Short extenders forcings – doing without preparations.
Dropping cofinalities.

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The basic issue with dropping cofinalities is that models of small sizes relatively to \( \kappa_n \)'s are supposed to be used (basically much less than \( \kappa_n \)'s). The number of possible types inside such models is limited. Even not every measure of the extender over \( \kappa_n \) is in a model. So we will need to specify in advanced which types are allowed. Let us start with choosing a set of permitted types.

1 Dropping cofinalities–gap 3.

We deal here with the first relevant case– \( 2^\kappa = \kappa^{+3} \) with the witnessing scale has points of cofinality \( \kappa^{++} \) dropping down from \( \kappa_n \)'s to smaller \( \lambda_n \)'s.

Fix \( n < \omega \). Let us define models that will be permitted to use over \( \kappa_n \) in order to allow a cofinality drop to \( \lambda_n \), where \( \lambda_0 < \kappa_0 \) and \( \kappa_{n-1} < \lambda_n < \kappa_n \), for every \( n, 0 < n < \omega \), and \( \lambda_n, \kappa_n \) carry extenders \( E^{\lambda_n}_n, E^{\kappa_n}_n \).

We deal with a simplest case of a single drop. Assume that the length of \( E^{\kappa_n}_n \) is \( \kappa_n^{+n+2} \) and \( E^{\lambda_n}_n \) is \( \lambda_n^{+n+2} \).

Fix some \( \chi_n \) large enough. Let \( \eta < \kappa_n^{+n+2} \) be such that every type of an ordinal \( < \kappa_n^{+n+2} \) is realized below \( \eta \) and for every \( \xi \geq \eta \) the type \( tp_m(\xi) \) is realized unboundedly often below \( \kappa_n^{+n+2} \), for each \( m < \omega \).

Define by induction for every \( \nu < \lambda_n \) two \( \in \)-increasing continuous sequences \( \{ \mathfrak{M}_i | i < \nu^{+n+2} \} \), \( \{ \mathfrak{N}_i | i < \nu^{+n+2} \} \) of elementary submodels of \( H(\chi_n^{+\omega+1}) \) such that

1. \( | \mathfrak{M}_i | = \kappa_n^{+n+1} \),
2. \( \mathfrak{M}_i \cap \kappa^{+n+2} \) is an ordinal above \( \eta \) of cofinality \( \nu^{+n+2} \),
3. \( | \mathfrak{N}_i | = \nu^{+n+1} \),
4. $\mathcal{M}_i \in \mathcal{N}_i$, if $i = 0$ or $i$ is a successor ordinal,

5. $(\mathcal{M}_j)_{j \leq i}, (\mathcal{N}_j)_{j \leq i} \in \mathcal{M}_{i+1}$,

6. $(\mathcal{M}_j)_{j \leq i}, (\mathcal{N}_j)_{j \leq i} \in \mathcal{N}_{i+1}$,

7. $\nu^{i+n+1} \mathcal{M}_i \subseteq \mathcal{M}_i$, if $i = 0$ or $i$ is a successor ordinal,

8. $\nu^{i+n} \mathcal{N}_i \subseteq \mathcal{N}_i$, if $i = 0$ or $i$ is a successor ordinal,

9. if $\nu < \nu'$, then $(\mathcal{M}_i \mid i < \nu^{i+n+2}), (\mathcal{N}_i \mid i < \nu^{i+n+2}) \in \mathcal{M}_{\nu'} \cap \mathcal{N}_{\nu'}$.

The set of permitted types will be the set of all types of models $\mathcal{M}_i, \mathcal{N}_i$. Formally set 

$$PT^n_\nu = \{tp_m(\mathcal{M}_i) \mid i < \nu^{i+n+2}, 2 < m < \omega\}, \quad PT^\lambda_\nu = \{tp_m(\mathcal{N}_i) \mid i < \nu^{i+n+2}, 2 < m < \omega\},$$

$$PT_\nu = PT^n_\nu \cup PT^\lambda_\nu.$$ 

The idea behind the above is that once $\nu$ is an indiscernible (a member of one element Prikry sequence) for the normal measure of $E_\nu^\lambda$, then models with types in $PT_\nu$ are allowed to be used over $\kappa_n$. Note that types of models $\mathcal{N}_i$’s are inside $\mathcal{M}_i$ by the choice of $\eta$ and the item (2).

