

ON NORMAL PRECIPITOUS IDEALS

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ABSTRACT. An old question of T. Jech and K. Prikry asks if an existence of a precipitous ideal implies necessary existence of a normal precipitous ideal. The aim of the paper is to prove some results in the positive direction. Thus, it is shown that under some mild assumptions, an existence of a precipitous ideal over \aleph_1 implies an existence of a normal precipitous ideal over \aleph_1 once a Cohen subset is added to \aleph_2 .

1. INTRODUCTION

The notion of precipitous ideal was introduced by T. Jech and K. Prikry in [4]. They asked if the existence of a precipitous ideal necessary imply the existence of a normal precipitous ideal. Given a precipitous ideal it is naturally to look at the normal ideal below it in the Rudin-Kiesler order. It turned out that those may not exist (see [2]) or even if it exists it can be not precipitous as was shown by R. Laver in [5] (see also the M. Foreman handbook's chapter, also a construction over a cardinal can be found in [2]).

Let F is a precipitous ideal over a cardinal κ and $j : V \rightarrow M$ be the corresponding generic embedding induced by $G(F^+)$, a generic set for F^+ . Consider

$$G(F_{normal}^+) := \{\pi \text{ `` } X \mid X \in G(F^+) \}.$$

It is a normal ultrafilter over $\mathcal{P}(\kappa) \cap V$. $G(F_{normal}^+)$ may be not generic for F_{normal}^+ once F_{normal} is not precipitous. But still we have $i : V \rightarrow (V \cap {}^\kappa V)/G(F_{normal}^+)$ and $k : (V \cap {}^\kappa V)/G(F_{normal}^+) \rightarrow M$, where $k([f]_{G(F_{normal}^+)}) = j(f)(\kappa)$. Clearly, k insures well foundedness of $(V \cap {}^\kappa V)/G(F_{normal}^+)$. Further let us identify it with the transitive collapse which we denote by M_{normal} . Clearly, $i(\kappa) \geq (\kappa^+)^V$. If there is a function $f : \kappa \rightarrow \kappa$ with $\|f\| = \kappa^+$ (in V) or there is no too large cardinals in inner models, then $i(\kappa) > (\kappa^+)^V$.

The aim of this paper is to prove the following somewhat surprising positive results related to existence of a normal precipitous filters:

Theorem 1.1. *Assume that F is a precipitous filter over \aleph_1 and $2^{\aleph_1} = \aleph_2$. Suppose that*

$$\kappa \Vdash_{F^+} i(\kappa) > (\kappa^+)^V.$$

Then for each $\tau < \aleph_3$ there is a normal filter over \aleph_1 with generic ultrapower well founded up to the image of τ .

Note that the above falls shortly before precipitousness. Thus, well foundedness up to the image of \aleph_3 implies the full well foundedness. It is a well known result, but for a convenience of a reader let us present a proof.

Lemma. *Suppose κ is a cardinal, $2^\kappa = \kappa^+$, and E is a σ -complete filter over κ such that any generic embedding induced by E is well-founded up to the image of κ^{++} , then E is precipitous.*

Proof. Suppose otherwise. Let $\langle f_n \mid n < \omega \rangle$ be a sequence of names such that:

$$(\star) \quad \kappa \Vdash_{E^+} \forall n < \omega \check{\omega}(f_n : \check{\kappa} \rightarrow On) \wedge \left[f_n \right]_{\check{G}(E^+)} > \left[f_{n+1} \right]_{\check{G}(E^+)}.$$

For $n < \omega$, we may assume that f_n is a *nice name* of the form $\{\langle g_{n,\alpha}, A_{n,\alpha} \rangle \mid \alpha < \delta_n\}$ where $\delta_n \leq \kappa^+$. For a large enough regular θ , pick $N \prec \mathcal{H}_\theta$ with $|N| = \kappa^+$ and:

$$\kappa^+ \cup \mathcal{P}(\kappa) \cup \{E, \langle f_n \mid n < \omega \rangle\} \cup \{g_{n,\alpha} \mid n < \omega, \alpha < \delta_n\} \subseteq N.$$

Consider the transitive collapse $\pi : N \rightarrow N^*$. Clearly $\pi(\kappa^+) = \kappa^+$ and $\pi(\kappa^{++}) < \kappa^{++}$. By $\mathcal{P}(\kappa) \subseteq N$ and $E \in N$, we get that $E^+ \subseteq N$, and then:

$$\pi(f_n) = \{\langle \pi(g_{n,\alpha}), A_{n,\alpha} \rangle \mid \alpha < \delta_n\} \quad (n < \omega).$$

Notice that for any relevant (n, α) , we have that $\pi(g_{n,\alpha})$ is a function from κ to $\tau := On \cap N^*$, and that τ is some ordinal $< \kappa^{++}$.

It follows that for all $n < \omega$, $\pi(f_n)$ is a nice name of a function from κ to an ordinal $< \kappa^{++}$, thus, to meet a contradiction, it suffices to show that:

$$\kappa \Vdash_{E^+} \forall n < \omega \check{\omega} \left[\pi(f_n) \right]_{\check{G}(E^+)} > \left[\pi(f_{n+1}) \right]_{\check{G}(E^+)}.$$

To see that, fix an arbitrary $A \in E^+$.

Let $n < \omega$. Find some $B \in E^+$ stronger than A and $h_n, h_{n+1} \in N$ such that:

$$B \Vdash_{E^+} f_n = \check{h}_n, \quad B \Vdash_{E^+} f_{n+1} = \check{h}_{n+1}.$$

Let $h^n := \pi(h_n)$, $h^{n+1} := \pi(h_{n+1})$. Since $\pi(E^+) = E^+$ and $\pi(B) = B$, we get from (\star) that:

$$B \Vdash_{E^+} \left[\check{h}^n \right]_{\check{G}(E^+)} > \left[\check{h}^{n+1} \right]_{\check{G}(E^+)},$$

and hence we may find some $C \in E^+ \cap \mathcal{P}(B)$ with $h^n(\nu) > h^{n+1}(\nu)$ for all $\nu \in C$. Then:

$$C \Vdash_{E^+} \left[\pi(f_n) \right]_{\mathcal{G}(E^+)} > \left[\pi(f_{n+1}) \right]_{\mathcal{G}(E^+)}.$$

By a density argument, now

$$\kappa \Vdash_{E^+} \left[\pi(f_n) \right]_{\mathcal{G}(E^+)} > \left[\pi(f_{n+1}) \right]_{\mathcal{G}(E^+)}.$$

But recall that E is σ -complete. Hence

$$\kappa \Vdash_{E^+} \forall n < \check{\omega} \left[\pi(f_n) \right]_{\mathcal{G}(E^+)} > \left[\pi(f_{n+1}) \right]_{\mathcal{G}(E^+)}.$$

□

In [3], Magidor and the author starting with an ω_1 - Erdős cardinal showed that it is possible to construct a model satisfying the conclusion of the theorem, e.g. in which for every $\tau < \aleph_3$ there is a normal filter over \aleph_1 with generic ultrapower well founded up to the image of τ . In particular, this property does not necessary implies precipitousness.

Theorem 1.2. *Assume that there is a precipitous filter F over \aleph_1 . Suppose that*

$$\kappa \Vdash_{F^+} i(\kappa) > (\kappa^+)^V.$$

Then, after adding a Cohen subset to \aleph_2 , there will be a normal precipitous filter on \aleph_1 .

In addition the method of the proof of 1.1 will allow to deduce the following:

Theorem 1.3. *Suppose that there is no inner model satisfying $(\exists \alpha \ o(\alpha) = \alpha^{++})$. Assume that F is a precipitous filter over \aleph_1 and $2^{\aleph_1} = \aleph_2$. If \aleph_3 is not a limit of measurable cardinals of the core model, then there exists a normal precipitous filter on \aleph_1 .*

The situation with larger cardinals is less clear. Still the following weaker analogs of the theorems above hold:

Theorem 1.4. *Assume that F is a precipitous filter over a successor cardinal κ . Suppose the following:*

- (1) $2^\kappa = \kappa^+$
- (2) $\kappa \Vdash_{F^+} i(\kappa) > (\kappa^+)^V$
- (3) $(\kappa^-)^{<\kappa^-} = \kappa^-$, where κ^- denotes the immediate predecessor of κ
- (4) the projection of F to a normal exists and concentrates on

$$\{\nu < \kappa \mid \text{cof}(\nu) = \kappa^-\}.$$

Then for each $\tau < \kappa^{++}$ there is a normal filter over κ with generic ultrapower well founded up to the image of τ .

Theorem 1.5. Assume that F is a precipitous filter over a successor cardinal κ . Suppose the following:

- (1) $\kappa \Vdash_{F^+} i(\kappa) > (\kappa^+)^V$
- (2) $(\kappa^-)^{<\kappa^-} = \kappa^-$, where κ^- denotes the immediate predecessor of κ
- (3) the projection of F to a normal exists and concentrates on

$$\{\nu < \kappa \mid \text{cof}(\nu) = \kappa^-\}.$$

Then, after adding a Cohen subset to κ^+ , there will be a normal precipitous filter on κ .

Theorem 1.6. Suppose that there is no inner model satisfying $(\exists \alpha \text{ } o(\alpha) = \alpha^{++})$. Assume that F is a precipitous filter over a successor cardinal κ and the following hold:

- (1) $2^\kappa = \kappa^+$
- (2) $(\kappa^-)^{<\kappa^-} = \kappa^-$, where κ^- denotes the immediate predecessor of κ
- (3) the projection of F to a normal exists and concentrates on

$$\{\nu < \kappa \mid \text{cof}(\nu) = \kappa^-\}.$$

If κ^{++} is not a limit of measurable cardinals of the core model, then there exists a normal precipitous filter on κ .

2. PROOFS

Let us start with a simple observation showing that if one is able to replace i by j in the statements of the theorems above, then it is possible to remove the corresponding requirements at all.

Proposition 2.1. Suppose that some $X \in F^+$ forces

$$j_{G(F^+)}(\kappa) = (\kappa^+)^V.$$

Then the projection of $F + X$ to a normal filter is isomorphic to $F + X$ and so is precipitous.

Proof. Assume for simplification of the notation that $X = \kappa$.

Let (in V) $\langle h_\nu \mid \nu < \kappa^+ \rangle$ be a sequence of canonical functions. They are defined as follows: for each limit $\alpha < \kappa^+$ fix in advance a cofinal sequence $\langle \alpha_\rho \mid \rho < \text{cof}(\alpha) \rangle$. Now,

$$h_\alpha(\tau) = \sup\{h_{\alpha_\rho}(\tau) \mid \rho < \min(\tau, \text{cof}(\alpha))\},$$

for each $\tau < \kappa$.

