An other model with tree property and not SCH.

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May 28, 2018

Abstract

We will use the method for blowing up the power of a singular cardinal of [3] in order to get models with tree property on successors of singulars and not SCH. The advantage of the present technique is that no cardinal is collapsed or changes its cofinality. A question by O. Ben-Neria, C. Lambie-Hanson, S. Unger from [2] is answered.

1 A model in which SCH fail at a singular cardinal, but the tree property holds at its successor.

Such a model was first constructed by I. Neeman [7] for a singular of countable cofinality and later was generalized to uncountable one by D. Sinapova [10].

The forcing used in [7] is based on the forcing of [5]. We will use here the forcing of [3] instead and deal with countable and uncountable cofinalities simultaniously.

Fix a regular cardinal η. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals and let $\langle E_\alpha \mid \alpha < \eta \rangle$ be a sequence of extenders such that for every $\alpha < \eta$

1. $\eta < \kappa_0$,
2. $E(\alpha)$ is a $(\kappa_\alpha, \bar{\kappa}_\eta^{++})-$extender, where $\bar{\kappa}_\eta = \bigcup_{\alpha<\eta} \kappa_\alpha$,
3. $E(\alpha) \lhd E(\alpha + 1)$,
4. there is a supercompact cardinal in the interval (η, κ_0),
5. for every $\alpha < \eta$ there is a supercompact cardinal in the interval $(\kappa_\alpha, \kappa_{\alpha+1})$.

*The work was partially supported by Israel Science Foundation Grant No. 58/14. We are grateful to Spencer Unger for reading a draft of the paper, his corrections and comments.

Let $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq, \leq^* \rangle$ be the forcing of Section 2 of [3].
For every limit $\alpha \leq \eta$ denote $\bar{\kappa}_\alpha = \bigcup_{\alpha'<\alpha} \kappa_{\alpha'}$.
By [3], Section 2, it has the following properties:

1. $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq, \leq^* \rangle$ is a Prikry type forcing,
2. the forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$:
   (a) blows up the power of $\bar{\kappa}_\eta$ to $\bar{\kappa}_\eta^{++}$,
   (b) blows up the power of $\bar{\kappa}_\alpha$ above $\bar{\kappa}_\alpha^{+}$, for every limit $\alpha < \eta$,
   (c) preserves cardinals and cofinalities,
   (d) preserves strong limitness of each of κ_α's, for every $\alpha \leq \eta$, and $\bar{\kappa}_\alpha$'s, for every limit $\alpha \leq \eta$
   (e) does not add new subsets to κ_0.
3. The forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq^* \rangle$ is equivalent to the product of Cohen forcings $Cohen(\kappa_\alpha^+, \bar{\kappa}_\eta^{++})$ with full support for $\leq^*$ $-$extension of $0_{\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}}$.

We force first with the Laver type preparation forcings to ensure indestructibility of the relevant supercompact cardinals under directed closed forcings which preserve cardinals, as it is done in A. Apter [1]. Let H be a corresponding generic set.
It is easy to extend the extender $E(\alpha)$ and its elementary embedding in $V[H]$. Let us abuse the notation a bit and still denote the extension of $E(\alpha)$ in $V[H]$ by $E(\alpha)$.

Force with $\langle \mathcal{P}_{\langle E(\alpha)|\alpha<\eta\rangle}, \leq \rangle$. Let G be a generic. We claim that $V[H*G]$ is as desired, i.e. it satisfies $TP_{\bar{\kappa}_\eta^+}$ and $2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$.
$2^{\bar{\kappa}_\eta} = \bar{\kappa}_\eta^{++}$ follows by [3]. Let deal with the tree property.

The argument below follows [6] and [7].

Suppose that T is a $\bar{\kappa}_\eta^+$$-$tree in $V[H*G]$.
We can assume that for every $\alpha < \bar{\kappa}_\eta^+$, the level α of T is $\{(\alpha,\xi) \mid \xi < \bar{\kappa}_\eta\}$.

