Simpler Short Extenders Forcing- Preserving Strong Cardinals

Moti Gitik
School of Mathematical Sciences
Raymond and Beverly Sackler Faculty of Exact Science
Tel Aviv University
Ramat Aviv 69978, Israel

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Abstract

Our aim is to define a version of a simpler short extenders forcing preserving strong cardinals.

1 The Main Preparation Forcing

In this section we will redefine the preparation forcing of [6] in order to allow eventually to preserve strong cardinals. The definition will follow those of [6] with certain additions.

Fix two cardinals κ and θ such that $\kappa < \theta$ and θ is regular.

Definition 1.1 The set $\mathcal{P}'$ consists of all sequences of triples.

$$\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$$

such that

1. s is a closed set of cardinals from the interval $[\kappa^+, \theta)$ satisfying the following:

   (a) $|s \cap \delta| < \delta$ for each inaccessible $\delta \in [\kappa^+, \theta)$

   (b) $\kappa^+ \in s$

   (c) if $\rho^+ \in s$ and $\rho \geq \kappa^+$, then $\rho \in s$

   (d) if $\rho \in s$ is singular, then s is unbounded in ρ and $\rho^+ \in s$.

2. For every $\tau \in s$, $A^{0\tau}$ is a subset of $H(\theta)$ closed under certain basic operations specified below.

   (a) $|A^{0\tau}| = \tau$ and $A^{0\tau} \supseteq \tau$

   (b) ${}^{cf\tau>}A^{0\tau} \subseteq A^{0\tau}$

3. If $\tau, \tau' \in s$ and $\tau < \tau'$ then $A^{0\tau} \subseteq A^{0\tau'}$

4. If τ is a limit point of s, then $A^{0\tau} = \cup\{A^{0\rho} \mid \rho \in s \cap \tau\}$.

5. For every $\tau \in s$, $A^{1\tau}$ is a set of at most τ many elementary submodels of $A^{0\tau}$ such that

   (a) $A^{0\tau} \in A^{1\tau}$ and each element of $A^{1\tau}\backslash\{A^{0\tau}\}$ belongs to $A^{0\tau}$

   (b) if $B \in A^{1\tau}$, then $\tau \subseteq B$

   (c) if $B \in A^{1\tau}$ then $\tau \in B$

   (d) if $A, B \in A^{1\tau}$ and $B \subset A$, then $B \in A$

   In particular, the above condition (d) imply that $\langle A^{1\tau}, \subseteq \rangle$ is well founded.

   Let $A \in A^{1\tau}$. We define $otp_\tau(A)$ to be $\sup\{otp(C) \mid C \subseteq \mathcal{P}(A) \cap A^{1\tau}$ and C is a chain under the inclusion relation$\}$.

   Further, we shall list more properties of $A^{1\tau}$. Let us now turn to C^τ.

6. For every $\tau \in s$, $C^\tau : A^{1\tau} \to \mathcal{P}(A^{1\tau})$ is a function such that

   (a) (Closure and maximality condition) for each $A \in A^{1\tau}$, $C^\tau(A)$ is a closed chain (under inclusion) of elements of $\mathcal{P}(A) \cap A^{1\tau}$ of the length $otp_\tau(A)$ and there is no chain in $\mathcal{P}(A) \cap A^{1\tau}$ that properly includes $C^\tau(A)$.

      In particular, this means that there are chains of the maximal length (i.e. $otp_\tau(A)$ which was defined as a supremum is really maximum) and $C^\tau(A)$ is one of them.

   (b) (Coherency condition) if $B \in C^\tau(A)$ then $C^\tau(B)$ is the initial segment of $C^\tau(A)$ which starts with B.

   (c) (Unboundedness condition) If $otp_\tau(A) - 1$ is a limit ordinal (note that A itself is always the last member of $C^\tau(A)$, hence $otp_\tau(A)$ is always a successor ordinal) then $C^\tau(A)\backslash\{A\}$ is unbounded in A, i.e. $\cup(C^\tau(A)\backslash\{A\}) = A$.
   We call A in such a case **a limit model** and otherwise **a successor** one. Note that if $B \in A^{1\tau}$, $B \subsetneq A$ then $B \in A$ and hence B is included in a member of $C^\tau(A)\backslash\{A\}$.

(d) if $A \in A^{1\tau}$ is a successor model, then ${}^{cf(\tau)>}A \subseteq A$

7. If $A, B \in A^{1\tau}$ then $otp(A) = otp(B)$ iff $otp_\tau(A) = otp_\tau(B)$.

Let us introduce one basic notion - Δ - system type.

Let $F_0, F_1, F \in A^{1\mu}$ for some $\mu \in s$. We say then that F_0, F_1, F are of a **Δ-system type** iff

(a) F is a successor model

(b) F_0, F_1 are its immediate predecessors (under the inclusion relation) and $otp_\mu F_0 = otp_\mu F_1$ (in particular, the conclusion of (27) above holds and in particular F_0, F_1 are isomorphic over $F_0 \cap F_1$)

(c) $F \in C^\mu(A^{0\mu})$

(d) one of F_0, F_1 is in $C^\mu(F)$

(e) there are $G_0, G_0^*, G_1, G_1^*, G^* \in C^\tau(A^{0\tau})$,
for $\tau = \min(s \backslash \mu + 1)$ such that

(i) $G_0 \in F_0$

(ii) $G_1 \in F_1$

(iii) $F_0 \cap F_1 = F_0 \cap G_0 = F_1 \cap G_1$

(iv) $F_0 \in G_0^*, F_1 \in G_1^*, F \in G^*$ and G_0^*, G_1^*, G^* are the least under the inclusion elements of $A^{1\tau}$ including F_0, F_1, F respectively

(v) $G_0 \in F_0 \in G_0^* \in G_1 \in F_1 \in G_1^* \in F \in G^*$.
Note that $\tau = \mu^+$ unless it is an inaccessible.

We will say that F_0, F_1, F are of a **Δ-system type according to a chain** X if the conditions (a) - (e) above are satisfied, only in (e) we have $C^\tau(A^{0\tau})$ replaced by X.

Let us call a triple $F_0, F_1, F \in A^{1\mu}$ **a suitable for switching** iff

(a) F_0, F_1, F are of a Δ-system type

(b) for each $\tau \in s \cap \mu$, $F \in A^{0\tau}$ and if $A \in C^\tau(A^{0\tau})$ is the first with $F \in A$, then its immediate predecessor A^- in $C^\tau(A^{0\tau})$ is in F. Moreover, if there are $A_0, A_1 \in A^{1\tau}$ such that the triple A_0, A_1, A is a is of a Δ- system type, then $\sup(A_0) < \sup(A_1)$, implies $A_0 \in F \in A_1$.

Note that in the last case, i.e. if there are $A_0, A_1 \in A^{1\tau}$ such that the triple A_0, A_1, A is of a Δ- system type and $\sup(A_0) < \sup(A_1)$, then it will be impossible to have $A_1 \in F \in A$ by 1.1(11). Also, by 1.1(29), we will must to have $F_0, F_1 \in A_1$ as well.

Let us say that F_0, F_1, F are **suitable for switching according to a chain** X if the above conditions are satisfied, with C^μ replaced by X.

Let us state some preliminary definitions.

Definition 1.2 Suppose now that a triple F_0, F_1, F is a suitable for switching, $F \in C^\mu(A^{0\mu}), F_0 \in C^\mu(F)$. Define

$$\langle C^\nu(A^{0\nu})_F | \nu \in s\rangle,$$

the switch of

$$\langle C^\nu(A^{0\nu}) | \nu \in s\rangle,$$

by F as follows:

$$C^\nu(A^{0\nu})_F = C^\nu(A^{0\nu}),$$

for each $\nu \in s \backslash \mu + 1$,

$$C^\mu(A^{0\mu})_F = (C^\mu(A^{0\mu}) \backslash C^\mu(F_0))^\frown C^\mu(F_1),$$

$$C^\nu(A^{0\nu})_F = (C^\nu(A^{0\nu}) \backslash C^\nu(A))^\frown C^\nu(\pi_{F_0 F_1}[A]),$$

for each $\nu \in s \cap \mu$, where $A \in C^\nu(A^{0\nu})$ is the maximal element of $C^\nu(A^{0\nu})$ contained in F_0.

Note that for $\nu \in s \cap \mu$, $C^\nu(A^{0\nu})_F$ is still continuous. It is also increasing due to the choice of F_0, F_1, F as a suitable for switching pair and further condition 1.1(29).

Definition 1.3 Let us call

$$\bigcup\{C^\nu(A^{0\nu}) | \nu \in s\}$$

the **central line**.

Suppose now that a triple B_0, B_1, B is a suitable for switching, $B \in C^\mu(A^{0\mu}), B_0 \in C^\mu(B)$. Define the **line 1 generated by** B to be

$$\bigcup\{C^\nu(A^{0\nu})_B | \nu \in s\}.$$

Continue, let a triple B^1_0, B^1_1, B^1 be a suitable for switching according to the line 1, i.e. according to increasing parts of $C^{\mu^1}(A^{0\mu^1})_B$, for some $\mu^1 \in s$, $B^1 \in C^{\mu^1}_B(A^{0\mu^1}), B^1_0 \in C^{\mu^1}(B^1)$. Define the **line 2 generated by** B, B^1.

It will be

$$\bigcup\{C^\nu(A^{0\nu})_{BB^1}|\nu \in s\},$$

where

$$C^\nu(A^{0\nu})_{BB^1} = C^\nu(A^{0\nu})_B,$$

for each $\nu \in s \backslash \mu^1 + 1$,

$$C^{\mu^1}(A^{0\mu^1})_{BB_1} = (C^{\mu^1}(A^{0\mu^1})_B \backslash C^{\mu^1}(B^1_0))^\frown C^{\mu^1}(B^1_1),$$

$$C^\nu(A^{0\nu})_{BB^1} = (C^\nu(A^{0\nu})_B \backslash C^\nu(A))^\frown C^\nu(\pi_{B^1_0 B^1_1}[A]),$$

for each $\nu \in s \cap \mu^1$, where $A \in C^\nu(A^{0\nu})_B$ is the maximal element of $C^\nu(A^{0\nu})_B$ contained in B^1_0.

Continue by induction and define **line** n for each $n < \omega$.

Definition 1.4 (General distance)

Let $A \in A^{1\nu}$ for some $\nu \in s$. Define $gd(A)$ **the general distance from the central line** to be 0 if $A \in C^\nu(A^{0\nu})$. If $A \notin C^\nu(A^{0\nu})$ then let $gd(A)$ be the least $n < \omega$ such that there exist $B^1, B^2, ..., B^n$ with $C^\nu(A^{0\nu})_{B^1,...,B^n}$ defined and with $A \in C^\nu(A^{0\nu})_{B^1,...,B^n}$.

Note that by further generation condition 1.1(17) and 1.1(31), $gd(A)$ will always be defined.

Let us formulate a similar to the Δ - system type (but a bit weaker) notion. The only difference will be that we replace in the clause (e) of the definition of a Δ - system type $C^\tau(A^{0\tau})$ by the k -line version for some $k < \omega$.

Let $F_0, F_1, F \in A^{1\mu}$ for some $\mu \in s$. We say then that F_0, F_1, F are of a **weak Δ-system type** iff F_0, F_1, F are of a Δ-system type

or the following holds:

(a) F is a successor model

(b) F_0, F_1 are its immediate predecessors and $otp_\mu F_0 = otp_\mu F_1$ (in particular, the conclusion of (28) above holds)

(c) F_0, F_1 are isomorphic over $F_0 \cap F_1$

(d) $gd(F)$ is defined and equal to some $k, 0 < k < \omega$.

(e) there is a sequence of models $B^1, ..., B^k$ witnessing $gd(F) = k$ such that

(i) $F \in C^\mu(A^{0\mu})_{B^1,...,B^k}$

(ii) one of F_0, F_1 in $C^\mu(F)$

(iii) there are $G_0, G_0^*, G_1, G_1^*, G^*$ all in $C^\tau(A^{0\tau})_{B_1,...,B_k}$, for $\tau = \min(s \backslash \mu + 1)$ such that

(α) $G_0 \in F_0$

(β) $G_1 \in F_1$

(γ) $F_0 \cap F_1 = F_0 \cap G_0 = F_1 \cap G_1$

(δ) $F_0 \in G_0^*, F_1 \in G_1^*, F \in G^*$ and G_0^*, G_1^*, G^* are the least under the inclusion elements of $A^{1\tau}$ including F_0, F_1, F respectively

(ϵ) $G_0 \in F_0 \in G_0^* \in G_1 \in F_1 \in G_1^* \in F \in G^*$.

Note that $\tau = \mu^+$ unless it is an inaccessible.

Further we shall require that a small adjustment turns a weak Δ-system type into a Δ-system type.

Let us call F for which there are F_0, F_1 with F_0, F_1, F of a Δ-system type or of a weak Δ-system type— a **splitting point**.

The next condition guarantees the uniqueness for triples as above.

8. (Immediate predecessors condition)

Let F be in $A^{1\mu}$ for some $\mu \in s$. Suppose that there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a weak Δ-system type with F being the largest model, then F_0, F_1 are unique.

Let us state now a condition that deals with extensions of a Δ-system type models.

9. (Bigger models condition)

Let F be in $C^\mu(A^{0\mu})$ for some $\mu \in s$. Suppose that there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a Δ-system type with F being the largest model. Let $\tau = \min(s \backslash \mu + 1)$.

If F' is one of F_0, F_1, F and G' is the smallest element of $C^\tau(A^{0\tau})$ including F' then the following hold

(a) if G' is not the first element of $C^\tau(G')$, then the immediate predecessor $\hat{G}'$ of G' in $C^\tau(G')$ belongs to F' as well as $C^\tau(\hat{G}')$. In particular, $\tau \in F'$

(b) if $H \in C^\tau(A^{0\rho})$ and $H \supseteq F'$, for some $\rho \in s \backslash \mu + 1$, then $H \supseteq G'$.

(c) if $H \in C^\tau(A^{0\rho})$, $H \supseteq F'$, for some $\rho \in s \backslash \mu + 1$ and H is the first like this in $C^\tau(A^{0\rho})$, then then the immediate predecessor of H in $C^\tau(A^{0\rho})$ (if exists) is in F'

The following condition says that once we have models of a Δ-system type then it is impossible to have models of smaller cardinalities in between.

10. (No small models condition) Let F_0, F_1, F be as in (8) and $F_0 \in C^\mu(F)$. If for some $\xi \in s \cap \mu$ we have $A \in C^\xi(A^{0\xi})$ with $A \subseteq F$, then $A \in F_0$.

    Further it will be shown that the above is true for $A^{1\mu}$ replacing C^μ and $A^{1\xi}$ replacing C^ξ.

11. (No splittings between a model and its immediate predecessor of maximal supremum) Let $F_0, F_1, F \in A^{1\mu}$ be of a weak Δ-system type. Suppose that $\sup(F_0) < \sup(F_1)$. Then there is no splitting points between F_1 and F, i.e. there is no $\rho \in s$ and a splitting point $B \in A^{1\rho}$ with $F_1 \in B \in F$. But there may (and actually will be many) splitting points B with $F^- \in B \in F_1$.

    Let $F \in A^{1\mu}$ be a successor model. We denote by F^- its immediate predecessor in $C^\mu(F)$. Let us define now the set $Pred(F)$.

    Suppose first that there are no $F_0, F_1 \in A^{1\mu} \cap F$ such that F_0, F_1, F are of a weak Δ-system type. Assume that $gd(F)$ is defined (actually, the generation condition (17) will guarantee that this is always the case). Let $gd(F) = k$. Fix the smallest (or simplest) sequence of models $B^1, ..., B^k$ witnessing this. Then $F, F^- \in C^\mu(A^{0\mu})_{B^1,...,B^k}$.

    Set then

$$Pred_0(F) = \{F^-\},$$

$$Pred_{n+1}(F) = \bigcup_{i<\omega} Pred_{n+1,i}(F),$$

where

$$Pred_{n+1,0}(F) = Pred_n(F)$$

and

$$Pred_{n+1,i+1}(F) = Pred_{n+1,i}(F) \cup \{\pi_{B_0 B_1}[G] | G \in Pred_{n,i}(F), B_0, B_1, B \in F \cap A^{1\rho}$$

are of a weak $\Delta-$system type for some $\rho \in s\backslash\mu+1$ and $G \subset B_0(G \in (A^{1\mu})^{B_0})$,

the general distance of B relatively to F is at most i with a witnessing sequence inside F

(i.e. relatively to $C^\mu(A^{0\mu})_{B^1,...,B^k}$, or in other words $k-i \leq gd(B) \leq k+i)\}$,

for each $n<\omega$.

Suppose now that there are $F_0, F_1 \in A^{1\mu} \cap F$ such that F_0, F_1, F are of a weak Δ-system type. Assume that $\sup(F_1) > \sup(F_0)$, otherwise just switch between them.

Assume that $gd(F_1)$ is defined (actually, the generation condition (17) will guarantee that this is always the case). Let $gd(F_1)=k$. Fix the smallest (or simplest) sequence of models $B^1,...,B^k$ witnessing this.

Set

$$Pred_0(F) = \{F_0, F_1\}$$

$$Pred_{n+1}(F) = \bigcup_{i<\omega} Pred_{n+1,i}(F),$$

where

$$Pred_{n+1,0}(F) = Pred_n(F)$$

and

$$Pred_{n+1,i+1}(F) = Pred_{n+1,i}(F) \cup \{\pi_{B_0B_1}[G] | G \in Pred_{n,i}(F), B_0, B_1, B \in F_1 \cap A^{1\rho}$$

are of a weak $\Delta-$system type for some $\rho \in s\backslash\mu+1$ and $G \subset B_0(G \in (A^{1\mu})^{B_0})$,

the general distance of B relatively to F_1 is at most i with a witnessing sequence inside F_1

(i.e. relatively to $C^\mu(A^{0\mu})_{B^1,...,B^k}$, or in other words $k-i \leq gd(B) \leq k+i)\}$,

for each $n<\omega$.