Let $G \subseteq F^+$ be generic and $j : V \rightarrow M \simeq V \cap {}^{\kappa}V/G$ be the corresponding elementary

embedding. We claim that for every $\alpha < j(\kappa) = (\kappa^+)^V$, $\alpha = j(h_\alpha)(\kappa)$. It is easy to check by induction on α . Thus, for example, let α be an ordinal of cofinality κ . By elementarity and the definition of h_α , we have

$$j(h_\alpha)(\kappa) = h_{j(\alpha)}(\kappa) = \sup\{h_{j(\alpha)_\rho}(\kappa) \mid \rho < \kappa\}.$$

But for $\rho < \kappa$ we have $j(\alpha)_\rho = j(\alpha_\rho)$. So

$$j(h_\alpha)(\kappa) = \sup\{j(h_{\alpha_\rho})(\kappa) \mid \rho < \kappa\}.$$

Now we can use the induction. Hence,

$$j(h_\alpha)(\kappa) = \sup\{j(h_{\alpha_\rho})(\kappa) \mid \rho < \kappa\} = \sup\{\alpha_\rho \mid \rho < \kappa\} = \alpha.$$

In particular, $[id]_G = j(h_\nu)(\kappa)$ for some $\nu < (\kappa^+)^V$.

Let F_{normal} be the normal filter below F in the Rudin - Keisler order and let π be the projection. Consider

$$G_{normal} := \{\pi^*X \mid X \in G\}.$$

Let $i : V \rightarrow M_{normal} \cong V \cap^{\kappa} V / G_{normal}$ and $k : M_{normal} \rightarrow M$, $k([g]_{G_{normal}}) = k(i(g)(\kappa)) = j(g)(\kappa)$. Every $\alpha < (\kappa^+)^V$ will be represented in M_{normal} by h_α .

But then $[id]_G$ will be in the range of k . So k must be the identity map and hence $M = M_{normal}$. Which imply in turn that F and F_{normal} are isomorphic. In particular, F_{normal} is precipitous.

□

From now we can assume that

$$\kappa \Vdash_{F^+} j_{G(F^+)}(\kappa) > (\kappa^+)^V.$$

2.1. Proof of Theorem 1.1. Denote \aleph_1 (of V) by κ . F is a precipitous ideal, so let $I := \check{F}$ denote its dual ideal, and let $j : V \rightarrow M$ denote the generic embedding induced by $G(F^+)$, a generic set for F^+ .

Shrink F if necessary to one of its positive sets in order to insure that there is a normal (probably not precipitous) filter below in the Rudin-Kiesler ordering, i.e., find $\pi : \kappa \rightarrow \kappa$ in V and $A \in F^+$ such that $A \Vdash [\hat{\pi}]_{G(F^+)} = \hat{\kappa}$ and consider $F + A$.¹ Suppose for simplicity that F already has this property. Let π be a projection to the normal, and consider the normal filter defined as the projection of F by π :

$$F_{normal} = \pi_*F = \{X \subseteq \kappa \mid \pi^{-1}[X] \in F\}.$$

¹Recall that $F + A = \{X \subseteq \kappa \mid (X \cap A) \cup (\kappa \setminus A) \in F\}$. In particular, $X \in (F + A)^+$ iff $X \cap A \in F^+$, and $X \in F + A$ iff $A \setminus X \in \check{F}$.

Notice that the normality of F_{normal} implies that:

$$\nabla_{\beta < \kappa}^\pi A_\beta := \{\nu < \kappa \mid \exists \beta < \pi(\nu), \nu \in A_\beta\}$$

is in I whenever $\{A_\beta \mid \beta < \kappa\} \subseteq I$.

Since $\kappa = \aleph_1$, we get in $V[G(F)]$ that $j(\kappa) = (\aleph_1)^M$. Let us use

$$\kappa \Vdash_{F^+} j_{G(F^+)}(\kappa) > (\kappa^+)^V.$$

It implies that the cardinality of $(\kappa^+)^V$ is ω in M .

Consider

$$G(F_{normal}^+) := \{\pi \text{ ``} X \mid X \in G(F^+) \}.$$

It is a normal ultrafilter over $\mathcal{P}(\kappa) \cap V$. $G(F_{normal}^+)$ may be not generic for F_{normal}^+ once F_{normal} is not precipitous. But still we have $i : V \rightarrow (V \cap {}^\kappa V)/G(F_{normal}^+)$ and $k : (V \cap {}^\kappa V)/G(F_{normal}^+) \rightarrow M$, where $k([f]_{G(F_{normal}^+)}) = j(f)(\kappa)$. Clearly, k insures well foundedness of $(V \cap {}^\kappa V)/G(F_{normal}^+)$. Further let us identify it with the transitive collapse which we denote by M_{normal} .

By the assumption of the theorem

$$\kappa \Vdash_{F^+} i(\kappa) > (\kappa^+)^V.$$

Which means that $|(\kappa^+)^V| = \omega$ in M_{normal} . Then there is a function $H \in V$ such that

$$[H]_{G(F_{normal}^+)} : \omega \xrightarrow{onto} (\kappa^+)^V.$$

So, in M , we have

$$k([H]_{G(F_{normal}^+)}) = j(H)(\kappa) : \omega \xrightarrow{onto} (\kappa^+)^V.$$

Hence, in V , there is $A \in F^+$ such that:

$$A \Vdash_{F^+} j(H)(\kappa) : \omega \xrightarrow{onto} (\kappa^+)^V.$$

By shrinking A , if necessary, let us assume that for each $\nu \in A$, $H(\pi(\nu))$ maps ω onto some ordinal below κ . Without loss of generality we can assume that $A = \kappa$, just otherwise replace F by $F + A$. For each $\alpha < \kappa^+$, let $h_\alpha : \kappa \rightarrow \kappa$ be the canonical function representing α in a generic ultrapower, i.e. α -th canonical function.

Now, for each $\alpha < \kappa^+$ and $n < \omega$, consider the set:

$$A_{n\alpha} = \{\nu < \kappa \mid H(\pi(\nu))(n) = h_\alpha(\pi(\nu))\}.$$

Our proof will not refer to the function H . Instead, we will be using the following three properties of the matrix $\langle A_{n\alpha} \mid n < \omega, \alpha < \kappa^+ \rangle$:

Lemma 2.2. *For any $\alpha < \kappa^+$ and $n < \omega$, $\pi^{-1}[\pi[A_{n\alpha}]] = A_{n\alpha}$.*

Lemma 2.3. *For every $\alpha < \kappa^+$ and $X \in F^+$, there is $n < \omega$ so that $A_{n\alpha} \in (F + X)^+$.*

Lemma 2.4. *Let $n < \omega$ and $X \in F^+$. Then the set:*

$$\{A_{n\alpha} \mid \alpha < \kappa^+ \text{ and } A_{n\alpha} \in (F + X)^+\}$$

is a maximal antichain in $(F + X)^+$ (i.e., it is a maximal antichain in F^+ below X).

Proof of Lemma 2.3. Fix $\alpha < \kappa^+$ and $X \in F^+$. By:

$$\kappa \Vdash_{F^+} j(H)(\kappa) : \omega \xrightarrow{\text{onto}} (\kappa^+)^V,$$

there will be $Y \in F^+ \cap \mathcal{P}(X)$ and $n < \omega$ such that

$$Y \Vdash_{F^+} (j(H)(\kappa))(n) = \alpha = j(h_\alpha)(\kappa).$$

Hence $Y \Vdash_{F^+} A_{n\alpha} \in G(F^+)$, where $G(F^+)$ is the canonical name of a generic subset of F^+ (i.e. a generic ultrafilter extending \tilde{F}). \square

Proof of Lemma 2.4. Note that if $A_{n\alpha} \in F^+$, then $A_{n\alpha} \Vdash_{F^+} (j(H)(\kappa))(n) = \alpha$. In particular, $A_{n\alpha} \cap A_{n\beta} \in I$ for any distinct $\alpha, \beta < \kappa^+$. To see maximality, let $Z \in (F + X)^+$. We have:

$$Z \Vdash_{(F+X)^+} [id] \in j(Z) \wedge j(H)(\kappa) : \omega \xrightarrow{\text{onto}} (\kappa^+)^V.$$

Hence there are $\alpha < \kappa^+$ and $Y \in (F + X)^+ \cap \mathcal{P}(Z)$ such that

$$Y \Vdash_{(F+X)^+} [id] \in j(Z) \text{ and } (j(H)(\kappa))(n) = j(h_\alpha)(\kappa).$$

In particular, Y will be contained in $Z \cap A_{n\alpha} \text{ mod } F + X$. \square

The next lemma was suggested and proved by A. Rinot. It allows to simplify further the indexation.

Lemma 2.5. *Suppose that $\mathcal{D} \subseteq \mathcal{P}(F^+)$ is a family of κ^+ pre-dense subsets of F^+ .*

There exists a sequence $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$ such that for all $Z \in F^+$ and $n < \omega$, if $Z_n = \{\alpha < \kappa^+ \mid A_{n\alpha} \cap Z \in F^+\}$ has cardinality κ^+ , then:

- (1) *For any $X \in F^+$, there exists $\alpha \in Z_n$ with $X_\alpha = X$.*
- (2) *For any $D \in \mathcal{D}$, there exists $\alpha \in Z_n$ with $X_\alpha \cap A_{n\alpha} \cap Z \in F^+$, and $X_\alpha \in D$.*

Proof. Let $\{S_i \mid i < \kappa^+\} \subseteq [\kappa^+]^{\kappa^+}$ be a partition of κ^+ , let $\{D_\alpha \mid \alpha < \kappa^+\}$ be an enumeration of \mathcal{D} , and let \triangleleft be a well-ordering of $\kappa^+ \cup \kappa^+ \times \kappa^+$ of order-type κ^+ .

By $2^\kappa = \kappa^+$, let $\{Y_\alpha \mid \alpha < \kappa^+\}$ be an enumeration of F^+ , and fix a function $\varphi : \kappa^+ \xrightarrow{\text{onto}} \{(Z, n) \in F^+ \times \omega \mid |Z_n| = \kappa^+\}$.

We would like to define a function $\psi : \kappa^+ \rightarrow \kappa^+ \times \kappa^+$ and the sequence $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$. Define by recursion on $\alpha < \kappa^+$ two sequences of ordinals $\{L_\alpha \mid \alpha < \kappa^+\}, \{R_\alpha \mid \alpha < \kappa^+\}$ and the values of ψ and the sequence on the intervals $[L_\alpha, R_\alpha]$.

For the base case, set $L_0 = R_0 = 0$ and put $\psi(0) = 0, X_0 = Y_0$.