Let $\underset{\sim}{T}$ be a $\langle \mathcal{P}, \leq \rangle$$-$name of a $\bar{\kappa}_\eta^+$$-$tree T.
Suppose for simplicity that

$$0_{\mathcal{P}} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{T} \text{ is a } \bar{\kappa}_\eta^+ - \text{tree}).$$

Let $\kappa, \eta < \kappa < \kappa_0$ be a supercompact.
Now, in $V[H]$, using the indestructibility of supercompactness of κ under the forcing $\langle \mathcal{P}, \leq^* \rangle$, let us pick $N \prec \langle H(\bar{\kappa}_\eta^{+++}), \in \rangle$[1] such that

1. $N \cap \kappa \in \kappa$,
2. $|N| < \kappa$,
3. $\underset{\sim}{T} \in N$,
4. for every $A \subseteq N \cap \bar{\kappa}_\eta^+$ there is $B \in N$ such that $B \cap N = A$.

Let $\bar{N}$ be the transitive collapse of N, and let $\pi : \bar{N} \to N$. Denote $\kappa \cap N$ by $\kappa(N)$. Now the assumption that κ was forced to be indestructible applied to the forcing $\langle \mathcal{P}, \leq^* \rangle$, provides a $\bar{N}-$generic set. Its image under π can be easily turned into a condition in $\langle \mathcal{P}, \leq^* \rangle$. Let $p(N)$ be such a condition. Then for every G^* generic for $\langle \mathcal{P}, \leq^* \rangle$ with $p(N) \in G^*$, $N[p(N)]$ will be a generic extension of N and an elementary submodel of $(H(\bar{\kappa}_\eta^{+++}))[G^*]$, satisfying the same properties as N.
Fix some G^* like this.

Let $\delta = \sup(N[p(N)] \cap \bar{\kappa}_\eta^+) = \sup(N \cap \bar{\kappa}_\eta^+)$ and let $t_\delta \in Lev_\delta(T)$.
For every $\alpha \in N \cap \bar{\kappa}_\eta^+$ and $\eta' < \eta$ consider the following statement:

$$\sigma_\alpha^{\eta'} \equiv \exists \xi < \kappa_{\eta'}(t_\delta >_{\underset{\sim}{T}} (\alpha, \xi)).$$

Then, by the Prikry property and η^+-closure of $\langle \mathcal{P}, \leq^* \rangle$, there is $\eta_\alpha < \eta$ and $p^\alpha \geq^* p(N)$, $p^\alpha \in G^*$ such that

$$p^\alpha \Vdash_{\langle \mathcal{P}, \leq \rangle} \exists \xi < \kappa_{\eta_\alpha}(t_\delta >_{\underset{\sim}{T}} (\alpha, \xi)).$$

Since $|N| < \kappa$, there will be $I(N) \subseteq \delta$ unbounded in δ, $I(N) \in V[H]$, $\eta^* < \eta$ and $p^*(N) \geq^* p(N)$,
$p^*(N) \in G^*$ such that for every $\alpha \in I(N)$,

$$p^*(N) \Vdash_{\langle \mathcal{P}, \leq \rangle} \exists \xi < \kappa_{\eta^*}(t_\delta >_{\underset{\sim}{T}} (\alpha, \xi)).$$

Now let $\alpha < \beta, \alpha, \beta \in I(N)$.
Consider the following set:

$$D_{\alpha\beta} = \{q \geq^* 0_{\mathcal{P}} \mid q \parallel_{\langle \mathcal{P}, \leq \rangle} \exists \xi_\alpha, \xi_\beta < \kappa_{\eta^*}((\alpha, \xi_\alpha) <_{\underset{\sim}{T}} (\beta, \xi_\beta))\}.$$

[1]Alternatively, we can use a supercompact embedding and to work in the ultrapower.

It is dense in $\langle \mathcal{P}, \leq^* \rangle$ above $0_{\mathcal{P}}$.
So there is $q \leq^* p(N), q \in D_{\alpha\beta}$. Then $q \leq p^*(N)$, and hence, q must force the existence of such ξ_α, ξ_β.
So, $p(N)$ forces this as well.

