An other model with tree property and not SCH.

Moti Gitik*

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Abstract

We will use the method for blowing up the power of a singular cardinal of [3] in order to get models with tree property on successors of singulars and not SCH. The advantage of the present technique is that no cardinal is collapsed or changes its cofinality. A question by O. Ben-Neria, C. Lambie-Hanson, S. Unger from [2] is answered.

1 A model in which SCH fail at a singular cardinal, but the tree property holds at its successor.

Such a model was first constructed by I. Neeman [7] for a singular of countable cofinality and later was generalized to uncountable one by D. Sinapova [10].

The forcing used in [7] is based on the forcing of [5]. We will use here the forcing of [3] instead and deal with countable and uncountable cofinalities simultaneously.

Fix a regular cardinal $\eta$. Let $\langle \kappa_\alpha | \alpha < \eta \rangle$ be an increasing sequence of cardinals and let $\langle E_\alpha | \alpha < \eta \rangle$ be a sequence of extenders such that for every $\alpha < \eta$

1. $\eta < \kappa_0$,
2. $E(\alpha)$ is a $(\kappa_\alpha, \bar{\kappa}_\eta^++)$–extender, where $\bar{\kappa}_\eta = \bigcup_{\alpha < \eta} \kappa_\alpha$,
3. $E(\alpha) \lhd E(\alpha + 1)$,
4. there is a supercompact cardinal in the interval $(\eta, \kappa_0)$,
5. for every $\alpha < \eta$ there is a supercompact cardinal in the interval $(\kappa_\alpha, \kappa_\alpha + 1)$.

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Let \(<\mathcal{P}(E(\alpha)|\alpha<\eta),\leq,\leq^*>)\) be the forcing of Section 2 of [3].
For every limit \(\alpha \leq \eta\) denote \(\bar{\kappa}_\alpha = \bigcup_{\alpha'<\alpha} \kappa_{\alpha'}\).
By [3], Section 2, it has the following properties:

1. \(<\mathcal{P}(E(\alpha)|\alpha<\eta),\leq,\leq^*>)\) is a Prikry type forcing,
2. the forcing \(<\mathcal{P}(E(\alpha)|\alpha<\eta),\leq>)\):
   (a) blows up the power of \(\bar{\kappa}_\eta\) to \(\bar{\kappa}_\eta^{++}\),
   (b) blows up the power of \(\bar{\kappa}_\alpha\) above \(\bar{\kappa}_\alpha^{+}\), for every limit \(\alpha < \eta\),
   (c) preserves cardinals and cofinalities,
   (d) preserves strong limitness of each of \(\kappa_\alpha\)'s, for every \(\alpha \leq \eta\), and \(\bar{\kappa}_\alpha\)'s, for every limit \(\alpha \leq \eta\)
   (e) does not add new subsets to \(\kappa_0\).
3. The forcing \(<\mathcal{P}(E(\alpha)|\alpha<\eta),\leq^*>)\) is equivalent to the product of
   Cohen forcings \(\text{Cohen}(\kappa_\alpha^+,\bar{\kappa}_\eta^{++})\) with full support for \(\leq^*\) – extension of \(0_{\mathcal{P}(E(\alpha)|\alpha<\eta)}\).

We force first with the Laver type preparation forcings to ensure indestructibility of the relevant supercompact cardinals under directed closed forcings which preserve cardinals, as it is done in A. Apter [1]. Let \(H\) be a corresponding generic set.
It is easy to extend the extender \(E(\alpha)\) and its elementary embedding in \(V[H]\). Let us abuse the notation a bit and still denote the extension of \(E(\alpha)\) in \(V[H]\) by \(E(\alpha)\).

Force with \(<\mathcal{P}(E(\alpha)|\alpha<\eta),\leq>)\). Let \(G\) be a generic. We claim that \(V[H \ast G]\) is as desired, i.e. it satisfies \(TP_{\bar{\kappa}_\eta^{++}}\) and \(2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}\).