Now, suppose that $\alpha < \kappa^+$ and $\{L_\beta, R_\beta \mid \beta < \alpha\}$ and $\psi \upharpoonright \bigcup_{\beta < \alpha} [L_\beta, R_\beta]$ has been defined. Take i to be the unique index such that $\alpha \in S_i$. Let $(Z, n) = \varphi(i)$. Set $L_\alpha = \min(\kappa^+ \setminus \bigcup_{\beta < \alpha} [L_\beta, R_\beta])$ and $R_\alpha = \min(Z_n \setminus L_\alpha)$. Now, for each $\gamma \in [L_\alpha, R_\alpha]$, put $\psi(\gamma) = t$, where:

$$t = \min_{\triangleleft} (\kappa^+ \cup \{i\} \times \kappa^+) \setminus \psi''(Z_n \cap L_\alpha).$$

If $t \in \kappa^+$, then set $X_\gamma = Y_t$ for all $\gamma \in [L_\alpha, R_\alpha]$.

Otherwise, $t = (i, \delta)$ for some $\delta < \kappa^+$. $R_\alpha \in Z_n$ implies $A_{nR_\alpha} \cap Z \in F^+$. D_δ is a pre-dense subset. Pick some $X \in D_\delta$ such that $X \cap A_{nR_\alpha} \cap Z \in F^+$ and let $X_\gamma = X$ for all $\gamma \in [L_\alpha, R_\alpha]$. This completes the construction.

Let us check that the construction works. Fix $Z \in F^+$ and $n < \omega$ with $|Z_n| = \kappa^+$. Let $i < \kappa^+$ be such that $\varphi(i) = (Z, n)$, then the construction insures that $\psi''Z_n = \kappa^+ \cup \{i\} \times \kappa^+$.

(1) Let $X \in F^+$. There exists $t \in \kappa^+$ with $Y_t = X$. Let $\alpha \in Z_n$ be such that $\psi(\alpha) = t$, then $X_\alpha = Y_t = X$.

(2) Let $D \in \mathcal{D}$. There exists $\delta < \kappa^+$ such that $D = D_\delta$, and $\alpha \in Z_n$ such that $\psi(\alpha) = (i, \delta)$. Then, by the construction, $X_\alpha \in D_\delta$ and $X_\alpha \cap A_{n\alpha} \cap Z \in F^+$. \square

Fix τ with $\kappa^+ \leq \tau < \kappa^{++}$. Let $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$ be a sequence given by the preceding lemma for $\mathcal{D} = \{D_f \mid f \in {}^{(\kappa\tau)}V\}$, where for any function $f : \kappa \rightarrow On$:

$$D_f = \left\{ X \in F^+ \mid \begin{array}{l} \exists \zeta \in On. X \Vdash_{F^+} j(\check{f})([id]) = \check{\zeta} \\ \exists \eta \in On. X \Vdash_{F^+} j(\check{f})(\kappa) = \check{\eta} \end{array} \right\}.$$

We now turn to an inductive process on $n < \omega$ of extending F , yielding an increasing chain of filters $\{F_n \mid n \leq \omega\}$, so that the generic ultrapower by F_ω , would be well-founded up to the image τ . Further more, this property would also hold to its projection by π , but in addition this projection would be normal.

Start with $n = 0$. We would like construct a sequence of F -positive sets $\langle C_{\langle \alpha \rangle} \mid \alpha < \kappa^+ \rangle$.

Let $\alpha < \kappa^+$. There are three cases:

Case I: If $|\{\xi < \kappa^+ \mid A_{0\xi} \in F^+\}| = \kappa^+$ and $A_{0\alpha} \cap X_\alpha \in F^+$, let $C_{\langle \alpha \rangle} = A_{0\alpha} \cap X_\alpha$, and

$$F_{\langle \alpha \rangle} = F + C_{\langle \alpha \rangle}.$$

Case II: If Case I does not apply, but $A_{0\alpha} \in F^+$, let $C_{\langle \alpha \rangle} = A_{0\alpha}$, and $F_{\langle \alpha \rangle} = F + C_{\langle \alpha \rangle}$.

Case III: If $A_{0\alpha} \in I$, then $C_{\langle \alpha \rangle}$ and $F_{\langle \alpha \rangle}$ would not be defined.

This completes the description of the construction. Notice that by Lemma 2.4, there exists some $\alpha < \kappa^+$ with $A_{0\alpha} \notin I$, thus $\{\alpha < \kappa^+ \mid F_{\langle \alpha \rangle} \text{ is defined}\}$ is non-empty,

Definition 2.6. Set $F_0 = \bigcap \{F_{\langle \alpha \rangle} \mid \alpha < \kappa^+, F_{\langle \alpha \rangle} \text{ is defined}\}$, and denote the corresponding dual ideals by $I_{\langle \alpha \rangle}$ and I_0 .

Clearly, $I_0 = \bigcap \{I_{\langle \alpha \rangle} \mid \alpha < \kappa^+, I_{\langle \alpha \rangle} \text{ is defined}\}$. Also, $F_0 \supseteq F$ and $I_0 \supseteq I$, since each $F_{\langle \alpha \rangle} \supseteq F$ and $I_{\langle \alpha \rangle} \supseteq I$.

The next two lemmas follow from the definition of I_0 , since it is an intersection of proper κ -complete ideals over κ which are normal below π .

Lemma 2.7. *I_0 is a proper κ -complete ideal over κ .*

Similarly, I_0 is normal below π , i.e. the following holds:

Lemma 2.8. *If $\langle Y_\beta \mid \beta < \kappa \rangle$ is a sequence of sets in I_0 then the set*

$$Y = \nabla_{\beta < \kappa}^\pi Y^\beta = \{\nu < \kappa \mid \exists \beta < \pi(\nu) \quad \nu \in Y_\beta\}$$

is in I_0 as well.

We now turn to the successor step of the construction, i.e., $m = n + 1$.

For any function $\sigma : m \rightarrow \kappa^+$ with F_σ defined, we would like to define a sequence of F_σ -positive sets $\langle C_{\sigma \frown \alpha} \mid \alpha < \kappa^+ \rangle$. Fix such σ and $\alpha < \kappa^+$. There are three cases:

Case I: If $|\{\xi < \kappa^+ \mid A_{m\xi} \in F_\sigma^+\}| = \kappa^+$ and $A_{m\alpha} \cap X_\alpha \in F_\sigma^+$, let $C_{\sigma \frown \alpha} = C_\sigma \cap A_{m\alpha} \cap X_\alpha$, and $F_{\sigma \frown \alpha} = F_\sigma + C_{\sigma \frown \alpha} = F + C_{\sigma \frown \alpha}$.

Case II: If Case I does not apply, but $A_{m\alpha} \in F_\sigma^+$, let $C_{\sigma \frown \alpha} = C_\sigma \cap A_{m\alpha}$, and $F_{\sigma \frown \alpha} = F + C_{\sigma \frown \alpha}$.

Case III: If $A_{m\alpha} \in I_\sigma$, then $C_{\sigma \frown \alpha}$ and $F_{\sigma \frown \alpha}$ would not be defined.

This completes the description of the construction.

Definition 2.9. *Let $F_{n+1} = \bigcap \{F_\sigma \mid \sigma : n + 2 \rightarrow \kappa^+, F_\sigma \text{ is defined}\}$, and define the corresponding dual ideals I_{n+1}, I_σ in the obvious way.*

The analogs of Lemma 2.7 and Lemma 2.8 are true for all I_n as well.

Lemma 2.10. *Let $n < \omega$. I_n is a proper κ -complete ideal over κ .*

Lemma 2.11. *Let $n < \omega$. If $\langle Y_\beta \mid \beta < \kappa \rangle$ is a sequence of sets in I_n then the set*

$$Y = \{\nu < \kappa \mid \exists \beta < \pi(\nu) \quad \nu \in Y_\beta\}$$

is in I_n as well.

We now describe the limit stage of the construction.

Definition 2.12. *Let F_ω be the closure under ω intersections of $\bigcup_{n < \omega} F_n$.*

Let $I_\omega =$ the closure under ω unions of $\bigcup_{n < \omega} I_n$.

Lemma 2.13. *$F \subseteq F_0 \subseteq \dots \subseteq F_n \subseteq \dots \subseteq F_\omega$ and $I \subseteq I_0 \subseteq \dots \subseteq I_n \subseteq \dots \subseteq I_\omega$, and I_ω is the dual ideal to F_ω .*

Lemma 2.14. *Let $s : m \rightarrow \kappa^+$ with F_s defined; then:*

- (1) $\{\alpha < \kappa^+ \mid F_{s \frown \alpha} \text{ is defined}\} = \{\xi < \kappa^+ \mid A_{m\xi} \in F_s^+\}$;
- (2) *There exists an extension $\sigma \supseteq s$ such that F_σ is defined and:*

$$|\{\xi < \kappa^+ \mid A_{\text{dom}(\sigma)\xi} \in F_\sigma^+\}| = \kappa^+.$$

Proof. (1) $F_{s \frown \alpha}$ is defined iff Case III of the construction does not hold, i.e., iff $A_{m\alpha} \in F_s^+$.

(2) Suppose not, then in particular $|\{\xi < \kappa^+ \mid A_{m\xi} \in F_s^+\}| \leq \kappa$, and hence:

$$|\{\alpha < \kappa^+ \mid F_{s \frown \alpha} \text{ is defined}\}| \leq \kappa.$$

From the same reason, for any $\rho : m+1 \rightarrow \kappa^+$ extending s with F_ρ defined, we have that $|\{\alpha < \kappa^+ \mid F_{\rho \frown \alpha} \text{ is defined}\}| \leq \kappa$. It follows that:

$$|\{\rho \in {}^{<\omega}\kappa^+ \mid s \subseteq \rho, F_\rho \text{ is defined}\}| \leq \kappa,$$

and hence if ρ extends s and C_ρ is defined, then it is defined according to Case II.

Pick $\delta \in \kappa^+ \setminus \bigcup\{\text{rng } \rho \mid s \subseteq \rho, F_\rho \text{ is defined}\}$. By Lemma 2.3, let us find some $n < \omega$ with $A_{n\delta} \cap C_s \in F^+$. We divide into two cases, both leading to a contradiction.

If $n < m$, then by $C_s \subseteq A_{ns(n)}$, Lemma 2.4 and $A_{n\delta} \cap C_s \in F^+$, we must conclude that $\sigma(n) = \delta$, contradicting the choice of δ .

If $n > m$, then we may recursively appeal to Lemma 2.4 and find $\beta_{n-1}, \dots, \beta_m$ so that $A_{m\beta_m} \cap \dots \cap A_{n-1\beta_{n-1}} \cap A_{n\delta} \cap C_s \in F^+$. By the previous item, $A_{m\beta_m} \cap C_s \in F^+$ implies that $C_{s \frown \beta_m}$ was defined, and so it was defined according to Case II, which means that $A_{m\beta_m} \cap C_s = C_{s \frown \beta_m}$.