Appeal now to the supercompactness in $V[H * G^*]$. So, there will be an unbounded in $\bar{\kappa}_\eta^+$ set $I \in V[H * G^*]$ such that for every $\alpha < \beta, \alpha, \beta \in I$ there is $q \in G^*$,

$$q \Vdash_{\langle \mathcal{P}, \leq \rangle} \exists \xi_\alpha, \xi_\beta < \kappa_{\eta^*}((\alpha, \xi_\alpha) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi_\beta)).$$

Then there is $q^* \in G^*$ such that (in $V[H]$)

$$(*) \qquad q^* \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \forall \alpha, \beta \in \underset{\sim}{I} (\alpha < \beta \to (\exists q \in \underset{\sim}{G}(\langle \mathcal{P}, \leq^* \rangle)$$
$$(q \Vdash_{\langle \mathcal{P}, \leq \rangle} \exists \xi_\alpha, \xi_\beta < \kappa_{\eta^*}((\alpha, \xi_\alpha) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi_\beta))))).$$

But then,
$(**)$ for every $\alpha < \beta$ and $\bar{q}_\sim \geq^* q^*$, [2] if $\bar{q}_\sim \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \alpha, \beta \in \underset{\sim}{I}$, then already for some choice sets of measures one for coordinates of $\bar{q}_\sim$ we will have

$$\bar{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} \exists \xi_\alpha, \xi_\beta < \kappa_{\eta^*}((\alpha, \xi_\alpha) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi_\beta)).$$

Since otherwise there will be $q' \geq^* \bar{q}$ such that

$$q' \Vdash_{\langle \mathcal{P}, \leq \rangle} \neg(\exists \xi_\alpha, \xi_\beta < \kappa_{\eta^*}((\alpha, \xi_\alpha) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi_\beta))).$$

Pick G' to be a generic for $\langle \mathcal{P}, \leq^* \rangle$ with q', and so, $\bar{q}, q^*$ inside. Then there is no $q \in G'$ such that

$$q \Vdash_{\langle \mathcal{P}, \leq \rangle} \exists \xi_\alpha, \xi_\beta < \kappa_{\eta^*}((\alpha, \xi_\alpha) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi_\beta)),$$

since every $q \in G'$ is $\leq^*$ $-$compatible with q', and hence also, $\leq$ $-$compatible with it.
But this contradicts $(*)$ above.

Note that $\underset{\sim}{I}$ is a $\langle \mathcal{P}, \leq^* \rangle-$canonical name of a subset of $\bar{\kappa}_\eta^+$, so it can be viewed as a $\langle \mathcal{P}, \leq \rangle$ as well. Let us argue its interpretation cannot be too small.

Claim 1 There is $\tilde{q} \geq^* q^*$ such that $\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{I}$ is unbounded in $\bar{\kappa}_\eta^+)$.

Proof. Consider the statement

$$\sigma \equiv \underset{\sim}{I} \text{ is unbounded in } \bar{\kappa}_\eta^+$$

[2]For $r \geq^* 0_{\mathcal{P}}$, we denote by $r_\sim$ the equivalence class or simply r with measure one sets removed.

of the forcing language $\langle \mathcal{P}, \leq \rangle$.
By the Prikry condition, there is $\tilde{q} \geq^* q^*$ which decides σ.
If $\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} \sigma$, then we are done.
Suppose that $\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} \neg\sigma$.
Then $\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{I}$ is bounded in $\bar{\kappa}_\eta^+)$.
Let $\underset{\sim}{\zeta}$ be a $\langle \mathcal{P}, \leq \rangle$−name of $\sup(\underset{\sim}{I})$.
Then, by the Prikry condition type argument showing that $\bar{\kappa}_\eta^+$ is preserved after the forcing $\langle \mathcal{P}, \leq \rangle$, there will be $q' \geq^* \tilde{q}$ and $\mu < \bar{\kappa}_\eta^+$ such that

$$q' \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{\zeta} < \mu).$$

But

$$q^* \Vdash_{\langle \mathcal{P}, \leq^* \rangle} (\underset{\sim}{I} \text{ is unbounded in } \bar{\kappa}_\eta^+).$$

Hence, there are $q'' \geq^* q'$ and $\tau, \mu < \tau < \bar{\kappa}_\eta^+$ such that $\langle q''', \tau \rangle \in \underset{\sim}{I}$, for some $q''' \leq^* q''$. Then, clearly, $q'' \Vdash_{\langle \mathcal{P}, \leq \rangle} (\tau \in \underset{\sim}{I})$, which is impossible. Contradiction.
$\square$ of the claim.
Assume for simplicity that already $0_{\mathcal{P}} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{I}$ is unbounded in $\bar{\kappa}_\eta^+)$.

Denote $\bar{\kappa}_\eta$ by λ.

Pick a supercompact cardinal κ, $\kappa_{\eta^*} < \kappa < \kappa_{\eta^*+1}$.