We required in (11) that in this case there is no splittings between F_1 and F, i.e. there is no splitting point B with $F_1 \in B \in F$. But there may (and actually will be many) splitting points B with $F_0 \in B \in F_1$. Also we require in (15) that F^- is in $Pred_n(F)$ for some $n<\omega$.

Consider now an additional possibility that was not allowed in [6].
For some inaccessible $\alpha \in s\backslash\mu+1$ we have

(a) $V_\alpha \in F$

(b) there is unique immediate predecessor F' of F inside V_α $F' \in V_\alpha$

(c) either

- there is $X \in A^{1\mu} \backslash V_\alpha$ which is an immediate predecessor of F under the inclusion and X is isomorphic over $X \cap V_\alpha$ to an element of $C^\mu(F')$, but not to F' itself
  or
- there is a directed (under inclusion) sequence $\vec{F}$ of the length less than μ of elements of $A^{1\mu} \backslash V_\alpha$ with limit not in $A^{1\mu}$ and with F being the least under the inclusion including all of its members, such that every element of the sequence is isomorphic over its intersection with V_α to an element of $A^{1\mu}(F')$. Moreover, $C^\mu(F')$ passes via an element of $\vec{F}$ intersected with V_α.

Let $Pred_0(F)$ be the set consisting of F^- and X or the sequence $\vec{F}$, as above. Set then

$$Pred_n(F) = Pred_0(F),$$

for each $n < \omega$.
Intuitively this means that moving via isomorphisms not allowed in such situation. Further we shall refer to the above case as **a special models case** and will call the models involved **special models**. Note that here in contrast to [6] we allow A's in $A^{1\mu}$ with $\mu \notin A$. Specially, our interest will be in models obtained by applying elementary embeddings $j : V \to M$ with critical point μ. Thus, if $B \in A^{1\mu}$, $\mu \in B$, then we may need $A = j''B$ to be in $A^{1\mu}$. It is crucial to allow such models in order to preserve strong cardinals.

Let us define in all three cases

$$Pred(F) = \bigcup_{n<\omega} Pred_n(F).$$

Note also that in a special models case models in $Pred(F)$ are not isomorphic any more. This cases a small complication in the argument used in the Intersection Lemma of [6].

The next condition requires a kind of a weak homogeneity.

12. (The weak homogeneity) Let $B \in C^\rho(A^{0\rho})$ be a splitting point as witnessed by B_0, B_1, for some $\rho \in s$ and let $\mu \in s \cap \rho$. Suppose that for some successor model $F \in C^\mu(A^{0\mu})$ the triple B_0, B_1, B is as in the definition of $Pred(F)$. Then for each $\eta \in s \cap \mu + 1$ we have $X \in A^{1\eta} \cap \mathcal{P}(B_0)$ iff $\pi_{B_0 B_1}[X] \in A^{1\eta} \cap \mathcal{P}(B_1)$.

Intuitively, this means that everything of cardinality at most μ is copied by the isomorphism $\pi_{B_0B_1}$ from B_0- side to B_1 - side and vise verse. This condition is crucial for preserving GCH.

The next condition describes the structure of bigger models inside a splitting.

13. (Bigger models over splitting points) Let F_1 be as in (11) with $F \in C^\mu(A^{0\mu})$ and $\rho \in s\backslash\mu+1$. Suppose that B is the least element of $C^\rho(A^{0\rho})$ including F. Then B is a successor point, moreover B^- is a successor point as well, $Pred(B) = \{B^-\}, Pred(B^-) = \{(B^-)^-\}$ and $F \in B, F_1 \in B^-$. In addition, if $\rho \in F_1$ then

$$(B^-)^- \in F_1 \in B^- \in F \in B.$$

If $\rho \in F\backslash F_1$ then

$$F_1 \in B^- \in F \in B.$$

14. (No splittings at limits) If $\rho \in s$ is a limit point of s, then no model in $A^{1\rho}$ can be a splitting model.

15. Let $F \in C^\mu(A^{0\mu})$ be a successor model and F^- be its immediate predecessor in $C^\mu(A^{0\mu})$, for some $\mu \in s$. Then $F^- \in Pred(F)$.

    Note that this condition is relevant only when F splits, otherwise $F^- \in Pred_0(F)$ by the definition.

16. (No small models condition 2)

    Let F be a successor point in $C^\mu(A^{0\mu})$ and $A \in A^{1\xi} \cap F$, for some $\xi \in s \cap \mu$. Then there is $G \in Pred(F)$ with $A \in G$.

    Let us define the sets $A^{1\mu}_k(A)$, for $A \in A^{1\mu}$ and $1 \leq k < \omega$.

$$\begin{aligned}
&A^{1\mu}_k(A) = \bigcup_{n<\omega} C^\mu_{kn}(A) \ , \quad \text{where} \\
&C^\mu_{k0}(A) = C^\mu(A) \\
&C^\mu_{k2n}(A) = \{E \mid \exists F \in C^\mu_{k2n-1}(A) \quad E \in C^\mu(F)\} \\
&C^\mu_{k2n+1}(A) = \{E \in A^{1\mu} \mid \exists F \in C^\mu_{k2n}(A) \quad E \in Pred(F)\backslash C^\mu(F) \\
&\ \text{and the generalized distance of } E \text{ from } C^\mu(F) \text{ is at most } k\}.
\end{aligned}$$

We define $A^{1\mu}_0(A)$ similar only with

$$C^\mu_{02n+1}(A) = \{E \in A^{1\mu} \mid \exists F \in C^\mu_{k2n}(A) \quad E \in Pred(F)\backslash C^\mu(F)$$

and $E, F,$ the immediate predecessor F^- of F in $C^\mu(F)$ are of a Δ- system type$\}$.

In particular, $A_0^{1\mu}(A)$ is defined using only models of cardinality μ.

Set $A^{1\mu}(A) = \bigcup_{k<\omega} A_k^{1\mu}(A)$. It is possible to define $A^{1\mu}(A)$ also as follows:

$$A^{1\mu}(A) = \bigcup_{n<\omega} C_n^\mu(A) \ , \quad \text{where}$$
$$C_0^\mu(A) = C^\mu(A)$$
$$C_{2n}^\mu(A) = \{E \mid \exists F \in C_{2n-1}^\mu(A) \quad E \in C^\mu(F)\}$$
$$C_{2n+1}^\mu(A) = \{E \in A^{1\mu} \mid \exists F \in C_{2n}^\mu(A) \quad E \in Pred(F) \backslash C^\mu(F)\}.$$

The next condition describes the way in which elements of $A^{1\mu}$ are generated.

17. (Generation condition)

    Let $\mu \in s$. Then $A^{1\mu} = A^{1\mu}(A^{0\mu})$.

    Set also $A_k^{1\mu} = A_k^{1\mu}(A^{0\mu})$ and $C_k^\mu = C_k^\mu(A^{0\mu})$ for each $k < \omega$.

    This condition implies that we can reconstruct everything just from the top models (i.e. $A^{0\xi}$'s), $C^\xi(A^{0\xi})$'s and the splitting points over $C^\xi(A^{0\xi})$'s.

    The next condition provides a weak form of elementarity.

18. If for some $\tau, \xi \in s$ we have $A \in C^\tau(A^{1\tau})$ and $B \in C^\xi(A^{1\xi}) \cap A$, then $C^\xi(B) \in A$, $A^{1\xi}(B) \in A$ as well. Also for each $E \in C^\tau(A)$, if there is an element of $C^\xi(B)$ including E, then the first such element is in A.

19. Let A be a set in $C^\tau(A^{0\tau})$ and $F \in C^\mu(A^{0\mu})$ be a member of $C^\mu(A^{0\mu})$ including A, for some $\tau, \mu \in s, \tau < \mu$. Then for each $\xi \in s$, $\tau < \xi \leq \mu$ implies that there is $G \in C^\xi(A^{0\xi})$ such that
$$A \subseteq G \subseteq F.$$

20. Let $\rho < \tau$ be in s and $A \in C^\rho(A^{0\rho})$ be a successor model. Suppose $B \in C^\tau(A^{0\tau})$ is the least with $A \subset B$. Then B is a successor model. Suppose that B is not the least element of $C^\tau(A^{0\tau})$. Let B^- be the immediate predecessor of B in $C^\tau(A^{0\tau})$. If $\tau \in A$ then $B^- \in A$. Moreover, if A is the least in $C^\rho(A^{0\rho})$ with B^- inside then $C^\rho(A) \backslash \{A\} \subseteq B^-$.

21. Let $\rho < \tau$ be in s and $A \in C^\rho(A^{0\rho})$ be a limit model. Suppose $B \in C^\tau(A^{0\tau})$ is the least with $A \subset B$. Then B is a limit model. In addition, if $\tau \in A$, then $A \cap (C^\tau(B) \setminus \{B\})$ is cofinal in B.

    Intuitively the last two conditions mean that the sequences $C^\tau(A^{0\tau})$ and $C^\rho(A^{0\rho})$ mix together nicely. Note that $C^\rho(A^{0\rho})$ is closed. Hence always, if $F \cap C^\rho(A^{0\rho})$ is not empty, then there is a maximal $A \in C^\rho(A^{0\rho})$ which is a subset of F.

22. (Least model including a successor one must be a successor model) Let $\rho < \tau$ be in s, $A \in C^\rho(A^{0\rho})$ be a successor model and $B \in C^\tau(A^{0\tau})$ be the least with $A \subset B$. Then B must be a successor model and $A \in B$.

23. (Local maximal models) Let $\rho < \tau$ be in s, $A \in C^\rho(A^{0\rho})$ be a successor model, $\tau \in A$ and $B \in C^\tau(A^{0\tau})$ be the least with $A \subset B$. Suppose that B is not the least element of $C^\tau(A^{0\tau})$. Let B^- be the immediate predecessor of B in $C^\tau(A^{0\tau})$. Then for every $X \in A \cap A^{1\tau}$ we have $X \in A^{1\tau}(B^-)$.

    This means that B^- is a local (relatively to A) version of $A^{0\tau}$.

    The next three conditions provide a kind of linearity over the central line.

24. Let $\rho < \mu < \tau$ be in s and $A \in C^\rho(A^{0\rho})$. Suppose that F, G are the least elements of $C^\mu(A^{0\mu})$ and $C^\tau(A^{0\tau})$ respectively including A. Then G includes F and it is the least such element of $C^\tau(A^{0\tau})$.

25. Let $\rho < \mu < \tau$ be in s, $F \in C^\tau(A^{0\tau})$, $F_1 \in C^\mu(A^{0\mu})$ be the maximal element of $C^\mu(A^{0\mu})$ contained in F (if exists) and F_2 be the maximal element of $C^\rho(A^{0\rho})$ contained in F_1, if exists. Then, if F_1, F_2 exist, then F_2 is the maximal element of $C^\rho(A^{0\rho})$ contained in F.

26. (Continuity at limit points) Suppose that ρ is a limit point of s. Let $\langle F_{\rho\alpha} \mid \alpha < \delta \rangle$ be an increasing enumeration of $C^\rho(A^{0\rho})$. For each $\alpha < \delta$ and $\xi \in s \cap \rho$ let $F_{\xi\alpha}$ be the largest element of $C^\xi(A^{0\xi})$ included in $F_{\rho\alpha}$, if it exists.

    Then for each $\alpha < \delta$ the following hold

    (a) $F_{\xi\alpha}$ exists for all but boundedly many $\xi \in \rho \cap s$

    (b) the sequence

    $$\langle F_{\xi\alpha} \mid \xi \in \rho \cap s, F_{\xi\alpha} \text{ exists} \rangle$$

    is increasing continuous with limit $F_{\rho\alpha}$

27. (Isomorphism condition) If A, B, C are of a Δ-system type for some $A \in C^\tau(C), C \in C^\tau(A^{0\tau})$ then the structures

$$\langle A, <, \in, \subseteq, \kappa, \tau, C^\tau(A), A^{1\tau}(A), f_A \ , \ \langle A^{1\rho} \cap A \mid \rho \in s \backslash \tau \rangle, \langle C^\rho \restriction A^{1\rho} \cap A \mid \rho \in s \backslash \tau \rangle \rangle$$

and

$$\langle B, <, \in, \subseteq \kappa, \tau, C^\tau(B), A^{1\tau}(B), f_B, \ \langle A^{1\rho} \cap B | \rho \in s \backslash \tau \rangle, \langle C^\rho \restriction A^{1\rho} \cap B | \rho \in s \backslash \tau \rangle \rangle$$

are isomorphic over $A \cap B$, where $f_A : \tau \leftrightarrow A$, $f_B : \tau \leftrightarrow B$ are some fixed in advance enumerations (for example, least such is the well-ordering $<$).

Let π_{AB} denotes the unique isomorphism. Note that, in particular, $A \cap \tau^+ = B \cap \tau^+$, since both are ordinals and, so π_{AB} is the isomorphism between them.

Let us state a similar condition. The main difference will be that $\langle C^\rho \restriction A^{1\rho} \cap A \mid \rho \in s \backslash \tau \rangle$ will not be mentioned. The reason is that the switching, which will be defined later, may change C^ρ's which are in one of the models without effecting an other model at all (unless $\tau^+ = \theta$, for example in Gap 4 case). For τ with $\tau^+ = \theta$ (27) suffice.

28. (General Isomorphism Condition) If $A, B \in A^{1\tau}$ and $otp_\tau(A) = otp_\tau(B)$ (equivalently, by (7) $otp(A) = otp(B)$) then the structures

$$\langle A, <, \in, \subseteq, \kappa, \tau, C^\tau(A), A^{1\tau}(A), f_A \ , \ \langle A^{1\rho} \cap A \mid \rho \in s \backslash \tau \rangle, C^\tau \restriction A^{1\tau} \cap A \rangle$$

and

$$\langle B, <, \in, \subseteq \kappa, \tau, C^\tau(B), A^{1\tau}(B), f_B, \ \langle A^{1\rho} \cap B | \rho \in s \backslash \tau \rangle, C^\tau \restriction A^{1\tau} \cap B | \rangle$$

are isomorphic where $f_A : \tau \leftrightarrow A$, $f_B : \tau \leftrightarrow B$ are some fixed in advance enumerations (for example, least such is the well-ordering $<$).

The next condition is weak version of elementarity.

29. (Weak elementarity condition)

Let $\tau, \mu \in s$, $A \in A^{1\tau}$ and $B \in A^{1\mu}$. If $B \in A$, then $A^{1\mu}(B)$ and $C^\mu(B)$ are in A. In addition, if $x \in A$ and for some $C \in C^\mu(B)$ we have $x \in C$, then the first member of $C^\mu(B)$ with this property is in A. Also require that if $B \in A$, then the function f_B as in (27) is in A. If $B' \in A$ and $otp(B) = otp(B')$ then the isomorphism $\pi_{BB'}$ is in A as well.

Let define now one more basic notion and then use it to state the requirement on a weak Δ-system type.

Let B_0, B_1, B with $B \in C^\rho(A^{0\rho}), B_0 \in C^\rho(B)$ be a suitable for switching triple, for some $\rho \in s$. We define the **switch by** B or $sw(B)$ of the functions $C^\tau, \tau \in s \cap \rho+1$ as follows: $C^\rho_B(B) = C^\rho(B_1) \cup \{B\}$ and for each $E \in C^\rho(A^{0\rho}) \backslash C^\rho(B)$ let $C^\rho_B(E) = (C^\rho(E) \backslash C^\rho(B)) \cup C^\rho_B(B)$.
Let now $\tau \in s \cap \rho$. Pick the first element A of $C^{0\tau}(A^{0\tau})$ with $B \in A$. Its immediate predecessor A^- in $C^\tau(A^{0\tau})$ is in B, by our assumption. Then $A^- \subset B_0$. Leave $C^\tau(A^-)$ unchanged as well all its initial segments. Set $C^\tau_B(A^{0\tau}(q)) = (C^\tau(A^{0\tau}) \backslash C^\tau(A^-)) \cup \pi_{B_0 B_1}[C^\tau(A^-)]$. In order to obtain the full function C^τ_B we just move the defined already portions via isomorphisms of the models in $A^{1\tau}$.
Remember that $B \in A$, hence $\pi_{B_0B_1}[A^-]$ remains inside $Pred(A)$.
Note that the above definition extends the definition 1.2, where we dealt only with $C^\tau(A^{0\tau})$.

We define now $sw(B^0, ..., B^n)$ by induction to be the result of the application of B^n to $sw(B^0, ..., B^{n-1})$.

Note that the application of a same B twice leaves the functions C^τ unchanged, i.e $C^\tau_{BB} = C^\tau$.

Let us require the following:

30. Suppose that $F_0, F_1, F \in A^{1\mu}$ are of a weak Δ -system type with $\sup(F_1) > \sup(F_0)$, for some $\mu \in s$. Then there are $B^0, ..., B^n$ with each B^i either in F_1 or $F \in B^i$, such that $sw(B^0, ..., B^n)$ turns $F_0, F_1, F \in A^{1\mu}$ into a Δ -system type triple with all the relevant conditions above satisfied according to the new C^τ's, i.e. $C^\tau_{B^0,...,B^n}$'s.

31. Let $A \in A^{1\mu}$, for some $\mu \in s$. Then there are $B^0, ..., B^n$ such that $sw(B^0, ..., B^n)$ moves A to the central line, i.e. $A \in C^\mu_{B^0,...,B^n}(A^{0\mu})$.

$\square$ of the definition.

Lemma 1.5 *For each $\mu \in s$ and $A \in A^{1\mu}$ which is not a special model there is $C \in C^\mu(A^{0\mu})$ with $otp(A) = otp(C)$.*

Proof. We prove the statement by induction. Let n be the least with $A \in C^\mu_n$. If $n = 0$ then take $C = A$.

If $n > 0$ is even, then there is $B \in C^\mu_{n-1}$ with $A \in C^\mu(B)$. By induction, then there is $D \in C^\mu(A^{0\mu})$ of the order type equal to $otp(B)$. Now use 1.1(7) for B and D. Let $A' = \pi_{BD}[A]$. Then $A' \in C^\mu(D)$ which is an initial segment of $C^\mu(A^{0\mu})$. So we are done.