Continuing the same way, we get that $C_{s \frown \beta_m \dots \frown \beta_{n-1}} \cap A_{n\delta} \in F^+$, that is, $A_{n\delta} \in F_{s \frown \beta_m \dots \frown \beta_{n-1}}^+$. It follows that $\rho = s \frown \beta_m \dots \frown \beta_{n-1} \frown \delta$ is a sequence extending s with F_ρ defined, and so the previous item yields a contradicting the choice of δ . \square

Now that the construction is finished, we begin to study the forms of elements from the constructed ideals. We begin with the I_n 's.

Lemma 2.15. *For all $n < \omega$ and $X \subseteq \kappa$: $X \in I_n$ iff $X \subseteq \nabla_{\beta < \kappa}^\pi Y_\beta \text{ mod } F$ for some sequences $\langle Y_\beta \mid \beta < \kappa \rangle$, $\langle \sigma_\beta \mid \beta < \kappa \rangle$ such that for each $\beta < \kappa$:*

- $\sigma_\beta \in {}^{\leq n+1}\kappa^+$;
- $Y_\beta \in I_{\sigma_\beta}$;
- $Y_\beta \subseteq \bigcap \{A_{m, \sigma_\beta(m)} \mid m < \text{dom}(\sigma_\beta)\}$.

Proof. \Leftarrow To see that $X \in I_\sigma$ for all $\sigma : n+1 \rightarrow \kappa^+$ with I_σ defined, let us fix such σ .

For all $m < n + 1$, consider the sets L_m and N_m :

$$L_m = \left\{ \beta < \kappa \mid \begin{array}{l} \text{dom}(\sigma_\beta) \supseteq m + 1 \\ \sigma_\beta \upharpoonright m = \sigma \upharpoonright m \\ \sigma_\beta \upharpoonright m + 1 \neq \sigma \upharpoonright m + 1 \end{array} \right\}.$$

$$N_m = \nabla_{\beta \in L_m}^\pi Y_\beta.$$

Now, for all $m < n + 1$:

$$A_{m\sigma(m)} \cap N_m \subseteq \nabla_{\beta \in L_m}^\pi (A_{m\sigma(m)} \cap A_{m\sigma_\beta(m)}).$$

Recalling Lemma 2.4, we notice that the right hand side is a π -diagonal union of sets from I , and hence $A_{m\sigma(m)} \cap N_m \in I$. Consequently, $N_m \in I_{\sigma \upharpoonright m+1}$ and $N_m \in I_\sigma$.

Let $L_* = \kappa \setminus \bigcup_{m < n+1} L_m$. Then for each $\beta \in L_*$, we have $\sigma_\beta = \sigma \upharpoonright \text{dom}(\sigma_\beta)$, and it follows that $Y_\beta \in I_{\sigma_\beta} = I_{\sigma \upharpoonright \text{dom}(\sigma_\beta)} \subseteq I_\sigma$. Thus, $N_* = \nabla_{\beta \in L_*}^\pi Y_\beta$ is a π -diagonal union of sets from I_σ , and hence $N_* \in I_\sigma$. Finally, we get:

$$\nabla_{\beta < \kappa}^\pi Y_\beta \subseteq (\nabla_{\beta \in L_*}^\pi Y_\beta) \cup \left(\bigcup_{m < n+1} \nabla_{\beta \in L_m}^\pi Y_\beta \right) = N_* \cup \left(\bigcup_{m < n+1} N_m \right).$$

The latter being a finite union of sets from I_σ , thus $X \in I_\sigma$.

\Rightarrow The result follows from the next, more informative, lemma. \square

Lemma 2.16. *For any $X \subseteq \kappa$, define the following rootless tree:*

$$T_X = \left\{ \sigma \frown \beta \mid \begin{array}{l} \sigma : m \rightarrow \kappa^+, m < \omega, \\ \beta < \kappa^+, X \cap A_{m\beta} \in F_\sigma^+ \end{array} \right\}.$$

For any $n < \omega$, if $X \in I_n$ then:

- (1) $T_X \subseteq {}^{\leq n+1}\kappa^+$;
- (2) $|T_X| \leq \kappa$;
- (3) If $\{\sigma_\beta \mid \beta < \kappa\}$ enumerates T_X and $Y_\beta = \bigcap \{A_{m,\sigma_\beta(m)} \mid m < \text{dom}(\sigma_\beta)\} \setminus C_{\sigma_\beta}$ for all $\beta < \kappa$ then $X \subseteq \nabla_{\beta < \kappa}^\pi Y_\beta \text{ mod } F$.

Remark. *In the definition of T_X , for σ with $\text{dom}(\sigma) = 0$, we regard F_σ^+ as F^+ .*

Proof. To avoid trivialities, we only deal with X which is F -positive.

(1) Suppose that σ is some sequence with F_σ defined. If $\text{dom}(\sigma) > n + 1$, then by $X \in I_n$, we get that $X \in I_{\sigma \upharpoonright (n+1)}$, i.e., $X \notin F_{\sigma \upharpoonright (n+1)}^+$. Then $\sigma \upharpoonright (n+2) \notin T_X$ and $\sigma \notin T_X$.

(2) Assume indirectly that $|T_X| > \kappa$, then let us pick some $\sigma \in {}^{<\omega}\kappa^+$ such that $B = \{\beta < \kappa^+ \mid \sigma \frown \beta \in T_X\}$ has cardinality κ^+ . Evidently, $B = \{\beta < \kappa^+ \mid A_{m\beta} \cap (X \cap C_\sigma) \in F^+\}$, where $m = \text{dom}(\sigma)$, and so, by Lemma 2.5, we may find some $\alpha < \kappa^+$ with $A_{m\alpha} \cap (X \cap C_\sigma) \in F^+$ such that $X = X_\alpha$. Trivially, $A_{m\alpha} \cap X_\alpha \in F_\sigma^+$ and hence $F_{\sigma \frown \alpha}$ is defined according to Case I, and $X \in F_{\sigma \frown \alpha}$. Pick $\rho \in {}^{<\omega}\kappa^+$ such that $\rho(0) = \alpha$, $\text{dom}(\sigma \frown \rho) = k > n$ and $F_{\sigma \frown \rho}$ is defined, then $X \in F_{\sigma \frown \rho}$, concluding that $X \notin I_{\sigma \frown \rho}$, in contradiction with $X \in I_n \subseteq I_k$.

(3) We prove by induction on $n < \omega$. For the base case, pick $X \in I_0$ and let $\{\sigma_\beta \mid \beta < \kappa\}$, $\{Y_\beta \mid \beta < \kappa\}$ be as above. Put $D = \nabla_{\beta < \kappa}^\pi Y_\beta$. Now, if $X \setminus D \notin I$, then we may appeal to Lemma 2.4 and find some $\xi < \kappa^+$ such that $(X \setminus D) \cap A_{0\xi} \in F^+$. In particular $X \cap A_{0\xi} \in F^+$ and $\langle \xi \rangle \in T_X$. Let $\beta < \kappa$ be such that $\langle \xi \rangle = \sigma_\beta$. By $\text{dom}(\sigma_\beta) = 1$ and $X \in I_0$, we have $X \cap C_{\sigma_\beta} \in I$, and hence $(X \setminus D) \cap (A_{0\sigma_\beta(0)} \setminus C_{\sigma_\beta}) \in F^+$, yielding a contradiction:

$$(X \setminus D) \cap (A_{0\sigma_\beta(0)} \setminus C_{\sigma_\beta}) = (X \setminus D) \cap Y_\beta \subseteq Y_\beta \setminus D \subseteq \pi^{-1}(\beta) \in I.$$

Inductive step. Suppose that n is some successor ordinal $< \omega$. Pick $X \in I_n$ and let $\{\sigma_\beta \mid \beta < \kappa\}$, $\{Y_\beta \mid \beta < \kappa\}$ be as above. Put $D = \nabla_{\beta < \kappa}^\pi Y_\beta$ and let us show that $(X \setminus D) \in I$. We start with showing that $(X \setminus D) \in I_{n-1}$. Suppose not, then there exists some $\sigma : n \rightarrow \kappa^+$ such that $(X \setminus D) \cap C_\sigma \in F^+$. As before, appeal to Lemma 2.4 to find some δ such that $(X \setminus D) \cap C_\sigma \cap A_{n\delta} \in F^+$, then in particular $X \cap A_{n\delta} \in F_\sigma^+$ and $\sigma \frown \delta \in T_X$. Consider $\beta < \kappa$ with $\sigma \frown \delta = \sigma_\beta$. By $X \in I_n$ and $\text{dom}(\sigma_\beta) = n + 1$, we have that $X \cap C_{\sigma_\beta} \in I$, and hence $(X \setminus D) \cap (A_{n\sigma_\beta(n)} \setminus C_{\sigma_\beta}) \in F_{\sigma_\beta}^+$, contradicting:

$$(X \setminus D) \cap (A_{n\sigma_\beta(n)} \setminus C_{\sigma_\beta}) \subseteq Y_\beta \setminus D \subseteq \pi^{-1}(\beta) \in I.$$

Thus, $X \setminus D \in I_{n-1}$, and by the inductive hypothesis, $(X \setminus D) \subseteq \nabla_{\beta < \kappa}^\pi Z_\beta \text{ mod } F$, where $Z_\beta = \bigcap \{A_{m,\rho_\beta(m)} \mid m < \text{dom}(\rho_\beta)\} \setminus C_{\rho_\beta}$ for all $\beta < \kappa$, and $\{\rho_\beta \mid \beta < \kappa\}$ is some enumeration of $T_{(X \setminus D)}$. Finally, by $T_{(X \setminus D)} \subseteq T_X$, we get that $\nabla_{\beta < \kappa}^\pi Z_\beta \subseteq D \text{ mod } F$, so we are done. \square

We now turn to study the form of elements from F_ω . First notice that since F_ω is the least σ -complete filter containing all the F_n 's, and since each F_n is, by itself, σ -complete, we have:

Lemma 2.17. $F_\omega = \{X \subseteq \kappa \mid \exists \langle X^n \mid n < \omega \rangle \forall n < \omega X^n \in F_n \quad X = \bigcap_{n < \omega} X^n\}$, and:

$$I_\omega = \{X \subseteq \kappa \mid \exists \langle X^n \mid n < \omega \rangle \forall n < \omega X^n \in I_n \quad X = \bigcup_{n < \omega} X^n\}.$$

Lemma 2.18. Let $s \in {}^{<\omega}\kappa^+$ with F_s defined, and $X \in I_\omega$.

Then there exists an extension $\sigma \supseteq s$ such that F_σ is defined and $C_\sigma \cap X \in I$.

