Combining Short extenders forcings with Extender based Prikry.

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January 3, 2013

Let κ be $< \kappa^{+\omega}$ strong cardinal. For every $n < \omega$ fix an extender E_n which witness κ^{+n} strongness of κ. We would like to change the cofinality of κ to ω and to add $\kappa^{+\omega+2}$ many cofinal ω-sequences to κ.
Let us describe three ways of doing this.

1 Way 1.

Change first the cofinality of κ to ω by adding a Prikry sequence $\langle \kappa_n \mid n < \omega \rangle$ such that each κ_n is κ_n^{+n+2}–strong. Then use the short extenders forcing which adds $\kappa^{+\omega+2}$ many cofinal ω-sequences to κ (after the preparation or simultaneously with it). All cardinals will be preserved then.
If one does not care about falling of κ^{+n}, then preform Gap 2 short extenders forcing. As a result $\kappa^{+\omega+2}$ will turn into κ^{++} of the extension.

2 Way 2.

We combine the extender based Prikry forcing with Gap 2 short extenders forcing here.

2.1 Types.

Fix some $n < \omega$. Let $1 < k < \omega$. Let χ be a regular cardinal large enough. Consider a structure

$$\mathfrak{A}_k = \langle H(\chi^{+k}), \in, <, \langle E_m \mid m < \omega \rangle, \langle \chi^{+i} \mid i \leq k \rangle, \kappa, 0, 1, ..., \alpha, ... \mid \alpha < \kappa^{+k} \rangle$$

in an appropriate language which we denote $\mathfrak{L}_k$.

For an ordinal $\xi < \chi$. Denote by $tp_k(\xi)$ the $\mathfrak{L}_k$–type realized by ξ in $\mathfrak{A}_k$.

Let $\mathfrak{L}'_k$ be the language obtained from $\mathfrak{L}_k$ by adding a new constant c'. For $\delta < \chi$ let $\mathfrak{A}_{k\delta}$ be the $\mathfrak{L}'_k$–structure obtained from $\mathfrak{A}_k$ by interpreting c' as δ. The type $tp_k(\delta, \xi)$ is the $\mathfrak{L}'_k$–type realized by ξ in $\mathfrak{A}_{k\delta}$. Further we shall identify types with ordinals corresponding to them in some fixed in advance well ordering.

Note that the total number of types tp_k, with $1 \leq k < \omega$, is $2^{\kappa^{+k}}$, and under GCH it is κ^{+k+1}. We are going to use here models of sizes κ^{+n}, $0 < n < \omega$. So not all m–types for $m > n$ can be inside a model of cardinality κ^{+n}.

Definition 2.1 Let $1 < k \leq n+2$ and $\beta < \kappa^{+n+2}$.
β is called k–good iff

1. for every $\gamma < \beta$, $tp_k(\gamma, \beta)$ is realized unboundedly often below κ^{+n+2},
2. for every bounded $z \subseteq \beta$ of cardinality $\leq \kappa$ there is $\alpha < \beta$ which corresponds to z in the enumeration of $[\kappa^{n+2}]^{\leq \kappa}$.

Lemma 2.2 *The set $\{\beta < \kappa^{+n+2} \mid \beta$ is k–good $\}$ contains a club, for every $1 < k \leq n+2$.*

Lemma 2.3 *The set $\{\beta < \kappa^{+n+2} \mid \forall 1 < k \leq n+2 \quad \beta$ is k–good $\}$ contains a club.*

Lemma 2.4 *Let $2 < k \leq n+2$ and $\beta < \kappa^{+n+2}$ be k–good. Then there are arbitrary large $k-1$–good ordinals below β.*

We are going to use models of sizes κ^{+n}, $0 < n < \omega$. So not all m–types for $m > n$ can be inside a model of cardinality κ^{+n}, however, if M is a model of size κ^{+n+1} with $\kappa^{+n+1} \subseteq M$, all tp_n's are in M (GCH is assumed). Hence, for every ξ there is $\xi' \in M$ such that $tp_n(\xi) = tp_n(\xi')$.

2.2 Forcing.

Basically an assignment functions $\langle a_n \mid n < \omega \rangle$ are added and a_n acts (partially) from $\kappa^{+\omega+2}$ to κ^{+n+2}.
In the extension $\kappa^{+\omega+2}$ will turn into κ^{++} and κ^{+n}'s will be collapsed for every $n, 1 < n < \omega$, as well as $\kappa^{+\omega+1}$.

We will follow Merimovich [8],[9].
Let us review the basic settings of [8] with adoptions made here.

2.2.1 Merimovich's type setting.

Denote the interval $[\kappa, \kappa^{+n+2})$ by $\mathfrak{D}_n$.

Let $d \in \mathcal{P}_{\kappa^+}\mathfrak{D}_n$. Then $\nu \in OB(d)$ iff

1. $\nu : \mathrm{dom}(\nu) \to \kappa$,
2. $\kappa \in \mathrm{dom}(\nu) \subseteq d$,
3. $|\nu| \leq (\nu(\kappa))_0^{+n+2}$,
   where ρ_0 denotes the largest inaccessible cardinal $< \rho$, if it exists and 0 otherwise.
4. $\forall \alpha, \beta \in \mathrm{dom}(\nu)(\alpha < \beta \Longrightarrow \nu(\alpha) < \nu(\beta))$.

