note that such  $n_\alpha$  exists since otherwise  $F_{\langle \alpha \rangle}^+$  will force that  $\widetilde{H}$  is bounded in  $\kappa^{++}$ , but the filter  $F_{\langle \alpha \rangle}$  is obtained from  $F_2$  basically by deciding the function which corresponds to  $\kappa_{\xi_{\langle \alpha \rangle}}$ .

Let  $\langle A_{\alpha\mu} \mid \mu < \kappa^{++} \rangle$  be a maximal antichain below  $X_\alpha \cap A_{\xi_\alpha}$  in  $F_{\langle \alpha \rangle}^+$  consisting of sets which decide  $\widetilde{H}(n_\alpha)$ .

Repeat the procedure above and define  $\xi_{\langle \alpha\gamma \rangle}, F_{\langle \alpha\gamma \rangle}$ , for  $\gamma < \kappa^{++}$ .

Thus, if there is  $\xi < \kappa^{++}$  such that  $\xi \neq \xi_{\langle \alpha\beta \rangle}$ , for each  $\beta < \gamma$  and  $X_\gamma \cap A_{\alpha\xi} \in F_{\langle \alpha \rangle}^+$ , then let  $\xi_{\langle \alpha\gamma \rangle}$  be the least such  $\xi$ . Extend  $F_{\langle \alpha \rangle}$  to  $F_{\langle \alpha \rangle} + X_\alpha \cap A_{\alpha\xi_{\langle \alpha\gamma \rangle}}$ . Then pick  $\beta_{\langle \alpha\gamma \rangle}$  to be the least  $\beta < \kappa^+$  such that for each  $k < \omega, \beta_1, \dots, \beta_k \in \kappa^+ \setminus \{\beta\}$  and  $t \in {}^{\kappa^k} \kappa \cap V$  the set

$$\{\langle \nu_0, \nu_1 \rangle \mid f_{\xi_{\langle \alpha\gamma \rangle}\beta}(\nu_0) < t(f_{\xi_{\langle \alpha\gamma \rangle}\beta_1}(\nu_0), \dots, f_{\xi_{\langle \alpha\gamma \rangle}\beta_k}(\nu_0))\} \in F_{\langle \alpha \rangle} + X_\alpha \cap A_{\alpha\xi_{\langle \alpha\gamma \rangle}}.$$

Note that always there is such  $\beta_{\langle \alpha\gamma \rangle}$ , since  $F_{\langle \alpha \rangle} + X_\alpha \cap A_{\alpha\xi_{\langle \alpha\gamma \rangle}}$  cannot decide which of the functions  $f_{\xi_{\langle \alpha\gamma \rangle}\beta'}$ , for  $\beta' < \kappa^+$ , is the least one. Now let  $F_{\langle \alpha\gamma \rangle}$  be the  $\aleph_1$ -complete filter generated by  $F_{\langle \alpha \rangle} + X_\alpha \cap A_{\alpha\xi_{\langle \alpha\gamma \rangle}}$  together with all the sets  $\{\langle \nu_0, \nu_1 \rangle \mid f_{\xi_{\langle \alpha\gamma \rangle}\beta_{\langle \alpha\gamma \rangle}}(\nu_0) < t(f_{\xi_{\langle \alpha\gamma \rangle}\beta_1}(\nu_0), \dots, f_{\xi_{\langle \alpha\gamma \rangle}\beta_k}(\nu_0))\}$ .

If there is no  $\xi$  as above then we leave  $\xi_{\langle \alpha\gamma \rangle}$  and  $F_{\langle \alpha\gamma \rangle}$  undefined.

Note that for each  $\mu < \kappa^{++}$ ,  $A_{\alpha\mu}$  appears in the list  $\langle X_\tau \mid \tau < \kappa^{++} \rangle$ . Hence,  $\{\xi_{\langle \alpha\gamma \rangle} \mid \gamma < \kappa^{++}, \xi_{\langle \alpha\gamma \rangle} \text{ is defined}\} = \kappa^{++}$ . In particular each  $\kappa_\mu (\mu > 1)$  has a chance to get a corresponding function.

Set  $F(1) = \bigcap \{F_{\langle \alpha\gamma \rangle} \mid F_{\langle \alpha\gamma \rangle} \text{ is defined}\}$ . Let  $I(1)$  be the dual ideal.

Continue further and define in a similar fashion  $F_\sigma, I_\sigma, F(n), I(n), \sigma \in {}^{\omega > \kappa^{++}}, n < \omega$ .

Finally set

$$F(\omega) = \text{the closure under } \omega \text{ intersections of } \bigcup_{n < \omega} F_n$$

and

$$I(\omega) = \text{the closure under } \omega \text{ unions of } \bigcup_{n < \omega} I_n.$$

The next two lemmas follow easily from the definitions.

**Lemma 4.2**  $F_2 \subseteq F(0) \subseteq \dots \subseteq F(n) \subseteq \dots \subseteq F(\omega)$  and  $I_2 \subseteq I(0) \subseteq \dots \subseteq I(n) \subseteq \dots \subseteq I(\omega)$ .