\(2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}\) follows by [3]. Let deal with the tree property.

The argument below follows [6] and [7].
Suppose that \(T\) is a \(\bar{\kappa}_\eta^{++}\) – tree in \(V[H \ast G]\).
We can assume that for every \(\alpha < \bar{\kappa}_\eta^{+}\), the level \(\alpha\) of \(T\) is \(\{(\alpha, \xi) \mid \xi < \bar{\kappa}_\eta\}\).
Let \(T\) be a \(<\mathcal{P},\leq>)\) – name of a \(\bar{\kappa}_\eta^{+}\) – tree \(T\).
Suppose for simplicity that

\[0_\mathcal{P} \models _{\langle \mathcal{P}, \leq \rangle} (\dot{T} \text{ is a } \bar{\kappa}_\eta^{+} \text{ – tree}).\]
Let $\kappa, \eta < \kappa < \kappa_0$ be a supercompact. Now, in $V[H]$, using the indestructibility of supercompactness of $\kappa$ under the forcing $\langle P, \leq^* \rangle$, let us pick $N < \langle H(\kappa_0^{++}), \in \rangle$ such that

1. $N \cap \kappa \in \kappa$,
2. $|N| < \kappa$,
3. $T \in N$,
4. for every $A \subseteq N \cap \kappa$ there is $B \in N$ such that $B \cap N = A$.

Let $\tilde{N}$ be the transitive collapse of $N$, and let $\pi : \tilde{N} \to N$. Denote $\kappa \cap N$ by $\kappa(N)$.

Now the assumption that $\kappa$ was forced to be indestructible applied to the forcing $\langle P, \leq^* \rangle$, provides a $\tilde{N}$-generic set. Its image under $\pi$ can be easily turned into a condition in $\langle P, \leq^* \rangle$.

Let $p(N)$ be such a condition. Then for every $G^*$ generic for $\langle P, \leq^* \rangle$ with $p(N) \in G^*$, $N[p(N)]$ will be a generic extension of $N$ and an elementary submodel of $(H(\kappa_0^{++}))[G^*]$, satisfying the same properties as $N$.

Fix some $G^*$ like this.

Let $\delta = \sup(N[p(N)] \cap \kappa^+_\eta) = \sup(N \cap \kappa^+_\eta)$ and let $t_\delta \in \text{Lev}_\delta(T)$.

For every $\alpha \in N \cap \kappa^+_\eta$ and $\eta' < \eta$ consider the following statement:

$$\sigma_{\alpha, \eta'}^\eta \equiv \exists \xi < \kappa_{\eta'}(t_\delta > \chi (\alpha, \xi)).$$

Then, by the Prikry property and $\eta^+$-closure of $\langle P, \leq^* \rangle$, there is $\eta_\alpha < \eta$ and $p^\alpha \geq^* p(N)$, $p^\alpha \in G^*$ such that

$$p^\alpha \parallel \langle P, \leq \rangle \exists \xi < \kappa_{\eta_\alpha}(t_\delta > \chi (\alpha, \xi)).$$

Since $|N| < \kappa$, there will be $I(N) \subseteq \delta$ unbounded in $\delta$, $I(N) \in V[H]$; $\eta^* < \eta$ and $p^*(N) \geq^* p(N)$, $p^*(N) \in G^*$ such that for every $\alpha \in I(N)$,

$$p^*(N) \parallel \langle P, \leq \rangle \exists \xi < \kappa_{\eta^*}(t_\delta > \chi (\alpha, \xi)).$$

Now let $\alpha < \beta$, $\alpha, \beta \in I(N)$.