Consider $R = G^* \restriction \langle \mathcal{P}_{\langle E(\alpha) | \eta^* < \alpha < \eta \rangle}, \leq^* \rangle$, i.e. the Cohen functions above $\kappa_{\eta^*}^+$.
Clearly, R is $V[H]$ generic for $\langle \mathcal{P}_{\langle E(\alpha) | \eta^* < \alpha < \eta \rangle}, \leq^* \rangle$.
We will be interested in $\underset{\sim}{I}_R$, i.e. the interpretation (partial) of $\langle \mathcal{P}_{\langle E(\alpha) | \alpha < \eta \rangle}, \leq^* \rangle$−name by R.

Use the indestructibility of κ and find $j : V[H * R] \to M[H^*, R^*]$ witnessing λ^+−supercompactness of κ.
Let $\delta = \sup(j''\lambda^+) < j(\lambda^+)$.
Pick some $s \in j(\mathcal{P}_{\langle E(\alpha) | \alpha \leq \eta^* \rangle}), s \geq^* j(0_{\mathcal{P}}) \restriction j(\mathcal{P}_{\langle E(\alpha) | \alpha \leq \eta^* \rangle})$ and a $j(\langle \mathcal{P}_{\langle E(\alpha) | \alpha \leq \eta^* \rangle}, \leq \rangle)$−name $\underset{\sim}{\gamma}$ of an ordinal such that

$$s \Vdash_{j(\langle \mathcal{P}_{\langle E(\alpha) | \alpha \leq \eta^* \rangle}, \leq \rangle)} (\underset{\sim}{\gamma} \geq \delta \wedge \underset{\sim}{\gamma} \in j(\underset{\sim}{I})_{R^*}).$$

Note that $s \geq^* j(0_{\mathcal{P}}) \restriction j(\mathcal{P}_{E(\alpha) | \alpha \leq \eta^* \rangle})$ implies in particular that

$$s \Vdash_{j(\langle \mathcal{P}_{\langle E(\alpha) | \alpha \leq \eta^* \rangle}, \leq^* \rangle)} (|j''(\underset{\sim}{I})_{R^*})| = \lambda^+).$$

Claim 2 There is $s^* \in j(\mathcal{P}_{\langle E(\alpha) | \alpha \leq \eta^* \rangle}), s \leq^* s^*$ such that

$$s^* \Vdash_{j(\langle \mathcal{P}_{\langle E(\alpha) | \alpha \leq \eta^* \rangle}, \leq \rangle)} (|\{\alpha < \lambda^+ \mid j(\alpha) \in j(\underset{\sim}{I})_{R^*}\}| = \lambda^+).^3$$

[3]Note that we have two orderings $\leq^*$ and $\leq$. The claim is about the later one.

Proof. Suppose otherwise. Let

$$\sigma \equiv |\{\alpha < \lambda^+ \mid j(\alpha) \in j(\underset{\sim}{I})_{R^*}\}| = \lambda^+.$$

Then for every $s' \in j(\mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}), s \leq^* s'$ there is $s^* \in j(\mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}), s' \leq^* s^*$ which forces $\neg\sigma$, which means that the set $\{\alpha < \lambda^+ \mid j(\alpha) \in j(\underset{\sim}{I})_{R^*}\}$ is bounded in λ^+.
The forcing $\langle \mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}, \leq \rangle$ satisfies $\kappa_{\eta^*}^{++}-$c.c., hence there is $\tau < \lambda^+$ such that for every $s^* \in j(\mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}), s \leq^* s^*$, if s^* decides σ, then

$$s^* \Vdash_{j(\langle \mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}, \leq \rangle)} (\{\alpha < \lambda^+ \mid j(\alpha) \in j(\underset{\sim}{I})_{R^*}\} \subseteq \tau).$$

But we have

$$s \Vdash_{j(\langle \mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}, \leq^*)} (|j''(\underset{\sim}{I})_{R^*})| = \lambda^+).^{4}$$

So, there are $s' \in j(\mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}), s \leq^* s'$ and $\tau', \tau \leq \tau' < \lambda^+$ such that

$$s' \Vdash_{j(\langle \mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}, \leq^*)} (j(\tau') \in j(\underset{\sim}{I})_{R^*}).$$

But this is impossible. Contradiction.
$\square$ of the claim.

