On Q-points and GCH

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Abstract

We construct models with $2^\kappa > \kappa^+$ and ultrafilters U over κ such that $U \supseteq Cub_\kappa$ and $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U$.

1 Introduction

Let κ be a measurable cardinal and $2^\kappa > \kappa^+$. Then, by a classical result of D. Scott, for unboundedly many $\nu < \kappa$, $2^\nu > \nu^+$. Moreover, if U is a normal ultrafilter over κ, then

$$\{\nu < \kappa \mid 2^\nu > \nu^+\} \in U.$$

But what will happend if we drop the normality assumption? It is easy to construct a model in which $2^\kappa > \kappa^+$ and for some $\kappa-$complete ultrafilter U over κ,

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U.$$

Just start with a sufficiently large cardinal κ, say supercompact or 2-strong. Assume GCH. Force with an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle,$$

where $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal. For an inaccessible $\beta \leq \kappa$, let Q_β be the Cohen forcing $Cohen(\beta, \beta^{++})$ which adds $\beta^{++}-$many subsets to β. Let $G \subseteq P_{\kappa+1}$ be a generic. κ will remain a measurable in $V[G]$ and $2^\kappa = \kappa^{++}$. Let W be a normal ultrafilter over κ. Consider its ultrapower embedding $j_W : V[G] \to M_W$. Note that in $V[G]$ for every

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successor cardinal $\nu < \kappa$, $2^\nu = \nu^+$. Set $\delta = (\kappa^+)^{M_W}$. Then, by elementarity, $M_W \models 2^\delta = \delta^+$. Define

$$U = \{X \subseteq \kappa \mid \delta \in j_W(X)\}.$$

Then

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U.$$

In this example U is Rudin-Keisler equivalent to a normal ultrafilter and concentrate on a non-stationary set.
But what will happen if we require that U contains all closed unbounded subsets of κ?

The answer is that it is still possible.

We will present several constructions. The first one will an easy one from a supercompact. The second will be from a strong and with some additional properties. However, the gap between κ and 2^κ which we can reach is only 2.
It turns out that the situation here is surprisingly similar to those of the previous sections. The third construction will again use a supercompactness.

2 The first construction

Assume GCH and let κ be a $\kappa^{++}-$supercompact cardinal.
Fix a normal ultrafilter W over $\mathcal{P}_\kappa(\kappa^{++})$. Let $j_W : V \to M_W$ be the corresponding elementary embedding. Then ${}^{\kappa^{++}}M_W \subseteq M_W$.

As above, we force with an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa\rangle,$$

where $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal. For an inaccessible $\beta \leq \kappa$, let Q_β be the Cohen forcing $Cohen(\beta, \beta^{++})$ which adds $\beta^{++}-$many subsets to β. Let $G \subseteq P_{\kappa+1}$ be a generic. κ will remain a $\kappa^{++}-$supercompact cardinal in $V[G]$ and $2^\kappa = \kappa^{++}$. Moreover W extends to a normal ultrafilter W^* and j_W extends to $j_{W^*} : V[G] \to M_{W^*} = M_W[G^*]$. Still we have ${}^{\kappa^{++}}M_{W^*} \subseteq M_{W^*}$.
Let $\langle C_\alpha \mid \alpha < \kappa^{++}\rangle$ be an enumeration of all clubs of κ in $V[G]$. Then, the set $Y = \{j_{W^*}(C_\alpha) \mid \alpha < \kappa^{++}\}$ is in M_{W^*} and consists of less than $j_W(\kappa)-$many clubs their. Hence, $C = \bigcup Y$ is a club of $j_W(\kappa)$ in M_{W^*}.

Note that in $V[G]$ for every singular cardinal $\nu < \kappa$, $2^\nu = \nu^+$.
Working in M_{W^*}, pick a singular cardinal $\delta \in C$ of cofinality ω. Then, by elementarity,

$M_{W^*} \models 2^\delta = \delta^+$. Define

$$U = \{X \subseteq \kappa \mid \delta \in j_{W^*}(X)\}.$$

Then

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in U.$$

Moreover, by the choice of δ, we will have that $U \supseteq Cub_\kappa$.