Consider the following set:

$$D_{\alpha, \beta} = \{ q \geq^* 0 \parallel \langle P, \leq \rangle \exists \alpha, \xi_\beta < \kappa_{\eta^*}((\alpha, \xi_\alpha) < \chi (\beta, \xi_\beta)) \}.$$
It is dense in \( \langle P, \leq^* \rangle \) above 0

So there is \( q \leq^* p(N), q \in D_{a, \beta} \). Then \( q \leq p^*(N) \), and hence, \( q \) must force the existence of such \( \xi_\alpha, \xi_\beta \).

So, \( p(N) \) forces this as well.

Appeal now to the supercompactness in \( V[H \ast G^+] \). So, there will be an unbounded in \( \bar{\kappa}_{\eta}^+ \) set \( I \in V[H \ast G^+] \) such that for every \( \alpha < \beta, \alpha, \beta \in I \) there is \( q \in G^* \),

\[ q \vdash_{\langle P, \leq \rangle} \exists \xi_\alpha, \xi_\beta < \kappa^*(\langle \alpha, \xi_\alpha \rangle < \mathcal{L} (\beta, \xi_\beta)) \]

Then there is \( q^* \in G^* \) such that (in \( V[H] \))

\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
of the forcing language $\langle P, \leq \rangle$.

By the Prikry condition, there is $\tilde{q} \geq^* q^*$ which decides $\sigma$.

If $\tilde{q} \Vdash \langle P, \leq \rangle \sigma$, then we are done.

Suppose that $\tilde{q} \Vdash \langle P, \leq \rangle \neg \sigma$.

Then $\tilde{q} \Vdash \langle P, \leq \rangle (I_\gamma$ is bounded in $\kappa^+_\eta)$.

Let $\zeta$ be a $\langle P, \leq \rangle$-name of sup($I_\gamma$).

Then, by the Prikry condition type argument showing that $\kappa^+_\eta$ is preserved after the forcing $\langle P, \leq \rangle$, there will be $q' \geq^* \tilde{q}$ and $\mu < \kappa^+_\eta$ such that

$$q' \Vdash \langle P, \leq \rangle (\zeta < \mu).$$

But

$$q^* \Vdash \langle P, \leq\rangle^{(I_\gamma$ is unbounded in $\kappa^+_\eta)}.$$ 

Hence, there are $q'' \geq^* q'$ and $\tau, \mu < \tau < \kappa^+_\eta$ such that $q'' \Vdash \langle q'', \tau \rangle \in I_\gamma$, for some $q'' \leq^* q''$. Then, clearly, $q'' \Vdash \langle P, \leq \rangle (\tau \in I_\gamma)$, which is impossible. Contradiction.

$\Box$ of the claim.

Assume for simplicity that already $0_P \Vdash \langle P, \leq \rangle (I_\gamma$ is unbounded in $\kappa^+_\eta)$.

Denote $\bar{\kappa}_\eta$ by $\lambda$.

Pick a supercompact cardinal $\kappa$, $\kappa^+ < \kappa < \kappa^*_\eta + 1$.

Consider $R = G^* \upharpoonright \langle P(\langle E(\alpha)\rangle_{\eta^* < \alpha < \eta^*}, \leq^* \rangle, \leq^* \rangle$, i.e. the Cohen functions above $\kappa^*_\eta$.

Clearly, $R$ is $V[H]$ generic for $\langle P(\langle E(\alpha)\rangle_{\eta^* < \alpha < \eta^*}, \leq^* \rangle, \leq^* \rangle$.

We will be interested in $I_{\gamma R}$, i.e. the interpretation (partial) of $\langle P(\langle E(\alpha)\rangle_{\eta^* < \alpha < \eta^*}, \leq^* \rangle, \leq^* \rangle$-name by $R$.

Use the indestructibility of $\kappa$ and find $j : V[H \ast R] \rightarrow M[H^*, R^*]$ witnessing $\lambda^+$—supercompactness of $\kappa$.

Let $\delta = \sup(j''\lambda^+) < j(\lambda^+)$.