Let us turn to the assignment functions $a$ of the level $n$ (the isomorphisms function between the suitable structures) for $\kappa^{++}$ and those of $\lambda_n$, and $b$ of the level $n$ for $\kappa^{+3}$ and those of $\kappa_n$.

We require that each model $A$ be in the domain of $a$ is of the form $A' \cap \kappa^{++}$, for some $A' \in \text{dom}(b)$. The rest of the requirements on $a$ are as in [2].

Turn to $b$. Let $A$ be in the domain of $b$. If $A$ has cardinality $\kappa^{++}$, then $b(A)$ is a name of a model with type in $PT^n_\nu$ depending on an indiscernible $\nu$ for the normal measure of $E_\nu^\lambda$.

If $A$ has cardinality $\kappa^+$, then $a(A) \cap \lambda_\nu^{i+n+2}$ is an ordinal and $b(A)$ is a name of a model with type as those of $\mathcal{N}_i$, where $\nu$ is an indiscernible for the normal measure and $i$ is the indiscernible for the measure $a(A) \cap \lambda_\nu^{i+n+2}$ of $E_\nu^\lambda$. Again, the rest of the requirements are as in [2].

**Lemma 1.1** The forcing $\mathcal{P}$ is $\kappa^+$-proper (and even $\kappa^+$-strongly proper).

**Proof.** Let $p \in \mathcal{P}$ and $M < H(\chi)$ with $|M| = \kappa^+$, $^*M \subseteq M$, $p, \mathcal{P} \in M$. $a_n(M \cap \kappa^+)$ is some $\alpha < \lambda_\nu^{i+n+2}$. Run the corresponding argument of [2]. We will get finally some $\beta < \alpha$ that corresponds to the part of the extension which belongs to $M$. Now we will have that
on a set of measure one $\beta^* \in \alpha^*$, where $\beta^*$ denotes an indiscernible for $\beta$ and $\alpha^*$ denotes an indiscernible for $\alpha$. Then $\mathfrak{M}_{\beta^*} \in \mathfrak{M}_{\alpha^*}$, where $\nu$ is an indiscernible for the normal measure. Hence we have no problem in getting the needed type inside $b(M \cap \kappa^+)$.

The argument of the next lemma is as those of [2], since models of big cardinality ($\kappa_n^{+n+1}$) are used here.

**Lemma 1.2** The forcing $\mathcal{P}$ is $\kappa^{++}$-proper (and even $\kappa^{++}$-strongly proper).

### 2 Dropping cofinalities–gap 4.

We like to blow up the power of $\kappa$ to $\kappa^{+4}$ with drops in cofinalities.

Split into two cases according to places of drops.

#### 2.1 $\kappa^{+3}$ drops down to $\lambda_n$’s.

We deal here with the case $2\kappa = \kappa^{+4}$ and the witnessing scale has points of cofinality $\kappa^{+3}$ dropping down from $\kappa_n$’s to smaller $\lambda_n$’s.

The main difference (related to the dropping cofinality) here from the previous section is that there are two sizes $\kappa^+$ and $\kappa^{++}$ of models witnessing the drop. Their images to $\kappa_n$’s has sizes below $\kappa_n$. The issue of having enough types inside such models becomes a bit more delicate.

Fix $n < \omega$. Let $\lambda_n < \kappa_n, \eta < \kappa_n^{+n+2}$ be as above. The length of the extender $E^{\lambda_n}_n$ will be now $\lambda_n^{+n+3}$ in order to accommodate three cardinals $\kappa^+, \kappa^{++}$ and $\kappa^{+3}$. The assignment function $a$ will act between $\kappa^{+3}$ and $\lambda_n^{+n+3}$.