Proof. Appealing to Lemma 2.14, find some $\sigma : t \rightarrow \kappa^+$ extending s with:

$$|\{\alpha < \kappa^+ \mid A_{t\alpha} \in F_\sigma^+\}| = \kappa^+.$$

Suppose that $\langle X^n \mid n < \omega \rangle$ is a sequence of subsets of κ such that $X^n \in I_n$ for each $n < \omega$. We shall find some $\delta < \kappa^+$ with $F_{\sigma \frown \delta}$ defined and $C_{\sigma \frown \delta} \cap X^n \in I$ for all $n < \omega$. The conclusion will then follow from the additivity degree of I .

Apply Lemma 2.15 and fix sequences $\langle Y_\beta^n \mid \beta < \kappa \rangle$, $\langle \sigma_\beta^n \mid \beta < \kappa \rangle$ such that for all $n < \omega, \beta < \kappa$:

$$(1) \text{ dom}(\sigma_\beta^n) \leq n + 1 \text{ and } Y_\beta^n \in I_{\sigma_\beta^n};$$

- (2) $Y_\beta^n \subseteq \bigcap \left\{ A_{m, \sigma_\beta^n(m)} \mid m < \text{dom}(\sigma_\beta^n) \right\};$
(3) $X^n \subseteq \nabla_{\beta < \kappa}^\pi Y_\beta^n \text{ mod } F.$

Fix $n < \omega$. For all $m < t$, consider the following sets:

$$L_m^n = \left\{ \beta < \kappa \mid \begin{array}{l} \text{dom}(\sigma_\beta^n) \supseteq m+1 \\ \sigma_\beta^n \upharpoonright m = \sigma \upharpoonright m \\ \sigma_\beta^n \upharpoonright m+1 \neq \sigma \upharpoonright m+1 \end{array} \right\}.$$

$$N_m^n = \nabla_{\beta \in L_m^n}^\pi Y_\beta^n.$$

Now, for all $m < t$:

$$C_\sigma \cap N_m^n \subseteq \nabla_{\beta \in L_m^n}^\pi (A_{m\sigma(m)} \cap A_{m\sigma_\beta^n(m)}),$$

Recalling Lemma 2.4, we get that the right hand side is a π -diagonal union of sets from I , and hence $C_\sigma \cap N_m^n \in I$. Now, Consider the following sets:

$$L_*^n = \{ \beta < \kappa \mid \sigma_\beta^n = \sigma \upharpoonright \text{dom}(\sigma_\beta^n) \},$$

$$N_*^n = \nabla_{\beta \in L_*^n}^\pi Y_\beta^n.$$

Then for each $\beta \in L_*^n$, we have $Y_\beta^n \in I_{\sigma_\beta^n} = I_{\sigma \upharpoonright \text{dom}(\sigma_\beta^n)} \subseteq I_\sigma$. Thus, N_*^n is a π -diagonal union of sets from I_σ , and hence $N_*^n \in I_\sigma$, that is, $C_\sigma \cap N_*^n \in I$. Finally, let:

$$L^n = \kappa \setminus \left(\bigcup_{m < t} L_m^n \cup L_*^n \right),$$

$$N^n = \nabla_{\beta \in L^n}^\pi Y_\beta^n.$$

Then, for each $\beta \in L^n$ we have that $\text{dom}(\sigma_\beta^n) > t$ and $\sigma_\beta^n \upharpoonright t = \sigma$. Consider the set:

$$\Gamma = \{ \sigma_\beta^n(t) \mid n < \omega, \beta \in L^n \}.$$

By the choice of σ , let us pick some $\delta \in \kappa^+ \setminus \Gamma$ such that $A_{t\delta} \in F_\sigma^+$.

Then for all $n < \omega$, we have:

$$C_{\sigma \frown \delta} \cap N^n \subseteq \nabla_{\beta \in L^n}^\pi (C_{\sigma \frown \delta} \cap Y_\beta^n) \subseteq \nabla_{\beta \in L^n}^\pi (A_{t\delta} \cap A_{t\sigma_\beta^n(t)}),$$

the latter being a π -diagonal union of sets from I , and hence $C_{\sigma \frown \delta} \cap N^n \in I$.

Altogether, we get that:

$$C_{\sigma \frown \delta} \cap \nabla_{\beta < \kappa}^\pi Y_\beta^n \subseteq C_{\sigma \frown \delta} \cap (N_m^n \cup N_*^n \cap N^n) \in I. \quad \square$$

Let us show that I_ω shares the properties of the I_n 's.

Lemma 2.19. I_ω is a proper κ -complete ideal over κ .

Proof. To see properness, pick $X \in I_\omega$. Then, by Lemma 2.18, we may find some $\sigma \in {}^{<\omega}\kappa^+$ with F_σ defined and $X \cap C_\sigma \in I$. Since $C_\sigma \in F^+$, we must conclude that $X \neq \kappa$.

Let us turn now to the κ -completeness. Fix $\{X_\alpha \mid \alpha < \mu\} \subseteq I$ for some $\mu < \kappa$. Then we may find $\langle X_\alpha^n \mid n < \omega, \alpha < \mu \rangle$ such that $X_\alpha^n \in I_n$ and $X_\alpha = \bigcup_{n < \omega} X_\alpha^n$ for all $\alpha < \mu, n < \omega$. It follows that $\bigcup_{\alpha < \mu} X_\alpha = \bigcup_{\alpha < \mu} \bigcup_{n < \omega} X_\alpha^n = \bigcup_{n < \omega} \bigcup_{\alpha < \mu} X_\alpha^n = \bigcup_{n < \omega} Y^n$, where $Y^n = \bigcup_{\alpha < \mu} X_\alpha^n$ is the union of $< \kappa$ many sets from I_n , and hence in I_n . \square

Lemma 2.20. *If $\langle Y_\beta \mid \beta < \kappa \rangle$ is a sequence of sets in I_ω , then the set*

$$Y = \{\nu < \kappa \mid \exists \beta < \pi(\nu) \quad \nu \in Y_\beta\}$$

is in I_ω as well.

Proof. For each $\beta < \kappa$, fix a sequence $\langle Y_\beta^n \mid n < \omega \rangle$ such that:

- (1) $Y_\beta^n \in I_n$;
- (2) $Y_\beta = \bigcup_{n < \omega} Y_\beta^n$.

For each $n < \omega$, by Lemma 2.11, the following set is in I_n :

$$Z^n = \{\nu < \kappa \mid \exists \beta < \pi(\nu) \quad \nu \in Y_\beta^n\}.$$

Finally, note that $Y \subseteq \bigcup_{n < \omega} Z^n \in I_\omega$. \square

Lemma 2.21. $F_\omega^+ = \bigcup \{F_\sigma \mid \sigma \in {}^{<\omega}\kappa^+, F_\sigma \text{ is defined}\}$.

Proof. (\supseteq) Pick a relevant σ with $X \in F_\sigma$. Suppose indirectly that $X \in I_\omega$, then by Lemma 2.18, we may find some ρ extending σ with $C_\rho \cap X \in I$, and hence $X \in I_\rho$. On the other hand, $X \in F_\sigma \subseteq F_\rho$, and so we meet a contradiction.

(\subseteq) Suppose that $X \in F_\omega^+$ is a counter-example. Consider the following rootless tree:

$$T_X = \left\{ \sigma \frown \beta \mid \begin{array}{l} \sigma : m \rightarrow \kappa^+, m < \omega, \\ \beta < \kappa^+, X \cap A_{m\beta} \in F_\sigma^+ \end{array} \right\}.$$

Notice that $|T_X| \leq \kappa$, because otherwise there would be some $\sigma \in {}^{<\omega}\kappa^+$ with κ^+ many successors in T_X and then $F_{\sigma \frown \alpha}$ will be defined according to Case I for some α with $X = X_\alpha$, yielding that $X \in F_{\sigma \frown \alpha}$, contradicting the choice of X .

Fix $n < \omega$. Let $T_n = \{\sigma \in T_X \mid \text{dom}(\sigma) \leq n + 1\}$, $\{\sigma_\beta^n \mid \beta < \kappa\}$ be an enumeration of T_n , and put $D_n = \bigcap_{\beta < \kappa} Y_\beta^n$, where, for each $\beta < \kappa$:

$$Y_\beta^n = \bigcap \left\{ A_{m, \sigma_\beta^n(m)} \mid m < \text{dom}(\sigma_\beta^n) \right\} \setminus C_{\sigma_\beta^n},$$

Then, by Lemma 2.15, we have $D_n \in I_n$, and hence $D = \bigcup_{n < \omega} D_n$ is in I_ω . Recall that $X \in F_\omega^+$. So we shall meet a contradiction by showing that $X \setminus D \in I$.

Suppose not, then $Y = X \setminus D \in F^+$ and we may force below it.

Pick $\delta \in \kappa^+ \setminus \{\beta < \kappa^+ \mid \exists \sigma \in {}^{<\omega}\kappa^+ (\sigma \frown \beta \in T_X)\}$. By Lemma 2.3, let us find some $n < \omega$ with $A_{n\delta} \cap Y \in F^+$. We divide into two cases, both showing that $A_{n\delta} \cap Y$ cannot be in F^+ .

Case I: If there exists some $\sigma : n \rightarrow \kappa$ such that $A_{n\delta} \cap Y \cap C_\sigma \in F^+$, then in particular, $A_{n\delta} \cap X \in F_\sigma^+$, and hence $\sigma \frown \delta \in T_X$, contradicting the choice of δ .

Case II: If there exists no $\sigma : n \rightarrow \kappa$ with F_σ defined and $A_{n\alpha} \cap Y \cap C_\sigma \in F^+$, then $A_{n\alpha} \cap Y \in I_n \cap F^+$. By Lemma 2.16, then $A_{n\alpha} \cap Y$ is covered (modulo F) by the π -diagonal union of the rootless tree $T_{A_{n\alpha} \cap Y}$. By $A_{n\alpha} \cap Y \subseteq X$, we then get that $T_{A_{n\alpha} \cap Y} \subseteq T_n$, and hence, $A_{n\alpha} \cap Y \subseteq D_n \text{ mod } F$, a contradiction to $Y \cap D_n = \emptyset$. \square

Definition 2.22. Let $I_\omega^* = \{X \subseteq \kappa \mid \pi^{-1}[X] \in I_\omega\}$.

Lemma 2.23. I_ω^* is a normal ideal.

Proof. We show that I_ω^* is closed under diagonal unions of its elements.