Let

$$U_\kappa = \{X \subseteq \kappa \mid \kappa \in j_{W^*}(X)\}.$$

Then, using commutativity of embeddings,

$$U_\kappa = \{X \subseteq \kappa \mid \kappa \in j_U(X)\},$$

and it is the projection of U to the least normal below it in the Rudin-Keisler ordering. Let $k : M_{U_\kappa} \to M_U$ be the corresponding embedding. Then $\mathrm{crit}(k) < [id]_U$, since $M_U \models \mathrm{cof}([id]_U) = \omega$.

3 Construction from the optimal assumptions

Assume GCH and suppose that a cardinal κ is $2-$strong. Let E be a witnessing extender. Instead, we can start from $o(\kappa) = \kappa^{++}$ which is optimal and, using [7], arrange the same type of a situation.

Define an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa\rangle.$$

Let $\underset{\sim}{Q}_\beta$ be trivial unless β is an inaccessible cardinal.

If $\beta < \kappa$ is an inaccessible cardinal then set $Q_\beta = Q^0_\beta * Q^1_\beta$, where Q^0_β is an atomic forcing which consists of two incompatible elements 0 and 1.

If a generic for Q^0_β is 0, then Q^1_β is the Cohen forcing for adding β^{++} subsets to β, $Cohen(\beta, \beta^{++})$.

Otherwise, i.e. if a generic for Q^0_β is 1, then $Q^1_\beta = Cohen(\beta, \beta^{++}) * Col(\beta, \beta^+)$.

Set $Q_\kappa = Cohen(\kappa, (\kappa^{++})^{M_{E(\kappa)}}) * (Cohen(\kappa^+, \kappa^{++}) \times Cohen(\kappa, [(\kappa^{++})^{M_{E(\kappa)}}, \kappa^{++}))$.

Let G be generic subset of $P_{\kappa+1}$. Then, in $V[G]$, $2^\kappa = \kappa^{++}$.

By Woodin's arguments, see [5], κ will remain a measurable in $V[G]$ and the embedding $j_E : V \to M_E$ extends to $j^* : V[G] \to M_E[G^*]$.

Set

$$U^* = \{X \subseteq \kappa \mid \kappa \in j^*(X)\}.$$

Then

1. $U^* \supseteq U$,

2. $j_{U^*} = j^*$

3. $M_{U^*} = M_E[G^*]$.

We have

$$j_E(P) = P_\kappa * Cohen(\kappa, \kappa^{++}) * P_{(\kappa, j_E(\kappa))} * Cohen(j_E(\kappa), (j_E(\kappa)^{++})^{M_{j_E(E(\kappa))}})$$
$$*(Cohen(j_E(\kappa^+), j_E(\kappa^{++})) \times Cohen(j_E(\kappa), [(j_E(\kappa)^{++})^{M_{j_E(E(\kappa))}}, j_E(\kappa^{++}))).$$

Consider now $U^* \times U^*$. We have that $j_{U^* \times U^*}$ extends $j_{E \times E} = j_{j_E(E)} \circ j_E$. Denote $j_E(\kappa)$ by κ_1 and $j_{E\times E}(\kappa) = j_{j_E(E)}(\kappa_1)$ by κ_2. Then $j_{j_E(E)} : M_E \to M_{E\times E}$ and κ_1 is its critical point.

Apply $j_{E\times E}$ to $P_{\kappa+1}$. In $M_{E\times E}$,

$$P_{\kappa_2+1} = P_\kappa * \underset{\sim}{Q}_\kappa * P_{(\kappa,\kappa_1)} * \underset{\sim}{Q}_{\kappa_1} * P_{(\kappa_1,\kappa_2)} * \underset{\sim}{Q}_{\kappa_2}.$$

We have, in $M_{E\times E}[G^* \cap P_{\kappa_1}]$, $Q_{\kappa_1} = Q^0_{\kappa_1} * Q^1_{\kappa_1}$. Let us take a generic for $Q^0_{\kappa_1}$ to be 1. Then $Q^1_{\kappa_1}$ will be $Cohen(\kappa_1, \kappa_1^{++}) * Col(\kappa_1, \kappa_1^+)$. Still $j_{j_E(E)}$ extends. Let $k : M_E[G^*] \to M_{E\times E}[G^{**}]$ be such extension. We will have also $j^{**} : V[G] \to M_{E\times E}[G^{**}]$ such that $j^{**} = k \circ j^*$.