Pick some $s \in j(\langle P(\langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle)$, $s \geq^* j(0_\eta) \upharpoonright j(\langle P(\langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle)$ and a $j(\langle \langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle)$-name $\gamma$ of an ordinal such that

$$s \Vdash j(\langle (P(\langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle)) (\gamma \geq \delta \land \gamma \in j(I_\gamma)_{R^*}).$$

Note that $s \geq^* j(0_\eta) \upharpoonright j(\langle P(\langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle)$ implies in particular that

$$s \Vdash j(\langle P(\langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle) (j''(I_\gamma)_{R^*}) = \lambda^+).$$

Claim 2 There is $s^* \in j(\langle P(\langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle)$, $s \leq^* s^*$ such that

$$s^* \Vdash j(\langle P(\langle E(\alpha)\rangle_{\eta^* < \eta^*}, \leq^* \rangle, \leq^* \rangle) (\{\alpha < \lambda^+ \mid j(\alpha) \in j(I_\gamma)_{R^*}\} = \lambda^+).$$

Note that we have two orderings $\leq^*$ and $\leq$. The claim is about the later one.
Proof. Suppose otherwise. Let

\[ \sigma \equiv \{ \alpha < \lambda^+ \mid j(\alpha) \in j(\mathcal{L})_{R^*} \} = \lambda^+. \]

Then for every \( s' \in j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*})) \), \( s \leq^* s' \) there is \( s^* \in j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*})) \), \( s' \leq^* s^* \) which forces \( \neg \sigma \), which means that the set \( \{ \alpha < \lambda^+ \mid j(\alpha) \in j(\mathcal{L})_{R^*} \} \) is bounded in \( \lambda^+ \).

The forcing \( (\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*}), \leq) \) satisfies \( \kappa_{\eta^*}^+ - \text{c.c.} \), hence there is \( \tau < \lambda^+ \) such that for every \( s^* \in j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*})) \), \( s \leq^* s^* \), if \( s^* \) decides \( \sigma \), then

\[ s^* \models j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*}), \leq)) \{ \alpha < \lambda^+ \mid j(\alpha) \in j(\mathcal{L})_{R^*} \} \subseteq \tau. \]

But we have

\[ s \models j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*}), \leq) (|j''(\mathcal{L})_{R^*}| = \lambda^+). \]

So, there are \( s' \in j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*})) \), \( s \leq^* s' \) and \( \tau', \tau \leq \tau' < \lambda^+ \) such that

\[ s' \models j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*}), \leq) (j(\tau') \in j(\mathcal{L})_{R^*}). \]

But this is impossible. Contradiction.

\( \Box \) of the claim.

Fix \( s^* \) as in the claim. Let \( S^* \) be \( j(\mathcal{P}(E(\alpha)_{\alpha \leq \eta^*}), \leq)) - \text{generic over } M[H^*, R^*] \) with \( s^* \in S^* \). Then, by Claim 2, in \( M[H^*, R^*, S^*] \) there is a set \( J' \subseteq \lambda^+ \) unbounded such that the following holds:

for every \( \alpha \in J' \), there are \( \zeta_\alpha, \xi_\alpha < \kappa_{\eta^*} \), \( s_\alpha \in S^*, r_\alpha \in R^* \) such that

\[ (s_\alpha, r_\alpha) \models j(\mathcal{P}(E(\beta)_{\beta < \eta^*}), \leq) ((j(\alpha), \xi_\alpha) < j(\mathcal{L}) (\gamma, \zeta_\alpha)). \]

We use here the \( \kappa_{\eta^*}^+ - \text{closure of } \leq^* \) – of \( j(\mathcal{P}(E(\beta)_{\beta < \eta^*}), \leq) \) in order to obtain \( r_\alpha \).