The measure $E_n(d)$ for $d \in \mathcal{P}_{\kappa^+}\mathfrak{D}_n$ is defined as follows:

$$\forall X \subseteq OB(d)(X \in E_n(d) \Longleftrightarrow mc(d) \in j_{E_{n+2}}(d)),$$

where

$$mc(d) = \{\langle j_{E_{n+2}}(\alpha), \alpha\rangle \mid \alpha \in d\}.$$

Turn now to $\mathbb{P}^*_{\vec{E}}$. Here instead of a single function $f : d \to \kappa^{<\omega}$ we will use a sequence $\vec{f} = \langle f^i \mid \ell(\vec{f}) \leq i < \omega\rangle$ such that for every $\ell(\vec{f}) \leq i \leq i' < \omega$:

1. $f^i : d^i \to \kappa^{<\omega}$,
2. $\kappa \in d^i \in \mathcal{P}_{\kappa^+}\mathfrak{D}_i$,
3. $f^{i'}$ extends f^i, i.e. $d^{i'} \subseteq d^i$ and $\forall \alpha \in d^i \quad f^i(\alpha) = f^{i'}(\alpha)$;
4. for each $\alpha \in d^i$, $f^i(\alpha) = \langle f^i_0(\alpha), ..., f^i_{k-1}(\alpha)\rangle$ is an increasing sequence of ordinals below κ,
5. the length of the sequence $f^i(\kappa)$ is $\ell(\vec{f})$.

Let $\vec{f}, \vec{g} \in \mathbb{P}^*_{\vec{E}}$. We say that $\vec{f}$ is an extension of $\vec{g}$ ($\vec{f} \geq_{\mathbb{P}^*_{\vec{E}}} \vec{g}$) iff $\ell(\vec{f}) = \ell(\vec{g})$ and for every $\ell(\vec{f}) \leq i < \omega, \quad f^i \supseteq g^i$.

Let $\vec{f} \in \mathbb{P}^*_{\vec{E}}$ and $\nu \in OB(\mathrm{dom}(f_{\ell(\vec{f})}))$. Define $\vec{g} = \vec{f}_{\langle\nu\rangle}$ to be $\langle g^m \mid \ell(\vec{f}) < m < \omega\rangle$ where for every $m, \ell(\vec{f}) < m < \omega$,

1. $\mathrm{dom}(g^m) = \mathrm{dom}(f^m)$,

2. for every $\alpha \in \mathrm{dom}(g^m)$

$$g^m(\alpha) = \begin{cases} f^m(\alpha)^\frown\langle\nu(\alpha)\rangle, & \alpha \in \mathrm{dom}(\nu), \nu(\alpha) > f^m_{|f^m(\alpha)|-1}(\alpha) \\ f^m(\alpha), & \text{otherwise.} \end{cases}$$

I.e. $\nu(\alpha)$ is added once it is bigger than every element of the finite sequence $f^m(\alpha)$.

A condition p in the forcing notion $\mathbb{P}_{\vec{E}}$ is of the form $\langle \vec{f}, \vec{A}\rangle$ where:

1. $\vec{f} \in \mathbb{P}^*_{\vec{E}}$,
2. $\vec{A} = \langle A^i \mid \ell(\vec{f}) \leq i < \omega\rangle$ such that
   (a) $A^i \in E_i(\mathrm{dom}(f^i))$, for every $i, \ell(\vec{f}) \leq i < \omega$,
   (b) for each $\langle\nu\rangle \in A^i$ and each $\alpha \in \mathrm{dom}(\nu)$,

$$f^i_{|f^i(\alpha)|-1}(\alpha) < \nu(\alpha).$$

We refer further to $\ell(\vec{f})$ also as $\ell(p)$.

Note that we use here a sequence of sets of measures one instead of a single set in the intersections of the measures in Merimovich [8].

Further let us write $\vec{f}^p, \vec{A}^p$ for $\vec{f}, \vec{A}$ etc.

Let $p, q \in \mathbb{P}_{\vec{E}}$. Set $p \geq^*_{\mathbb{P}_{\vec{E}}} q$ iff

1. $\vec{f}^p \geq_{\mathbb{P}^*_{\vec{E}}} \vec{f}^q$, in particular $\ell(\vec{f}) = \ell(\vec{g})$;
2. $A^{p,i} \restriction \mathrm{dom}(f^{q,i}) \subseteq A^{q,i}$, for every $i, \ell(\vec{f}) \leq i < \omega$.

Assume $q \in \mathbb{P}_{\vec{E}}$ and $\langle\nu\rangle \in A^{q,\ell(q)}$. The condition $p \in \mathbb{P}_{\vec{E}}$ is the one point extension of of q by $\langle\nu\rangle$ ($p = q_{\langle\nu\rangle}$) if the following holds:

1. $\vec{f}^p = \vec{f}^q_{\langle\nu\rangle}$,
2. $\vec{A}^p = \langle A^{q,m} \mid m > \ell(q)\rangle$.

$q_{\langle\nu_0,\ldots,\nu_n\rangle}$ is defined recursively by $(q_{\langle\nu_0,\ldots,\nu_{n-1}\rangle})_{\langle\nu_n\rangle}$.