Define

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j^{**}(X)\}.$$

Then W will be a $\kappa-$complete ultrafilter over κ which extends Cub_κ. Since for every club $C \subseteq \kappa$, $j^*(C)$ is a club in κ_1 in $M_E[G^*]$, and so, $\kappa_1 \in k(j^*(C)) = j^{**}(C)$.

Now, in $M_{E\times E}[G^{**}]$, $2^{\kappa_1} = \kappa_1^+$, since $Col(\kappa_1, \kappa_1^+)$ was applied to restore GCH over κ_1. Hence,

$$\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W,$$

and we are done.

In addition, $U^* \leq_{R-K} W$, and if $k : M_{U^*} \to M_W$ is a corresponding embedding, then $\text{crit}(k) = \kappa_1$.

Remark 3.1 *Note that as in the Woodin construction, the above gives a gap two and not more. We will see the reasons for such phenomenon in the next section.*

4 On a strength

Suppose that $W \supseteq Cub_\kappa$, $2^\kappa = \kappa^{++}$ and $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$.

Assume that there is no inner model with a strong cardinal. Let $\mathcal{K}$ be the core model. By Mitchell[16] or by Schindler [17], $j_W \upharpoonright \mathcal{K}$ is an iterated ultrapower of $\mathcal{K}$.

An ordinal $\alpha < j_W(\kappa)$ is called *a generator* of j_W, if there is no $n < \omega$ and $f : [\kappa]^n \to \kappa$ such that for some $\alpha_1 < ... < \alpha_n < \alpha$, $j_W(f)(\alpha_1, ..., \alpha_n) = \alpha$.

Let us call an ordinal $\alpha < j_W(\kappa)$ *a principal generator* of j_W, if there is no $n < \omega$ and $f : [\kappa]^n \to \kappa$ such that for some $\alpha_1 < ... < \alpha_n < \alpha$, $j_W(f)(\alpha_1, ..., \alpha_n) \geq \alpha$.

Then, κ and $[id]_W$ are principle generators, but probably there are more.

Denote by P the set of all generators of j_W.

Consider

$$\{j_W(f)(\alpha_1, ..., \alpha_n) \mid n < \omega, \alpha_1 < ... < \alpha_n, \alpha_1, ..., \alpha_n \in P, f : [\kappa]^n \to V\}.$$

Let M be the transitive collapse of it and let $j : V \to M$ be the corresponding elementary embedding.

Define $k : M \to M_W$ by setting $k(j(f)(\alpha_1, ..., \alpha_n)) = j_W(f)(\alpha_1, ..., \alpha_n)$.

Then $j_W = k \circ j$ and $[id]_W = j(\kappa)$ will be the critical point of k.

Note that in the construction of the previous section, we have $P = \{\kappa, [id]_W\}$ and M is the ultrapower by the normal ultrafilter generated by κ.

Denote $[id]_W$ by κ_1 and $j_W(\kappa)$ by κ_2.

We have then that $M \models 2^{\kappa_1} = \kappa_1^{++}$, by the elementarity of j, and $M_W \models 2^{\kappa_1} = \kappa_1^+$, since $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$.

Lemma 4.1 *$(\mathcal{P}(\kappa_1))^M \subseteq (\mathcal{P}(\kappa_1))^{M_W}$, and so, $(\kappa_1^+)^M \leq (\kappa_1^+)^{M_W}$.*

Proof. Let $A \subseteq \kappa_1, A \in M$. Then $k(A) \in M_W$. in We have $\kappa_1 = \text{crit}(k)$, so $k(A) \cap \kappa_1 = A$. Hence, $A \in M_W$.

□

Assume now that** $(\kappa_1)^M$ **is not collapsed in** M_W**.

Let $\vec{A} = \langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ be an enumeration of $\mathcal{P}(\kappa)$. Consider the images of $\vec{A}$ under j and j_W. Let $\vec{A}^1 = j(\vec{A}) = \langle A^1_\beta \mid \beta < (\kappa_1^{++})^M\rangle$ and $\vec{A}^2 = j_W(\vec{A}) = \langle A^2_\gamma \mid \gamma < (\kappa_2^{++})^{M_W}\rangle$.