Then there will be \( \zeta^*, \xi^* < \kappa_{\eta^*} \) which work for an unbounded subset \( J \) of \( J' \), i.e., for every \( \alpha \in J \), there are \( s_\alpha \in S^*, r_\alpha \in R^* \) such that

\[ (s_\alpha, r_\alpha) \models j(\mathcal{P}(E(\beta)_{\beta < \eta^*}), \leq) ((j(\alpha), \xi^*) < j(\mathcal{L}) (\gamma, \zeta^*)). \]

Let \( S \) be the restriction of \( S^* \) to \( \mathcal{P}(E(\mu)_{\mu \leq \eta^*}) \).

By elementarity, then

for every \( \alpha, \beta \in J, \alpha < \beta \), there are \( s_\alpha \in S, r_\alpha \in R \) such that

\[ (s_\alpha, r_\alpha) \models (\mathcal{P}(E(\beta)_{\beta < \eta^*}), \leq) (\alpha, \xi^*) < j(\mathcal{L}) (\beta, \xi^*). \]

\( \footnote{Forces with respect to } \leq^*. \)
Note that $J$ need not be in $V[H, R, S]$, but rather in the further extension $V[H, R, S^+]$ by the forcing $j(\langle P(\alpha)_{\alpha \leq \eta^*}, \leq \rangle)/S$, which is basically the forcing for adding more Cohen subsets to $\kappa_{\eta^*}^+$ and cardinals below. \footnote{Note that the forcing $\langle P(\alpha)_{\alpha < \eta^*}, \leq \rangle$ is defined using one element Prikry sequences. Namely, once a non-direct extension was made over a coordinate $\beta$ (i.e. using $E(\beta)$), then the rest of the forcing over coordinate $\beta$ will be just adding Cohen subsets. In particular, if $\eta^* < \omega$, then it will be just a product Cohen forcings, after non-direct extension was made at each coordinate $k \leq \eta^*$. If $\eta^* \geq \omega$, then a non-direct extension over $\eta^*$ will bound the size of the forcing over smaller coordinates.}

Also, if $\beta \in J, \alpha < \beta$ and for some $s \in S, r \in R$,

$$\langle s, r \rangle \Vdash_{\langle P(\mu)_{\mu < \eta^*}, \leq \rangle} ((\alpha, \xi^*) < \mathcal{L}(\beta, \xi^*)),$$

then $\alpha \in J$.

Now, by Neeman [7], 3.4.3.7, we must have that $J \in V[H, R, S]$.

Back to $V[H]$, we assume for simplicity that already the weakest conditions (in $S, R$) decide $\xi^*$.

Let now $\kappa_{\eta^*} < \kappa$ be a supercompact below $\lambda$ with a corresponding embedding

$$j : V[H] \rightarrow M[H^*].$$

Assume for simplicity that $\eta = \omega$.

Let $n < \omega$ be the maximal such that $\kappa_n < \kappa$.

Split $G^* \upharpoonright \mathcal{P}(E(\mu)_{\mu^* < \eta})$ into $G_{\leq n}$ and $G_{> n}$.

$\kappa$ remains a supercompact in $V[H, G_{> n}]$.

Moreover, there are $H^* \ast G_{> n}^*$ in $V[H, G_{> n}]$ which are $M -$generic and $j$ extends to

$$j^{H^*, G_{> n}^*} : V[H, G_{> n}] \rightarrow M[H^*, G_{> n}^*].$$

Let $U(H^*, G_{> n}^*)$ be the corresponding normal ultrafilter over $\mathcal{P}_\kappa(\lambda^+)$. Recall that $J \in V[H, S, G^* \upharpoonright \mathcal{P}(E(\mu)_{\eta^* < \mu < \eta})]$. So, in order to have it, we need to add $S, G_{\leq n}^* \subseteq \mathcal{P}(E(\kappa)_{k \leq \eta^*})$.

As a result, a generic should be picked on the $M -$side. Now it should be forced.