Let $p, q \in \mathbb{P}_{\vec{E}}$. Then p is stronger than q ($p \geq q$) if there are $n < \omega$ and $\langle\nu_{\ell(q)}, \ldots, \nu_{n-1}\rangle \in \prod_{\ell(q)\leq k<n} A^{q,k}$ such that $p \geq^* q_{\langle\nu_{\ell(q)},\ldots,\nu_{n-1}\rangle}$.

The following was proved in [8]:

1. $\langle \mathbb{P}_{\vec{E}}, \leq, \leq^* \rangle$ is Prikry type forcing notion,

2. $\langle \mathbb{P}_{\vec{E}}, \leq \rangle$ satisfies κ^{++}-c.c.,

3. $\langle \mathbb{P}_{\vec{E}}, \leq^* \rangle$ is κ–closed, and hence no new bounded subsets are added to κ,

4. κ changes its cofinality to ω,

5. $2^\kappa > \kappa^{+\omega}$.

2.2.2 A short extenders setting.

Add now short extenders forcings ingredients.

We will add over κ an assignment function a_n which connects between $\kappa^{+\omega+2}$ and κ^{+n+2} and a partial function $f_n : \kappa^{+\omega+2} \to \kappa$, which purpose is to hide a_n once κ_n is decided,, for every $n < \omega$.

So let $p \in \mathbb{P}_{\vec{E}}$ and $n < \omega$. Consider $f^{p,n}$. We would like to add to it new components a_n and f_n which satisfy the following:

1. $\mathrm{rng}(a_n) \subseteq \mathrm{dom}(f^{p,n})$,

2. a_n is partial order preserving function from $\kappa^{+\omega+2}$ to κ^{+n+2},

3. $f_n : \kappa^{+\omega+2} \to \kappa$ is a partial function of cardinality $\leq \kappa$,

4. $\mathrm{dom}(f_n) \cap \mathrm{dom}(a_n) = \emptyset$,

5. $n < m$ implies that $\mathrm{dom}(a_n) \subseteq \mathrm{dom}(a_m)$,

6. let $\zeta \in \mathrm{dom}(a_n)$, for some $n < \omega$. Then for every $k < \omega$ for all but finitely many $m < \omega$, $a_m(\zeta)$ is k–good.

Definition 2.5 The forcing notion $\mathbb{P}^{se}_{\vec{E}}$ (*se* for short extenders) consists of elements t of the form $\langle p, \langle a_n \mid \ell(p) \leq n < \omega\rangle\rangle, \langle f_n \mid n < \omega\rangle\rangle$, where

1. $p \in \mathbb{P}_{\vec{E}}$,

2. for every $n < \omega$, a_n and f_n satisfy the conditions 1-6 above.

Let us describe a one point extension of conditions in $\mathbb{P}^{se}_{\vec{E}}$.

Definition 2.6 Let $t = \langle p, \langle a_n \mid \ell(p) \leq n < \omega\rangle\rangle, \langle f_n \mid n < \omega\rangle\rangle \in \mathbb{P}^{se}_{\vec{E}}$ and $\langle \nu \rangle \in A^{p,\ell(p)}$. Define $t_{\langle \nu \rangle}$ to be of the form $\langle p_{\langle \nu \rangle}, \langle b_n \mid \ell(p) < n < \omega\rangle, \langle g_n \mid n < \omega\rangle\rangle$, where

1. for every $n > \ell(p)$, $b_n = a_n$ and $g_n = f_n$,

2. for every $n < \ell(p)$, $g_n = f_n$,

3. $g_{\ell(p)} = f_{\ell(p)} \cup \{\langle \eta, \xi \rangle \mid \eta \in \mathrm{dom}(a_{\ell(p)}), \xi = \nu(a_{\ell(p)}(\eta))$ and $\xi > f^{p,\ell(p)}_{|f^{p,\ell(p)}(a_{\ell(p)}(\eta))|-1}(a_{\ell(p)}(\eta))\}$.

Now the order $\leq$ on $\mathbb{P}^{se}_{\vec{E}}$ is defined recursively as $\mathbb{P}_{\vec{E}}$ was defined using one step extensions and $\leq^*$.

The arguments of [8] adopt here straightforwardly to show that $\langle \mathbb{P}^{se}_{\vec{E}}, \leq, \leq^* \rangle$ is a Prikry type forcing notion. Cardinals structure of a generic extension changes due to the assignment functions. Thus, $\kappa^{+\omega+2}$ is collapsed to κ^+.

Define now the equivalence relation $\longleftrightarrow$ and a forcing order $\longrightarrow$ which will allow us to preserve $\kappa^{+\omega+2}$.

Definition 2.7 (Equivalence of conditions) Let $t, s \in \mathbb{P}^{se}_{\vec{E}}$. Let $t = \langle p, \langle a_n \mid \ell(p) \leq n < \omega \rangle \rangle, \langle f_n \mid n < \omega \rangle \rangle$ and $s = \langle q, \langle p, \langle b_n \mid \ell(q) \leq n < \omega \rangle \rangle, \langle g_n \mid n < \omega \rangle \rangle$. Set $t \longleftrightarrow s$ iff

1. $p = q$,

2. $f_n = g_n$, for every $n < \omega$,

3. for every $\ell(p) \leq n < \omega$,

   (a) $\mathrm{dom}(a_n) = \mathrm{dom}(b_n)$,

   (b) for every $k < \omega$, $\mathrm{rng}(a_n)$ and $\mathrm{rng}(b_n)$ realize the same k–type, for all but finitely many n's. Moreover they always realize the same 4-type.