Lemma 4.2 *If $\vec{A}^1 \in M_W$, then $((\kappa_1)^{++})^M$ is be collapsed to $(\kappa_1^+)^{M_W}$ in M_W.*

Proof. This follows from $M_W \models 2^{\kappa_1} = \kappa_1^+$.

□

Lemma 4.3 $((\kappa_1)^{++})^M$ *cannot be collapsed to* $(\kappa_1^+)^{M_W}$.

Proof. By the anti-large cardinal assumption made, we have

$$\mathrm{cof}(((\kappa_1)^+)^{M_U}) = \kappa^+ \text{ and } \mathrm{cof}(((\kappa_1)^{++})^{M_U}) = \kappa^{++},$$

since the pointwise image of κ^+, or of κ^{++}, is unbounded in $((\kappa_1)^+)^M$, $((\kappa_1)^{++})^M$ respectively.

If the cofinality of $((\kappa_1)^{++})^M$ is changed to something below, in M_W, then $((\kappa_1)^{++})^M$ will be measurable in $\mathcal{K}^M$ and $M \models 2^{\kappa_1} = \kappa_1^{++}$, so, by [12], there must be an extender in $\mathcal{K}^M$ of a measurable length. Such possibility is ruled out as well by the anti-large cardinal assumption made.

□

Remark 4.4 Similar, if $2^\kappa > \kappa^{++}$, then there will be no room for such type of collapses.

Let us analyze the situation in which no such collapses occur.
We are unable to exclude such possibility, unless some additional assumptions are made.

Assume that

(א) $\vec{A}^1 \in M_W$,

or

(ב) $(\mathcal{P}(\kappa_1))^{M_W} \subseteq (\mathcal{P}(\kappa_1))^M$,

or

(ג) $\neg\diamondsuit^-_{par}(W)$, see [3],

or

(ד) W has the Galvin property.

If (א) holds, then we can use 4.2 and 4.3.

Suppose that (ב) holds.

Let $\langle B_\xi \mid \xi < \kappa_1^+\rangle$ be an enumeration of subsets of κ_1 in M_W. Remember that $j''\kappa^+$ is unbounded in κ_1^+ of M_W or the same of M. So, there will be $S \subseteq \kappa^{++}, |S| = \kappa^{++}$ and $\eta < \kappa_1^+$ such that every $A^1_\alpha, \alpha \in j''S$ appears below η in the enumeration $\langle B_\xi \mid \xi < \kappa_1^+\rangle$ of subsets of κ_1 in M_W. Fix, in M_W, a function $e_\eta : \kappa_1 \leftrightarrow \eta_1$. Then there will be $\mu < \kappa_1$ and $S' \subseteq S$, of cardinality κ^{++}, if $\mathrm{cof}(\kappa_1) = \kappa^+$ (which usually the case), or of κ^+, otherwise such that $e_\eta''\mu$ includes the indexes of all $A^1_\alpha, \alpha \in j''S'$.

Now, using (ב), it is not hard to find $X \in M, |X| < \kappa_1$ such that $A^1_{j(\beta)} \in X$, for every $\beta \in S'$. Then, there is $Y \in M, |Y| < \kappa_1$ such that $Y \supseteq j''S'$, which is impossible.

Note that the family $\{B_\xi \mid \xi \in e_\eta''\mu\}$ will be a cover in M_W of $\{A^1_\alpha \mid \alpha \in j''S'\}$. This

witnesses $\diamondsuit^-_{par}(W)$, which is impossible by ($\beth$).
By [3], the Galvin property implies $\neg\diamondsuit^-_{par}(W)$. So we are done.

5 Construction - gaps above 2

We will use supercompactness in order to get gaps bigger than 2 and $k \restriction \kappa_1 = id$.

Let us deal with a gap 3. The general case uses the same idea.

Assume GCH and let κ be a κ^+-supercompact cardinal. Fix a normal ultrafilter U over $\mathcal{P}_\kappa(\kappa^{+3})$ witnessing this. Let $j_U : V \to M_U \simeq V^{\mathcal{P}_\kappa(\kappa^{+3})}/U$ be the correspondent elementary embedding. Denote j_U by j and M_U by M. Let $\kappa_1 = j(\kappa)$.