Denote by $Q_{\leq n}$ the quotient forcing $j(\langle \langle \mathcal{P}(E(\kappa)_{k \leq \eta^*}), \leq \rangle \rangle)/S \times j(\langle \langle \mathcal{P}(E(\kappa)_{\eta^* < k \leq n}), \leq \rangle \rangle)/G_{\leq n}^*$, which is just a further Cohen forcing.

We work below with $Q_{\leq n} -$names which are assumed to have the desired properties as forced by the weakest condition in $Q_{\leq n}$.
Pick $\gamma \in j^{H*,G_{\leq n}^*}(J)$, $\gamma \geq \sup(j''\lambda^+)$. Then for every $\alpha \in J$ there will be $\langle s^\alpha, r^\alpha \rangle \in H^*, G_{\leq n}^*$, and $s^\alpha_{\leq n} \in Q_{\leq n}$ such that

$$\langle s^\alpha_{\leq n}, s^\alpha, r^\alpha \rangle \Vdash j^*(\langle P, \leq \rangle) \left( ((j(\bar{\alpha}), \bar{\xi}^*) <_{j^*H^*,G_{\leq n}^*}(\mathcal{U}) \langle \gamma, \xi^* \rangle) \right).$$

Pick a function $h^\alpha : \mathcal{P}_n(\lambda^+) \rightarrow \mathcal{P}$ which represents $\langle s^\alpha_{\leq n}, s^\alpha, r^\alpha \rangle$ in the ultrapower.

Now, if $\alpha < \beta, \alpha, \beta \in J$, then

$$\{P \in \mathcal{P}_n(\lambda^+) \mid h^\alpha(P) \land h^\beta(P) \Vdash \langle \langle (\alpha, \xi^*) <_{\mathcal{U}} (\beta, \xi^*) \rangle \rangle \} \in U(H^*, G_{\leq n}^*).$$

Turn now to $V[H]$. Let $q^\alpha \geq_0 0_\mathcal{P}, \alpha < \lambda^+$ and $q^\alpha \Vdash \langle \langle \rangle \rangle \alpha \in J$.

**Claim 3** Let $n < \omega$. There is a direct extension $q^{\alpha*}$ of $q^\alpha$ such that for every $\bar{v}$ from sets of measure one of $q^{\alpha*}_{\leq n}$ (i.e. from the first $n$-coordinates of $q^{\alpha*}$)

$j(q^{\alpha*} \sim \bar{v})$ is compatible with $s^\alpha_{\leq n}$.

**Proof.** Suppose otherwise. Then there is a direct extension $q^{\alpha*}$ of $q^\alpha$ such that for every $\bar{v}$ from sets of measure one of $q^{\alpha*}_{\leq n}$, $j(q^{\alpha*} \sim \bar{v})$ is incompatible with $s^\alpha_{\leq n}$.

Note that the critical point $\kappa$ of $j$ is above $\kappa_n$ and each coordinate $q^{\alpha*}(k), k \leq n$ has cardinality $\leq \kappa_k$. So, $j(q^{\alpha*}(k))$ is the pointwise image of $q^{\alpha*}(k)$, for every $k \leq n$. Moreover, $j$ does not move the set of measure one of $q^{\alpha*}(k)$, since it is a inside $V_\kappa$.

So, we can intersect this sets with those of $s^\alpha_{\leq n}$ (over the corresponding places). The result still in $V_\kappa$, and hence does move under $j$. This leads to contradiction. Namely, pick $\bar{v}$ from such intersections. Then, $j(q^{\alpha*} \sim \bar{v})$ will be compatible with $s^\alpha_{\leq n}$.

$\square$ of the claim.

Applying the claim for every $n < \omega$ and shrinking corresponding measure one sets, we will find a direct extension

$q^{\alpha**}$ of $q^\alpha$ such that for every $n < \omega$ and for every $\bar{v}$ from sets of measure one of $q^{\alpha**}_{\leq n}$,

$j_n(q^{\alpha**} \sim \bar{v})$ is compatible with $s^\alpha_{\leq n}$.