Definition 2.8 (The new order of $\mathbb{P}^{se}_{\vec{E}}$)
Let $t, s \in \mathbb{P}^{se}_{\vec{E}}$. Set $t \longrightarrow s$ iff there exists a finite sequence $\langle r_i \mid i \leq k \rangle$ of elements of $\mathbb{P}^{se}_{\vec{E}}$ such that

1. $t = r_0$,

2. $s = r_k$,

3. for every $i < k$ either

   - $r_i \leq r_{i+1}$
     or
   - $r_i \longleftrightarrow r_{i+1}$.

The next lemmas are standard:

Lemma 2.9 *$\langle \mathbb{P}^{se}_{\vec{E}}, \longrightarrow \rangle$ is a nice subforcing of $\langle \mathbb{P}^{se}_{\vec{E}}, \leq \rangle$.*

Lemma 2.10 *The forcing $\langle \mathbb{P}^{se}_{\vec{E}}, \longrightarrow \rangle$ satisfies $\kappa^{+\omega+2}$–c.c.*

Lemma 2.11 *In $V^{\langle \mathbb{P}^{se}_{\vec{E}}, \longrightarrow \rangle}$, $\mathrm{cof}(\kappa) = \omega$ and $2^\kappa = (\kappa^{+\omega+2})^V = \kappa^{++}$.*

3 Way 3.

We will proceed as in Gap $\omega + 2$ -doing without preparation [7] (or make the preparation preserving strongness of κ).
Instead of the chain condition properness will be proved.

We will have an isomorphism a_n between a $\leq \kappa$–suitable structures over κ at a level $n < \omega$. Both will be here over κ and actually over $\kappa^{+\omega+2}$. Models of the size $\kappa^{+\omega+1}$ will correspond to those of the size κ^{+n+1}. So κ^{+n+2} will correspond to $\kappa^{+\omega+2}$. Actually if E_n was a $(\kappa, \kappa^{+\omega+1})$–extender, then every regular cardinal of the interval $[\kappa^{+n+2}, \kappa^{+\omega+1}]$ will correspond to $\kappa^{+\omega+2}$.

A difference here from the usual short extenders forcings setting is that the number of types may be beyond a size of a model in the domain of an assignment function. This may effect the arguments that show the properness. Thus, for example, suppose that we like to prove η–properness for some $\eta = \kappa^{+n}$. A model M of size η is picked with a condition p inside is picked. First we extend p by adding M as a top model of the domain. Then we need to argue that the resulting condition in M-generic. For this purpose extensions are taken, but here there is no guarantee that their ranges realize types inside M.

Let us split into two settings which deal with this problem differently and also produce different PCF configuration.

3.1 First setting.

We restrict the number of types which will be allowed to use.
Fix $\mathfrak{A}^* \prec \langle H(\chi^{+\omega+2}), \in, <, \kappa, \langle E_n \mid n < \omega \rangle, ... \rangle$ of cardinality κ^+.

Definition 3.1 A set t is called *an allowed type* if for some $k < \omega$, t is a $\mathfrak{L}_k$–type realized by an ordinal $\xi < \chi$ in $\mathfrak{A}_k$ and $t \in \mathfrak{A}^*$.

Lemma 3.2

1. *Let t be an allowed type. Then t is realized in $\mathfrak{A}^*$.*

2. *Let $\xi \in \mathfrak{A}^* \cap \chi$, then for every $k < \omega$, $tp_k(\xi)$ is an allowed type.*

Proof. Both items follow since $\mathfrak{A}^* \prec \langle H(\chi^{+\omega+2}), \in, <, \kappa, \langle E_n \mid n < \omega \rangle, ... \rangle$ and so $\mathfrak{A}_k \in \mathfrak{A}^*$.
□

Lemma 3.3 *Suppose $M, A \in \mathfrak{A}^*$ are such that*

1. $M, A \prec H(\kappa^{+\omega+2})$,

2. $M \in A$,

3. $|M| = \kappa^{n+1} > |A| \geq \kappa$, *for some* $n < \omega$,

4. $M \supseteq \kappa^{+n+1}$,

5. ${}^{\kappa_n}M \subseteq M$.

Then there is $A' \in M \cap \mathfrak{A}^*$ which realizes the same n–type as A does over $A \cap M$, i.e. $tp_n(A \cap M, A) = tp_n(A \cap M, A')$.

Proof. All n–types are in M, since $M \supseteq \kappa^{+n+1}$, $A \cap M \in M$, since ${}^{\kappa_n}M \subseteq M$. Hence there is $A' \in H(\chi^{+\omega+2}) \cap M$ which realizes the same n–type as A does over $A \cap M$. But $\mathfrak{A}^* \prec H(\chi^{+\omega+2})$ and all the needed parameters $H(\chi^{+n+2}), A, M$ are in $\mathfrak{A}^*$. Hence there is such A' inside $\mathfrak{A}^*$.
□

We define now the forcing.