Fix s^* as in the claim. Let S^* be $j(\langle \mathcal{P}_{\langle E(\alpha)|\alpha \leq \eta^* \rangle}, \leq \rangle)-$generic over $M[H^*, R^*]$ with $s^* \in S^*$. Then, by Claim 2, in $M[H^*, R^*, S^*]$ there is a set $J' \subseteq \lambda^+$ unbounded such that the following holds:
for every $\alpha \in J'$, there are $\zeta_\alpha, \xi_\alpha < \kappa_{\eta^*}$, $s_\alpha \in S^*, r_\alpha \in R^*$ such that

$$(s_\alpha, r_\alpha) \Vdash_{j(\langle \mathcal{P}_{\langle E(\beta)|\beta < \eta \rangle}, \leq \rangle)} ((j(\alpha), \xi_\alpha) <_{j(\underset{\sim}{\mathcal{I}})} (\gamma, \zeta_\alpha)).$$

We use here the $\kappa_{\eta^*}^+-$closure of $\leq^* -$ of $j(\mathcal{P}_{E(\beta)|\eta^* < \beta < \eta \rangle})$ in order to obtain r_α.

Then there will be $\zeta^*, \xi^* < \kappa_{\eta^*}$ which work for an unbounded subset J of J', i.e.
for every $\alpha \in J$, there are $s_\alpha \in S^*, r_\alpha \in R^*$ such that

$$(s_\alpha, r_\alpha) \Vdash_{j(\langle \mathcal{P}_{\langle E(\beta)|\beta < \eta \rangle}, \leq \rangle)} ((j(\alpha), \xi^*) <_{j(\underset{\sim}{\mathcal{I}})} (\gamma, \zeta^*)).$$

Let S be the restriction of S^* to $\mathcal{P}_{\langle E(\mu)|\mu \leq \eta^* \rangle}$.
By elementarity, then
for every $\alpha, \beta \in J, \alpha < \beta$, there are $s_\alpha \in S, r_\alpha \in R$ such that

$$(s_\alpha, r_\alpha) \Vdash_{\langle \mathcal{P}_{\langle E(\beta)|\beta < \eta \rangle}, \leq \rangle} ((\alpha, \xi^*) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi^*)).$$

[4] Forces with respect to $\leq^*$.

Note that J need not be in $V[H,R,S]$, but rather in the further extension $V[H,R,S^*]$ by the forcing $j(\langle \mathcal{P}_{\langle E(\alpha)|\alpha\leq\eta^*\rangle}, \leq \rangle)/S$, which is basically the forcing for adding more Cohen subsets to $\kappa^+_{\eta^*}$ and cardinals below. [5]
Also, if $\beta \in J, \alpha < \beta$ and for some $s \in S, r \in R$,

$$\langle s,r\rangle \Vdash_{\langle \mathcal{P}_{\langle E(\mu)|\mu<\eta\rangle},\leq\rangle} ((\alpha,\xi^*) <_{\underset{\sim}{\mathcal{I}}} (\beta,\xi^*)),$$

then $\alpha \in J$.
Now, by Neeman [7], 3.4,3.7, we must have that $J \in V[H,R,S]$.

Back to $V[H]$, we assume for simplicity that already the weakest conditions (in S,R) decide ξ^*.

Let now $\kappa_{\eta^*} < \kappa$ be a supercompact below λ with a corresponding embedding

$$j : V[H] \to M[H^*].$$

Assume for simplicity that $\eta = \omega$.
Let $n < \omega$ be the maximal such that $\kappa_n < \kappa$.
Split $G^* \restriction \mathcal{P}_{\langle E(\mu)|\eta^*<\mu<\eta\rangle}$ into $G^*_{\leq n}$ and $G^*_{>n}$.
κ remains a supercompact in $V[H,G^*_{>n}]$.
Moreover, there are $H^* * G^{**}_{>n}$ in $V[H,G^*_{>n}]$ which are $M-$generic and j extends to