Force with $\langle \mathcal{P}, \leq \rangle$. Let $G$ be a generic. Suppose that $\alpha < \beta < \lambda^+$ and $q^{\alpha**}, q^{\beta**} \in G$.

Now back to $V[H]$, let $p \geq q^{\alpha**} \land q^{\beta**}$.

**Claim 4** There is $p' \geq p$ such that

$$p' \Vdash \langle \mathcal{P}, \leq \rangle \left( ((\alpha, \xi^*) <_{\mathcal{U}} (\beta, \xi^*)) \right).$$

**Proof.** There is $q \geq_0 q^{\alpha**} \land q^{\beta**}$ and $\bar{v}$ from sets of measures one of $q^{\alpha**} \land q^{\beta**}$ such that $p = q^\sim \bar{v}$.

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Let \( n < \omega \) be the number of coordinates involved in \( \bar{\nu} \). Pick a supercompact \( \kappa \) to be the least above \( \kappa_n \) and let \( j \) denotes the corresponding embedding.

Let us use the freedom that we have in choosing \( G^*, H^*, G^{**} \). Thus, assume that \( q \in G^* \). Then, \( j^{H^* G^{**}}(q) \in G^{**} \). So, it is compatible with \( \langle s^\alpha, r^\alpha \rangle \) and \( \langle s^\beta, r^\beta \rangle \).

Using the previous claim (Claim 3), we obtain a compatibility of \( j(q^{**} \wedge q^{**\sim} \bar{\nu}) \) with \( s^\alpha \sim \alpha \leq n \wedge s^\beta \sim \beta \leq n \), which implies those of \( j(q^{\sim} \bar{\nu}) \).

Pick a condition \( x \) to be stronger than \( j(q^{\sim} \bar{\nu}), s^\sim \alpha \leq n \wedge s^\sim \beta \leq n, \langle s^\alpha, r^\alpha \rangle \) and \( \langle s^\beta, r^\beta \rangle \).

Pick a function \( h : \mathcal{P}_\kappa(\lambda^+) \to \mathcal{P} \) which represents \( x \).

Then, \( \{ P \in \mathcal{P}_\kappa(\lambda^+) \mid h(P) \geq q^{\sim} \bar{\nu} \text{ and } h(P) \forces_{\langle P, \leq \rangle} ((\alpha, \xi^*) < \mathcal{X}(\beta, \xi^*)) \} \in U(H^*, G^{**}) \).

Now any \( h(P) \) with \( P \) from this set will be as desired.

\( \Box \) of the claim.

Now, by density, there is \( p' \in G \) such that

\[ p' \forces_{\langle P, \leq \rangle} (\langle \alpha, \xi^* \rangle < \mathcal{X}(\beta, \xi^*)). \]

Note that the only requirements on \( q^\alpha \) were that \( q^\alpha \geq^* 0_P, \alpha < \lambda^+ \) and \( q^\alpha \forces_{\langle P, \leq \rangle} \alpha \in \mathcal{J} \).

Consider all possibilities, i.e. let \( Y^\alpha \) be the set consisting of \( q^{**} \) given by Claim 3 with \( q^\alpha \) arbitrary such that \( q^\alpha \geq^* 0_P, \alpha < \lambda^+ \) and \( q^\alpha \forces_{\langle P, \leq \rangle} \alpha \in \mathcal{J} \).

The next claim completes the argument.

**Claim 5** There is \( \tilde{q} \geq^* 0_P \) such that

\[ \tilde{q} \forces_{\langle P, \leq \rangle} (\{ \alpha < \lambda^+ \mid Y^\alpha \cap G \neq \emptyset \} \text{ is unbounded in } \lambda^+) \]

**Proof.** Consider the statement:

\[ \sigma \equiv (\{ \alpha < \lambda^+ \mid Y^\alpha \cap G \neq \emptyset \} \text{ is unbounded in } \lambda^+) \]

of the forcing language \( \langle P, \leq \rangle \).