At a level n we will have an isomorphism a_n between a $\leq \kappa$–suitable structures over κ. Both will be here over κ. Recall that E_n is a (κ, κ^{+n})–extender. Let us change enumeration a bit and assume that E_n is a $(\kappa, \kappa^{+n+1+n+1})$–extender. Models of the size $\kappa^{+\omega+1}$ will correspond to those of the size $\kappa^{+n+1+n+1}$. So κ^{+2n+3} will correspond to $\kappa^{+\omega+2}$. Actually if E_n was a $(\kappa, \kappa^{+\omega+1})$–extender, then every regular cardinal of the interval $[\kappa^{+2n+3}, \kappa^{+\omega+1}]$ will correspond to $\kappa^{+\omega+2}$.

Let us turn to formal definitions.

Definition 3.4 The forcing notion $\mathbb{P}^{se}_{\vec{E}}$ consists of elements t of the form $\langle p, \langle a_n \mid \ell(p) \leq n < \omega\rangle\rangle, \langle f_n \mid n < \omega\rangle\rangle$, where

1. $p \in \mathbb{P}_{\vec{E}}$,
2. for every $\ell(p) \leq n < \omega$,
   (a) a_n is an isomorphism between a κ–suitable structure $\mathfrak{X}$ over κ with models of cardinalities $\{\kappa^+, ..., \kappa^{+n}, \kappa^{+\omega+1}\}$ (see [7]) which include $\mathfrak{A}^* \cap H(\kappa^{+\omega+2})$ and a κ–suitable structure $\mathfrak{X}'$ over κ with models in $\mathfrak{A}^*$
   of cardinalities $\{\kappa^{+n+1}, \kappa^{+n+2}, ..., \kappa^{+n+1+n}, \kappa^{+n+1+n+1}\}$, such that models of cardinality $\kappa^{+k}, k \leq n$ are moved to models of cardinality κ^{+n+1+k}. Models of size $\kappa^{+\omega+1}$ are moved to those of size $\kappa^{+n+1+n+1}$. This way $\kappa^{+\omega+2}$ will correspond over the level n to κ^{+2n+3}.
   We can allow models of sizes $\kappa^{+k}, n < k < \omega$ in $\mathfrak{X}'$ and move them to models of cardinality $\kappa^{+n+1+n+1}$ the same way as models of size $\kappa^{+\omega+1}$ are moved.
   (b) $\text{rng}(a_n) \subseteq \text{dom}(f^{p,n})$ under some fixed codding,
   (c) types of members of $\text{rng}(a_n)$ are allowed types only,
3. for every $n < \omega$,$f_n : \kappa^{+\omega+2} \to \kappa$ is a partial function of cardinality $\leq \kappa$,
4. for every $\ell(p) \leq n \leq m < \omega$, $\text{dom}(a_n) \leq \text{dom}(a_m)$ in the order of suitable structures of [7],
5. if $\ell(p) \leq n \leq m$, then $\max(\text{dom}(a_n)) = \max(\text{dom}(a_m))$.
6. for every n, $\ell(p) \leq n < \omega$, and $X \in \text{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).
7. For every $n \geq \ell(p)$ and $\alpha \in \text{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \text{dom}(a_m) \setminus \text{dom}(f_m)$.
8. There is a κ-structure with pistes $\mathfrak{p}$ over κ such that
   (a) $\mathfrak{p} \geq \text{dom}(a_n)$, for every $n, \ell(p) \leq n < \omega$,
   (b) if a model A appears in $\mathfrak{p}$, then A appears in $\text{dom}(a_n)$ for some $n, \ell(p) \leq n < \omega$ (and then in a final segment of them),

(c) $\max(\mathrm{dom}(a_n)) = \max(\mathfrak{p})$ (actually this follows from the previous condition).

The relations $\leq, \leq^*, \longleftrightarrow, \longrightarrow$ are defined on $\mathbb{P}^{se}_{\vec{E}}$ as in the previous section.

The next lemmas are standard:

Lemma 3.5 *$\langle \mathbb{P}^{se}_{\vec{E}}, \longrightarrow \rangle$ is a nice subforcing of $\langle \mathbb{P}^{se}_{\vec{E}}, \leq \rangle$.*

Lemma 3.6 *In $V^{\langle \mathbb{P}^{se}_{\vec{E}}, \longrightarrow \rangle}$, $\mathrm{cof}(\kappa) = \omega$ and $2^\kappa = (\kappa^{+\omega+2})^V = \kappa^{++}$.*

Let us deal with the cardinals preservation.
We will follow closely [7]

Our tusk will be to prove the following two lemmas:

Lemma 3.7 *$\langle \mathcal{P}, \rightarrow \rangle$ is κ^+-proper.*

Lemma 3.8 *$\langle \mathcal{P}, \rightarrow \rangle$ is η-proper, for every regular $\eta, \kappa^+ \leq \eta \leq \kappa^{+\omega+2}$.*

Proof of 3.7. Let $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \kappa^+, {}^\kappa M \subseteq M$, $P, p, \mathfrak{A}^* \in M$.

Set $M' := M \cap H(\kappa^{+\omega+2})$. Extend p by adding M' to $\mathrm{dom}(a_n(p))$ as the largest model, make it potentially limit point. Pick a sequence of models with increasing goodness and cardinalities realizing allowed types and which is in M' to be a sequence of images of M'.