$$j^{H^*,G^{**}_{>n}} : V[H,G^*_{>n}] \to M[H^*,G^{**}_{>n}].$$

Let $U(H^*,G^{**}_{>n})$ be the corresponding normal ultrafilter over $\mathcal{P}_\kappa(\lambda^+)$.
Recall that $J \in V[H,S,G^* \restriction \mathcal{P}_{\langle E(\mu)|\eta^*<\mu<\eta\rangle}]$. So, in order to have it, we need to add $S, G^*_{\leq n} \subseteq \mathcal{P}_{\langle E(k)|k\leq n\rangle}$.
As a result, a generic should be picked on the $M-$side. Now it should be forced.
Denote by $Q_{\leq n}$ the quotient forcing $j(\langle \mathcal{P}_{\langle E(k)|k\leq\eta^*\rangle}, \leq^* \rangle)/S \times j(\langle \mathcal{P}_{\langle E(k)|\eta^*<k\leq n\rangle}, \leq^* \rangle)/G^*_{\leq n}$, which is just a further Cohen forcing.
We work below with $Q_{\leq n}-$names which are assumed to have the desired properties as forced by the weakest condition in $Q_{\leq n}$.

[5]Note that the forcing $\langle \mathcal{P}_{\langle E(\beta)|\beta<\eta^*\rangle}, \leq, \leq^* \rangle$ is defined using one element Prikry sequences. Namely, once a non-direct extension was made over a coordinate β (i.e. using $E(\beta)$), then the rest of the forcing over coordinate β will be just adding Cohen subsets. In particular, if $\eta^* < \omega$, then it will be just a product Cohen forcings, after non-direct extension was made at each coordinate $k \leq \eta^*$. If $\eta^* \geq \omega$, then a non-direct extension over η^* will bound the size of the forcing over smaller coordinates.

Pick $\underset{\sim}{\gamma} \in j^{H^*,G^{**}_{>n}}(\underset{\sim}{J}), \gamma \geq \sup(j''\lambda^+)$.
Then for every $\alpha \in J$ there will be $\langle s^\alpha, r^\alpha\rangle \in H^*, G^{**}_{>n}$ and $\underset{\sim}{s}^\alpha_{\leq n} \in Q_{\leq n}$ such that

$$\langle \underset{\sim}{s}^\alpha_{\leq n}, s^\alpha, r^\alpha\rangle \Vdash_{j^*(\langle \mathcal{P},\leq\rangle)} ((j(\check{\alpha}),\xi^*) <_{j^{H^*,G^{**}_{>n}}(\underset{\sim}{\mathcal{I}})} (\gamma,\xi^*)).$$

Pick a function $h^\alpha : \mathcal{P}_\kappa(\lambda^+) \to \mathcal{P}$ which represents $\langle \underset{\sim}{s}^\alpha_{\leq n}, s^\alpha, r^\alpha\rangle$ in the ultrapower.

Now, if $\alpha < \beta, \alpha, \beta \in J$, then

$$\{P \in \mathcal{P}_\kappa(\lambda^+) \mid h^\alpha(P) \wedge h^\beta(P) \Vdash_{\langle \mathcal{P},\leq\rangle} ((\alpha,\xi^*) <_{\underset{\sim}{\mathcal{I}}} (\beta,\xi^*))\} \in U(H^*, G^{**}_{>n}).$$

Turn now to $V[H]$. Let $q^\alpha \geq^* 0_{\mathcal{P}}, \alpha < \lambda^+$ and $q^\alpha \Vdash_{\langle \mathcal{P},\leq^*\rangle} \alpha \in \underset{\sim}{J}$.

Claim 3 Let $n < \omega$. There is a direct extension $q^{\alpha *}$ of q^α such that for every $\vec{\nu}$ from sets of measure one of $q^{\alpha *}_{\leq n}$ (i.e. from the first $n-$coordinates of $q^{\alpha *}$)
$j(q^{\alpha *\frown}\vec{\nu})$ is compatible with $\underset{\sim}{s}^\alpha_{\leq n}$.

Proof. Suppose otherwise. Then there is a direct extension $q^{\alpha *}$ of q^α such that for every $\vec{\nu}$ from sets of measure one of $q^{\alpha *}_{\leq n}$, $j(q^{\alpha *\frown}\vec{\nu})$ is incompatible with $\underset{\sim}{s}^\alpha_{\leq n}$.