Define by induction for every $\nu < \lambda_n$ two $\in$-increasing continuous sequences $\langle \mathfrak{M}_{\nu^i} \mid i < \nu^{+n+3} \rangle$, $\langle \mathfrak{M}_{\nu^i} \mid i < \nu^{+n+3} \rangle$ and a sequence $\langle \mathfrak{G}_{x\nu} \mid x \in [\nu^{+n+3}]^{\leq \nu^{+n+1}} \rangle$ of elementary submodels of $H(\lambda_n^{\omega+1})$ such that

1. $|\mathfrak{M}_{\nu^i}| = \kappa_n^{+n+1}$,
2. $\mathfrak{M}_{\nu^i} \cap \kappa_n^{+n+2}$ is an ordinal above $\eta$ of cofinality $\nu^{+n+3}$,
3. $|\mathfrak{M}_{\nu^i}| = \nu^{+n+2}$,
4. $\mathfrak{M}_{\nu^i} \cap \nu^{+n+3}$ is an ordinal,
5. $|\mathfrak{G}_{x\nu}| = \nu^{+n+1}$,
between the suitable structures) for

\[ S \langle M \langle x \langle y \langle z \rangle \rangle \rangle \]

\[ n \]

rest of the requirements are as in \[2\].

We require that each model \( A \) be in the domain of \( a \) is of the form \( A' \cap \kappa'^+ \), for some \( A' \in \text{dom}(b) \). The rest of the requirements on \( a \) are as in [2].

Turn to \( b \). Let \( A \) be in the domain of \( b \). If \( A \) has cardinality \( \kappa'^+ \), then \( b(A) \) is a name of a model with type in \( PT^{\kappa'}_{\nu} \) depending on an indiscernible \( \nu \) for the normal measure of \( E_n^{\kappa'} \). If \( A \) has cardinality \( \kappa'^+ \), then \( a(A) \cap \lambda^{n+3} \) is an ordinal and \( b(A) \) is a name of a model with type as those of \( M_{i\nu} \), where \( \nu \) is an indiscernible for the normal measure and \( i \) is the indiscernible for the measure \( a(A) \cap \lambda^{n+3} \) of \( E_n^{\lambda_n} \). The rest of the requirements are as in [2]. If \( A \) has cardinality \( \kappa'^+ \), then \( a(A) \cap \lambda^{n+3} \) is a set of cardinality \( \lambda^{n+1} \) and \( b(A) \) is a name of a model with type as those of \( \mathfrak{S}_{x\nu} \), where \( \nu \) is an indiscernible for the normal measure and \( x \in [\nu^{n+3}]^{\leq x^{n+1}} \) is the indiscernible for the measure \( a(A) \cap \lambda^{n+3} \) of \( E_n^{\lambda_n} \). Again, the rest of the requirements are as in [2].
2.2 \( \kappa^{+3} \) does not drop down to \( \lambda_n \)'s.

We deal here with the case \(- 2^\kappa = \kappa^{+4} \) and the witnessing scale has points of cofinality \( \kappa^{++} \) dropping down from \( \kappa_n \)'s to smaller \( \lambda_n \)'s, but those of cofinality \( \kappa^{+3} \) do not drop down. Here only models of the size \( \kappa^+ \) will witness the drop. Their images to \( \kappa_n \)'s will have sizes below \( \lambda_n \).

Fix \( n < \omega \). Let \( \lambda_n < \kappa_n \), \( \eta < \kappa_n^{+n+2} \) be as above. The length of the extender \( E_n^{\lambda_n} \) will be now \( \lambda_n^{+n+2} \) and of \( E_n^{\kappa_n} \) will be \( \kappa_n^{+n+3} \). The assignment function \( a \) will act between \( \kappa^{++} \) and \( \lambda_n^{+n+2} \).

Define by induction for every \( \nu < \lambda_n \) two \( \in \)-increasing continuous sequences \( \langle M_{i\nu} \mid i < \nu^{+n+2} \rangle \), \( \langle B_{i\nu} \mid i < \nu^{+n+2} \rangle \) and a sequence \( \langle N_{i\nu} \mid i < \nu^{+n+2} \rangle \) of elementary submodels of \( H(\lambda_n^{+\omega+1}) \) such that