Let p' be the resulting condition. We claim that p' is (M, P)–generic.
Let $q \geq p'$ and $D \in M$ be a dense open. Let us show that there is an element of $D \cap M$ which is compatible with q. Consider $\mathfrak{q}$ the κ–structure with pistes over κ of q. Now, $\mathfrak{q} \restriction M'$ is κ–structure with pistes over κ.
Pick some $M'' \prec H(\kappa^{+\omega+2})$ of size κ^+, $M'' \in M'$ and such that $\mathfrak{q} \restriction M'$ with M' removed is in M''[1]. Add M'' to $\mathfrak{q} \restriction M'$ both to domains of $a_n(\mathfrak{q} \restriction M')$ and pick a sequence of models with increasing goodness and cardinalities realizing allowed types and which is in M'' to be a sequence of images of M''. Denote the result by $\mathfrak{q}'$ and a corresponding condition by q' (i.e. we extend q in order to incorporate M'').
Set $\mathfrak{q}'' = \mathfrak{q}' \restriction M''$. It is a κ–structure with pistes over κ. Let $q'' \in M$ be a corresponding condition. Pick $r \in M \cap D$ above q''. Combine r with q. The result will be as desired. Notice that there is no need to pass to an equivalent condition here.
□
Proof of 3.8.

[1]This is possible since only allowed types are used in the range and hence all of them are in M' and here a crucial use of allowed types is made.

Let η be a regular cardinal such that $\kappa^+ < \eta \leq \kappa^{+\omega+2}$. Suppose that $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \eta, {}^{\eta>}M \subseteq M$, $P, p, \mathfrak{A}^* \in M$.

Set $M' := M \cap H(\kappa^{+\omega+2})$. If $\eta = \kappa^{+n}$, for some $n < \omega$, then we can assume without loss of generality that $\ell(p) > n$. Just otherwise replace p by its a non-direct extension $p' \in M$ of with $\ell(p') > n$.

Extend p by adding M' to $\mathrm{dom}(a_n(p))$ as the largest model, make it potentially limit point. If $\eta = \kappa^{+k}$, for some $k < \omega$, then pick a sequence of models with increasing goodness realizing allowed types and which is in M' to be a sequence of images of M'.

Let p' be the resulting condition. We claim that p' is (M, P)–generic.

Let $q \geq p'$ and $D \in M$ be a dense open. Extending if necessary, we can assume that $q \in D$. Let us show that some condition in $D \cap M$ which is compatible with q.

Consider $\mathfrak{q}$ the κ–structure with pistes over κ of q. Extending if necessary, we can assume that $A^{0\kappa^+}(\mathfrak{q})$ is the maximal model of $\mathfrak{q}$. Consider also $\mathfrak{q} \restriction M'$. Note that it need not be κ–structure with pistes over κ, since there may be no single maximal model of size κ^+ inside. Let us reflect $A^{0\kappa^+}(\mathfrak{q})$ and q down to M over $A^{0\kappa^+}(\mathfrak{q}) \cap M$, i.e. we pick some $A' \in M$ and q' which realizes the same k-type (for some $k < \omega$ sufficiently big) over $A^{0\kappa^+}(\mathfrak{q}) \cap M$ as $A^{0\kappa^+}(\mathfrak{q})$ and q do in a rich enough language which includes D as well. [2] In particular $q' \in D \cap M$[3]. Now q' is compatible with q. Just pick some model A of cardinality κ^+ which includes all relevant information, i.e. $A^{0\kappa^+}(\mathfrak{q}), A', q, q', M'$ etc. The triple $A, A^{0\kappa^+}(\mathfrak{q}), A'$ will form a Δ–system triple relatively to M' and the model which corresponds to M' in A'. Combine q, q' together adding A as the maximal model and replacing models in the range of q by equivalent ones in order to fit with the range of q'.

□

Finally, combining together the lemmas, we obtain the following:

Theorem 3.9 *Let G be a generic subset of $\langle \mathcal{P}, \longrightarrow \rangle$. Then $V[G]$ is a cardinal preserving extension of V in which $\mathrm{cof}(\kappa) = \omega$ and $2^\kappa = \kappa^{+\omega+2}$.*

Let us specify the resulting PCF structure in $V[G]$.

Theorem 3.10 *Let G be a generic subset of $\langle \mathcal{P}, \longrightarrow \rangle$ and $\langle \kappa_n \mid n < \omega \rangle$ be the Prikry sequence for the normal measures of the extenders. Then $V[G]$ satisfies the following:*

1. *for every $k, 1 \leq k < \omega$,* $\quad \mathrm{cof}(\langle \prod_{n<\omega} \kappa_n^{+k}, <_{bounded} \rangle) = \kappa^{+k}$,

[2] We follow here a suggestion by Carmi Merimovich to include D into the language.

[3] Note that $\mathrm{rng}(q) \in M$ since only allowed types are used there and all of them are in M, since $\mathfrak{A}^* \in M$.

2. $\mathrm{cof}(\langle \prod_{n<\omega} \kappa_n^{+n+1+k}, <_{bounded} \rangle) = \kappa^{+k}$,

3. $\mathrm{cof}(\langle \prod_{n<\omega} \kappa_n^{+n+1+n+1}, <_{bounded} \rangle) = \kappa^{+\omega+1}$,

4. $\mathrm{cof}(\langle \prod_{n<\omega} \kappa_n^{+n+1+n+2}, <_{bounded} \rangle) = \kappa^{+\omega+2}$.