Note that the critical point κ of j is above κ_n and each coordinate $q^{\alpha *}(k), k \leq n$ has cardinality $\leq \kappa_k$. So, $j(q^{\alpha *}(k))$ is the pointwise image of $q^{\alpha *}(k)$, for every $k \leq n$. Moreover, j does not move the set of measure one of $q^{\alpha *}(k)$, since it is a inside V_κ.
So, we can intersect this sets with those of $\underset{\sim}{s}^\alpha_{\leq n}$ (over the corresponding places). The result still in V_κ, and hence does move under j. This leads to contradiction. Namely, pick $\vec{\nu}$ from such intersections. Then, $j(q^{\alpha *\frown}\vec{\nu})$ will be compatible with $\underset{\sim}{s}^\alpha_{\leq n}$.
$\square$ of the claim.

Applying the claim for every $n < \omega$ and shrinking corresponding measure one sets, we will find a direct extension
$q^{\alpha **}$ of q^α such that for every $n < \omega$ and for every $\vec{\nu}$ from sets of measure one of $q^{\alpha **}_{\leq n}$, $j_n(q^{\alpha **\frown}\vec{\nu})$ is compatible with $\underset{\sim}{s}^\alpha_{\leq n}$.

Force with $\langle \mathcal{P}, \leq\rangle$. Let G be a generic. Suppose that $\alpha < \beta < \lambda^+$ and $q^{\alpha **}, q^{\beta **} \in G$. Now back to $V[H]$, let $p \geq q^{\alpha **} \wedge q^{\beta **}$.

Claim 4 There is $p' \geq p$ such that

$$p' \Vdash_{\langle \mathcal{P},\leq\rangle} ((\alpha,\xi^*) <_{\underset{\sim}{\mathcal{I}}} (\beta,\xi^*)).$$

Proof. There is $q \geq^* q^{\alpha **} \wedge q^{\beta **}$ and $\vec{\nu}$ from sets of measures one of $q^{\alpha **} \wedge q^{\beta **}$ such that $p = q^\frown\vec{\nu}$.

Let $n < \omega$ be the number of coordinates involved in $\vec{\nu}$. Pick a supercompact κ to be the least above κ_n and let j denotes the corresponding embedding.
Let us use the freedom that we have in choosing G^*, H^*, G^{**}. Thus, assume that $q \in G^*$. Then, $j^{H^*,G^{**}}(q) \in G^{**}$. So, it is compatible with $\langle s^\alpha, r^\alpha \rangle$ and $\langle s^\beta, r^\beta \rangle$.
Using the previous claim (Claim 3), we obtain a compatibility of $j(q^{\alpha **} \wedge q^{\beta **}{}^\frown \vec{\nu})$ with $\underset{\sim}{s}^\alpha_{\leq n} \wedge \underset{\sim}{s}^\beta_{\leq n}$, which implies those of $j(q^\frown \vec{\nu})$.
Pick a condition x to be stronger than $j(q^\frown \vec{\nu})$, $\underset{\sim}{s}^\alpha_{\leq n} \wedge \underset{\sim}{s}^\beta_{\leq n}$, $\langle s^\alpha, r^\alpha \rangle$ and $\langle s^\beta, r^\beta \rangle$.
Pick a function $h : \mathcal{P}_\kappa(\lambda^+) \to \mathcal{P}$ which represents x.
Then,

$$\{P \in \mathcal{P}_\kappa(\lambda^+) \mid h(P) \geq q^\frown \vec{\nu} \text{ and } h(P) \Vdash_{\langle \mathcal{P}, \leq \rangle} ((\alpha, \xi^*) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi^*))\} \in U(H^*, G^{**}_{>n}).$$

Now any $h(P)$ with P from this set will be as desired.
$\square$ of the claim.

Now, by density, there is $p' \in G$ such that

$$p' \Vdash_{\langle \mathcal{P}, \leq \rangle} ((\alpha, \xi^*) <_{\underset{\sim}{\mathcal{I}}} (\beta, \xi^*)).$$

Note that the only requirements on q^α were that $q^\alpha \geq^* 0_{\mathcal{P}}, \alpha < \lambda^+$ and $q^\alpha \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \alpha \in \underset{\sim}{J}$. Consider all possibilities, i.e. let Y^α be the set consisting of $q^{\alpha **}$ given by Claim 3 with q^α arbitrary such that $q^\alpha \geq^* 0_{\mathcal{P}}, \alpha < \lambda^+$ and $q^\alpha \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \alpha \in \underset{\sim}{J}$.

The next claim completes the argument.