1. \(|M_{i\nu}| = \kappa_n^{+n+3}\),
2. \(M_{i\nu} \cap \kappa_n^{+n+3} \) is an ordinal above \( \eta \),
3. \(|B_{i\nu}| = \kappa_n^{+n+2}\),
4. \(B_{i\nu} \cap \kappa_n^{+n+2} \) is an ordinal above \( \eta \) of cofinality \( \nu^{+n+2} \),
5. \(|N_{i\nu}| = \nu^{+n+1}\),
6. \(N_{i\nu} \cap \nu^{+n+2} \) is an ordinal,
7. \(M_{i\nu} \in B_{i\nu} \in N_{i\nu} \), if \( i = 0 \) or \( i \) is a successor ordinal,
8. \(\langle M_{j\nu} \mid i \leq j \rangle, \langle B_{j\nu} \mid i \leq j \rangle, \langle N_{j\nu} \mid i \leq j \rangle \in M_{i+1\nu},\)
9. \(\langle M_{j\nu} \mid i \leq j \rangle, \langle B_{j\nu} \mid i \leq j \rangle, \langle N_{j\nu} \mid i \leq j \rangle \in B_{i+1\nu},\)
10. \(\langle M_{j\nu} \mid i \leq j \rangle, \langle B_{j\nu} \mid i \leq j \rangle, \langle N_{j\nu} \mid i \leq j \rangle \in N_{i+1\nu},\)
11. for each \( x \in [N_{i+1\nu} \cap \nu^{+n+3}] \leq \nu^{+n+1}, \mathcal{E}_{x\nu} \in N_{i+1\nu},\)
12. \(\nu^{+n+2}B_{i\nu} \subseteq B_{i\nu} \), if \( i = 0 \) or \( i \) is a successor ordinal,
13. \(\nu^{+n+1}N_{i\nu} \subseteq N_{i\nu} \), if \( i = 0 \) or \( i \) is a successor ordinal,
14. \(\kappa^{+n+1}M_{i\nu} \subseteq M_{i\nu} \), if \( i = 0 \) or \( i \) is a successor ordinal,
15. if \( \nu < \nu' \), then \( \langle M_{i\nu} \mid i < \nu^{+n+3} \rangle, \langle N_{i\nu} \mid i < \nu^{+n+3} \rangle \in M_{0\nu'} \cap B_{0\nu'} \cap N_{0\nu'} \).
The set of permitted types will be the set of all types of models $\mathcal{M}_\nu$, $\mathcal{B}_\nu$, with parameters ordinals bigger than $\kappa^{++}$ types of models $\mathcal{N}_\nu$ with parameters ordinals in $\nu^{++}$. Formally set

$$PT^\lambda_n = \{ tp_m(\mathcal{N}_\nu) \mid i < \nu^{++}, \ 2 < m < \omega \},$$

$$PT^{\kappa+1}_n = \{ tp_m(\mathcal{B}_\nu) \mid x \in [\nu^{++}, \nu^{++}], 2 < m < \omega \},$$

$$PT^{\kappa+2}_n = PT^\lambda_n \cup PT^{\kappa+1}_n \cup PT^{\kappa+2}_n.$$

Let us turn to the assignment functions $a$ of the level $n$ (the isomorphisms function between the suitable structures) for $\kappa^{++}$ and those of $\lambda_n$, and $b$ of the level $n$ for $\kappa^{+3}, \kappa^{+4}$ and those of $\kappa_n$.

We require that each model $A$ be in the domain of $a$ is of the form $A' \cap \kappa^{++}$, for some $A' \in \text{dom}(b)$. The rest of the requirements on $a$ are as in [2].

Turn to $b$. Let $A$ be in the domain of $b$. If $A$ has cardinality $\kappa^{+3}$, then $b(A)$ is a name of a model with type in $PT^{\kappa+1}_n$.

If $A$ has cardinality $\kappa^{++}$, then $b(A)$ is a name of a model with type in $PT^{\kappa+2}_n$ depending on an indiscernible $\nu$ for the normal measure of $E^\lambda_n$.

If $A$ has cardinality $\kappa^+$, then $a(A) \cap \lambda^{++}$ is an ordinal and $b(A)$ is a name of a model with type as those of $\mathcal{N}_\nu$, where $\nu$ is an indiscernible for the normal measure and $i$ is the indiscernible for the measure $a(A) \cap \lambda^{++}$ of $E^\lambda_n$.

The rest of the requirements are as in [2].

### 3 General case.

The treatment is similar to those used in Gap 4 case. We are free to choose a point of splitting between cardinals that go to $\lambda_n$’s and to $\kappa_n$’s as it was done in 2.1, 2.2.

### References

[1] M. Gitik, Short extenders forcings I,

http://www.math.tau.ac.il/~gitik/short%20extenders%20forcings%201.pdf