Proof. Follows by the construction. Only note that the first item is true since the equivalence $\longleftrightarrow$ does not effect κ_n^{+k} for a fixed k.

□

Remark 3.11 1. A new element here that does not hold in the usual Short extenders settings is that all indiscernibles correspond here to a unique cardinal κ.

2. It is possible to replace $\kappa^{+\omega+2}$ by any $\kappa^{+\alpha+2}$ with $\omega \leq \alpha < \omega_1$.

3. Using stronger initial assumptions it is possible obtain arbitrary higher gaps in a similar setting.

3.2 Second setting.

Here all types will be allowed but assignment function will act on models of sizes $< \kappa^{+\omega}$ almost as the identity. In particular models of cardinality κ^{+k} will be moved to models of the same cardinality, for every $k, 1 \leq k < \omega$. Properness for each size below $< \kappa^{+\omega}$ will be proved as those for the smallest size κ^{+} without the reflection into a model. Only for $\kappa^{+\omega}$-properness the reflection argument will be used.

Let us turn to formal definitions.

Definition 3.12 The forcing notion $\mathbb{P}^{se}_{\vec{E}}$ consists of elements t of the form $\langle p, \langle a_n \mid \ell(p) \leq n < \omega\rangle\rangle, \langle f_n \mid n < \omega\rangle\rangle$, where

1. $p \in \mathbb{P}_{\vec{E}}$,

2. for every $\ell(p) \leq n < \omega$,

   (a) a_n is an isomorphism between a κ–suitable structure $\mathfrak{X}$ over κ with models of cardinalities $\{\kappa^{+}, ..., \kappa^{+n}, \kappa^{+\omega+1}\}$ (see [7]) and a κ–suitable structure $\mathfrak{X}'$ over κ with models of cardinalities $\{\kappa^{+}, \kappa^{++}, ..., \kappa^{+n}, \kappa^{+n+1}\}$, such that models of cardinality $\kappa^{+k}, k \leq n$ are moved to models of cardinality κ^{+k}. Models of size $\kappa^{+\omega+1}$ are moved to those of size κ^{+n+1}. This way $\kappa^{+\omega+2}$ will correspond over the level n to κ^{+n+2}.

(b) $\mathrm{rng}(a_n) \subseteq \mathrm{dom}(f^{p,n})$ under some fixed codding.

3. For every A in $\mathfrak{X}$ there is a non-decreasing converging to infinity sequence of natural numbers $\langle k_n \mid n < \omega \rangle$ such that for every $n < \omega$, A and $a_n(A)$ realize the same k_n–type. Note that once $|A| < \kappa^{+\omega}$ then starting with n_0 such that $|A| < \kappa^{+k_{n_0}}$ we will have $A \cap |A|^+ = a_n(A) \cap |A|^+$, for every $n \geq n_0$, since they realize the same $|A|^+$–type.

4. for every $n < \omega$,$f_n : \kappa^{+\omega+2} \to \kappa$ is a partial function of cardinality $\leq \kappa$,

5. for every $\ell(p) \leq n \leq m < \omega$, $\mathrm{dom}(a_n) \leq \mathrm{dom}(a_m)$ in the order of suitable structures of [7],

6. if $\ell(p) \leq n \leq m$, then $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$.

7. for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).

8. For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.

9. There is a κ-structure with pistes $\mathfrak{p}$ over κ such that

   (a) $\mathfrak{p} \geq \mathrm{dom}(a_n)$, for every $n, \ell(p) \leq n < \omega$,

   (b) if a model A appears in $\mathfrak{p}$, then A appears in $\mathrm{dom}(a_n)$ for some $n, \ell(p) \leq n < \omega$ (and then in a final segment of them),

   (c) $\max(\mathrm{dom}(a_n)) = \max(\mathfrak{p})$ (actually this follows from the previous condition).

The relations $\leq, \leq^*, \longleftrightarrow, \longrightarrow$ are defined on $\mathbb{P}^{se}_{\vec{E}}$ as before.
The next lemmas are standard:

Lemma 3.13 *$\langle \mathbb{P}^{se}_{\vec{E}}, \longrightarrow \rangle$ is a nice subforcing of $\langle \mathbb{P}^{se}_{\vec{E}}, \leq \rangle$.*

Lemma 3.14 *In $V^{\langle \mathbb{P}^{se}_{\vec{E}}, \longrightarrow \rangle}$, $\mathrm{cof}(\kappa) = \omega$ and $2^\kappa = (\kappa^{+\omega+2})^V = \kappa^{++}$.*

Let us deal with the cardinals preservation.
We will follow closely [7]

The next two lemmas are proved exactly as before. In the first lemma note that the restriction of a condition to the main model will be inside the model (up to passing to equivalent one) (i.e. $q \restriction M \in M$) due to the item 3 of Definition 3.12. For the second lemma only note that for every $n < \omega$ the number of n–types is κ^{+n+1} and it is below the size of the main model of the argument which is $\kappa^{+\omega+1}$.

Lemma 3.15 *$\langle \mathcal{P}, \rightarrow \rangle$ is κ^+-proper.*

Lemma 3.16 *$\langle \mathcal{P}, \rightarrow \rangle$ is $\kappa^{+\omega+1}$-proper.*

Let us turn to a lemma that requires now a new argument.