If n is odd then there is $F \in C^\mu_{n-1}$ with $A \in Pred(F)\backslash C^\mu(F)$. Now, if F is not a splitting a point, then $otp(F^-) = otp(A)$, where F^- is the immediate predecessor of F in $C^\mu(F)$. Now we apply the induction to F and use 1.1(28).

If F is a splitting point, then let $F_0 \in C^\mu(F), F_1$ be witnessing this. Again, $otp(A) = otp(F_0)$ and we can apply the induction to F and use 1.1(28).

$\square$

This lemma together with 1.1(28) allow to transfer the conditions of 1.1 stated for elements of $C^\mu(A^{0\mu})$ to those of $A^{1\mu}$. Thus for example the following general version of 1.1(23) holds:

Lemma 1.6 *Let $\rho < \tau$ be in s, $A \in A^{1\rho}$ be a successor model. Suppose that $A \cap A^{1\tau} \neq \emptyset$ Then there is $E \in A \cap A^{1\tau}$ such that for every $X \in A \cap A^{1\tau}$ we have $X \in A^{1\tau}(E)$.*

Proof. Using 1.5 find $A' \in C^\rho(A^{0\rho})$ of the same order type as those of A. By 1.1(23), there is the maximal element E' of $A' \cap A^{1\tau}$. Then we can use 1.1(28) to move it to A, i.e. set $E = \pi_{A'A}[E']$.

$\square$

Notation 1 Denote further the maximal model of $A \cap A^{1\tau}$ by $(A^{0\tau})^A$.

Lemma 1.7 *Let A be as in the lemma 1.6. Suppose that $A^{1\rho}(A) \cap (A^{0\tau})^A \neq \emptyset$, then for each $\tau' \in s \cap [\rho, \tau] \cap A$ the maximal model $(A^{0\tau'})^A$ exists.*

Proof. It follows by 1.5 and 1.1(19).

$\square$

Lemma 1.8 *Let $\rho < \tau$ be in s, $A \in A^{1\rho}$ be a successor model. Suppose that*

$$A \cap \bigcup \{A^{1\tau} | \tau \in s \backslash \rho\} \neq \emptyset.$$

Then there are $n < \omega$, $\tau_n > ... > \tau_0$ in $A \cap (s \backslash \rho)$ and the maximal models $(A^{0\tau_n})^A \in ... \in (A^{0\tau_0})^A$ such that for each $\tau \in s \backslash \rho$ we have $(A^{0\tau})^A \subseteq (A^{0\tau_k})^A$ (if defined), for some $k \leq n$.

Proof. By 1.5 and 1.1(28), it is enough to deal with $A \in C^{\rho}(A^{0\rho})$. Now it follows by 1.1(4, 25).
$\square$

The next lemma follows easily from the definition of $Pred$.

Lemma 1.9 *Let $\rho \in s, A \in A^{1\rho}$ be a successor model. Then $A^{1\rho}(A) = \bigcup\{A^{1\rho}(X) | X \in Pred(A)\}$.*

Lemma 1.10 *Let $\rho < \tau$ be in s, $A \in A^{1\rho}$ be a successor model. Then there are $n < \omega$, $\tau_n > ... > \tau_0$ in $A \cap (s \backslash \rho)$ and the maximal models $(A^{0\tau_n})^A \in ... \in (A^{0\tau_0})^A$ such that for each $B \in A^{1\rho}(A) \backslash \{A\}$ we have $B \in (A^{0\tau_k})^A$ for some $k \leq n$.*

Proof. Let B be in $A^{1\rho}(A)$. Then by 1.9 there is $X \in Pred(A)$ with $B \in A^{1\rho}(X)$. Let $n < \omega$, $\tau_n > ... > \tau_0$ in $A \cap (s \backslash \rho)$ and the maximal models $(A^{0\tau_n})^A \in ... \in (A^{0\tau_0})^A$ be as in 1.8. Now, by the definition of $Pred$ and 1.8, $B \in (A^{0\tau_k})^A$ for some $k \leq n$.
$\square$

The following is a consequence of 1.1(7), (8) and the previous lemma.

Lemma 1.11 *Let F be in $A^{1\mu}$ for some $\mu \in s$. Suppose that there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a Δ-system type with F being the largest model, then F_0, F_1 are unique.*

The following lemmas follow easily from the definition of $A^{1\mu}(A)$.

Lemma 1.12 *Let $\mu \in s, F \in A^{1\mu}$ be a successor model with unique immediate predecessor F^- in $C^{1\mu}(F)$. Then $A_0^{1\mu}(F) = A_0^{1\mu}(F^-) \cup \{F\}$*

and

$A^{1\mu}(F) = \bigcup\{A^{1\mu}(X) | X \in Pred(F)\}$.

Lemma 1.13 *Let $\mu \in s, F \in A^{1\mu}$ be a successor model which is a splitting point in $A^{1\mu}(F)$ i.e. there are $F_0, F_1 \in A^{1\mu}$ such that F_0, F_1, F are of a Δ-system type with F being the largest model. Then $A^{1\mu}(F) = A^{1\mu}(F_0) \cup A^{1\mu}(F_1) \cup \{F\}$.*

Lemma 1.14 *Let $\mu \in s, F \in A^{1\mu}$ be a limit model. Then $A^{1\mu}(F) = \bigcup\{A^{1\mu}(D) | D \in C^{\mu}(F)\}$.*

Lemma 1.15 *(Identity on the common part) Suppose $\mu \in s, A, B \in A_0^{1\mu}$ and $otp(A) = otp(B)$. Then π_{AB} is the identity on $A \cap B$.*

Proof. Suppose that $A \neq B$. Consider the walks from $A^{0\mu}$ to A and to B. Let G be the last common model of the walks. Then it must be a splitting point. Let G_0, G_1 be its immediate predecessors witnessing this with $G_0 \in C^\mu(G)$. So, G_0, G_1, G are of a weak Δ-system type. In particular $\pi_{G_0 G_1}$ is the identity on $G_0 \cap G_1$. Suppose that $A \in A^{1\mu}(G_0)$ and $B \in A^{1\mu}(G_1)$. Set $B_0 = \pi_{G_1 G_0}[B]$. If $x \in A \cap B$, then $x \in G_0 \cap G_1$ and so in B_0. B_0 is simpler then B so we can apply induction to A, B_0. Hence, π_{AB_0} is the identity on $A \cap B_0$. In particular, $\pi_{AB_0}(x) = x$. But $x \in G_0 \cap G_1$. So $\pi_{G_0 G_1}(x) = x$. Then $\pi_{B_0 B}(x) = x$, since $\pi_{G_0 G_1}$ extends $\pi B_0 B$. Now

$$\pi_{AB}(x) = \pi_{B_0 B}(\pi_{AB_0}(x)) = \pi_{B_0 B}(x) = x.$$

$\square$

Remark 1.16 (1) Note that in the gap 4 case we have $A_0^{1\mu} = A^{1\mu}$, for $\mu = \kappa^{++}$. Hence, any two elements of $A^{1\mu}$ of the same order type are isomorphic over their common intersection. This breaks down for $\mu = \kappa^+$ even in the gap 4 case.
(2) The argument of the lemma can be used in more general situations. Once having a splitting point G we can replace B by $\pi_{G_1 G_0}[B]$. The crucial is that $\pi_{G_1 G_0}$ is the identity on $G_0 \cap G_1$ and this is true always for splitting points.

Definition 1.17 (The general walk between models) Let $\nu \in s$. Define a function gwk on elements A of $A^{1\nu}$. We will call $gwk(A)$ a **general walk from $A^{0\nu}$ to A** . The definition is by induction on the general distance of A from the central line, i.e. on $gd(A)$ simultaneously on each $\nu \in s$ and $A \in A^{1\nu}$.

(a) if $gd(A) = 0$ then set $gwk(A) = \langle A \rangle$

(b) if $gd(A) = n > 0$, then, by 1.1(31) there are models $B^1, ..., B^n$ such that $A \in C^\nu(A^{0\nu})_{B^0,...,B^n}$. We pick simplest in the general walk sense models $B^1, ..., B^n$ such that $A \in C^\nu(A^{0\nu})_{B^0,...,B^n}$ Consider the triple B_0^n, B_1^n, B^n (forming a Δ - system type as in 1.3). Set $A_0 = \pi_{B_1^n B_0^n}[A]$. Note that $A \subseteq B_1^n$, since otherwise there will be now need in B^n. Set

$$gwk(A) = gwk(A_0)^\frown gwk(B^n)^\frown B_0^{n\frown} B_1^{n\frown} A.$$

Let us make now one technical definition which relates to intersections of models.

Definition 1.18 Let $\xi, \zeta \in s, A \in A^{1\xi}$ and $B \in A^{1\zeta}$. We say that A **satisfies the intersection property with respect to** B or shortly $ip(A, B)$ iff either

(1) $\xi > \zeta$

or

(2) $\xi \leq \zeta$ and $A \subseteq B$

or

(3) $\xi = \zeta$ and $B \subseteq A$

or

(4) $\xi \leq \zeta, A \not\subseteq B, B \not\subseteq A$ and then there are $A' \in A \cup \{A\}$ and $D_1 \in (A^{1\rho_1})^A, ..., D_n \in (A^{1\rho_n})^A$, for some $\rho_1, ..., \rho_n \in s \backslash \xi + 1$ such that

(a) $A' = A$ unless $\xi = \zeta$ and $otp_\xi(A) > otp_\xi(B)$.

If this is the case (i.e. $otp_\xi(A) > otp_\xi(B)$), then $otp(A') = otp(B)$ and ($A' \in (A^{1\xi})^A$ or A' is an image of an element of $(A^{1\xi})^A$ under isomorphisms $\pi_{G_0 G_1}$ for models $G_0, G_1 \in A$).

(b) $A \cap B = A \cap A' \cap D_1 \cap D_2 \cap ... \cap D_n$.

(c) $A' \in A^{1\xi}$

or

$A' = \pi_{IJ}[A'' \cap H_1 ... \cap H_k]$, for some $A'' \in A^{1\xi}(A), H_1 \in (A^{1\eta_1})^A, ..., H_k \in (A^{1\eta_k})^A, I, J \in (A^{1\eta})^A$, for some $\eta, \eta_1, ..., \eta_k \in s \backslash \xi + 1$.

Let $ipb(A, B)$ denotes that both $ip(A, B)$ and $ip(B, A)$ hold.

Lemma 1.19 *(General Intersection Lemma) Let $\xi, \zeta \in s, A \in A^{1\xi}$ and $B \in A^{1\zeta}$. Then $ipb(A, B)$.*

Proof. At least one of A and B is not on the central line. Without loss of generality we can assume that one of A, B is on the central line. Otherwise make finitely many switches that lead to this situation. We put the model of the least cardinality between A and B on the central line. Let A be such a model. We like to show $ip(A, B)$.

Consider the walk from $A^{0\zeta}$ to B. Let Z be the last model in $C^\zeta(A^{0\zeta})$ of this walk. Then Z must be a successor model. Let Z^- be the immediate predecessor of Z in $C^\zeta(A^{0\zeta})$ and

$Z_1 \in Pred(Z)$ be the next point in the walk leading to B. If Z^-, Z_1 are isomorphic over $Z^- \cap Z_1$ then we would like to use $\pi_{Z_1 Z^-}$ to move B to a simpler (according to the generalized distance gd) model B_0 and from $ip(A, B_0)$ deduce $ip(A, B)$. Also in general case we would like to replace B by a simpler model. Proceed as follows. If Z^-, Z_1, Z are of a weak Δ - system type, then denote Z_1 by G_B and let F_A be the smallest model in $C^\zeta(A^{0\zeta})$ including A. If Z^-, Z_1, Z are not of a weak Δ - system type, then let $G_B \in A^{1\rho}$ be the last model used to generate Z_1 in $Pred(Z)$ with some $\rho \in s \backslash \zeta$. Let F_A be the smallest model in $C^\zeta(A^{0\rho})$ including A.

Compare now G_B and F_A.

Case 1. $F_A \notin A^{1\rho}(G_B)$ and $G_B \notin A^{1\rho}(F_A)$.

Consider the last common point of the walks to G_B and to F_A from $A^{0\rho}$. Let E denotes this point. Then it must be a successor point.

Subcase 1.1. E does not have immediate predecessors of a weak Δ - system type or it does but at least one of F_A, G_B is not in $A^{1\rho}$ of them.

Suppose that F_A is such. Then there are $\eta \in s \backslash \rho + 1$ and a model $H_A \in A^{1\eta}$ with immediate predecessors H_{A0}, H_{A1} of a weak Δ - system type such that F_A is on the H_{A1} - side. Pick the smallest model K_B in the moved (according the way of moving to (or generating) G_B) $C^\eta(A^{0\eta})$ with G_B inside $(A^{1\rho})^{K_B}$.

Now again we compare H_A and K_B according to the walks from $A^{0\eta}$. Note that the models under the consideration are simpler than F_A, G_B since they are more close to the central (beginning) line, i.e. gd decreases. So we can reduce the situation (either induction or finitely many applications of the process used above) to the negation of the present subcase.

Subcase 1.2. E has immediate predecessors E_0, E_1 of a weak Δ - system type with $F_A \in A^{1\rho}(E_0)$ and $G_B \in A^{1\rho}(E_1)$.

By the definition of a Δ - system type, there will be $D_{01} \in E_0 \cap A^{1\zeta}, D_{10} \in E_1 \cap A^{1\zeta}$ such that

$$E_0 \cap E_1 = E_0 \cap D_{01} = E_1 \cap D_{10}$$

and E_0, E_1 are isomorphic over $E_0 \cap E_1$. Let E_0 be the one in $C^\rho(E)$.

Now we move G_B and B to E_0 side. Set $G^0_B = \pi_{E_1 E_0}[G_B]$ and $B^0 = \pi_{E_1 E_0}[B]$. Then

$$A \cap B = A \cap F_A \cap B \cap G_B = A \cap F_A \cap E_0 \cap E_1 \cap G_B \cap B =$$

$$A \cap F_A \cap E_0 \cap D_{01} \cap G^0_B \cap B^0 = A \cap B^0 \cap D_{01}.$$

Induction can be applied to A, B^0, D_{01}, since at least B^0 and D_{01} are simpler than B again according to the distance from the basic central line, i.e. gd.

Case 2. $F_A \in A^{1\rho}(G_B)$ or $G_B \in A^{1\rho}(F_A)$.

Let $G_B \in A^{1\rho}(F_A)$.

Subcase 2.1. $G_B \notin A$.

Denote by G_B^0 the model used at the last step together with G_B to move (construct) B. Then there is $G \in A^{1\rho}$ such that G_B^0, G_B, G are of a weak Δ - system type. Here we have $G_B^0 \in C^\rho(G)$ and $G \in A^{1\rho}(F_A)$.

Subsubcase 2.1.1 $\rho \in A$.

By minimality of F_A, then also $G \in A^{1\rho}(F')$, for some $F' \in A \cap C^\rho(F)$. We use here 1.1(20) or (21). If F_A is a successor model, then F_A^- exists, it is in A and is equal to $(A^{0\rho})^A$. Consider the walk from F' to G. We assume that no models of bigger than ρ cardinalities are involved here (otherwise we are back in the situation considered in Case 1) and so the walk is entirely in $A_0^{1\rho}$. Let F be the last point of this walk in A and E the very next point of this walk. Then F must be a limit point. Let

$$\tilde{F} = \bigcup \{X | X \in A \cap C^\rho(F) \backslash \{F\}\}.$$

Suppose first that E is a splitting point with two immediate predecessors E_0, E_1 of a Δ - system type, $E_0 \in C^\rho(E), G \in A^{1\rho}(E_1)$. We would like to move to E_0 side simplifying the situation. By the definition of a Δ - system type, there will be $D_{01} \in E_0 \cap A^{1\zeta}, D_{10} \in E_1 \cap A^{1\zeta}$ such that

$$E_0 \cap E_1 = E_0 \cap D_{01} = E_1 \cap D_{10}$$

and E_0, E_1 are isomorphic over $E_0 \cap E_1$. Set $G_B^0 = \pi_{E_1 E_0}[G_B]$ and $B^0 = \pi_{E_1 E_0}[B]$. Then

$$A \cap B = A \cap F_A \cap B \cap G_B = A \cap F_A \cap E \cap E_1 \cap B = A \cap F' \cap E \cap E_1 \cap B =$$

$$A \cap \tilde{F} \cap E_1 \cap B = A \cap E_0 \cap E_1 \cap B = A \cap F' \cap E_0 \cap E_1 \cap G_B \cap B =$$

$$A \cap F' \cap D_{01} \cap G_B^0 \cap B^0 = A \cap F' \cap B^0 \cap D_{01}.$$

Induction can be applied to A, B^0, D_{01}, since at least B^0 and D_{01} are simpler than B again according to the distance from the basic central line.

In contrast to [6], we need to consider here one more case - E is a special model. Suppose that this is the case. Then there should be unboundedly many such models below F in $C^\rho(F)$. Just otherwise, using the weak elementarity condition (1.1(29)) we will be able to go down further than F. Assume that we have the same α witnessing the speciality of this models and $\alpha \in A$. Otherwise we deal in a similar fashion with the least $\alpha^* \in A$ above such

α's.