GCH and the closure of M under $\kappa^{+3}-$sequences of its elements imply the following:

1. for every $k \leq 4, (\kappa^{+k})^M = \kappa^{+k}$,
2. $|j(\kappa)| = \kappa^{+4}$,
3. $\mathrm{cof}(j(\kappa)) = \kappa^{+4}$,
4. $j(\kappa^{+4}) = \bigcup j''\kappa^{+4}$,
5. $j(\kappa^{+5}) = \kappa^{+5}$,
6. for every $1 \leq k \leq 4, \mathrm{cof}(j(\kappa^{+k})) = \kappa^{+4}$.

Define an Easton support iteration

$$\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle,$$

as in the gap 2, only replace $Cohen(\beta, \beta^{++})$ by $Cohen(\beta, \beta^{+3})$
and $Col(\beta, \beta^+)$ by $Col(\beta^+, \beta^{+3})$.
Let G be generic subset of $P_{\kappa+1}$. Then, in $V[G]$, $2^\kappa = \kappa^{+3}$. Using the closure of M under $\kappa^{+3}-$sequences of its elements we construct $G^* \subseteq P_{j(\kappa)+1}$ which $M-$generic. Then j extends to $j^* : V[G] \to M[G^*]$ and U to U^*.
Now we proceed as in the gap 2 construction and use $U \times U$. Finally, over κ_1 in $M_{U\times U}[G^*]$, $Col(\kappa_1^+, \kappa_1^{+3})$ is used. Note that we have a generic for it in $V[G]$, by items (1)-(6).
The rest is as in the gap 2 construction.

6 Construction gaps above 2 from optimal assumptions

We would like here to show that the limitations of Section 4. Ideas of [4] which alow to create kind of many Cohen functions from a few will be used.

The following is a typical theorem under this lines:

Theorem 6.1 *Assume GCH. Let κ be a 3-strong cardinal. Then in a generic extension which satisfies $2^\kappa = \kappa^{+3}$ there is a $\kappa-$complete ultrafilter W over κ which includes Cub_κ such that $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$.*

Proof. Let E be a $(\kappa, \kappa^{+3})-$extender. Let $j_1 = j_E : V \to M_1 = M_2$. Consider also the second ultrapower, i.e. $k = j_{j(E)} : M_1 \to M_2$. Also, let $j_2 = j_{E\times E} : V \to M_2$ be the ultrapower by $E \times E$. Denote $j_1(\kappa)$ by κ_1 and $j_2(\kappa)$ by κ_2.

Force with the Cohen forcing $Cohen(\nu, \nu^{+3})$, for every $\nu \leq \kappa$. Let G be a generic.
Then in $V[G]$, let j_1 extends to an embedding $j_1^* : V[G] \to M_1[G^*]$.
Set $U^* = \{X \subseteq \kappa \mid \kappa \in j_1^*(X)\}$. Let $j_2^* : V[G] \to M_2[G^{**}]$ be the extension of j_2 and $k^* : M_1[G^*] \to M_2[G^{**}]$ of k which correspond to $U^* \times U^*$.

Now we would like to extend j_2 and k differently.
Let g_1 denotes $G^* \cap Cohen(\kappa_1, \kappa_1)$.
Use [4] and reorganize g_1 in order to generate $g^* \subseteq Cohen(\kappa_1, \kappa_1^{+3})^{M_1[G^*\restriction\kappa_1]}$ which is $M_1[G^* \restriction \kappa_1]-$generic. Then $\mathcal{P}(\kappa_1)^{M_1[G^*\restriction\kappa_1, g^*]} \subseteq \mathcal{P}(\kappa_1)^{M_1[G^*\restriction\kappa_1, g_1]}$.
Extend now k to some $k' : M_1[G^* \restriction \kappa_1, g^*] \to M_2[G^{**} \restriction \kappa_2, g^{**}]$.
Then,

$$\{A^2_\alpha \cap \kappa_1 \mid \alpha < (\kappa_2^{+3})^{M_2}\} \subseteq \{B_\beta \mid \beta < (\kappa_1^+)^{M_2}\},$$