Lemma 3.17 *$\langle \mathcal{P}, \rightarrow \rangle$ is η-proper, for every regular $\eta, \kappa^+ < \eta < \kappa^{+\omega}$.*

Proof. Let η be a regular cardinal such that $\kappa^+ < \eta < \kappa^{+\omega}$. Suppose that $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \eta, {}^{\eta>}M \subseteq M,\ P, p, \mathfrak{A}^* \in M$.

Set $M' := M \cap H(\kappa^{+\omega+2})$. We can assume without loss of generality that $\ell(p) > n$. Just otherwise replace p by its a non-direct extension $p' \in M$ of with $\ell(p') > n$.
Extend p by adding M' to $\mathrm{dom}(a_n(p))$ as the largest model, make it potentially limit point.

Pick a sequence of models $\langle M'_n \mid n < \omega \rangle$ with increasing goodness realizing types as those of M' and let it be the sequence of images of M'.
Let p' be the resulting condition. We claim that p' is (M, P)–generic.
Let $q \geq p'$ and $D \in M$ be a dense open.
Note that models of the range of q nor their types need to be in M due to its relatively small cardinality. Namely η^+–types and up. So the previous reflection argument does not apply in the present situation.
Let us proceed a bit differently.
Without loss of generality assume that the largest model of q has cardinality η^-, i.e. the immediate predecessor of η. Reflect in the domain. Thus consider $\mathfrak{X}(q)$. Let A be its largest model. Reflect it to M' over $M' \cap A$ and find $A' \in M'$ which realizes the same η^-–type over $M' \cap A$ as A does. Require also the measures which correspond to A and A' are the same (this is automatic once $\eta^- > \kappa^+$). Then A, A' will be of a Δ–system type relatively to M'.
Now, for every $n < \omega$, find $A'_n \in M'_n$ which corresponds to A in M'_n (say have the same index in some fixed in advance enumeration etc.). Due to the similarity of types of M and M'_n's (the item 3 of Definition 3.12) also $a_n(q)(A), A'_n$ will be of a Δ–system type relatively to M'_n.

Replace models of size $\kappa^{+\omega+1}$ by those of size κ^{+n+1} in A'_n in the obvious fashion according to $a_n(q)(A)$. Denote the resulting reflection by q'.

Then $q' \in M$ and so it is possible to extend it to some $q'' \in M \cap D$. Let B'' denotes the largest model of q''. Again we can assume that $|B''| = \eta^-$.
Now let us reflect back. Find B which realizes the same η^{--}–type over A as B'' does over A' relatively to M. Require also the corresponding measures to be the same (it is possible even if η was κ^{++}, since $M \supseteq \kappa^{++}$). Then B, B'' will be of a Δ–system type relatively to M. Do the same for each $n < \omega$ and define B_n which corresponds to B''_n, where B''_n is the model which corresponds to B at the level n.
This way a common extension r of q and q'' is constructed. So $r \in D$ and we are done.
□

Finally, combining together the lemmas, we obtain the following:

Theorem 3.18 *Let G be a generic subset of $\langle \mathcal{P}, \longrightarrow \rangle$. Then $V[G]$ is a cardinal preserving extension of V in which* $\mathrm{cof}(\kappa) = \omega$ *and* $2^\kappa = \kappa^{+\omega+2}$.

Let us specify the resulting PCF structure in $V[G]$.

Theorem 3.19 *Let G be a generic subset of $\langle \mathcal{P}, \longrightarrow \rangle$ and $\langle \kappa_n \mid n < \omega \rangle$ be the Prikry sequence for the normal measures of the extenders. Then $V[G]$ satisfies the following:*

1. *for every $k, 1 \leq k < \omega$,* $\mathrm{cof}(\langle \prod_{n<\omega} \kappa_n^{+k}, <_{bounded} \rangle) = \kappa^{+k}$,
2. *moreover, for every $k, 1 \leq k < \omega$,* $\mathfrak{b}_{\kappa^{+k}}[a] = \{\kappa_n^{+k} \mid n < \omega\}$,
   where a is a set of regular cardinals which includes $\{\kappa_n^{+k} \mid n < \omega\}$ and $\mathfrak{b}_{\kappa^{+k}}[a]$ is a pcf–generator corresponding to κ_n^{+k},
3. $\mathrm{cof}(\langle \prod_{n<\omega} \kappa_n^{+n+1}, <_{bounded} \rangle) = \kappa^{+\omega+1}$,
4. $\mathrm{cof}(\langle \prod_{n<\omega} \kappa_n^{+n+2}, <_{bounded} \rangle) = \kappa^{+\omega+2}$.

Proof. Follows by the construction. The first and the second items hold since models of cardinality κ^{+k} correspond always to models of the same cardinality, for every $k, 1 \leq k < \omega$. The last two items follow from the construction.
□

Remark 3.20
1. The PCF–structure here is more similar to the standard one (like Silver-Prikry or Extender based Prikry). Short extenders component effects solely $\kappa^{+\omega+2}$.
2. It is possible to replace $\kappa^{+\omega+2}$ by any $\kappa^{+\alpha+2}$ with $\omega \leq \alpha < \omega_1$.

3. Using stronger initial assumptions it is possible obtain arbitrary higher gaps in a similar setting.

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