Let again

$$\tilde{F} = \bigcup\{X|X \in A \cap C^{\rho}(F)\backslash\{F\}\}.$$

Consider now also

$$\langle H \cap V_{\alpha}|H \in C^{\rho}(F)\backslash\{F\}\rangle.$$

It is clearly also increasing continuous sequence unbounded in $F \cap V_{\alpha}$. For each $H \in C^{\rho}(F)\backslash\{F\}$ let H' will be the first special model in $C^{\rho}(F)$ above H. Let $Z \in V_{\alpha} \cap H'$ be the witness of speciality of H'. Then $Z \supset H \cap V_{\alpha}$, by the definition of a special model. Now we proceed similar to above. Let $E' \in V_{\alpha}, \vec{E}$ witness the speciality of E. Assume that the continuation of the walk to B goes via E' and B sits higher enough not allowing replace it by an isomorphic over the intersection with V_{α} model in $\vec{E}$. Assume first that $B \in C^{\rho}(E')$. Recall that $C^{\rho}(E')$ goes via an element of $\vec{E}$ intersected with V_{α}, by the definition of a special model. So, B contains the intersection of this element with V_{α}. Then

$$A \cap B = A \cap E' \cap B = A \cap F \cap E' \cap B = A \cap \tilde{F} \cap E' \cap B =$$

$$A \cap \tilde{F} \cap V_{\alpha} \cap E' \cap B = A \cap \tilde{F} \cap V_{\alpha} = A \cap F \cap V_{\alpha}.$$

Now, if $B \notin C^{\rho}(E')$, then we continue the walk to B and pick the last point of this walk $H \in C^{\rho}(E')$. Assume for simplicity that H is a splitting point with two immediate predecessors $H_0 \in C^{\rho}(H), H_1$ of a Δ-system type and the walk to B goes via H_1. Set $B_0 = \pi_{H_1H_0}[B]$. Then

$$A \cap B = A \cap H \cap B = A \cap F \cap V_{\alpha} \cap B = A \cap H \cap H_1 \cap B =$$

$$A \cap \tilde{F} \cap V_{\alpha} \cap H_0 \cap H_1 \cap B = A \cap \tilde{F} \cap D_{H_0H_1} \cap B_0.$$

It is crucial here that intersections with V_{α} of models on C^{ρ} between E^- and $\tilde{F}$ is contained in models from $C^{\rho}(E')$.

Subsubcase 2.1.2. $\rho \notin A$.

Let $\delta = \min(A \cap (s\backslash\rho))$. Let F_A^{δ} be the least element of $C^{\delta}(A^{0\delta})$ including A. Then by 1.1(24), $F_A \subseteq F_A^{\delta}$. The walk from $A^{0\rho}$ to G goes via F_A. Assume again that no models of cardinalities above ρ are involved in this walk. Let E be the last model of the walk inside $C^{\rho}(A^{0\rho})$. Now, $E \subseteq F_A$, since the walk passes F_A. Moreover, $E \in F_A$, since F_A is the least member of $C^{\rho}(A^{0\rho})$ including A and A is on the central line as well. Let H be the least element of $C^{\delta}(A^{0\delta})$ including E. Then $H \subset F_A^{\delta}$. By minimality of F_A^{δ}, then also $H \in C^{\delta}(F')$, for some $F' \in A \cap C^{\delta}(F_A^{\delta})$. We use here 1.1(20) or (21). If F_A^{δ} is a successor model, then

$(F^\delta_A)^-$ exists, it is in A and is equal to $(A^{0\delta})^A$. Pick the smallest $F \in A \cap C^\delta(F')$ with $H \subseteq F$. Note that in the present case F need not be a limit point. Thus it may be equal to H and since δ is a limit point of s, H will be an increasing continuous union of models smaller cardinalities in s. We set

$$\tilde{F} = \bigcup \{X | X \in A \cap C^\nu(A^{0\nu}) \cap F, \nu \in s \cap \delta\}.$$

Let E be a splitting point with two immediate predecessors E_0, E_1 of a Δ - system type, $E_0 \in C^\rho(E), G \in A^{1\rho}(E_1)$. The case of a special model is treated similar following the lines of 2.1.1. We would like to move to E_0 side simplifying the situation. By the definition of a Δ - system type, there will be $D_{01} \in E_0 \cap A^{1\zeta}, D_{10} \in E_1 \cap A^{1\zeta}$ such that

$$E_0 \cap E_1 = E_0 \cap D_{01} = E_1 \cap D_{10}$$

and E_0, E_1 are isomorphic over $E_0 \cap E_1$. Set $G^0_B = \pi_{E_1 E_0}[G_B]$ and $B^0 = \pi_{E_1 E_0}[B]$. Then, using 1.1(26), we obtain

$$A \cap B = A \cap F_A \cap B \cap G_B = A \cap F_A \cap E \cap E_1 \cap B = A \cap F \cap E \cap E_1 \cap B =$$

$$A \cap \tilde{F} \cap E_1 \cap B = A \cap E_0 \cap E_1 \cap B = A \cap F \cap E_0 \cap E_1 \cap G_B \cap B =$$

$$A \cap F' \cap D_{01} \cap G^0_B \cap B^0 = A \cap F' \cap B^0 \cap D_{01}.$$

Induction can be applied to A, B^0, D_{01}, since at least B^0 and D_{01} are simpler than B again according to the distance from the basic central line.

Subcase 2.2. $G_B \in A$.

Let G^0_B, G be as in the previous case. Then they also are in A. Now we deal with G^0_B and G_B exactly as in the appropriate case of the third intersection lemma (or see below). This allows to replace G_B (and so B) by a simpler (closer to the central line) model G^0_B (and B by $B_0 = \pi_{G_B G^0_B}[B]$). Let us reproduce the argument of the third intersection lemma. Denote for simplicity G_B by G_1 and G^0_B by G_0. Let $B_0 = \pi_{G_1 G_0}[B]$.

Recall that $G_0 = f_{G_0}[\rho]$ and $G_1 = f_{G_1}[\rho]$, where f_{G_0} and f_{G_1} are the fixed functions from ρ one to one onto G_0 and G_1 respectively. Also, they are respected by isomorphism $\pi_{G_0 G_1}$ of the structures and are in A by the elementarity condition 1.1(29). Set $T_0 = f^{-1}_{G_0}[B_0]$ and $T_1 = f^{-1}_{G_1}[B]$. Then $\pi_{G^i_0 G^i_1}[T_0] = T_1$, but $T_0, T_1 \subseteq \rho$ and $\pi_{G_0 G_1} \restriction \rho = id$, since $\rho \subseteq G^i_0 \cap G^i_1$. Hence $T_0 = T_1$. Note that $A \cap B = f_{G_1}[A \cap T_1]$, since $\alpha \in A \cap B$ iff $f^{-1}_{G_1}(\alpha) \in A$ and $f^{-1}_{G_1}(\alpha) \in T_1$ iff $f^{-1}_{G_1}(\alpha) \in A \cap T_1$, also $A \cap G_1 = f_{G_1}[A \cap \rho]$. Similar, $A \cap B_0 = f_{G_1}[A \cap T_0]$.

Now

$$A \cap B = f_{G_1}[A \cap T_1] = \pi_{G_0 G_1}(f_{G_0}[A \cap T_0]) =$$

$$\pi_{G_0G_1}[A \cap B_0],$$

since $\alpha \in f_{G_1}[A \cap T_1]$ iff $\alpha \in f_{G_1}[T_1]$ and $\alpha \in f_{G_1}[A \cap \rho]$ iff $\pi_{G_1G_0}(\alpha) \in f_{G_0}[T_0]$ and $\pi_{G_1G_0}(\alpha) \in f_{G_0}[A \cap \rho]$. iff $\pi_{G_1G_0}(\alpha) \in f_{G_0}[T_0 \cap A] = A \cap B_0$.

Note only that $\pi_{G_1G_0}(\alpha) \in A$ iff $\alpha \in A \cap G_1$, since $\pi_{G_1G_0} \in A$. It is crucial that $\pi_{G_1G_0} \restriction \rho = id$ and that $G_0, G_1 \in A$ implies $f_{G_1}[A \cap \rho] = A \cap G_1$, $f_{G_0^i}[A \cap \rho] = A \cap G_0$.
$\square$

The proof of the next lemma is similar to those of 1.19.

Lemma 1.20 *Let A, B be sets in $A^{1\tau}$ for some $\tau \in s$ and $B \subset A$. Then $B \in A^{1\tau}(A)$.*

Proof. Suppose otherwise. Without loss of generality we can assume that one of the models A, B is on the central line. Let then E be the last common model of the walks from $A^{0\tau}$ to A and to B (or just the last model of the walk to B in $C^\tau(A^{0\tau})$, if A is in the central line, i.e. $A \in C^\tau(A^{0\tau})$) . Then E must be a successor model. Suppose that E is a splitting point. The non splitting case is treated similar. Let E_0, E_1 be the immediate predecessors of E such that the triple E_0, E_1, E is of a weak Δ - system type. If $A \in A^{1\tau}(E_0)$ and $B \in A^{1\tau}(E_1)$ (or $A \in A^{1\tau}(E_1)$ and $B \in A^{1\tau}(E_0)$), then $B \subseteq E_0 \cap E_1$ and so $\pi_{E_0E_1}$ does not move B, since the triple E_0, E_1, E is of a weak Δ - system type. It is impossible to have now $E_0 \in C^\tau(E)$, since then the common walk can be continued further to E_0. Let us replace A by $A' = \pi_{E_0E_1}(A)$. Then $A' \supset B$. Applying induction, we will have $B \in A^{1\tau}(A')$. Now, moving back, B (which does not move) will be in $A^{1\tau}(A)$.

Suppose now that at least one of A, B is not in $A^{1\tau}(E_i)$ for $i \in 2$. Let $\sup(E_1) > \sup(E_0)$. Then there is $X \in Pred(E) \backslash Pred_0(E)$ with A or B inside $A^{1\tau}(X)$. Consider models $H_0, H_1, H \in E_1 \cap A^{1\rho}$ of a weak Δ - system type generating X as in the definition of $Pred$. If $H \in A$, then $\pi_{H_1H_0}[B] \subset A$. Induction applies then to A and $\pi_{H_1H_0}[B]$. Hence, $\pi_{H_1H_0}[B] \in A^{1\tau}(A)$. Then also $B \in A^{1\tau}(A)$.
Note that it is impossible to have in the present situation the following:

$$A = E_1, B = \pi_{H_0H_1}[E_0].$$

Since then $E_0 \subset A = E_1$. Which implies that $E_0 = E_1$.
Suppose now that $H \notin A$. Assume that $B \in A^{1\tau}(X)$. The case $A \in A^{1\tau}(X)$ is similar. Let F_A be the smallest model in the moved (according the way of moving to A from the central line) $C^\rho(A^{0\rho})$ with A inside $(A^{1\tau})^F_A$. Compare F_A with H.

Case 1. $F_A \notin A^{1\rho}(H)$ and $H \notin A^{1\rho}(F_A)$.

Consider the last common point K of the walks to F_A and to H. Proceeding as in 1.19, we can assume that K has immediate predecessors K_0, K_1 of a weak Δ - system type with $F_A \in A^{1\rho}(K_0)$ and $H \in A^{1\rho}(K_1)$. By the definition of a Δ - system type, there will be $D_{01} \in K_0 \cap A^{1\zeta}, D_{10} \in K_1 \cap A^{1\zeta}$ such that

$$K_0 \cap K_1 = K_0 \cap D_{01} = K_1 \cap D_{10}$$

and K_0, K_1 are isomorphic over $K_0 \cap K_1$. Let K_0 be the one in $C^\rho(K)$.

Now we move H and B to K_0 side. Set $H^0 = \pi_{K_1 K_0}[H]$ and $B^0 = \pi_{K_1 K_0}[B]$. But $B \subseteq K_0 \cap K_1$, since $B \subseteq A \subseteq F_A \subseteq K_0$. Hence $B_0 = B$. This contradicts the choice of H as the simplest possible, since we found a simpler replacement H^0.

Case 2. $F_A \in A^{1\rho}(H)$ or $H \in A^{1\rho}(F_A)$.

Let $H \in A^{1\rho}(F_A)$. Assume that $\rho \in A$. The case $\rho \notin A$ is similar and repeats Subsubcase 2.1.2 of 1.19. By minimality of F_A, then also $H \in A^{1\rho}(F')$, for some $F' \in A \cap C^\rho(F)$. We use here 1.1(20) or (21). If F_A is a successor model, then F_A^- exists, it is in A and is equal to $(A^{0\rho})^A$. Consider the walk from F' to H. We assume that no models of bigger than ρ cardinalities are involved here (otherwise we are back in the situation considered in Case 1) and so the walk is entirely in $A_0^{1\rho}$. Let F be the last point of this walk in A and Y the very next point of this walk. Then F must be a limit point. Let

$$\tilde{F} = \bigcup \{X | X \in A \cap C^\rho(F) \backslash \{F\}\}.$$

Y must be a splitting point with two immediate predecessors Y_0, Y_1 of a Δ - system type, $Y_0 \in C^\rho(Y), G \in A^{1\rho}(Y_1)$. We would like to move to Y_0 side simplifying the situation. By the definition of a Δ - system type, there will be $D_{01} \in Y_0 \cap A^{1\zeta}, D_{10} \in Y_1 \cap A^{1\zeta}$ such that

$$Y_0 \cap Y_1 = Y_0 \cap D_{01} = Y_1 \cap D_{10}$$

and Y_0, Y_1 are isomorphic over $Y_0 \cap Y_1$. Then

$$A \cap B = A \cap F_A \cap B \cap H = A \cap F_A \cap Y \cap Y_1 \cap B = A \cap F' \cap Y \cap Y_1 \cap B =$$

$$A \cap \tilde{F} \cap Y_1 \cap B = A \cap Y_0 \cap Y_1 \cap B = A \cap \tilde{F} \cap Y_0 \cap Y_1 \cap H \cap B =$$

$$A \cap \tilde{F} \cap D_{10} \cap B.$$

Now, since $B \subseteq A$ we must have $B \subseteq D_{10}$. So, $B \subseteq Y_0$ and we can move everything to the Y_0 - side simplifying the situation.

$\square$

The following two lemmas extend similar statements for $A_0^{1\tau}$. Their prove follows the lines of 1.19.

Lemma 1.21 *Let A be a set in $A^{1\tau}$ for some $\tau \in s$. Then the following holds: for each $\rho \in s \backslash \tau + 1$ there is $F \in A^{1\rho}$ such that*

(1) $A \subseteq F$

(2) $gd(F) \leq gd(A)$

(3) if $G \in A^{1\xi}$, for some $\xi \in s \backslash \rho$ and $G \supseteq A$, then $A \subseteq F \subseteq G$.

Proof. Suppose that $A \in C^{\tau}(A^{0\tau})$, otherwise just move it to the central line by doing finitely many switches. Pick F to be the least model in $C^{\rho}(A^{0\rho})$ including A. We claim that F is as desired. Thus let $G \in A^{1\xi}$, for some $\xi \in s \backslash \rho$ and $G \supseteq A$. Assume that $F \not\supseteq G$. By 1.19, we have $ip(F, G)$ and by the definition 1.18 of $ip(F, G)$ there will be $D \in F \cap A^{1\xi}$ with $F \cap D \supseteq F \cap G \supseteq A$, for some $\xi \in s \backslash \rho + 1$. Let $E \in C^{\xi}(A^{0\xi})$ be the least model including A. Then $E \supset F$, by 1.1(24),as $\xi > \rho$ and both models E and F are on the central line. Hence $D \subset E$. But $D \supseteq A$ and E was the least model of $C^{\xi}(A^{0\xi})$ including A this is impossible by 1.1(20, 21, 6(a)).
□

Lemma 1.22 *Let A be a set in $A^{1\tau}$ for some $\tau \in s$. Then the following holds: if $H \in A^{1\xi}$, for some $\xi \in s \backslash \tau + 1$, and $H \supseteq A$, then for each $\rho \in s, \tau < \rho < \xi$ there is $F \in A^{1\rho}$ with $A \subseteq F \subseteq H$.*

Proof. Pick $F \in A^{1\rho}$ and $E \in A^{1\xi}$ satisfying the conclusion of 1.21 with ρ and with ξ respectively. Then, by 1.21(3) (for F), we obtain

$$A \subset F \subset G.$$

But 1.21(3) for E implies $H \supseteq G$. So,

$$A \subseteq F \subseteq H$$

and we are done.
□

We turn now to the definition, of the order on $\mathcal{P}'$.

Let us give a preliminary definition.

Definition 1.23 Let $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}'$ and $B \in C^\rho(A^{0\rho})$ for some $\rho \in s$. Define **the switching of** p **by** B , or shortly- $swt(p, B)$ to be $q = \langle\langle A^{0\tau}(q), A^{1\tau}(q), C^\tau(q) \mid \tau \in s(q)\rangle$ so that $q = p$ unless the following condition is satisfied:

(*) B is a successor point having two immediate predecessors $B_0 \in C^\rho(B)$ and B_1 such that the triple B_0, B_1, B is suitable for switching (see 1.2) i.e.

for each $\tau \in s \cap \rho$, $B \in A^{0\tau}$ and if $A \in C^\tau(A^{0\tau})$ is the first with $B \in A$, then its immediate predecessor A^- in $C^\tau(A^{0\tau})$ is in B. Moreover, if A is a splitting point as witnessed by A_0, A_1 and $\sup(A_0) < \sup(A_1)$, then $A_0 \in B \in A_1$.

Note that in the last case, i.e. if A is a splitting point as witnessed by A_0, A_1 and $\sup(A_0) < \sup(A_1)$, then it is impossible to have $A_1 \in B \in A$ by 1.1(11). Also, by 1.1(29), we must have $B_0, B_1 \in A_1$ as well. It is not hard to construct B's that fail to satisfy the second part of (b). What is needed is a chain of models of the length $> \tau$ which splits more than τ many times and two successive models $A^-, A = A^{0\tau}$ with $A^- \in C^\tau(A)$, and the chain inside both A^- and A. Now any splitting point of this chain $B \in A$ which is above $\sup(A^- \cap \rho)$ will do the job.

If (*) holds then q will be obtained from p by switching B_0 and B_1. Thus $s(q) = s$, $A^{0\tau}(q) = A^{0\tau}, A^{1\tau}(q) = A^{1\tau}$ for each $\tau \in s$, $C^{1\tau}(q) = C^{1\tau}$ for every $\tau \in s\backslash\rho+1$. Only $C^\tau(q)$'s for $\tau \in s \cap \rho + 1$ may be different.

Let $C^\rho(q)(B) = C^\rho(B_1) \cup \{B\}$ and for each $E \in C^\rho(A^{0\rho})\backslash C^\rho(B)$ let $C^\rho(q)(E) = (C^\rho(E)\backslash C^\rho(B)) \cup C^\rho(q)(B)$.