Let $\langle Z_\alpha \mid \alpha < \kappa \rangle$ be a sequence of elements of I_ω^* and:

$$Z = \{\tau < \kappa \mid \exists \alpha < \tau \quad \tau \in Z_\alpha\}.$$

Set $Y_\alpha = \pi^{-1}[Z_\alpha]$ for each $\alpha < \kappa$. Then every Y_α is in I_ω . By Lemma 2.20, then

$$Y = \{\nu \mid \exists \alpha < \pi(\nu) \quad \nu \in Y_\alpha\} \in I_\omega.$$

But $\pi^{-1}[Z] = Y$, and hence $Z \in I_\omega^*$, so we are done. \square

Lemma 2.24. For any $f \in (\kappa^\tau)^V$, the following is a dense subset of F_ω^+ :

$$E_f = \{C_\sigma \mid \sigma \in {}^{<\omega}\kappa^+, C_\sigma \text{ is defined}\} \cap D_f.$$

Proof. Let $f \in (\kappa^\tau)^V$ and $X \in F_\omega^+$. By Lemma 2.21, there exists some $s \in {}^{<\omega}\kappa^+$ such that $X \in F_s$. Apply Lemma 2.14 and find $\sigma \supseteq s$ such that F_σ is defined and:

$$|\{\xi < \kappa^+ \mid A_{\text{dom}(\sigma)\xi} \in F_\sigma^+\}| = \kappa^+.$$

Let $n = \text{dom}(\sigma)$ and $Z = C_\sigma$. By Lemma 2.14, then

$$\{\alpha < \kappa^+ \mid F_{\sigma \frown \alpha} \text{ is defined}\} = \{\xi < \kappa^+ \mid A_{n\xi} \cap Z \in F^+\}.$$

We pick, using Lemma 2.5(1), some $\alpha \in Z_n$ with $X_\alpha = X$. Then $A_{n\alpha} \in F_\sigma^+$ and $X_\alpha \in F_s \subseteq F_\sigma$, and hence $A_{n\alpha} \cap X_\alpha \in F_\sigma^+$ and the definition of $C_{\sigma \frown \alpha}$ is made according to Case I, yielding that $C_{\sigma \frown \alpha} \subseteq X$.

Apply again Lemma 2.14 and find $\rho \supseteq \sigma \frown \alpha$ such that F_ρ is defined and:

$$|\{\xi < \kappa^+ \mid A_{m\xi} \cap C_\rho \in F^+\}| = \kappa^+,$$

where $m = \text{dom}(\rho)$. By Lemma 2.5(2), let us pick some $\beta < \kappa^+$ with $X_\beta \cap A_{m,\beta} \cap C_\rho \in F^+$, and $X_\beta \in D_f$. Thus, the definition of $C_{\rho \frown \beta}$ is made according to Case I, and:

$$C_{\rho \frown \beta} = C_\rho \cap A_{m,\beta} \cap X_\beta \subseteq C_\rho \subseteq C_{\sigma \frown \alpha} \subseteq X.$$

By Lemma 2.21, then $C_{\rho \frown \beta} \in F_\omega^+$. $C_{\rho \frown \beta} \in D_f$ follows now, since D_f is an open dense set. Altogether, we conclude that $C_{\rho \frown \beta} \in F_\omega^+ \cap D_f$ is as required. \square

Lemma 2.25. *Generic ultrapower by I_ω is well founded up to the image of τ .*

Proof. Suppose that $\langle \dot{g}_n \mid n < \omega \rangle$ is a sequence of F_ω^+ -names of old (in V) functions from κ to τ . We shall show it is not strictly decreasing. Let $G \subseteq F_\omega^+$ be a generic ultrafilter. Pick a condition $Y_0 \in G$ and a function $g_0 : \kappa \rightarrow \tau$ in V such that:

$$Y_0 \Vdash_{F_\omega^+} \dot{g}_0 = \check{g}_0.$$

Since G is generic, we may appeal to Lemma 2.24, and find some $\sigma_0 \in {}^{<\omega}\kappa^+$ and $\zeta_0 \in On$ such that $Y_0 \cap C_{\sigma_0} \in G$ and $C_{\sigma_0} \Vdash_{F^+} j(\check{g}_0)([id]) = \check{\zeta}_0$. In particular:

$$Y_0 \cap C_{\sigma_0} \Vdash_{F_\omega^+} j(\dot{g}_0)([id]) = \check{\zeta}_0.$$

Continue now and pick $Y_1 \in G \cap \mathcal{P}(C_{\sigma_0} \cap Y_0)$ and a function $g_1 : \kappa \rightarrow \tau$ in V so that:

$$Y_1 \Vdash_{F_\omega^+} \dot{g}_1 = \check{g}_1.$$

As before, find $\sigma_1 \in {}^{<\omega}\kappa^+$ and ζ_1 such that $Y_1 \cap C_{\sigma_1} \in G$ and $C_{\sigma_1} \Vdash_{F^+} j(\check{g}_1)([id]) = \check{\zeta}_1$.

Notice that $F_\omega^+ \ni Y_1 \cap C_{\sigma_1} \subseteq Y_0 \cap C_{\sigma_0}$ implies that σ_1 and σ_0 are compatible, and so, we may assume that σ_1 extends σ_0 .

In the same way, continue the process of extending $\sigma_0 \subseteq \sigma_1 \subseteq \dots \subseteq \sigma_m$ until we obtain $k < m < \omega$ such that $\zeta_k \leq \zeta_n$. Then $G \ni C_{\sigma_m} \cap Y_m \subseteq C_{\sigma_k} \cap Y_k$ and:

$$C_{\sigma_m} \cap Y_m \Vdash_{F_\omega^+} [\dot{g}_k] \leq [\check{g}_m]. \quad \square$$

Lemma 2.26. *For any $f \in (\kappa\tau)^V$, the following is a dense subset of $(I_\omega^*)^+$:*

$$E_f = \{ \pi[C_\sigma] \mid \sigma \in {}^{<\omega}\kappa^+, C_\sigma \text{ is defined and } C_\sigma \in D_f \}.$$

Proof. Let $f \in (\kappa\tau)^V$ and $Y \in (I_\omega^*)^+$. Then $\pi^{-1}[Y] \in F_\omega^+$, and so by Lemma 2.24, there exists some $\sigma \in {}^{<\omega}\kappa^+$ with $C_\sigma \in D_f$ and $C_\sigma \subseteq \pi^{-1}[Y]$. In particular, $\pi[C_\sigma] \subseteq Y$. \square

Lemma 2.27. *Generic ultrapower by I_ω^* is well founded up to the image of τ .*

Proof. The proof is similar to those of Lemma 2.25. Suppose that $\langle \dot{g}_n \mid n < \omega \rangle$ is a sequence of $(I_\omega^*)^+$ -names of old (in V) functions from $\kappa \rightarrow \tau$. Let $G \subseteq (I_\omega^*)^+$ be a generic ultrafilter. Pick a condition $Y_0 \in G$ and a function $g_0 : \kappa \rightarrow \tau$ in V such that:

$$Y_0 \Vdash_{(I_\omega^*)^+} \dot{g}_0 = \check{g}_0.$$

Since G is generic, we may appeal to Lemma 2.26, and find some $\sigma_0 \in {}^{<\omega}\kappa^+$ and $\eta_0 \in On$ such that $Y_0 \cap \pi[C_{\sigma_0}] \in G$ and $C_{\sigma_0} \Vdash_{F^+} j(\check{g}_0)(\kappa) = \check{\eta}_0$.

Continue now and pick $Y_1 \in G \cap \mathcal{P}(\pi[C_{\sigma_0}] \cap Y_0)$ and a function $g_1 : \kappa \rightarrow \tau$ in V so that:

$$Y_1 \Vdash_{F^+} \dot{g}_1 = \check{g}_1.$$

As before, find $\sigma_1 \in {}^{<\omega}\kappa^+$ and η_1 such that $Y_1 \cap \pi[C_{\sigma_1}] \in G$ and $C_{\sigma_1} \Vdash_{F^+} j(\check{g}_1)(\kappa) = \check{\eta}_1$.

The following simple observation is crucial here: $G \ni Y_1 \cap \pi[C_{\sigma_1}] \subseteq Y_0 \cap \pi[C_{\sigma_0}]$ implies that σ_1 and σ_0 are compatible. Otherwise there would be some $n \in \text{dom}(\sigma_0) \cap \text{dom}(\sigma_1)$ with $\alpha_0 = \sigma_0(n) \neq \sigma_1(n) = \alpha_1$; then from $C_{\sigma_0} \subseteq A_{n\alpha_0}$ and $C_{\sigma_1} \subseteq A_{n\alpha_1}$, we would get that $G \ni Y_1 \cap \pi[C_{\sigma_1}] \subseteq \pi[A_{n\alpha_0}] \cap \pi[A_{n\alpha_1}]$ and $\pi^{-1}[\pi[A_{n\alpha_0}] \cap \pi[A_{n\alpha_1}]] \in F^+$. In particular, $\pi^{-1}[\pi[A_{n\alpha_0}]] \cap \pi^{-1}[\pi[A_{n\alpha_1}]] \in F^+$, contradicting Lemmas 2.2 and 2.4.

So, we may assume that σ_1 extends σ_0 .

In the same way, continue the process of extending $\sigma_0 \subseteq \sigma_1 \subseteq \dots \subseteq \sigma_m$ until we obtain $k < m < \omega$ such that $\eta_k \leq \eta_m$. Then $G \ni \pi[C_{\sigma_m}] \cap Y_m \subseteq \pi[C_{\sigma_k}] \cap Y_k$ and:

$$\{\nu \in \pi[C_{\sigma_m}] \cap Y_m \mid g_k(\nu) \leq g_m(\nu)\} \in G,$$

and hence:

$$\pi[C_{\sigma_m}] \cap Y_m \Vdash_{(I_\omega^*)^+} [\dot{g}_k] \leq [g_m]. \quad \square$$

So, I_ω^* is a normal ideal over κ with generic ultrapower well founded up to the image of τ . This completes the proof of the theorem.

2.2. Proof of Theorem 1.2. We shall force with the Cohen forcing for adding a function from κ^+ to 2, i.e.:

$$\text{Cohen}(\kappa^+) = \{p \mid p \text{ is a partial function of cardinality at most } \kappa \text{ from } \kappa^+ \text{ to } 2\}.$$

Let $c : \kappa^+ \rightarrow 2$ be a generic function for $\text{Cohen}(\kappa^+)$.