where $\langle A^2_\alpha \mid \alpha < (\kappa_2^{+3})^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of subsets of κ_2 and $\langle B_\beta \mid \beta < (\kappa_1^+)^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of subsets of κ_1.
Now, since $\mathrm{cof}(\kappa_1) = \mathrm{cof}((\kappa_1^+)^{M_2} = \kappa^+$ and $\mathrm{cof}((\kappa_2^{++})^{M_2} = \mathrm{cof}((\kappa_1^{++})^{M_1} = \kappa^{++}$,
there will be $S \subseteq \kappa^{++}, |S| = \kappa^{++}$, $I \in M_2$ and $\eta < \kappa_1$, such that
$M_2 \models |I| = \eta$ and $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S\} \subseteq \{B_\beta \mid \beta \in I\}$.
We are exactly in a situation that was excluded in Section 4.
□

A similar construction, based on $(\kappa, \kappa^{++})-$extender, shows that the assumption about preservation of $(\kappa_1^+)^M$ made in Section 4 is possible to realize.

Theorem 6.2 *Assume GCH. Let κ be a 2-strong cardinal. Then in a generic extension which satisfies $2^\kappa = \kappa^{++}$ there is a $\kappa-$complete ultrafilter W over κ which includes Cub_κ such that $\{\nu < \kappa \mid 2^\nu = \nu^+\} \in W$ and $(j_U(\kappa)^+)^{M_U}$ is preserved,*
where as usual, $U = \{X \subseteq \kappa \mid \kappa \in j_W(X)\}$.

Let us deal now with the strength of the principle $\diamondsuit^-_{par}(W)$. It was introduced by T. Benhamou and G. Goldberg in [3]. The results above show that it consistency strength fall down to 2-strong, but let us push it further down to a measurable. The point is that there is no need here to blow up the power of κ.

Theorem 6.3 *Assume GCH. Let κ be a measurable cardinal. Then in a generic extension $\diamondsuit^-_{par}(W)$ holds for a $\kappa-$complete ultrafilter W over κ which includes Cub_κ.*

Proof. Fix a normal ultrafilter U over κ. Let $j_1 = j_U : V \to M_1 = M_2$. Consider also the second ultrapower, i.e. $k = j_{j(U)} : M_1 \to M_2$. Also, let $j_2 = j_{U\times U} : V \to M_2$ be the ultrapower by $U \times U$. Denote $j_1(\kappa)$ by κ_1 and $j_2(\kappa)$ by κ_2.

Force with the Cohen forcing $Cohen(\nu, \nu^+)$, for every $\nu \leq \kappa$. Let G be a generic.
Then in $V[G]$, let j_1 extends to an embedding $j_1^* : V[G] \to M_1[G^*]$.
Set $U^* = \{X \subseteq \kappa \mid \kappa \in j_1^*(X)\}$. Let $j_2^* : V[G] \to M_2[G^{**}]$ be the extension of j_2 and $k^* : M_1[G^*] \to M_2[G^{**}]$ of k which correspond to $U^* \times U^*$.

Now we would like to extend j_2 and k differently.
Let g_1 denotes $G^* \cap Cohen(\kappa_1, \kappa_1^+)$. Use [4] and reorganize $g_1 \restriction \kappa_1$(i.e. only first κ_1- many Cohen functions are used for this), in order to generate $g^* \subseteq Cohen(\kappa_1, \kappa_1^{++})^{M_1[G^* \restriction \kappa_1]}$ which is $M_1[G^* \restriction \kappa_1]-$generic. Then $\mathcal{P}(\kappa_1)^{M_1[G^*\restriction \kappa_1, g^*]} \subseteq \mathcal{P}(\kappa_1)^{M_1[G^* \restriction \kappa_1, g_1]}$.
Extend now k to some $k' : M_1[G^* \restriction \kappa_1, g^*] \to M_2[G^{**} \restriction \kappa_2, g^{**}]$.
Then,

$$\{A^2_\alpha \cap \kappa_1 \mid \alpha < (\kappa_2^{++})^{M_2}\} \subseteq \{B_\beta \mid \beta < (\kappa_1^+)^{M_2}\},$$