Let now $\tau \in s \cap \rho$. Pick the first element A of $C^{0\tau}(A^{0\tau})$ with $B \in A$. Its immediate predecessor A^- in $C^\tau(A^{0\tau})$ is in B, by (b). Then $A^- \subset B_0$. Leave $C^\tau(A^-)$ unchanged as well all its initial segments. Set $C^\tau(q)(A^{0\tau}(q)) = (C^\tau(A^{0\tau})\backslash C^\tau(A^-)) \cup \pi_{B_0B_1}[C^\tau(A^-)]$. In order to obtain the full function $C^\tau(q)$ we just move the defined already portions via isomorphisms of the models in $A^{1\tau}$. Remember that $B \in A$, hence $\pi_{B_0B_1}[A^-]$ remains inside $Pred(A)$.

It is not hard to see that such defined q is in $\mathcal{P}'$.

Note that in particular, $C^\tau(q)(A^-) = C^\tau(A^-)$. Also, if A is a splitting point as witnessed by A_0, A_1 and $\sup(A_0) < \sup(A_1)$, then, as it was pointed above, we have $A_0 \in B \in A_1$, by 1.1(11) and so, by 1.1(29), $B_0, B_1 \in A_1$ as well. Now, suppose that $A_0 \in C^\tau(A)$. Then $A_0 = A^-$ and, so $C^\tau(A_0)$ does not change. Then also $C^\tau(A_1)$ does not change, since the models A_0, A_1 are isomorphic. Note that in this situation $\langle A_0, C^\rho(q) \restriction A^{1\rho} \cap A_0\rangle = \langle A_0, C^\rho \restriction A^{1\rho} \cap A_0\rangle$ is not isomorphic to $\langle A_1, C^\rho(q) \restriction A^{1\rho} \cap A_1\rangle$, since B_0 and B_1 switched and both are in A_1.

$\square$ of Definition 1.23.

Remark 1.24 (1) It is problematic to deal here only with models for which being of the same order type implies isomorphism over a common part. The switches that preserve this condition are not suffice. Thus Strategic Closure and Chain Condition Lemmas below break down. Let us illustrate this in the gap 4 case. Suppose that we have $p \in \mathcal{P}'$ of the following form: $\langle A^{0\kappa^+}(p), A^{1\kappa^+}(p) = \{A^{0\kappa^+}(p), A\}, C^{\kappa^+}(p) = \{A^{0\kappa^+}(p), A\}, A^{0\kappa^{++}}(p), A^{1\kappa^{++}}(p) = \{A^{0\kappa^{++}}(p), G, G_0, G_1\}, C^{\kappa^{++}}(p) = \{A^{0\kappa^{++}}(p), G, G_0\}, ...\rangle$, with G_0, G_1, G of a Δ-system type and $G_0, G_1, G \in A^{0\kappa^+}(p), A \in G_0$. Then $swt(p, G) \in \mathcal{P}'$. Let $A' = \pi_{G_0 G_1}[A]$. But suppose that we like (in order to show κ^{+++}-c.c. of $\mathcal{P}'_{\leq\kappa^+}$) to combine p with a similar condition q but with $A^{0\kappa^+}(q) \subset G_0$ and $A^{0\kappa^+}(q) \not\subset G_1$. Let r be such combination. Now if we need to preform the switch of G in order to show the strategic closure (for example, if we need to replace A by A') , then there is a problem. Thus $swt(r, G) \notin \mathcal{P}'$, since $\pi_{G_0 G_1}[A^{0\kappa^+}(q)]$ will have the same order type as those of $A^{0\kappa^+}(p)$ but will not be isomorphic to it by the isomorphism which is the identity on the common part.

(2) Note that Chain Conditions Lemmas require switchings with models satisfying the condition $(*)$ of 1.23.

Note that $swt(swt(p, B), B) = p$, where swt of $swt(p, B)$ is defined as above in 1.23.

We define also $swt(p, B_0, \ldots, B_n)$. Just use an induction on the length of the finite sequence of models $B_0, \ldots, B_n$. Thus, if $r = swt(p, B_0, \ldots, B_m)$ is defined then set

$$swt(p, B_0, \ldots, B_m, B_{m+1}) = swt(r, B_{m+1})\ .$$

Definition 1.25 Let $p, r \in \mathcal{P}'$. Then $p \geq r$ iff there are $B_0, \ldots, B_n$ such that $q = swt(p, B_0, \ldots, B_n)$ is defined and the following holds:

(1) $s(q) \supseteq s(r)$

(2) for every $\tau \in s(r)$

  (a) $A^{1\tau}(q) \supseteq A^{1\tau}(r)$

  (b) $C^\tau(q) \restriction A^{1\tau}(r) = C^\tau(r)$

(c) $A^{0\tau}(r) \in C^\tau(q)(A^{0\tau}(q))$

(e) for each $A \in A^{1\tau}(r)$ we have $A^{1\tau}(r)(A) = A^{1\tau}(q)(A)$.

This means that no changes can be made inside models that were already chosen.

Remark 1.26 (1) Note that if $t = swt(p, B_0, \ldots, B_n)$, then $t \geq p$ and

$$p = swt(swt(p, B_0, \ldots, B_n), B_n, B_{n-1}, \ldots, B_0) = swt(t, B_n, \ldots, B_0) \geq t.$$

Hence the switching produces equivalent conditions.

(2) We need to allow $swt(p, B)$ for the Δ-system argument. Since in this argument two conditions are combined into one and so C^0 should pick one of them only.

(3) The use of finite sequences $B_0, \ldots, B_n$ is needed in order to insure transitivity of the order $\leq$ on $\mathcal{P}'$.

Let us start with a lemma that provides a simple way to extend conditions.

Lemma 1.27 *(Extension Lemma)*

Let $p = \langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle \mid \nu \in s\rangle \in \mathcal{P}'$. Suppose that $\langle B(\nu) \mid \nu \in s\rangle$ is an increasing continuous sequence such that

(*a*) $|B(\nu)| = \nu$

(*b*) $B(\nu) \supseteq \nu$

(*c*) ${}^{cf\nu>}B(\nu) \subseteq B(\nu)$

(*d*) $B(\nu) \prec H(\theta)$

(*e*) $p \in B(\kappa^+)$

Then the extension $p^\frown\langle B(\nu) \mid \nu \in s\rangle$, defined in the obvious fashion, is in $\mathcal{P}'$ and is stronger than p, where for $\nu \in s$ we just replace $A^{0\nu}$ by $B(\nu)$, add $B(\nu)$ to $A^{1\nu}$ and extend C^ν by adding $B(\nu)$.

Proof. All the conditions of 1.1 hold easily here. Also 1.25 is trivially satisfied.
□

The next lemma is needed (or is nontrivial) only if there are more than κ^+ cardinals between κ and θ or even if there are inaccessible cardinals between κ and θ. If the number of the cardinals between κ and θ is less than κ^{++}, then then the support of conditions can be fixed. Thus we can use always s to be the set of all regular cardinals of the interval $[\kappa^+, \theta]$ and require that each model of a condition includes s.

Lemma 1.28 *Let $p = \langle\langle A^{0\tau}, A^{1\tau}, C^\tau\rangle \mid \tau \in s\rangle$ be in $\mathcal{P}'$ and $\rho \in [\kappa^+, \theta]$ be a regular cardinal. Then there is $q = \langle\langle B^{0\tau}, B^{1\tau}, D^\tau\rangle \mid \tau \in t\rangle$ extending p and with $\rho \in t$.*

Proof. Clearly, we can assume that $\rho \notin s$. Let $\rho^* = \min(s\backslash\rho + 1)$. Recall that θ is always in support of any condition. So, $\rho^* \leq \theta$. By 1.1(1), ρ^* should be an inaccessible. Let $\rho' = \max(s \cap \rho)$. If ρ is itself an inaccessible or if $\rho = (\rho')^+$, then set $t = s \cup \{\rho\}$. Otherwise we are forced to add together with ρ some additional cardinals. If there are no inaccessibles in the interval $(\rho', \rho]$, then set $t = s \cup \{\xi \in (\rho', \rho] \mid \xi$ is a cardinal $\}$. If there are inaccessibles inside the interval (ρ', ρ), but ρ is not an inaccessible, then let $\rho'' = \sup\{\xi < \rho \mid \xi$ is an inaccessible $\}$. Now, if ρ'' itself is an inaccessible (i.e. if there is maximal inaccessible below ρ) then set $t = s \cup \{\xi \in [\rho'', \rho] \mid \xi$. If ρ'' is singular then pick a cofinal closed sequence

$\langle \rho_i \mid i < cf\rho'' \rangle$ in ρ'' such that for each $i, \rho_i \in (\rho', \rho'')$ and ρ_{i+1} is an inaccessible. Set then $t = s \cup \{\rho_i | cf\rho_i \geq \kappa^+\} \cup \{\xi \in [\rho'', \rho] | \xi \text{ is a cardinal }\}$.

Turn now to the definition of q. We concentrate on the central line. The full condition will be obtained by mapping it using isomorphisms over splitting points. So the issue will be to satisfy 1.1(19). Thus for each $A \in C^\tau(A^{0\tau})$, with $\tau \in s \cap \rho^*$, $F \in C^{\rho^*}(A^{0\rho^*})$ and $\xi \in t \backslash s$ we need to add a model G such that $A \subseteq G \subseteq F$ with $|G| = \xi$. It is enough to deal only with $A \in C^{\rho'}(A^{0\rho'}), F \in C^{\rho^*}(A^{0\rho^*})$ such that F is the least element of $C^{\rho^*}(A^{0\rho^*})$ including A and A on the other hand is the maximal element of $C^{\rho'}(A^{0\rho'})$ included in F. Denote by S the set of all such pairs $\langle A, F \rangle$. Clearly the cardinality of S is at most ρ'.
By induction let us pick for each $\langle A, F \rangle$ the smallest possible increasing continuous chain $\langle B_\mu | \mu \in t \backslash s \rangle$ of elementary submodels of $\langle F, p \cap F \rangle$ such that

(0) $A \in B_{(\rho')^+}$

(1) $|B_\mu| = \mu$ and $B_\mu \supseteq \mu$

(2) ${}^{cf\mu>}B_\mu \subseteq B_\mu$

(3) if μ is nonlimit then $\langle B_{\mu'} \mid \mu' < \mu \rangle \in B_\mu$

(4) $B_{(\rho')^+}$ includes models added (if any) for each pair $\langle A', F' \rangle \in S$ with $A' \in A$, as well as $A', F' \rangle$.

Let $q = \langle \langle B^{0\tau}, B^{1\tau}, D^\tau \rangle \mid \tau \in t \rangle$ be the set obtained from p by adding the sequences defined above to the central line and then mapping the result by isomorphisms over splitting points.
□

Now we turn to splittings of $\mathcal{P}'$.

Definition 1.29 Let $\tau \in (\kappa, \theta]$ be a cardinal. Set

$$\mathcal{P}'_{\geq\tau} = \{\langle\langle A^{0\rho}, A^{1\rho}, C^\rho \rangle \mid \rho \in s \backslash \tau \rangle \mid \exists \langle\langle A^{0\nu}, A^{1\nu}, C^\nu \rangle \mid \nu \in s \cap \tau \rangle \ \langle\langle A^{0\mu}, A^{1\mu}, C^\mu \rangle \mid \mu \in s \rangle \in \mathcal{P}\}\ .$$

Let $G(\mathcal{P}'_{\geq\tau})$ be generic. Define

$$\begin{aligned}\mathcal{P}'_{<\tau} =& \{\langle\langle A^{0\nu}, A^{1\nu}, C^\nu \rangle \mid \nu \in s \cap \tau \rangle \mid \exists \langle\langle A^{0\rho}, A^{1\rho}, C^\rho \rangle \mid \rho \in s \backslash \tau \rangle \in G(\mathcal{P}'_{\geq\tau}) \\ & \langle\langle A^{0\mu}, A^{1\mu}, C^\mu \rangle\rangle \mid \mu \in s \rangle \in \mathcal{P}'\}\ .\end{aligned}$$

Note that it is not immediate here that $\mathcal{P}'$ splits into $\mathcal{P}'_{\geq\tau} * \underset{\sim}{\mathcal{P}}'_{<\tau}$.

Let τ be a regular cardinal. If $p \in \mathcal{P}'$, then $p\backslash\tau$- the part of p above τ, is defined as follows:

$$p\backslash\tau = \langle\langle A^{0\xi}(p), A^{1\xi}(p), C^{\xi}(p)\rangle \mid \xi \in s(p)\backslash\tau\rangle$$

Similarly, define $p \upharpoonright \tau$ to be the part of p consisting of its elements below τ, i.e.

$$p \upharpoonright \tau = \langle\langle A^{0\xi}(p), A^{1\xi}(p), C^{\xi}(p)\rangle \mid \xi \in s(p) \cap \tau\rangle$$

Note that $\mathcal{P}'$ is not $\mathcal{P}'_{<\tau} \times \mathcal{P}_{\geq\tau}$ where $\mathcal{P}_{<\tau} = \{p \upharpoonright \tau \mid p \in \mathcal{P}'\}$. The complication here is due to the way of interconnections between models. So, instead of product let us deal with the iteration. Thus in $V^{\mathcal{P}'}_{\geq\tau}$ we define $\mathcal{P}'_{<\tau}$ to be the set of all $p \upharpoonright \tau$ for $p \in \mathcal{P}'$ such that $p\backslash\tau$ is in the generic set $G(\mathcal{P}'_{\geq\tau}) \subseteq \mathcal{P}'_{\geq\tau}$. The next lemma shows that the map $p \mapsto p\backslash\tau$ is a projection map and so $\mathcal{P}'_{\geq\tau}$ is a nice suborder of $\mathcal{P}'$.

For $p \in \mathcal{P}'$ and $q \in \mathcal{P}'_{\geq\tau}$ let $q{}^\frown p$ denotes the set obtained by combining p and q in the obvious fashion. Note that such a set need not be in general a condition in $\mathcal{P}'$, but in reasonable cases it will.

Lemma 1.30 *(The Splitting Lemma) Let $p \in \mathcal{P}'$, τ be a regular cardinal in $(\kappa, \theta] \cap s(p)$ and $q \in \mathcal{P}'_{\geq\tau}$. If $q \geq_{\mathcal{P}'_{\geq\tau}} p\backslash\tau$, then $q{}^\frown p \in \mathcal{P}'$ and extends p.*

Proof. Let $p = \langle\langle A^{0\xi}, A^{1\xi}\ ,\ C^{\xi}\rangle \mid \xi \in s\rangle$. Note that $q{}^\frown p$ need not be a condition since 1.1 may break badly. Thus for example, switching inside $\mathcal{P}'_{\geq\tau}$ may move models in a way that when adding back $A^{0\xi}$'s (for $\xi < \tau$) C^{ξ}'s cannot be moved. In order to deal with such situations, we first replace q by an equivalent condition (switching it into such condition) satisfying 1.25 (1,2) with $p\backslash\tau$ and only then add the full p. Once $A^{0\xi}(p) \in C^{\xi}(q)(A^{0\xi}(q))$ and $C^{\xi}(q)$ extends $C^{\xi}(p)$ for $\xi \in s\backslash\tau$ the problem above disappears.

The rest easily follows from 1.1.

□

Let us show now a strategic closure of the forcing.

Lemma 1.31 *(Strategic Closure Lemma) Let $\rho \in (\kappa, \theta]$ be a regular cardinal. Then $\langle\mathcal{P}'_{\geq\rho}, \leq\rangle$ is ρ^+ – strategically closed.*

Proof. We define a winning strategy for the player playing at even stages. Thus suppose $\langle p_j \mid j < i\rangle$ is a play according to this strategy up to an even stage i. Define p_i.

Let for each $j < i$

$$p_j = \langle \langle A_j^{0\tau}, A_j^{1\tau}, C_j^{\tau} \rangle \mid \tau \in s_j \rangle \ .$$

Case 1 i is a successor ordinal.
Pick a sequence $\langle B(\tau) \mid \tau \in s_{i-1} \rangle$ satisfying the conditions (a) – (d) of 1.27 with p replaced by p_{i-1}. Let p_i be the extension of p_{i-1} by $\langle B(\tau) \mid \tau \in s_{i-1} \rangle$.

Case 2 i is a limit ordinal.
Replacing each p_j $(j < i)$ by a switched condition if necessary, we can assume p_j's satisfy the conditions of (1),(2) of 1.25, i.e. one extends another in the natural sense. Define first $p = \langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \rangle$ as follows: set $s = \bigcup_{j<i} s_j$, $A^{0\tau} = \bigcup_{j<i, \tau \in s_j} A_j^{0\tau}$, $A^{1\tau} = \bigcup_{j<i, \tau \in s_j} A_j^{1\tau} \cup \{A^{0\tau}\}$ and $C^{\tau} = \bigcup_{j<i, \tau \in s_j} C_j^{\tau} \cup \{\langle A^{0\tau}, \cup \{C_j^{\tau}(A_j^{0\tau}) \mid j \text{ is even and } \tau \in s_j \rangle\}$, for $\tau \in s$.

Such defined p is not necessarily a condition. Thus, for example, 1.1(2(b)) may fail. We fix this by defining p_i from p as follows. Set $B(\rho) = A^{0\rho}$ and for each $\tau \in (\rho, \theta] \cap s$ we chose $B(\tau)$ to be a model such that

(i) $A^{0\tau} \in B(\tau)$

(ii) $|B(\tau)| = \tau$, $B(\tau) \supseteq \tau$

(iii) ${}^{cf\tau>}B(\tau) \subseteq B(\tau)$

(iv) if $\tau < \tau'$ then $B(\tau) \subseteq B(\tau')$

(v) if τ is a limit point of s then $B(\tau) = \cup \{B(\tau') \mid \tau' \in s \cap \tau\}$

(vi) $\langle p_j \mid j < i \rangle, p,\ B(\rho) \in B(\tau)$ for every $\tau \in (\rho, \theta] \cap s$.

Let p_i be obtained from p by adding the sequence $\langle B(\tau) \mid \tau \in [\rho, \theta) \cap s \rangle$. We define

$$C^{\tau}(p_i)(B(\tau)) = C^{\tau} \cup \{\langle B(\tau)\ ,\ C^{\tau}(A^{0\tau})^{\frown} B(\tau) \rangle\} \ .$$

Such defined p_i is a condition. The proof as those of 1.27 follows easily. Note that here we have $\{p_j \mid j < i\} \subseteq A^{0\tau}$ for each $\tau \in s$.
$\square$

Let us turn now to the chain conditions.