Claim 5 There is $\tilde{q} \geq^* 0_{\mathcal{P}}$ such that

$$\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\{\alpha < \lambda^+ \mid Y^\alpha \cap \underset{\sim}{G} \neq \emptyset\} \text{ is unbounded in } \lambda^+).$$

Proof. Consider the statement:

$$\sigma \equiv (\{\alpha < \lambda^+ \mid Y^\alpha \cap \underset{\sim}{G} \neq \emptyset\} \text{ is unbounded in } \lambda^+)$$

of the forcing language $\langle \mathcal{P}, \leq \rangle$.
By the Prikry condition, there is $\tilde{q} \geq^* 0_{\mathcal{P}}$ which decides σ.
If $\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} \sigma$, then we are done.
Suppose that $\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} \neg\sigma$.
Then $\tilde{q} \Vdash_{\langle \mathcal{P}, \leq \rangle} (\{\alpha < \lambda^+ \mid Y^\alpha \cap \underset{\sim}{G} \neq \emptyset\}$ is bounded in $\lambda^+)$.
Let $\underset{\sim}{\zeta}$ be a $\langle \mathcal{P}, \leq \rangle$−name of a bound.

Then, by the Prikry condition type argument showing that $\bar{\kappa}_\eta^+ = \lambda^+$ is preserved after the forcing $\langle \mathcal{P}, \leq \rangle$, there will be $q' \geq^* \tilde{q}$ and $\mu < \bar{\kappa}_\eta^+$ such that

$$q' \Vdash_{\langle \mathcal{P}, \leq \rangle} (\underset{\sim}{\zeta} < \mu).$$

We have

$$q' \geq^* 0_{\mathcal{P}} \Vdash_{\langle \mathcal{P}, \leq^* \rangle} (\underset{\sim}{J} \text{ is unbounded in } \bar{\kappa}_\eta^+).$$

Pick $\alpha, \mu < \alpha < \bar{\kappa}_\eta^+$ and $q'' \geq^* q'$ such that $q'' \Vdash_{\langle \mathcal{P}, \leq^* \rangle} \alpha \in \underset{\sim}{J}$. Then $q''^{**} \geq^* q''$ is in Y^α, by the definition of Y^α. Clearly, then

$$q''^{**} \Vdash_{\langle \mathcal{P}, \leq \rangle} (q''^{**} \in Y^\alpha \cap \underset{\sim}{G}).$$

Contradiction.
$\square$ of the claim.

2 ¬SCH and the tree property for a club.

In [2], O. Ben-Neria, C. Lambie-Hanson, S. Unger use the supercompact Radin forcing to constract a model in which both AP and SCH fail on a proper class club of singular cardinals. They asked whether it is possible to replace ¬AP by the tree property.
Here we would like to give an affirmative answer. Again, the initial assumption will be stronger than those used in [2], however no cardinals will change their cofinality.

Theorem 2.1 *Suppose that θ is the least inaccessible cardinal which is a limit of supercompact cardinals.*
Then there is cofinality preserving extension so that

- *θ remaining inaccessible,*
- *there is a club in θ consisting of singular strong limit cardinals ν such that*
  1. *$2^\nu > \nu^+$,*
  2. *ν^+ has the tree property.*

Proof. The construction of the previous section can be applied here, only replace η by an inaccessible cardinal θ.
Let $\langle \delta_\alpha \mid \alpha < \theta \rangle$ be an increasing sequence of supercompact cardinals. Set $\kappa_\alpha = \delta_{\alpha+1}$, for every $\alpha < \theta$. Clearly, each κ_α is strong. Repeat the previous construction using the sequence $\langle \kappa_\alpha \mid \alpha < \theta \rangle$.
Note that given a limit $\alpha < \theta$, we do not know in advance (i.e. without forcing with $E(\alpha)$) what will be $2^{\bar{\kappa}_\alpha}$, where, as before, $\bar{\kappa}_\alpha = \bigcup_{\beta<\alpha} \kappa_\beta$. So, if we have only boundedly many supercompacts below κ_α, then it is possible that there will be no supercompact in the interval $(2^{\bar{\kappa}_\alpha}, \kappa_\alpha)$. However, having a supercompact inside $(\kappa_\alpha, \kappa_{\alpha+1})$, we can repeat the argument of the previous section just using $\kappa_{\alpha+1}$ as the first strong in this argument.
□

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