**Lemma 4.3**

$$F(\omega) = \{X \subseteq \kappa^2 \mid \exists \langle X_n \mid n < \omega \rangle \forall n < \omega \quad X_n \in F(n) \text{ and } X = \bigcap_{n < \omega} X_n\}$$

and

$$I(\omega) = \{X \subseteq \kappa^2 \mid \exists \langle X_n \mid n < \omega \rangle \forall n < \omega \quad X_n \in I(n) \text{ and } X = \bigcup_{n < \omega} X_n\}.$$

**Lemma 4.4**  $I(\omega)$  is a proper  $\kappa$ -complete ideal over  $\kappa^2$ .

*Proof.* Let  $\langle X_n \mid n < \omega \rangle$  be a sequence such that  $X_n \in I_n$ , for every  $n < \omega$  and  $X = \bigcup_{n < \omega} X_n$ . Assume that each  $X_n$  is  $F_2$ -positive. Consider for every  $n < \omega$  the set

$$Z_n = \{\xi < \kappa^{++} \mid X_n \cap A_\xi \in F_2^+\}.$$

Then, by Lemma 4.1,  $|Z_n| \leq \kappa^+$ . Hence  $|\bigcup_{n < \omega} Z_n| \leq \kappa^+$ . Note that

$$Z := \{\xi < \kappa^{++} \mid X \cap A_\xi \in F_2^+\} = \bigcup_{n < \omega} Z_n$$

and so  $Z$  has cardinality at most  $\kappa^+$  as well.

Pick now any  $\xi \in \kappa^{++} \setminus Z$ . Then  $X \cap A_\xi \notin F_2^+$  which implies that  $I(\omega)$  is a proper ideal, since, in particular,  $X$  never can be  $\kappa^2$ .

□

**Lemma 4.5**  $X \in F(\omega)^+$  iff there is  $\sigma \in {}^\omega \kappa^{++}$  such that  $X \in F_\sigma$ .

*Proof.* Let  $X \in F(\omega)^+$ . Suppose that  $X \notin F_\sigma$ , for any  $\sigma \in {}^\omega \kappa^{++}$ . Set

$$Z_0 = \{\xi < \kappa^{++} \mid X \cap A_\xi \in F_2^+\}.$$

By lemma 4.1,  $|Z_0| \leq \kappa^+$ . Then for every  $\xi \in Z_0$ , set

$$Z_{1\xi} = \{\rho < \kappa^{++} \mid X \cap A_\xi \cap A_{\xi\rho} \in F_{\langle \xi \rangle}^+\}$$

and

$$Z_1 = \bigcup_{\xi \in Z_0} Z_{1\xi}.$$

Then  $|Z_1| \leq \kappa^+$ . Similar define  $Z_n$ , for each  $n < \omega$ .

There is  $\eta_0 < \kappa^{++}$  such that

$$X \Vdash_{F_2^+} \underline{H}(0) < \eta_0,$$

since  $|Z_0| \leq \kappa^+$ . Similar for each  $n < \omega$  there will be  $\eta_n < \kappa^{++}$  such that

$$X \Vdash_{F_2^+} \underline{H}(n) < \eta_n.$$

But then

$$X \Vdash_{F_2^+} \text{ran}(\underline{H}) \text{ is bounded in } \kappa^{++}.$$

Which is impossible by the choice of  $\underline{H}$ . Contradiction.

□

**Lemma 4.6**  $F(\omega)$  is a precipitous filter over  $\kappa^2$ .

*Proof.* It is enough to show that for each  $X \in F(\omega)^+$  and  $\eta < \kappa^{++}$  there is  $Y \subseteq X, Y \in F(\omega)^+$  deciding which function from  $\{f_{\eta\beta} \mid \beta < \kappa^+\}$  will correspond to  $\kappa_\eta$  (again, if  $\eta < \omega$ , then we move by 2). By Lemma 4.5 there is  $\sigma \in {}^\omega > \kappa^{++}$  such that  $X \in F_\sigma$ . Find  $\gamma < \kappa^{++}$  such that  $\xi_{\sigma \smallfrown \gamma} = \eta$ . Set  $Y = X \cap A_{\sigma \xi_{\sigma \smallfrown \gamma}}$ . It will be as desired.

□

## 5 Open problems

In conclusion let us state some problems on the subject that remain open.

**Question 1.** Is the assumption  $o(\kappa) = \kappa^{++}$  needed for a model with a precipitous ideal on  $\aleph_1$  but without a normal one?

We think that it is likely to be possible to show that if  $\aleph_1$  is  $\infty$ -semi precipitous with a witnessed forcing satisfying  $\aleph_3$ -c.c. and with image of  $\aleph_1$  under the corresponding generic embedding is at least  $\aleph_3$ , then  $o(\kappa) = \kappa^{++}$  in an inner model. But probably there is no need to go via a construction of such  $\infty$ -semi precipitous.

**Question 2.** Is it possible to have a GCH model with a precipitous ideal on  $\aleph_1$  but without a normal one?

By [7] large cardinals not far from  $o(\kappa) = \kappa^{++}$  are needed for such a model.

**Question 3.** Is it possible to generalize the present result to cardinals bigger than  $\aleph_1$ ? Simplest case: Is there a model with a precipitous ideal on  $\aleph_2$  but without a normal one?

The next question is well known with partial answers given by Schimmerling, Velickovic [13], Woodin [14](8.1 Condensation Principles) and recently by Wu.

**Question 4.** Is it consistent that there is a supercompact cardinal and  $\aleph_1$  does not carries a precipitous ideal?

The construction above can be carried out below a supercompact cardinal and so it provides a model with a supercompact and no precipitous filters on  $\aleph_1$  which extend  $Cub_{\aleph_1}$  restricted to a stationary set. It is natural so ask the following question.

**Question 5.** Is it consistent that there is a supercompact cardinal and  $\aleph_1$  does not carries a precipitous filters that are  $Q$ -points, i.e. isomorphic to filters which extend  $Cub_{\aleph_1}$ ?

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