Lemma 1.32 *(Chain Condition Lemma) Let τ be a regular cardinal in $[\kappa^+, \theta]$. Then, in $V^{\mathcal{P}'_{\geq\tau}}$ the forcing $\mathcal{P}_{<\tau}$ satisfies τ^+-chain condition.*

Proof. Suppose otherwise. Let us assume that

$$\phi \Vdash_{\mathcal{P}'_{\geq\tau}} (\underset{\sim}{p}_\alpha = \langle\langle \underset{\sim}{A}^{0\xi}_\alpha, \underset{\sim}{A}^{1\xi}_\alpha, \underset{\sim}{C}^{\xi}_\alpha\rangle | \xi \in \underset{\sim}{s}_\alpha\rangle | \alpha < \tau^+\rangle \text{ is an antichain in } \underset{\sim}{\mathcal{P}}'_{<\tau}) \ .$$

Define by induction, using the strategy of 1.4 for $\mathcal{P}'_{\geq\tau}$, an increasing sequence of conditions $\langle q_\alpha | \alpha < \tau^+\rangle$, $q_\alpha = \langle\langle A^{0\xi}_\alpha, A^{1\xi}_\alpha, C^{\xi}_\alpha\rangle | \xi \in t_\alpha\rangle$ and a sequence $\langle p_\alpha \mid \alpha < \tau^+\rangle, p_\alpha = \langle\langle A^{0\xi}_\alpha, A^{1\xi}_\alpha, C^{\xi}_\alpha\rangle | \xi \in s_\alpha\rangle$ so that for every $\alpha < \tau^+$

$$q_\alpha \Vdash_{\mathcal{P}'_{\geq\tau}} \langle\langle \underset{\sim}{A}^{0\xi}_\alpha, \underset{\sim}{A}^{1\xi}_\alpha, \underset{\sim}{C}^{\xi}_\alpha\rangle \mid \xi \in \underset{\sim}{s}_\alpha\rangle = \check{p}_\alpha \ .$$

For a limit $\alpha < \tau^+$ let

$$\overline{q}_\alpha = \langle\langle \overline{A}_\alpha^{0\xi}, \overline{A}_\alpha^{1\xi}, \overline{C}_\alpha^{\xi}\rangle \mid \xi \in \overline{t}_\alpha\rangle$$

be the condition produced by the strategy and q_α be its extension deciding p_α. We form a Δ-system now stabilizing as many parts of the conditions as possible. Note that $s_\alpha \subseteq \tau$ and $|s_\alpha| < \tau$ since τ is regular, for each $\alpha < \tau^+$. Hence we can assume that all s_α's are the same and equal to some s. Let $\alpha < \beta < \tau^+$, $cf\alpha = cf\beta = \tau$ be in the system. We like to show then the compatibility of $q_\alpha \frown p_\alpha$ and $q_\beta \frown p_\beta$ or since $q_\beta \geq q_\alpha$ the compatibility of $q_\beta \frown p_\alpha$ and $q_\beta \frown p_\beta$.

Let $\hat{\tau} = \max(\tau \cap s)$, which exists and is regular since τ is regular by the definition of a support. First pick $B^{\hat{\tau}}(0) \prec A^{0\tau}_{\beta+1}$ of cardinality $\hat{\tau}$ with $q_\beta, p_\alpha, p_\beta \in B^{\hat{\tau}}(0)$ and ${}^{\hat{\tau}>}B^{\hat{\tau}}(0) \subseteq B^{\hat{\tau}}(0)$. Then we define by induction on $\xi \in s$ sets B^ξ such that

(1) $|B^\xi| = \xi$, ${}^{cf\xi>}B^\xi \subseteq B^\xi$

(2) $B^{\hat{\tau}}(0) \in B^\xi$

(3) $B^\xi \prec A^{0\tau}_{\beta+1}$

(4) $\langle B^{\xi'} \mid \xi' \in s \cap \xi\rangle \in B^\xi$.

Define now a common extension

$$p = \langle\langle B^{0\xi}, B^{1\xi}, D^\xi\rangle \mid \xi \in s \cup t_\beta\rangle$$

as follows. For each $\xi \in s$ let

$$B^{0\xi} = B^\xi, B^{1\xi} = A^{1\xi}_\alpha \cup A^{1\xi}_\beta \cup \{B^\xi\} \ ,$$

if $\xi \neq \hat{\tau}$ and

$$B^{1\hat{\tau}} = A^{1\hat{\tau}}_\alpha \cup A^{1\hat{\tau}}_\beta \cup \{B^{\hat{\tau}}(0), B^{\hat{\tau}}\},$$

$$D^{\xi} = C^{\xi}_{\alpha} \cup C^{\xi}_{\beta} \cup \{\langle B^{\xi}, \langle C^{\xi}_{\beta}(A^{0\xi}_{\beta})^{\frown} B^{\xi}\rangle\}$$

(if $\xi = \hat{\tau}$, then we add also $B^{\hat{\tau}}(0)$).
For every $\xi \in t_{\beta}$ let

$$B^{0\xi} = A^{0\xi}_{\beta+1}, B^{1\xi} = A^{1\xi}_{\beta} \cup \{A^{0\xi}_{\beta+1}\} \text{ and } D^{\xi} = C^{\xi}_{\beta} \cup \{\langle A^{0\xi}_{\beta+1}\}, \langle C^{\xi}_{\beta}(A^{0\xi}_{\beta})^{\frown} A^{0\xi}_{\beta+1}\rangle\}.$$

We need to check that such defined p is in $\mathcal{P}'$.

Note that $B^{\hat{\tau}}(0)$ will be the immediate successor of $A^{0\hat{\tau}}_{\alpha}, A^{0\hat{\tau}}_{\beta}$ and the triple $A^{0\hat{\tau}}_{\beta}, A^{0\hat{\tau}}_{\alpha}, B^{\hat{\tau}}(0)$ will be of a Δ-system type over $C^{\tau}(A^{0\tau}_{\beta+1})$. Also, $B^{\hat{\tau}}(0) \in B^{0\xi}$ for each $\xi \in s \cup t_{\beta}$. Hence the requirements of 1.1 related to splittings of models are satisfied here, as well as the requirement (b) on switching of 1.23. The rest of the conditions hold trivially in the present context.
$\square$

The next lemma shows GCH in $V^{\mathcal{P}'}$. The forcing $\mathcal{P}'$ was designed specially to make this true.

Lemma 1.33 *(GCH Lemma) Let τ be a regular cardinal in $[\kappa^+, \theta]$. Then in $V^{\mathcal{P}'}$ we have $2^{\tau} = \tau^+$.*

Proof. Let $N \prec H((2^{\lambda})^+)$ for λ large enough such that $\mathcal{P}' \in N$, $|N| = \tau^+$ and ${}^{\tau}N \subseteq N$. Using τ^{++}-strategic closure of $\mathcal{P}'_{\geq\tau^+}$ we find $p^N_{\geq\tau^+} \in \mathcal{P}'_{\geq\tau^+}$ which is N-generic for $\mathcal{P}'_{\geq\tau^+}$. Let $G(\mathcal{P}'_{\geq\tau^+})$ be a generic subset of $\mathcal{P}'_{\geq\tau^+}$ with $p_{\geq\tau^+} \in G(\mathcal{P}'_{\geq\tau^+})$. Then, $N[p_{\geq\tau^+}] \prec V_{\lambda}[G(\mathcal{P}'_{\geq\tau^+})]$. By Lemma 1.8, $\mathcal{P}'_{<\tau^+}$ satisfies τ^{++}-c.c in $V[G(\mathcal{P}'_{\geq\tau^+})]$. In particular, $\mathcal{P}_{=\tau}$ satisfies τ^{++}-c.c. Let $G(\mathcal{P}'_{=\tau})$ be a generic subset of $\mathcal{P}_{=\tau}$ over $V[G(\mathcal{P}'_{\geq\tau^+})]$. Denote $N[p_{\geq\tau^+}]$ by N_1. Then $N_1[N_1 \cap G(\mathcal{P}'_{=\tau})] \prec V[G(\mathcal{P}'_{\geq\tau^+})][G(\mathcal{P}'_{=\tau})]$, since each antichain for $\mathcal{P}'_{=\tau}$ has cardinality at most τ^+. Hence, if it belongs to N_1 then it is also contained in N_1. Denote $N_1[N_1 \cap G(\mathcal{P}'_{=\tau})]$ by N_2. We now consider $\mathcal{P}'_{<\tau} \cap N_2$. Clearly this is a forcing of cardinality τ^+. We claim that it is equivalent to $\mathcal{P}'_{<\tau}$. Thus, by Lemma 1.8, $\mathcal{P}'_{<\tau}$ satisfies τ^+-c.c., so $\mathcal{P}'_{<\tau} \cap N_2$ is a nice suborder of $\mathcal{P}'_{<\tau}$. Let $G \subseteq \mathcal{P}'_{<\tau}$ be generic over $V[G(\mathcal{P}'_{\geq\tau^+})][G(\mathcal{P}'_{=\tau})]$ and $H = G \cap N_2$. Then H is $\mathcal{P}'_{<\tau} \cap N_2$ generic over $V[G(\mathcal{P}'_{\geq\tau^+})][G(\mathcal{P}'_{=\tau})]$. Thus, if $A \subseteq \mathcal{P}'_{<\tau} \cap N_2$ is a maximal antichain, then A is antichain also in $\mathcal{P}'_{<\tau}$, since N_2 is an elementary submodel. Hence $|A| \leq \tau$. But then $A \in N_2$, and so $N_2 \vDash$ (A is a maximal antichain in $\mathcal{P}'_{<\tau}$). By elementary, A is a maximal antichain in $\mathcal{P}'_{<\tau}$. So there is $p \in G \cap A$. Finally, $A \subseteq N_2$ implies that $p \in N_2$ and hence $p \in H$.

We claim that each subset of τ is already in $N_2[G]$. It is enough since $|N_2[G]| = |N| = \tau^+$. Let $\underset{\sim}{a}$ be a name of a function from τ to 2. Work in V. Define by induction (using the strategic closure of the forcings and τ^+-c.c. of $\mathcal{P}'_{<\tau}$) sequences of ordinals

$$\langle \delta_\beta | \beta < \tau \rangle, \langle \gamma(\alpha, \beta) | \beta < \tau, \alpha < \delta_\beta \rangle$$

and sequences of conditions

$$\langle p_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau), \langle p(\beta) | \beta < \tau \rangle$$

such that

(1) for each $\beta < \tau$, $\delta_\beta < \tau^+$

(2) for each $\beta < \tau$, $\langle p_\beta(\alpha)_{\geq\tau} | \alpha < \delta_\beta \rangle$ is increasing sequence of elements of $\mathcal{P}'_{\geq\tau}$ and $p(\beta)$ is its upper bound obtained as in the Strategic Closure Lemma

(3) $p_0(0)_{\geq\tau^+} \geq p^N_{\geq\tau^+}$

(4) the sequence $\langle p(\beta) | \beta < \tau \rangle$ is increasing

(5) for each $\beta < \tau$ and $\alpha < \delta_\beta$, $p_\beta(\alpha)$ forces "$\underset{\sim}{a}(\beta) = \gamma(\alpha, \beta)$"

(6) if some $p \in \mathcal{P}'$ is stronger than $p(\beta)_{\geq\tau}$ where top models of cardinalities below τ are viewed as empty or trivial, then there is $\alpha < \delta$ such that the conditions $p, p_\beta(\alpha)$ are compatible. (I.e. $\{p_\beta(\alpha)_{<\tau} | \alpha < \delta_\beta\}$ is a pre-dense set as forced by $p(\beta)_{\geq\tau}$).

Set $p(\tau)$ to be the upper bound of $\langle p(\beta) | \beta < \tau \rangle$ as in the Strategic Closure Lemma. Let L denotes the top model of cardinality τ of $p(\tau)_{\geq\tau}$, i.e. $A^{1\tau}(p(\tau)_{\geq\tau})$. Pick $K \in N$ realizing the same type as those of L in $H(\lambda)[G_{\geq\tau^+}]$. Let

$$\langle q(\beta) | \beta < \tau \rangle, \langle q_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau)$$

be the sequences corresponding to

$$\langle p_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau), \langle p(\beta) | \beta < \tau \rangle.$$

Define a name $\underset{\sim}{b}$ of a subset of τ to be

$$\{ < q_\beta(\alpha), \gamma(\alpha, \beta) > | \alpha < \delta_\beta, \beta < \tau \}.$$

Clearly, $\underset{\sim}{b}$ is in N. Combine now K, L into one condition making them a splitting point. Let M be a model of cardinality τ such that $K, L \in M$ as well as the sequences

$$\langle p_\beta(\alpha) | \alpha < \delta_\beta \rangle (\beta < \tau), \langle p(\beta) | \beta < \tau \rangle$$

and

$$\{< q_\beta(\alpha), \gamma(\alpha,\beta) > |\alpha < \delta_\beta, \beta < \tau\}.$$

Let $\langle A(\xi)|\xi \in s\rangle$ be an increasing continuous sequence of models with $|A^\xi| = \xi$ and $K, L, M, \langle p_\beta(\alpha)|\alpha < \delta_\beta\rangle(\beta < \tau), \langle p(\beta)|\beta < \tau\rangle$ and $\{< q_\beta(\alpha), \gamma(\alpha,\beta) > |\alpha < \delta_\beta, \beta < \tau\} \in A(\kappa^+)$. Put this sequence to be the top sequence of such combined condition which we denote by r.

Claim 1.33.1 $r \Vdash \underset{\sim}{a} = \underset{\sim}{b}$.

Proof. Let G be a generic subset of $\mathcal{P}'$ with $r \in G$. Then also $p(\tau)_{\geq\tau}, q(\tau)_{\geq\tau} \in G$. Now, for each $\beta < \tau$ there is $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$ (just otherwise there will be a condition t in G forcing that for some β there is no $\alpha < \delta_\beta$ with $p_\beta(\alpha) \in G$. Extend it to t' deciding the value $\underset{\sim}{a}(\beta)$. By (6) there is α such that $t', p_\beta(\alpha)$ are compatible). Let $r' \in G$ be a common extension of r and $p_\beta(\alpha)$. Now M will be a splitting point witnessed by L, K in r' and the isomorphism π_{LK} moves $p_\beta(\alpha)$ to $q_\beta(\alpha)$. Hence $q_\beta(\alpha) \leq r'$. But then $q_\beta(\alpha) \in G$.
$\square$ of the claim.
$\square$

2 Preserving Large Cardinals

We will need to make some minor changes in the previous setting. Thus, first it will be convenient to increase a bit a set of conditions by allowing to remove some maximal models (i.e. $A^{0\alpha}$) from elements of $\mathcal{P}'$. This way the original $\mathcal{P}'$ will be dense in the new one, so from the forcing point of view nothing changes. Second, we like to deal with elementarity. In 1.1, we had $H(\theta)$ and considered its elementary submodels. But once embeddings $j : V \to M$ are around, $j(H(\theta)) = (H(j(\theta)))^M$ may differ from $H(\theta)$ even if θ is not moved. So being elementary in sense of M will differ from being elementary in sense of V. We suggest below two ways to overcome this difficulty. The first one will be to assume that θ is a 2^θ-supercompact cardinal. Consider the following set

$$S = \{\alpha < \theta | \alpha \text{ is a superstrong cardinal with target } \theta$$

(i.e. there is $i : V \to M$, $crit(i) = \alpha$, $i(\alpha) = \theta$ and $M \supseteq V_\theta)\}$.

It is stationary (actually of measure one for a normal measure over θ), see for example [Kan, 26.11].

Now, $V_\alpha \prec V_\theta$ for every $\alpha \in S$. Hence, $V_\alpha \prec V_\beta$ for every $\alpha < \beta$, $\alpha, \beta \in S$. Also the following holds:

Lemma 2.1 *Let $\alpha \in S$ and $i : V \to N$ is an ultrapower by an (α, ν)-extender for some $\nu \leq \theta$. which is a part of superstrong (α, θ)-extender with target θ. Then $V_\beta \prec (V_{i(\alpha)})^N$ for every $\beta \in S$, $\alpha \leq \beta < \nu$.*

Proof. Let $j : V \to M$ be the ultrapower by a superstrong (α, θ)-extender with target θ extending the used (α, ν)-extender. Then the following diagram is commutative

$$\begin{tikzcd}
 & M \\
V \arrow[ur, "j"] \arrow[dr, "i"'] & \\
 & N \arrow[uu, "k"']
\end{tikzcd}$$

where k is defined in the obvious fashion.

Now, $k((V_{i(\alpha)})^N) = V_{j(\alpha)} = V_\theta$. Also $k(\beta) = \beta$ and $V_\beta \prec V_\theta$. Hence, $V_\beta \prec (V_{i(\alpha)})^N$.

□

Note also that by elementarity $(V_{i(\alpha)})^N \prec (V_{i(\theta)})^N = (V_\theta)^N$.

The second way will be to deal with just subsets (closed enough) and Σ_1 elementarity. Using this approach there will be no need in supercompacts cardinals- thus strongs alone suffice.

Lemma 2.2 *Suppose that $V_\delta \prec_{\Sigma_1} V_\theta$, α is δ-strong and $j : V \to M$ be an elementary embedding such that*

- $M \supseteq V_\delta$
- $j(\theta) = \theta$.

Then $V_\delta \prec_{\Sigma_1} (V_\theta)^M$.

Proof. Just note that

$$V_\delta \subset (V_\theta)^M \subset V_\theta.$$

Models $V_\theta, (V_\theta)^M$ agree about Σ_0 formulas. So each Σ_1 formula with parameters from $(V_\theta)^M$ true in $(V_\theta)^M$ is also true in V_θ. But $V_\delta \prec_{\Sigma_1} V_\theta$, hence $V_\delta \prec_{\Sigma_1} (V_\theta)^M$.