where $\langle A^2_\alpha \mid \alpha < (\kappa_2^{++})^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of subsets of κ_2 and $\langle B_\beta \mid \beta < (\kappa_1^+)^{M_2}\rangle$ is an enumeration in $M_2[G^{**} \restriction \kappa_2, g^{**}]$ of subsets of κ_1.
Now, since $\mathrm{cof}(\kappa_1) = \mathrm{cof}((\kappa_1^+)^{M_2} = \kappa^+$ and $\mathrm{cof}((\kappa_2^{++})^{M_2} = \mathrm{cof}((\kappa_1^{++})^{M_1} = \kappa^{++}$,
there will be $S \subseteq \kappa^{++}, |S| = \kappa^{++}$, $I \in M_2$ and $\eta < \kappa_1$, such that
$M_2 \models |I| = \eta$ and $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S\} \subseteq \{B_\beta \mid \beta \in I\}$.
Now pick some $S' \subseteq S, |S'| = \kappa^+$. Still, clearly, $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S'\} \subseteq \{B_\beta \mid \beta \in I\}$.
We have that $j_2''\kappa^{++}$ is unbounded in $(\kappa_2^{++})^{M_2}$, so there is some $\delta, \kappa^+ \leq \delta < \kappa^{++}$ such that $\sup(j_2{}''S') < j_2(\delta)$.

So, instead of adding $(\kappa_1^{++})^{M_1}-$ many Cohen functions over M_1 we can add and use only $j_1(\delta)-$many.
Clearly, the forcing $Cohen(\kappa,\kappa^+)$ is equivalent to $Cohen(\kappa,\delta)$. Hence, all the embeddings j_1, j_2, k extend now, but the property $\{A^2_\alpha \cap \kappa_1 \mid \alpha \in j_2{}''S'\} \subseteq \{B_\beta \mid \beta \in I\}$ remained valid.
Hence we will have $\diamondsuit^-_{par}(W)$ in $V[G]$.
□

Let us deal with two other similar principals which were introduced by T. Benhamou and G. Goldberg in [3].

Definition 6.4 (T. Benhamou and G. Goldberg) κ is called *non-Galvin cardinal* if there are elementary embeddings $j : V \to M, i : V \to N, k : N \to M$ such that

1. $k \circ i = j$,
2. $\mathrm{crit}(j) = \kappa, \mathrm{crit}(k) = i(\kappa)$,
3. ${}^\kappa N \subseteq N, {}^\kappa M \subseteq M$,
4. there is $A \in M$ such that $i''\kappa^+ \subseteq A$ and $M \models |A| < i(\kappa)$.

Proposition 6.5 *Suppose that there exists a non-Galvin cardinal. Then there exists an inner model with a strong cardinal (or an inner model with Woodin cardinal).*

Proof. Assume that the core model $\mathcal{K}$ exists. By Mitchell [16], Schindler [17] $i \restriction \mathcal{K}, j \restriction \mathcal{K}$ are iterated ultrapowers of $\mathcal{K}$.
For every club $C \subseteq \kappa$, we have $i(\kappa) \in j(C)$. Hence, the iteration $i \restriction \mathcal{K}$ continues by using an extender over $i(\kappa)$ in $\mathcal{K}^N$. Let us denote by $\sigma : \mathcal{K}^N \to \mathcal{K}_1$ this embedding. Then the iteration continues to the final $j \restriction \mathcal{K}$.
We have $i(\kappa^+) = (i(\kappa)^+)^{\mathcal{K}_1} = (i(\kappa)^+)^{\mathcal{K}_M}$.

By the definition of a non-Galvin cardinal, there is $A \subseteq i(\kappa^+), A \in M$ which covers $i''\kappa^+$ and $|A|^M < i(\kappa)$.
Apply the Covering Lemma to A over $\mathcal{K}^M$. It will imply existence of measurable cardinals in the interval $(i(\kappa), (i(\kappa)^+)^{\mathcal{K}_M}]$ which is impossible.
□

Another principle $\diamondsuit^-_{thin}(W)$, similar to $\diamondsuit^-_{par}(W)$, was defined in [3], Definition 3.4. Set $U = \{X \mid \kappa \in j_W(X)\}$. If λ of [3], Definition 3.4 is $j_U(\kappa)$, then κ is a non-Galvin cardinal. Hence, in the view of 6.5, this principle is rather strong as well.

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