Lemma 2.28. *In $V[c]$: let $\mathcal{D} = \{D \in V \mid D \subseteq F^+ \text{ is pre-dense}\}$.*

There exists a sequence $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$ such that for all $Z \in F^+$ and $n < \omega$, if $Z_n = \{\alpha < \kappa^+ \mid A_{n\alpha} \cap Z \in F^+\}$ has cardinality κ^+ , then:

- (1) *For any $X \in F^+$, there exists $\alpha \in Z_n$ with $X_\alpha = X$.*
- (2) *For any $D \in \mathcal{D}$, there exists $\alpha \in Z_n$ with $X_\alpha \cap A_{n\alpha} \cap Z \in F^+$, and $X_\alpha \in D$.*

Proof. In V , fix a sequence of injective functions from κ to κ^+ , $\langle s_\alpha \mid \alpha < \kappa^+ \rangle$, such that $\kappa^+ = \biguplus_{\alpha < \kappa^+} \text{rng}(s_\alpha)$. In $V[c]$, for each $\alpha < \kappa^+$, let $X_\alpha = \{\xi < \kappa \mid c(s_\alpha(\xi)) = 0\}$.

Back in V . Let $p \in \text{Cohen}(\kappa^+)$. Then there exists some $\gamma < \kappa^+$ such that $\text{dom}(p) \subseteq \gamma$. Suppose that $Z \in F^+$ and $n < \omega$ are such that $|Z_n| = \kappa^+$. Let $B = \{\beta < \kappa^+ \mid \text{rng}(s_\beta) \cap \gamma \neq \emptyset\}$. Then $|B| \leq |\gamma| < \kappa^+$, and we may pick some $\alpha \in Z_n \setminus B$.

Now use the standard density argument. Thus, for (1), let $X \in F^+$, simply extend p to a condition q with $\text{dom}(q) = \text{dom}(p) \cup \text{rng}(s_\alpha)$, by defining $q(s_\alpha(\xi)) = 0$ whenever $\xi \in X$, and $q(s_\alpha(\xi)) = 1$ whenever $\xi \in \kappa \setminus X$.

For (2) let $D \in \mathcal{D}$. Find (in V) some $X \in D$ with $X \cap A_{n\alpha} \cap Z \in F^+$, and extend p to a condition q with $\text{dom}(q) = \text{dom}(p) \cup \text{rng}(s_\alpha)$, by defining $q(s_\alpha(\xi)) = 0$ iff $\xi \in X$. \square

Define now F_ω and I_ω^* exactly as in the proof of the Theorem 1.1, appealing to the sequence $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$ given by Lemma 2.28, instead of the one given in Lemma 2.5. All results up to Lemma 2.23 are established in exactly the same way, and we get:

Lemma 2.29. *In $V[c]$: I_ω^* is a proper κ -complete, normal ideal.*

Lemma 2.30. *In $V[c]$: $F_\omega^+ = \bigcup \{F_\sigma \mid \sigma \in {}^{<\omega}\kappa^+, F_\sigma \text{ is defined}\}$, and for any open dense subset $D \subseteq F^+$ from V , the following is a dense subset of F_ω^+ :*

$$E = \{C_\sigma \mid \sigma \in {}^{<\omega}\kappa^+, C_\sigma \text{ is defined}\} \cap D.$$

Finally, to see that both I_ω and I_ω^* are precipitous, just notice that if $f \in ({}^\kappa \text{On})^V$, then:

$$D_f = \left\{ X \in F^+ \mid \begin{array}{l} \exists \zeta X \Vdash_{F^+} j(\check{f})([id]) = \check{\zeta} \\ \exists \eta X \Vdash_{F^+} j(\check{f})(\kappa) = \check{\eta} \end{array} \right\}$$

is an open dense subset in V , and hence $D_f \in \mathcal{D}$ of Lemma 2.28.

Remark. *It is possible to deduce Theorem 1.1 from Theorem 1.2. Thus let $\tau < \kappa^{++}$. Pick an elementary submodel M of $H(\chi)$ (with χ big enough) such that $|M| = \kappa^+$ and $M \cap \kappa^{++} > \tau$. Let c be a Cohen generic over M . Now we apply Theorem 1.2 to $M[c]$.*

2.3. Proof of Theorem 1.3. We assume that there is no inner model with α of the Mitchell order α^{++} . Let \mathcal{K} be the core model. Let $G \subseteq F^+$ be generic and

$$j : V \rightarrow M \cong V \cap {}^\kappa V / G$$

be the corresponding embedding. Consider $j \upharpoonright \mathcal{K}$. By [6] it is an iterated ultrapower of \mathcal{K} by its measures. Note that it need not be definable in \mathcal{K} or in V . Let us replace it by a bigger iterated ultrapower which is definable in V . Work in V . Proceed by recursion. Each time if certain F -positive set forces that some of the measures of \mathcal{K} is used apply it or its image under the embedding defined so far. Eventually we will finish with a kind of a maximal iterated ultrapower embedding $j^* : \mathcal{K} \rightarrow \mathcal{K}^*$ which is definable in V all possible generic embeddings defined by forcing with F^+ are parts of it.

Fix such j^* and similar find i^* which incorporates all the restrictions to \mathcal{K} of i , the embedding of the projection to the normal for each generic $G \subseteq F^+$. Clearly it is possible to choose both j^*, i^* with critical points κ . Let (in $V[G]$) $\sigma \circ j = j^*$ and $\vartheta \circ i = i^*$.

Let us point out the following fact (basically due to W. Mitchell) which is needed in order to apply Theorem 1.1.

Lemma 2.31. $\kappa^+ = (\kappa^+)^{\mathcal{K}}$

Proof. Suppose otherwise. Then $|(\kappa^+)^{\mathcal{K}}| = \kappa$. Let $G \subseteq F^+$ be a generic and $j : V \rightarrow M_G$ be the corresponding elementary embedding. Let U be the measure over κ in \mathcal{K} used first to move κ in the iteration $j \upharpoonright \mathcal{K}$. Let $\langle A_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of U in V . Then, clearly, $\langle j(A_\alpha) \mid \alpha < \kappa \rangle \in M_G$. But $A_\alpha = j(A_\alpha) \cap \kappa$, for each $\alpha < \kappa$. Hence $\langle A_\alpha \mid \alpha < \kappa \rangle \in M_G$. So, $U \in M_G$. But U is a normal $j(\mathcal{K})$ -ultrafilter. So it should be on the sequence of $j(\mathcal{K})$ over κ . This is impossible, since U was already used in the process of iteration to $j(\mathcal{K})$. A Contradiction. \square

The next lemma follows now easily.

Lemma 2.32. $\kappa \Vdash_{F^+} i(\kappa) > (\kappa^+)^V$.

Proof. Just the restriction of i to \mathcal{K} is an iterated ultrapower of \mathcal{K} starting with its measure over κ . So, the image of κ will be above $(\kappa^+)^{\mathcal{K}}$. But by the previous lemma we have

$$(\kappa^+)^V = (\kappa^+)^{\mathcal{K}}.$$

So, $(\kappa^+)^V$ is of cardinality κ in M_{normal} and hence cannot be moved by k . \square

Note that we have $2^\kappa = \kappa^+$, $|F^+| = \kappa^+$ and there are no measurable cardinals in \mathcal{K} in the interval $[\delta, \kappa^{++})$ for some $\delta < \kappa^{++}$. Then, by [6], every subset X of κ^{++} of cardinality less than κ^{++} there is $Y \in \mathcal{K}$ covering X and such that in \mathcal{K} its cardinality is at most δ . In particular, the set $\mathcal{P}_\delta(\kappa^{++}) \cap \mathcal{K}$ is unbounded in V . This would be essential below.

Lemma 2.33. *For every function $f : \kappa \rightarrow \kappa^{++}$ there are functions $t : \delta \rightarrow \kappa^{++}$, $t \in \mathcal{K}$ and $g : \kappa \rightarrow \delta$ such that $f = t \circ g$.*

Proof. Use the unboundedness of $\mathcal{P}_\delta(\kappa^{++}) \cap \mathcal{K}$, to cover $\text{rng}(f)$. \square

Now we shall use Theorem 1.1 in order to get well foundedness up to δ . All other relevant functions will be of the form $t \circ g$ for t and g as in Lemma 2.33. Then j^* (or i^*) will be applied to t 's to provide the desired well foundedness.

Thus, by the proof of Theorem 1.1 (taking $\tau = \delta$) to each function $g : \kappa \rightarrow \delta$ in V correspond two ordinals $\zeta(g)$ and $\eta(g)$ which are forced to be $j(g)([id])$ and $j(g)(\kappa)$.

Here we need rather to freeze their images under σ . It can be done by using the obvious easy modification of the construction of 1.1. Just use

$$D_g = \left\{ X \in F^+ \mid \begin{array}{l} \exists \zeta \in \text{On}.X \Vdash_{F^+} \sigma(j(\check{g})([id])) = \check{\zeta} \\ \exists \eta \in \text{On}.X \Vdash_{F^+} \sigma(j(\check{g})(\kappa)) = \check{\eta} \end{array} \right\}.$$

Denote $\sigma(\zeta(g))$ by $\zeta(g)^*$ and $\sigma(\eta(g))$ by $\eta(g)^*$.

Now, given a function $f : \kappa \rightarrow \kappa^{++}$. Apply Lemma 2.33. We will get functions $t : \delta \rightarrow \kappa^{++}$, $t \in \mathcal{K}$ and $g : \kappa \rightarrow \delta$ such that $f = t \circ g$. Then, in a generic ultrapower (by F^+), we have

$$j(f)([id]) = j(t \circ g)([id]).$$

Now, use σ . Then

$$\sigma(j(f)([id])) = \sigma(j(t \circ g)([id])) = ((j^*(t))(\sigma(j(g)([id]))) = j^*(t)(\sigma(\zeta(g))) = j^*(t)(\zeta(g)^*).$$

So, we can attach to f the ordinal $j^*(t)(\zeta(g)^*)$. This insures well foundedness of the part up to the image of κ^{++} , which implies in turn the full well-foundedness.

3. ABOVE \aleph_1 .

In this section we would like to extend the previous results to cardinals bigger than \aleph_1 . Our aim will be to prove Theorems 1.4,1.5,1.6 stated in the Introduction. The proofs mostly repeat those of the corresponding Theorems 1.4,1.5,1.6. Let us concentrate on Theorem 1.4. The difference here is that the construction of the filters does not necessary stops at the stage ω , but rather may run all the way up to κ^- the immediate predecessor of κ . Thus, Lemma 2.3 should be replaced by

Lemma 3.1. *For every $\alpha < \kappa^+$ and $X \in F^+$, there is $\tau < \kappa^-$ so that $A_{\tau\alpha} \in (F + X)^+$.*

Just now

$$\kappa \Vdash_{F^+} j(H)(\kappa) : (\kappa^-)^V \xrightarrow{\text{onto}} (\kappa^+)^V$$

and all the cardinals $\leq \kappa^-$ remain cardinals in M .

Let $\langle F_n \mid n \leq \omega \rangle$ be defined as in the proof of 1.1 (with obvious changes of ω to κ^-).