□

The crucial observation will be that $\mathcal{P}'$ breaks at each $\alpha \in S$ (or just for each $\alpha < \theta$ which is Mahlo and has δ's as in 2.2) into forcing $\mathcal{P}'(\alpha)$ which deals with elementary submodels (or just closed enough subsets) of V_α and $\mathcal{P}'_{\geq\alpha}$ which breaks in turn into $\mathcal{P}'_{>\alpha} * \mathcal{P}'_{\{\alpha\}} * Q_\alpha$.

Define $\mathcal{P}'(\alpha)$ the same way as $\mathcal{P}'$ but only with V_α replacing V_θ. Thus in this notation $\mathcal{P}'$ is actually $\mathcal{P}'(\theta)$.

Lemma 2.3 *Suppose that α is a Mahlo cardinal. Then $\mathcal{P}'(\alpha)$ satisfies α -c.c.*

Proof. Let $\langle p_\beta | \beta < \alpha \rangle$ be a sequence of conditions in $\mathcal{P}'(\alpha)$, $p_\beta = \langle\langle A^{0\tau}(p_\beta), A^{1\tau}(p_\beta), C^\tau(p_\beta)\rangle | \tau \in s(p_\beta)\rangle, \beta < \alpha$.
Consider their supports sequence $\langle s(p_\beta) | \beta < \alpha\rangle$. Recall that supports are of the Easton form. Hence we can find a stationary $X \subseteq \alpha$ and s such that $\langle s(p_\beta) | \beta \in X\rangle$ forms a Δ-system with support s. Moreover,

- each $\beta \in X$ is inaccessible
- $s(p_\beta) \cap \beta = s$
- if $\gamma < \beta$ is also in X then for each $\tau \in s(p_\gamma)$, then $A^{0\tau}(p_\gamma) \subset V_\beta$.

This implies that

$$A \subset V_\beta \subseteq B,$$

whenever $\gamma < \beta$ in X, $A \in A^{1\tau}, \tau \in s(p_\gamma)$ and $B \in A^{1\rho}, \rho \in s(p_\beta) \backslash s$.

Shrinking X more, if necessary we can insure that for each $\gamma, \beta \in X$ the following two structures

$$\langle A^{0\max(s)}(p_\beta), <, \in, \subseteq, \kappa, A^{0\max(s)}(p_\beta) \cap p_\beta\rangle$$

and

$$\langle A^{0\max(s)}(p_\gamma), <, \in, \subseteq \kappa, A^{0\max(s)}(p_\gamma) \cap p_\gamma\rangle$$

are isomorphic over $A^{0\max(s)}(p_\beta) \cap A^{0\max(s)}(p_\gamma)$.

Note that $A^{0\tau}(p_\beta)$'s may have elements above β.

Now we claim that such p_β and p_γ are compatible, say $\gamma < \beta$. The proof repeats 1.?. Note that models of cardinalities in $s_\gamma \backslash s$ should be added between models of p_β of cardinalities in s and those including them of cardinalities in $s(p_\beta) \backslash s$. In order to this, we work over the center line of p_β add models which include p_γ as a member and then such setting via isomorphisms.

□

Lemma 2.4 *Suppose that α is a Mahlo cardinal and $V_\alpha \prec V_\theta$. Then $\mathcal{P}' \gtrdot \mathcal{P}'(\alpha)$.*

Proof. Consider $\mathcal{P}' \cap V_\alpha$. By the definition of conditions 1.1 we must have $\mathcal{P}'(\alpha) = \mathcal{P}' \cap V_\alpha$. The cardinal α is an inaccessible. Hence ${}^{\alpha>}V_\alpha \subseteq V_\alpha$. In particular, each antichain of $\mathcal{P}'(\alpha)$ is in V_α, by the previous lemma. Hence , if $H \subseteq \mathcal{P}'(\alpha)$ is $\mathcal{P}'(\alpha)$-generic over V_α, then H will be full $\mathcal{P}'(\alpha)$-generic.
Note that $\mathcal{P}'(\alpha)$ is definable in V_α and using the same formula that defines $\mathcal{P}'$ in V_θ.
Let $A \subseteq \mathcal{P}'(\alpha)$ be a maximal antichain. Then $|A| < \alpha$ and, so $A \in V_\alpha$. In addition,

$$V_\alpha \models A \text{ is a maximal antichain in } \mathcal{P}'.$$

Then, by elementarity,

$$V_\theta \models A \text{ is a maximal antichain in } \mathcal{P}'.$$

So, $G \cap A \neq \emptyset$, for any generic $G \subseteq \mathcal{P}'$. Also, $V_\alpha[G \cap V_\alpha] \prec V_\theta[G]$.
$\square$

By the lemma above $\mathcal{P}'$ projects to $\mathcal{P}'(\alpha)$. We prefer to deal with an explicit projection rather then with the projection defined via the corresponding Boolean algebras. In order to define an explicit projection we consider the following dense subset of $\mathcal{P}'$:

$$D = \{\langle\langle A^{00\tau}, A^{01\tau}\rangle, A^{1\tau}, C^\tau\rangle | \tau \in s \cap \alpha\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle | \nu \in s\backslash\alpha\rangle \in \mathcal{P}' \quad |$$

$$\alpha \in s \& \forall \tau \in s \cap \alpha \quad A^{00\tau} \in V_\alpha \text{ and the structure}$$

$$\langle A^{00\max(s\cap\alpha)}, <, \in, \subseteq, \kappa, A^{00\max(s\cap\alpha)} \cap \langle\langle A^{00\tau}, A^{01\tau}\rangle, A^{1\tau}, C^\tau\rangle | \tau \in s \cap \alpha\rangle\rangle \text{ is isomorphic to}$$

$$\langle A^{01\max(s\cap\alpha)}, <, \in, \subseteq, \kappa, A^{01\max(s\cap\alpha)} \cap \langle\langle A^{00\tau}, A^{01\tau}\rangle, A^{1\tau}, C^\tau\rangle | \tau \in s \cap \alpha\rangle\rangle \text{ over } V_\alpha \cap A^{01\max(s\cap\alpha)}\}.$$

Here is the point where we prefer to allow two top models ($A^{00\tau}, A^{01\tau}$, $\tau \in s \cap \alpha$) instead of a single one. Using $V_\alpha \prec_{\Sigma_1} V_\theta$ it is easy to extend any standard (i.e. with single top model in each cardinality) condition in $\mathcal{P}'$ to one in D. We need just to intersect its part consisting of models of cardinality below α with V_α and then using elementarity of V_α to find inside V_α something isomorphic over this intersection.

Now, once we have $p = \langle\langle A^{00\tau}, A^{01\tau}\rangle, A^{1\tau}, C^\tau\rangle | \tau \in s \cap \alpha\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle | \nu \in s\backslash\alpha\rangle \in D$, then define $\sigma(p)$ to $\mathcal{P}'(\alpha)$ to be

$$\langle A^{00\tau}, A^{1\tau} \cap \mathcal{P}(A^{00\tau}), C^\tau \restriction \mathcal{P}(A^{00\tau})\rangle | \tau \in s \cap \alpha\rangle^\frown\langle\langle A^{0\nu}, A^{1\nu}, C^\nu\rangle | \nu \in s\backslash\alpha\rangle.$$

Let us check that such defined σ is indeed a projection map.

Lemma 2.5 *The map σ is a projection map from D to $\mathcal{P}'(\alpha)$.*

Proof. Let $p \in D$ be as above and $q \in \mathcal{P}'(\alpha)$ be an extension of $\sigma(p)$. Pick increasing continuous sequence $\langle B_\tau | \tau \in s\rangle$ such that for each $\tau \in s$ the following holds:

1. $B_\tau \prec V_\theta$
2. $|B_\tau| = \tau$
3. $p, q \in B_{\kappa^+}$.

Now let $r = \langle\langle A^{0\tau}(r), A^{1\tau}(r), C^\tau(r)\rangle | \tau \in s\rangle$ be defined as follows:

- $A^{0\tau}(r) = B_\tau$
- $A^{1\tau}(r) = A^{1\tau} \cup \{B_\tau\}$, if $\tau \in s\backslash\alpha$ and $A^{1\tau}(r) = A^{1\tau} \cup \{B_\tau\} \cup A^{1\tau}(q)$, if $\tau \in s \cap \alpha$
- $C^\tau(r) = C^\tau \cup \langle B_\tau, C^{\tau\frown}B^\tau\rangle$, if $\tau \in s\backslash\alpha$ and $C^\tau(r) = C^\tau \cup C^\tau(q) \cup \langle B_\tau, C^\tau(q)^\frown B^\tau\rangle$, if $\tau \in s \cap \alpha$.

Then r is an element of $\mathcal{P}'$ stronger than both p and q. Note that the situation as here was specially allowed in 1.1 in contrast with the parallel definition of [6]. It remains to extend r to some $r' \in D$ and then to take $\sigma(r')$ which will be above q.
$\square$

Lemma 2.6 *Suppose that α is a Mahlo cardinal and $V_\alpha \prec V_\theta$. Let $\gamma < \alpha$ be a regular cardinal. Then $\mathcal{P}'_{\geq\gamma} \gtrdot \mathcal{P}'(\alpha)_{\geq\gamma}$.*

The proof repeats those of Lemma 2.4.

Note that $\mathcal{P}'_{\geq\alpha}$ does not add new sets of cardinalities $\geq \alpha$ and $\mathcal{P}' = \mathcal{P}'_{\geq\alpha} * \mathcal{P}'_{<\alpha}$.

Lemma 2.7 *Let $V_\alpha \prec V_\theta$, α be a Mahlo and $\delta < \alpha$ be a regular. Then $\mathcal{P}' = \mathcal{P}'_{\geq\delta} * (\mathcal{P}'(\alpha))_{<\delta}$.*

Proof. Pick $M \prec V_\alpha$, $\delta^+ \subseteq M$ and $|M| = \delta^+$. By 2.4, we have $\mathcal{P}'_{\geq\delta} \gtrdot (\mathcal{P}'(\alpha))_{\geq\delta}$. Note that $M \cap \mathcal{P}' = M \cap \mathcal{P}'(\alpha)$, since $M \prec V_\alpha$. Pick $p \in V_\alpha \cap \mathcal{P}'_{\geq\delta^+}$ to be $(\mathcal{P}'_{\geq\delta^+}, M)$-generic. Then $p \in (\mathcal{P}'(\alpha))_{\geq\delta^+}$ and it is $((\mathcal{P}'(\alpha))_{\geq\delta^+}, M)$-generic. Pick now $G_{\geq\delta^+} \subseteq \mathcal{P}'_{\geq\delta^+}$ generic with $p \in G_{\geq\delta^+}$ and $G_{=\delta} \subseteq \mathcal{P}'_{=\delta}$ generic over $V[G_{\geq\delta^+}]$. Recall that $\mathcal{P}'_{=\delta}$ satisfies δ^{++}-c.c. Hence each antichain of $\mathcal{P}'_{=\delta}$ which belongs to $M[p]$ will be contained in $M[p]$. So, $G_{=\delta} \cap M[p]$ will be $(\mathcal{P}'_{=\delta}, M[p])$- generic. But $(G_{\geq\delta} * G_{=\delta}) \cap V_\alpha$ is $\mathcal{P}'_{\geq\delta}$ -generic over V_α, by 2.4. So $G_{=\delta} \cap M[p]$ will be $(\mathcal{P}'_{=\delta}, M[p])$-generic. Denote $G_{=\delta} \cap M[p]$ by G_M. Then $M[p, G_M] \prec V_\alpha[G_{\geq\delta^+} * G_{=\delta} \cap V_\alpha] \prec V_\theta[G_{\geq\delta^+}, G_{=\delta}]$.

Let us turn now to $\mathcal{P}'_{<\delta}$. By 1.last, $\mathcal{P}'_{<\delta}$ in $V_\theta[G_{\geq\delta}, G_{=\delta}]$ is equivalent to $\mathcal{P}'_{<\delta} \cap M[p, G_M]$. But $M[p, G_M] \prec V_\alpha[G_{\geq\delta^+} * G_{=\delta} \cap V_\alpha]$. Hence, $\mathcal{P}'_{<\delta} \cap M[p, G_M]$ is just the same as $(\mathcal{P}'(\alpha))_{<\delta} \cap M[p, G_M]$. But this is last forcing is equivalent to $(\mathcal{P}'(\alpha))_{<\delta}$. So we are done.

$\square$

Lemma 2.8 *Let $V_\alpha \prec V_\theta$, α be a Mahlo and $\delta < \alpha$ be a regular. Then $\mathcal{P}' = \mathcal{P}'(\alpha)_{\geq\delta} * (Q \times (\mathcal{P}'(\alpha))_{<\delta})$.*

Proof. By Lemma 2.6, $\mathcal{P}'_{\geq\delta} \gtrdot \mathcal{P}'(\alpha)_{\geq\delta}$. So let $\mathcal{P}'_{\geq\delta} = \mathcal{P}'(\alpha)_{\geq\delta} * Q$, for some Q. Now, $\mathcal{P}'(\alpha) = \mathcal{P}'(\alpha)_{\geq\delta} * \mathcal{P}'(\alpha)_{<\delta}$. By Lemma 2.7 we have $\mathcal{P}' = \mathcal{P}'_{\geq\delta} * (\mathcal{P}'(\alpha))_{<\delta}$. Hence

$$\mathcal{P}' = \mathcal{P}'(\alpha)_{\geq\delta} * Q * (\mathcal{P}'(\alpha))_{<\delta}.$$

But Q does not add new bounded subsets to α. So this can be written as follows:

$$\mathcal{P}' = \mathcal{P}'(\alpha)_{\geq\delta} * (Q \times (\mathcal{P}'(\alpha))_{<\delta}).$$

$\square$

Recall that $\mathcal{P}'_{\geq\alpha} * (\mathcal{P}'_{<\alpha})_{\geq\beta}$ is β-strategically closed, $\mathcal{P}'(\alpha)_{<\beta}$ satisfies β^+-c.c. and is actually isomorphic to a forcing of cardinality β^+, by **??**.

Lemma 2.9 *Let $\alpha \in S$, $\delta < \theta$, $(S \cap \delta)\backslash\alpha + 1 \neq \emptyset$ and*

$$\begin{tikzcd} & M \supseteq V_{j(\alpha)} = V_\theta \\ V \arrow[ur, "j"] \arrow[dr, "i"'] & \\ & N \arrow[u, "k"'] \end{tikzcd}$$

be a commutative diagram with N being the ultrapower by an (α, δ)-extender. Then i extends to

$$\hat{i} : V^{\mathcal{P}'} \longrightarrow N^{i(\mathcal{P}')}$$

Alternatively, using only strongs we can show that the following analog of this lemma holds:

Lemma 2.10 *Suppose that*

1. $\rho < \theta$ *is a Mahlo cardinal*
2. $V_\rho \prec_{\Sigma_1} V_\theta$
3. α *is ρ-strong, as witnessed by* $j : V \to M \supset V_\rho$
4. $\delta, \alpha < \delta < \rho$ *is a regular cardinal*
5. *there is* $\mu, \alpha < \mu < \delta$ *such that* $V_\mu \prec V_\rho$.

Let

$$\begin{tikzcd} & M \supseteq V_\rho \\ V \arrow[ur, "j"] \arrow[dr, "i"'] & \\ & N \arrow[uu, "k"'] \end{tikzcd}$$

be a commutative diagram with N being the ultrapower by an (α, δ)-extender derived from j, such that $\rho = k(\xi)$, for some ξ. Then i extends to

$$\hat{i} : V^{\mathcal{P}'} \to N^{i(\mathcal{P}')}.$$

The proofs of both lemmas are very similar. We concentrate on the proof of 2.9 and state the minor changes needed for those of 2.10

Proof. Note that by the definition of forcings $\mathcal{P}'(\xi)$ we have $\mathcal{P}' = \mathcal{P}'(\theta)$. Also, $i(\theta) = \theta$, since θ is an inaccessible. In N, hence $i(\mathcal{P}') = (\mathcal{P}'(i(\theta)))^N = (\mathcal{P}'(\theta))^N$. We split first $(\mathcal{P}'(\theta))^N$ into $(\mathcal{P}'(i(\alpha)) \times ((\mathcal{P}'(\theta)_{\geq i(\alpha)}) * (\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq \alpha}))^N$.

Let us deal first with $(\mathcal{P}'(i(\alpha)))^N$. Note that $V_\delta \subseteq N$. We split in N the forcing $\mathcal{P}'(i(\alpha))$ into $\mathcal{P}'(i(\alpha))_{\geq \delta} * \mathcal{P}'(i(\alpha))_{<\delta}$. The part $\mathcal{P}'(i(\alpha))_{\geq \delta}$ is δ^+-strategically closed. The extender used to form N has no generators above δ, so standard methods apply. Thus, we can find an N^*-generic set for $(\mathcal{P}'(i_{N^*}(\alpha))_{\geq \delta})^{N^*}$ move it then to N and in this way obtain an N-generic set for $(\mathcal{P}'(i(\alpha))_{\geq \delta})^N$, where N^* is the ultrapower by the measure $U = \{X \subseteq \alpha^2 \mid (\alpha, \delta) \in i(X)\}$. For 2.10, we include also ξ, i.e. $U = \{X \subseteq \alpha^3 \mid (\alpha, \delta, \xi) \in i(X)\}$.
Denote the corresponding embedding by i^* and those of N^* into N by k^*. Then we obtain the following commutative diagram:

$$
\begin{tikzcd}
 & M & \supseteq V_{j(\alpha)} = V_\theta \quad \text{(or just } M \supseteq V_\rho \text{ in 2.10)} \\
V \arrow[ur, "j"] \arrow[r, "i"] \arrow[dr, "i^*"'] & N \arrow[u, "k"'] & \\
 & N^* \arrow[u, "k^*"'] & \simeq V^\alpha/U
\end{tikzcd}
$$

Let δ^* be the preimage of δ under k^* (and ξ^* the preimage of ξ). Use α^+-strategic closure of $\mathcal{P}'(i^*(\alpha))_{\geq(\delta^*)^+}$ to build an N^*-generic subset of $(\mathcal{P}'(i^*(\alpha))_{\geq\delta^*})^{N^*}$. Then move it by k^* to obtain an N-generic subset of $(\mathcal{P}'(i^*(\alpha))_{\geq\delta})^N$.