By the Prikry condition, there is \( \tilde{q} \geq^* 0_P \) which decides \( \sigma \).

If \( \tilde{q} \forces_{\langle P, \leq \rangle} \sigma \), then we are done.

Suppose that \( \tilde{q} \forces_{\langle P, \leq \rangle} \neg \sigma \).

Then \( \tilde{q} \forces_{\langle P, \leq \rangle} (\{ \alpha < \lambda^+ \mid Y^\alpha \cap G \neq \emptyset \} \text{ is bounded in } \lambda^+) \).

Let \( \zeta \) be a \( \langle P, \leq \rangle \)-name of a bound.
Then, by the Prikry condition type argument showing that $\bar{\kappa}^\eta = \lambda^+$ is preserved after the forcing $\langle P, \leq \rangle$, there will be $q' \geq^* \tilde{q}$ and $\mu < \bar{\kappa}^\eta$ such that

$$q' \models_{\langle P, \leq \rangle} (\zeta < \mu).$$

We have

$$q' \geq^* 0_P \models_{\langle P, \leq \rangle^*} (J \text{ is unbounded in } \bar{\kappa}^\eta).$$

Pick $\alpha, \mu < \alpha < \bar{\kappa}^\eta$ and $q'' \geq^* q'$ such that $q'' \models_{\langle P, \leq \rangle^*} \alpha \in J$. Then $q''^** \geq^* q''$ is in $Y^\alpha$, by the definition of $Y^\alpha$. Clearly, then

$$q''^** \models_{\langle P, \leq \rangle} (q''^** \in Y^\alpha \cap G).$$

Contradiction.

$\square$ of the claim.
2 \ -\text{SCH and the tree property for a club.}

In [2], O. Ben-Neria, C. Lambie-Hanson, S. Unger use the supercompact Radin forcing to construct a model in which both AP and SCH fail on a proper class club of singular cardinals. They asked whether it is possible to replace \(\neg\text{AP}\) by the tree property.

Here we would like to give an affirmative answer. Again, the initial assumption will be stronger than those used in [2], however no cardinals will change their cofinality.

\textbf{Theorem 2.1} Suppose that \(\theta\) is the least inaccessible cardinal which is a limit of supercompact cardinals.

Then there is cofinality preserving extension so that

- \(\theta\) remaining inaccessible,
- there is a club in \(\theta\) consisting of singular strong limit cardinals \(\nu\) such that
  1. \(2^\nu > \nu^+\),
  2. \(\nu^+\) has the tree property.

\textbf{Proof.} The construction of the previous section can be applied here, only replace \(\eta\) by an inaccessible cardinal \(\theta\).

Let \(\langle \delta_\alpha \mid \alpha < \theta \rangle\) be an increasing sequence of supercompact cardinals. Set \(\kappa_\alpha = \delta_{\alpha+1}\), for every \(\alpha < \theta\). Clearly, each \(\kappa_\alpha\) is strong. Repeat the previous construction using the sequence \(\langle \kappa_\alpha \mid \alpha < \theta \rangle\).

Note that given a limit \(\alpha < \theta\), we do not know in advance (i.e. without forcing with \(E(\alpha)\)) what will be \(2^{\kappa_\alpha}\), where, as before, \(\kappa_\alpha = \bigcup_{\beta < \alpha} \kappa_\beta\). So, if we have only boundedly many supercompacts below \(\kappa_\alpha\), then it is possible that there will be no supercompact in the interval \((2^{\kappa_\alpha}, \kappa_\alpha)\). However, having a supercompact inside \((\kappa_\alpha, \kappa_{\alpha+1})\), we can repeat the argument of the previous section just using \(\kappa_{\alpha+1}\) as the first strong in this argument.

\(\square\)
References


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