Lemma 3.2. *If $X \in F_\omega^+$ then*

(1) *there is $\sigma \in {}^{<\omega}\kappa^+$ with F_σ defined and $X \in F_\sigma$*

or

(2) *there is a sequence $\langle \sigma_\xi \in {}^\omega \kappa^+ \mid \xi < \kappa^* \leq \kappa \rangle$ such that $F_{\sigma_\xi \upharpoonright n}$ is defined (for each $n < \omega, \xi < \kappa^*$), $\bigcap_{n < \omega} C_{\sigma_\xi \upharpoonright n} \in F^+$ and $\bigcup_{\xi < \kappa^*} \bigcap_{n < \omega} C_{\sigma_\xi \upharpoonright n} \supseteq X \text{ mod } F_\omega$.*

Proof. Suppose that (1) does not hold. We follow the proof of Lemma 2.21. Suppose that $X \in F_\omega^+$. Define $T_X, T_n, \{\sigma_\beta^n \mid \beta < \kappa_n\}$, (for some cardinal $\kappa_n \leq \kappa$) as in 2.21.

Let us turn the family $\{A_{0\gamma} \mid \gamma < \kappa^+, X \cap A_{0\gamma} \in F^+\} = \{A_{0\sigma_\beta^0(0)} \mid \beta < \kappa_0\}$ into a family of disjoint sets. Set $A'_{0\sigma_\beta^0(0)} = A_{0\sigma_\beta^0(0)} \setminus \{0\}$ and for each $\beta < \kappa_0$, let $A'_{0\sigma_\beta^0(0)} = A_{0\sigma_\beta^0(0)} \setminus (\bigcup_{\beta' < \beta} A_{0\sigma_{\beta'}^0(0)} \cup \beta + 1)$.

Clearly, $\langle X \cap A'_{0\sigma_\beta^0(0)} \mid \beta < \kappa_0 \rangle$ is still a maximal antichain in F^+ below X and, hence $Z_0 := X \setminus \bigcup_{\beta < \kappa_0} A'_{0\sigma_\beta^0(0)} \in I$. Also the replacement of $A_{0\sigma_\beta^0(0)}$ by $A'_{0\sigma_\beta^0(0)}$ and $C_{\sigma_\beta^0}$ by $C'_{\sigma_\beta^0} := C_{\sigma_\beta^0} \cap A'_{0\sigma_\beta^0(0)}$ will not effect F_σ 's, since we removed each time a set in I .

Deal now with the second level. Thus let $\sigma \in T_0$. Consider

$$\{A_{1\beta} \cap C'_\sigma \mid \beta < \kappa^+, X \cap C'_\sigma \cap A_{1\beta} \in F^+\} = \{A_{1\sigma_\beta^1(1)} \cap C'_\sigma \mid \sigma_\beta^1 \supseteq \sigma\}.$$

Turn this family into disjoint one $\langle A'_{1\sigma_\beta^1(1)} \mid \sigma_\beta^1 \supseteq \sigma \rangle$ as above. Then

$$(X \cap C'_\sigma) \setminus \bigcup \{A'_{1\sigma_\beta^1(1)} \mid \sigma_\beta^1 \supseteq \sigma\} \in I.$$

Equivalently,

$$X \cap A'_{0\sigma(0)} = ((X \cap (A'_{0\sigma(0)} \setminus C'_\sigma)) \cup (X \cap \bigcup \{A'_{1\sigma_\beta^1(1)} \mid \sigma_\beta^1 \supseteq \sigma\})) \cup Z_\sigma,$$

for some $Z_\sigma \in I$. Note that we have a disjoint union on the right side.

We do the above for each $\sigma \in T_0$. Set

$$Z_1 = \bigcup_{\sigma \in T_0} Z_\sigma.$$

Claim 3.2.1. $Z_1 \in I$.

Proof. Suppose otherwise. Recall that $Z_1 \subseteq X \cap \bigcup_{\beta < \kappa_0} A'_{0\sigma_\beta^0(0)}$. But $Z_1 \in F^+$, hence for some $\beta < \kappa_0$ we must have $Z \cap A'_{0\sigma_\beta^0(0)} \in F^+$. Let $\sigma := \sigma_\beta^0(0)$. The disjointness of the sets involved implies $Z_1 \cap A'_{0\sigma_\beta^0(0)} = Z_\sigma$. But $Z_\sigma \in I$. Contradiction.

□ of the claim.

Continue similar for each $n > 1$. We will obtain sets $Z_n \in I$. Set $Z_\omega = \bigcup_{n < \omega} Z_n$. Then, also $Z_\omega \in I$. Consider $Y = X \setminus Z_\omega$. We may assume that each $\nu \in Y$, $\pi(\nu)$ has cofinality κ^- . Just use the assumption of the theorem and remove a set in I if necessary.

Now we would like to truck the origin of each $\nu \in Y$. It is possible here since the appropriate sets are disjoint. So we either will have ν in the intersection of sets along some ω -branch σ or it may come from $A'_{n\sigma(n)} \setminus C'_\sigma$, for some $\sigma \in T_n, n < \omega$. In both cases we can freeze a result using the normality of F , $\text{cof}(\nu) = \kappa^-$ and $\kappa^- > \kappa^- = \kappa^-$.

The above provides a splitting of Y into F -positive sets $\langle Y_\xi, S_\mu \mid \xi < \kappa^* \leq \kappa, \mu < \kappa' \leq \kappa \rangle$, where Y_ξ 's are contained in intersections along ω -branches and S_μ 's are subsets of $A'_{n\sigma(n)} \setminus C'_\sigma$, for some $\sigma \in T_n, n < \omega$. By the definition of I_ω , we will have $\bigcup_{\mu < \kappa'} S_\mu \in I_\omega$. So, $\bigcup_{\xi < \kappa^*} Y_\xi \supseteq X \text{ mod } F_\omega$.

□

Remark. Note that we cannot, in general, claim that there is $\sigma \in {}^\omega \kappa^+$ such that for all $n < \omega$ $F_{\sigma \upharpoonright n}$ is defined and $\bigcap_{n < \omega} C_{\sigma \upharpoonright n} \in F^+$. Thus, for example, iterate the Namba forcing all the way below a measurable cardinal λ turning it to \aleph_2 and leaving cardinals in a set of measure one untouched by Namba, as it is done in [7] Chapter 11. Then we will have a precipitous filter which satisfies the conditions of the Theorem 1.4. In a generic ultrapower will be arbitrary large cardinals below $j(\kappa)$ that change their cofinality to ω . Take one of them and use a name \underline{a} of his ω -sequence. Let us decide, in addition, at the level n of the construction also the value of $\underline{a}(n)$. This will insure that there is no $\sigma \in {}^\omega \kappa^+$ such that for all $n < \omega$ $F_{\sigma \upharpoonright n}$ is defined and $\bigcap_{n < \omega} C_{\sigma \upharpoonright n} \in F^+$. Just otherwise, $\bigcap_{n < \omega} C_{\sigma \upharpoonright n}$ will decide \underline{a} completely, which is clearly impossible.

Lemma 3.3. Suppose that for some $\sigma \in {}^{>\omega} \kappa^+$ with F_σ defined, the following holds: for each $\rho \in {}^{>\omega} \kappa^+$ extending σ with F_ρ defined

$$|\{\alpha < \kappa^+ \mid F_{\rho \upharpoonright \alpha} \text{ defined}\}| \leq \kappa.$$

Then there is $\eta \in {}^\omega \kappa^+$ extending σ such that for all $n < \omega$ $F_{\eta \upharpoonright n}$ is defined and $\bigcap_{n < \omega} C_{\eta \upharpoonright n} \in F^+$.

Proof. This follows from the proof of the lemma 3.2 above. Just take $X = C_\sigma$ and run the argument. It is crucial that each time we split into at most κ many sets, which allows to turn them into disjoint ones. □

Consider the following set:

$$\Sigma = \{\sigma \in {}^\omega \kappa^+ \mid \forall n < \omega \text{ } F_{\sigma \upharpoonright n} \text{ is defined and } \bigcap_{n < \omega} C_{\sigma \upharpoonright n} \in F^+\}.$$

If there is $\sigma^* \in {}^{>\omega} \kappa^+$ with F_{σ^*} defined but without $\sigma \in \Sigma$ extending it, then we replace F_ω by $F_\omega + C_{\sigma^*}$. Continue as in 1.1 and show that the projection of $F_\omega + C_{\sigma^*}$ to a normal is already precipitous.

Suppose now that for each $\sigma^* \in {}^{>\omega} \kappa^+$ with F_{σ^*} defined, we have $\sigma \in \Sigma$ which extends σ^* . In this case we shall continue to extend filters. Thus, for each $\sigma \in \Sigma$, we set

$$C_\sigma = \bigcap_{n < \omega} C_{\sigma \upharpoonright n} \text{ and } F_\sigma = F + C_\sigma.$$

Define

$$F_{\omega+1} = \bigcap_{\sigma \in \Sigma} F_\sigma.$$

Deal now with $F_{\omega+1}, C_\sigma$'s ($\sigma \in \Sigma$) and $\langle A_{\omega, \alpha} \mid \alpha < \kappa^+ \rangle$. Define $F_{\omega+2}$ as in 1.1. Continue the construction. At successor stages we proceed as in 1.1 and at limit ones as above. At limit stage τ , we will have the following analog of Lemma 3.2:

Lemma 3.4. *If $X \in F_\tau^+$ then*

- (1) *there is $\sigma \in {}^{<\tau}\kappa^+$ with F_σ defined and $X \in F_\sigma$*
or
- (2) *there is a sequence $\langle \sigma_\xi \in {}^\tau\kappa^+ \mid \xi < \kappa^* \leq \kappa \rangle$ such that $F_{\sigma_\xi \upharpoonright \mu}$ is defined (for each $\mu < \tau, \xi < \kappa^*$), $\bigcap_{\mu < \tau} C_{\sigma_\xi \upharpoonright \mu} \in F^+$ and $\bigcup_{\xi < \kappa^*} \bigcap_{\mu < \tau} C_{\sigma_\xi \upharpoonright \mu} \supseteq X \text{ mod } F_\tau$.*

Either our construction will stop at some limit $\tau < \kappa^-$ or it will run all the way to κ^- . Let $\tau \leq \kappa^-$ be the terminal point of the construction. Then only the first clause of Lemma 3.4 will hold. Thus if $\tau < \kappa^-$, then just by the construction, since we stop at τ . If $\tau = \kappa^-$, then the argument of Lemma 2.21 applies. Note that by Lemma 3.3, we cannot stop before reaching a situation where for every $\sigma \in {}^{\tau > \kappa^+}$ with F_σ defined, there is $\rho \in {}^{\tau > \kappa^+}$ extending σ with F_ρ defined and

$$|\{\alpha < \kappa^+ \mid F_{\rho \upharpoonright \alpha} \text{ defined}\}| > \kappa.$$

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