We deal now with $(\mathcal{P}'(i(\alpha))_{<\delta})^N$. Let $A^* \in N^*$ be an elementary submodel of $(V_{i^*(\alpha)})^{N^*}$ (or of $(V_{\xi^*})^{N^*}$ in 2.10) of cardinality $((\delta^*)^+)^{N^*}$ closed under δ^*-sequences. Let $A \in N$ be $k^*(A^*)$. Then it is an elementary submodel of $(V_{i(\alpha)})^N$ of cardinality $(\delta^+)^N$ closed under δ-sequences. Let $k(A) = B$. Then, B will be an elementary submodel of $(V_{j(\alpha)})^M = V_{j(\alpha)}$ (or of $(V_\rho)^M = V_\rho$ correspondently) of cardinality δ^+. Recall that $k \restriction (\delta^+)^N = id$, $|(\delta^+)^N| = \delta, cf((\delta^+)^N) = \alpha^+$ and $k((\delta^+)^N) = \delta^+$.

Pick in N^* a condition $r_1 \in \mathcal{P}'(i^*(\alpha))_{\geq(\delta^*)^+}$ which is A^*-generic. Let G^* be an N^*-generic subset of $(\mathcal{P}'(i^*(\alpha))_{\geq\delta^*})^{N^*}$ with $r_1 \in G^*$, built using the α^+ strategic closure of the forcing.

Moving to N we set $q_1 = k^*(r_1)$. Then $q_1 \in \mathcal{P}'(i(\alpha))_{\geq\delta^+}$ will be A-generic. Set $p_1 = k(q_1)$. Then, by elementarity, p_1 will be B-generic for the real $\mathcal{P}'(j(\alpha))_{\geq\delta^+}$.

Let r_2 be $G^* \cap A^*[r_1]$ and q_2 be generated by $k^{*''} r_2$. Then q_2 will be $(A, \mathcal{P}'(i(\alpha))_{\{\delta\}})$-generic set (remember that $\mathcal{P}'(i(\alpha))_{\{\delta\}}$ is δ^+-strategically closed).

Consider $k'' q_2$. It contains an increasing cofinal subset of size α^+ - the image of r_2 under $k \circ k^*$. Now, $k'' A \in B$, since ${}^\delta B \subseteq B$, by elementarity. Let $p_2 \in \mathcal{P}'(j(\alpha))_{\{\delta\}}$ be the union of conditions in $k'' q_2$. It exists, due to this cofinal subset of size α^+.

Chose a generic over M (or,the same V) with (p_1, p_2) inside. Let $\tilde{p}_2$ be a $(B[p_1], \mathcal{P}'(j(\alpha))_{\{\delta\}})$-generic over M with $p_2 \in \tilde{p}_2$. Then $k \restriction A$ extends to an elementary embedding

$$\tilde{k} : A[q_1, q_2] \to B[p_1, \tilde{p}_2] \ .$$

By 1.9, $\mathcal{P}'(j(\alpha))_{<\delta}$ is equivalent to $\mathcal{P}'(j(\alpha))_{<\delta} \cap B[p_1, \tilde{p}_2]$ and the same is true in N replacing $B[p_1, \tilde{p}_2]$ by $A[q_1, q_2]$. Also, by 1.?, $\mathcal{P}'(j(\alpha))_{<\delta}$ satisfies δ^+-c.c. Hence $\tilde{k}$ will move maximal antichains to maximal antichains. This allows us to obtain $(\mathcal{P}'(i(\alpha)))^N_{<\delta}$ -generic set from $\mathcal{P}'(j(\alpha))_{<\delta}$ -generic one, just intersect the last one with $\tilde{k}'' A[q_1, \tilde{q}_2]$ and pull back the result to N using $\tilde{k}^{-1}$.

Putting together now the parts above and below δ we will obtain an N generic subset $G_{i(\alpha)}$ of $(\mathcal{P}'(i(\alpha)))^N$.

Let us turn now to the forcing $(\mathcal{P}'(\theta))^N$ and also deal with the master condition part.

Let $\mu \in (S \cap \delta)\backslash(\alpha+1)$ (or in 2.10, let μ we as in (5), i.e. $V_\mu \prec V_\rho$). We pick in V an elementary submodel $A \prec V_\mu \prec V_\theta$ (or V_ρ) of cardinality α^+ and closed under α-sequences of its elements. Let p be $\mathcal{P}'_{\geq\alpha^+}$-generic over A. It exists since $\mathcal{P}'_{\geq\alpha^+}$ is α^+-strategically closed. Fix an increasing continuous sequence $\langle A_\nu \mid \nu < \alpha^+\rangle$ of elementary submodels of A each of cardinality $\alpha, \langle A_\xi \mid \xi \leq \nu\rangle \in A_{\nu+1}$ and $V_\alpha \in A_0$. Without loss of generality for each $\nu < \alpha^+$ we may assume that $A_\nu[p \cap A_\nu] \prec A[p]$. Consider now the forcing $\mathcal{P}'_{=\alpha}$. It satisfies α^{++} -c.c. Hence each antichain in $\mathcal{P}'_{=\alpha}$ that belongs to $A[p]$ is contained in $A[p]$. Now working inside A it is easy to see for each $\xi < \alpha^+$ the set of conditions q in $\mathcal{P}'_{=\alpha}$ having A_ν for some ν, $\xi < \nu < \alpha^+$, as the maximal model, i.e. $A^{0\alpha}(q) = A_\nu$ is dense. Let us use $G_{i(\alpha)} \cap \mathcal{P}'_{=\alpha}(\delta^*)$ to produce $\mathcal{P}'_{=\alpha}$ -generic over A. Note that the set

$$T = \{\nu < \alpha^+ \mid A_\nu \text{ is the maximal model of a condition in this generic set}\}$$

is unbounded. Actually, using α^+ -strategic closure of $\mathcal{P}'_{=\alpha}$ it is not hard to see that T is stationry and fat.

Consider in N models

$$B = i(A), B_{i(\nu)} = i(A_\nu),\ B[i(p)], B_{i(\nu)}[i(p) \cap B_{i(\nu)}]\ .$$

We have $\cup(i''\alpha^+) = i(\alpha^+)$, hence

$$B = \bigcup_{\nu<\alpha^+} B_{i(\nu)} \text{ and } B[i(p)] = \bigcup_{\nu<\alpha^+} B_{i(\nu)}[i(p) \cap B_{i(\nu)}]\ .$$

Now we fix a list $\langle E_\nu \mid \nu < \alpha^+\rangle$ of dense open subsets of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\delta})^N$ in $B[i(p)]$ which are the images of all dense open subsets coming from the ultrapower by the normal measure of the extender i. Note that the forcing under the consideration is δ^+-strategically closed (in N) and the generators of i are below δ, so this can be done.

For each $\nu < \alpha^+$ let E'_ν be the dense open subset of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\alpha})^N$ obtained from E_ν by adding to each $q \in E_\nu$ models of cardinalities in the interval $[\alpha, \delta]$, i.e. $q^\frown r \in E'_\nu$ iff $q \in E_\nu$, $q^\frown r \in ((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\alpha})^N$ and r consists of models of cardinalities in the interval $[\alpha, \delta]$. We may assume that E_ν (and hence also E'_ν) is in $B_{i(\nu)}[i(p) \cap B_{i(\nu)}]$, just removing some of B_ν's if necessary.

Recall that $G_{i(\alpha)}$ is an N -generic subset of $(\mathcal{P}'(i(\alpha)))^N$ constructed above. Our next tusk will be to consider the projection of $(\mathcal{P}'(\theta))^N_{\geq\alpha}$ over $G_{i(\alpha)}$ and to claim that certain elements are in $(\mathcal{P}'(\theta))^N_{\geq\alpha}/G_{i(\alpha)}$.

Claim 2.9.1 *For each $\nu \in T$ of cofinality α we have $i'' A_\nu \in (\mathcal{P}'(\theta))^N_{\geq\alpha}/G_{i(\alpha)}$.*

Remark Note that $(G_{i(\alpha)})_{\geq\alpha} \cap A_\nu$ is a condition in $\mathcal{P}'$ (or just in $(\mathcal{P}'(i(\alpha)))^N$), due to 1.1(28?). Our interest is in $((G_{i(\alpha)})_{\geq\alpha} \cap A_\nu)^\frown A_\nu$. By putting in $i'' A_\nu$ we actually add all of $i''(((G_{i(\alpha)})_{\geq\alpha} \cap A_\nu)^\frown A_\nu)$. The claim basically deals with it rather then only with $i'' A_\nu$.

Proof. Consider $C^\alpha(A_\nu) \restriction A_\nu$. It is a closed unbounded sequence in A_ν and since $cof(\nu) = \alpha$, it has a cofinal subsequence $\langle A_{\nu,\beta} | \beta < \alpha \rangle$. Apply i. Then $i(\langle A_{\nu,\beta} | \beta < \alpha \rangle)$ will be a cofinal subsequence of $C^{\imath(\alpha)}(B_\nu) = i(C^\alpha(A_\nu))$. Denote $i(\langle A_{\nu,\beta} | \beta < \alpha \rangle)$ by $\langle B_{\nu,beta} | \beta < \imath(\alpha) \rangle$. Clearly, $i'' A_\nu \subset B_{\nu,\alpha}$.

It is enough to show that $i'' A_\nu$ is compatible with every element of $G_{i(\alpha)}$. Note that models of cardinalities $\geq \alpha$ are mapped to generic set over N for $(\mathcal{P}'(\theta))_{\geq i(\alpha)}$, just this set is generated by such images. Hence there is no problems with the images (i.e. $i(X)$) of elements of $A_\nu \cap (G_{i(\alpha)})_{\geq\alpha}$. We need only to take care of $i'' X$ for $X \in (A_\nu \cap (G_{i(\alpha)})_{=\alpha}) \cup \{A_\nu\}$.

Pick any element q of $(\mathcal{P}'(i(\alpha))^N$ with A_ν inside. Assume also that A_ν is on the central line of q. Consider $i(q)$. It will consists of models of cardinalities below α and those of cardinalities at least $i(\alpha)$ (remember that each condition has Easton support). Also B_ν appears in $i(q)$ on the central line. We would like to find a common extension of q and $i(q)$ which includes $i'' A_\nu$. Proceed as follows. Pick first some $\beta^*, \alpha < \beta^* < i(\alpha)$, such that B_{β^*} is a unique immediate predecessor of B_{β^*+1} and there is no models of cardinalities above $i(\alpha)$ (and so, no models at all) in between. Using elementarity and density argument it is possible to find such β^*. Now inside B_{ν,β^*} we pick an increasing continuous sequence $\langle X_\tau | \tau \in s(q) \rangle$ of models (elementary or Σ_1-elementary in B_{ν,β^*}) such that $q, i'' A_\nu, i(q) \cap B_{\nu,\beta^*+1} \in X_{\kappa^+}$. Then $q^\frown i'' A_\nu{}^\frown \langle X_\tau | \tau \in s(q) \rangle^\frown i(q)$ will be as desired.
$\square$ of the claim.

Let ν_0 be the first element of T of cofinality α. Consider $A_{\nu_0}{}^\frown i'' A_{\nu_0}$. By Claim 2.9.1, $A_{\nu_0}{}^\frown i'' A_{\nu_0} \in (\mathcal{P}'(\theta))^N / G_{i(\alpha)}$. Now inside B_{ν_0} we extend $A_{\nu_0}{}^\frown i'' A_{\nu_0}$ to a condition q_0 in E'_0 with the projection to $(\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ inside $G_{i(\alpha)}$.

Claim 2.9.2 $q_0{}^\frown B_{\nu_0} \in (\mathcal{P}'(\theta))^N / G_{i(\alpha)}$.

Proof. Again we need to show that $q_0{}^\frown B_{\nu_0}$ is compatible with every element of $G_{i(\alpha)}$. Let $t \in G_{i(\alpha)}$ There is a common extension q of q_0 and t with projection in $G_{i(\alpha)}$, since $q_0 \in (\mathcal{P}'(\theta))^N / G_{i(\alpha)}$. By elementarity, we can find such q inside B_{ν_0}. Thus

$$(\mathcal{P}'(i(\alpha)))^N \subseteq (V_{i(\alpha)})^N \subseteq B_{\nu_0}$$

and, hence

$$B_{\nu_0}[G_{i(\alpha)}] \prec B[G_{i(\alpha)}] \prec (V_\theta[G_{i(\alpha)}])^N.$$

Also, $B_{\nu_0}[G_{i(\alpha)}] \cap (\mathcal{P}'(\theta))^N = B_{\nu_0} \cap (\mathcal{P}'(\theta))^N$.

Consider $q^\frown B_{\nu_0}$. It is almost a condition in $(\mathcal{P}'(\theta))^N$ only with maximal models missing for lot of cardinalities. Extend it to some $r \in (\mathcal{P}'(\theta))^N$ for which the projection to $(\mathcal{P}'(\theta))^N$ is defined. Then $r \geq q$ implies that the projection r' of r is above the one of q. But then $r' \geq t$ in $(\mathcal{P}'(i(\alpha)))^N$. This means in particular that $q_0{}^\frown B_{\nu_0}$ is compatible with t.
$\square$ of the claim.

We proceed similar at each successor stage. Thus, if for $\xi < \alpha^+$, q_ξ, B_{ν_ξ} are defined $q_\xi \subseteq B_{\nu_\xi}$ and $q_\xi{}^\frown B_{\nu_\xi} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$, then we pick $\nu_{\xi+1}$ to be the least element of T above ν_ξ such that $cof(\nu_{\xi+1}) = \alpha$ and $A_{\nu_\xi} \in C^\alpha(A_{\nu_{\xi+1}})$. As in Claim 2.9.1, we will have $q = A_{\nu_{\xi+1}}{}^\frown q_\xi{}^\frown B_{\nu_\xi} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$.
Now inside $B_{\nu_{\xi+1}}$ we extend q to a condition $q_{\xi+1}$ in $E'_{\xi+1}$ with the projection to $(\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ inside $G_{i(\alpha)}$. Then, as in Claim 2.9.2, we will have $q_{\xi+1}{}^\frown B_{\nu_{\xi+1}} \in (\mathcal{P}'(\theta))^N/G_{i(\alpha)}$.

Let us turn to limit stages of the construction. Assume that ξ is a limit ordinal. Let $\nu_\xi = \cup_{\tau<\xi}\nu_\tau, \nu_{\xi+1}$ be the first element of $T\backslash\nu_\xi + 1$ of cofinality α and $q'_\xi = \cup\{q_\tau | \tau < \xi\}$. This q'_ξ is just the formal union of all q_τ's constructed at the previous stages. We do not take unions of the maximal models of q_τ's etc. Let q''_ξ be obtained from q'_ξ by adding $i''A_{\nu_{\xi+1}}$ and, if A_{ν_ξ} is in a condition in $G_{i(\alpha)}$, then also $i''A_{\nu_\xi}$.

Claim 2.9 *q''_ξ projects to an element of $G_{i(\alpha)}$.*

Proof. Let us show that for each $t_1 \in G_{i(\alpha)}$ above the projection of q'_τ the following holds:

if $t \in (\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ and $t \geq t_1$, then there is $q \geq q''_\tau$ with the projection to $(\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ stronger than t.

Let $t_1 \leq t$ be as above. Then intial seqments of q''_ξ project below t. Just q'_ξ projects to a condition in $G_{i(\alpha)}$ below $t_1 \leq t$. Also, the addition of $i''A_{\nu_{\xi+1}}, i''A_{\nu_\xi}$ is above $i(\alpha)$. So we can find a common extension $r \in B_{i(\nu_{\xi+1})}$ of t and q''_ξ. Using the elementarity of $V^N_{i(\alpha)}$, find $r' \in (V_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha))_{\geq\alpha})^N$ realizing the same type as r over $r \cap V^N_{i(\alpha)}$. Finally, let q be obtained from $r \cup r'$ by adding the maximal models including those of both r, r' and this models via $C^\rho(q)$'s to those of r'. Then the projection of q to $(\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ is $r' \geq t$ and we are done.
$\square$ of the claim.

Now we extend q''_ξ to $q_\xi \in E_\xi$ in $B_{i(\nu_{\xi+1})}$ with the projection to $(\mathcal{P}'(i(\alpha)))^N_{\geq\alpha}$ inside $G_{i(\alpha)}$.

This completes the construction.

Consider finally the resulting sequence $\langle q_\nu \mid \nu < \alpha^+\rangle$. Let $\langle q^*_\nu \mid \nu < \alpha^+\rangle$ be the sequence obtained from it by removing from each q_ν models of cardinalities below δ^+. Then, $q^*_\nu \in E_\nu$ for every $\nu < \alpha^+$. Hence $\langle q^*_\nu \mid \nu < \alpha^+\rangle$ generates a $B[i(p)]$-generic subset of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\delta^+})^N$. By the construction, the projections of q^*_ν's to $((\mathcal{P}'(i(\alpha)))_{\geq\delta^+})^N$ are in $G_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha)_{\geq\delta^+})^N$. The same is true (again by the construction) for q_ν's, i.e. projections to $((\mathcal{P}'(i(\alpha)))_{\geq\alpha})^N$ are in $G_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha)))_{\geq\alpha})^N$. Then q_ν's will be in $B[i(p)]$-generic subset of $((\mathcal{P}'(\theta)_{<i(\alpha)})_{\geq\alpha})^N$ generated by $G_{i(\alpha)} \cap (\mathcal{P}'(i(\alpha))_{\geq\alpha})^N$ and $\langle q^*_\nu \mid \nu < \alpha^+\rangle$. Moreover, models $i''(A_\nu)$ appear in q_ν's. Each $r \in \mathcal{P}'_{<\alpha}$ which is inside some A_ν will be moved by i to $i(r) \in (\mathcal{P}'(\theta)_{<\alpha})^N$ inside $i''A_\nu$. But $i''A_\nu$ is a model inside a condition in generic set, so $i(r)$ is such as well. Hence images of elements from $G_{i(\alpha)} \cap \mathcal{P}'_{<\alpha}$ are in the constructed this way N-generic subset of $(\mathcal{P}'(\theta)_{<\alpha})^N$. So we are done.
$\square$ of the lemma.

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