Short extenders forcings – doing without preparations. Dropping cofinalities.

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The basic issue with dropping cofinalities is that models of small sizes relatively to κ_n's are supposed to be used (basically much less than κ_n's). The number of possible types inside such models is limited. Even not every measure of the extender over κ_n is in a model. So we will need to specify in advanced which types are allowed. Let us start with choosing a set of permitted types.

1 Dropping cofinalities–gap 3.

We deal here with the first relevant case– $2^\kappa = \kappa^{+3}$ with the witnessing scale has points of cofinality κ^{++} dropping down from κ_n's to smaller λ_n's.
Fix $n < \omega$. Let us define models that will be permitted to use over κ_n in order to allow a cofinality drop to λ_n, where $\lambda_0 < \kappa_0$ and $\kappa_{n-1} < \lambda_n < \kappa_n$, for every $n, 0 < n < \omega$, and λ_n, κ_n carry extenders $E_n^{\lambda_n}, E_n^{\kappa_n}$.
We deal with a simplest case of a single drop. Assume that the length of $E_n^{\kappa_n}$ is κ_n^{+n+2} and $E_n^{\lambda_n}$ is λ_n^{+n+2}

Fix some χ_n large enough. Let $\eta < \kappa_n^{+n+2}$ be such that every type of an ordinal $< \kappa_n^{+n+2}$ is realized below η and for every $\xi \geq \eta$ the type $tp_m(\xi)$ is realized unboundedly often below κ_n^{+n+2}, for each $m < \omega$.
Define by induction for every $\nu < \lambda_n$ two $\in$–increasing continuous sequences $\langle \mathfrak{M}_{i\nu} \mid i < \nu^{+n+2}\rangle$, $\langle \mathfrak{N}_{i\nu} \mid i < \nu^{+n+2}\rangle$ of elementary submodels of $H(\chi_n^{+\omega+1})$ such that

1. $|\mathfrak{M}_{i\nu}| = \kappa_n^{+n+1}$,
2. $\mathfrak{M}_{i\nu} \cap \kappa_n^{+n+2}$ is an ordinal above η of cofinality ν^{+n+2},
3. $|\mathfrak{N}_{i\nu}| = \nu^{+n+1}$,

4. $\mathfrak{M}_{i\nu} \in \mathfrak{N}_{i\nu}$, if $i = 0$ or i is a successor ordinal,

5. $\langle \mathfrak{M}_{j\nu} \mid j \leq i \rangle, \langle \mathfrak{N}_{j\nu} \mid j \leq i \rangle \in \mathfrak{M}_{i+1\nu}$,

6. $\langle \mathfrak{M}_{j\nu} \mid j \leq i \rangle, \langle \mathfrak{N}_{j\nu} \mid j \leq i \rangle \in \mathfrak{N}_{i+1\nu}$,

7. ${}^{\nu^{+n+1}}\mathfrak{M}_{i\nu} \subseteq \mathfrak{M}_{i\nu}$, if $i = 0$ or i is a successor ordinal,

8. ${}^{\nu^{+n}}\mathfrak{N}_{i\nu} \subseteq \mathfrak{N}_{i\nu}$, if $i = 0$ or i is a successor ordinal,

9. if $\nu < \nu'$, then $\langle \mathfrak{M}_{i\nu} \mid i < \nu^{+n+2} \rangle, \langle \mathfrak{N}_{i\nu} \mid i < \nu^{+n+2} \rangle \in \mathfrak{M}_{0\nu'} \cap \mathfrak{N}_{0\nu'}$.

The set of permitted types will be the set of all types of models $\mathfrak{M}_{i\nu}, \mathfrak{N}_{i\nu}$. Formally set

$$PT^{\kappa_n}_\nu = \{tp_m(\mathfrak{M}_{i\nu}) \mid i < \nu^{+n+2}, 2 < m < \omega\}, PT^{\lambda_n}_\nu = \{tp_m(\mathfrak{N}_{i\nu}) \mid i < \nu^{+n+2}, 2 < m < \omega\},$$

$$PT_\nu = PT^{\kappa_n}_\nu \cup PT^{\lambda_n}_\nu .$$

The idea behind the above is that once ν is an indiscernible (a member of one element Prikry sequence) for the normal measure of $E^{\lambda_n}_n$, then models with types in PT_ν are allowed to be used over κ_n.
Note that types of models $\mathfrak{N}_{i\nu}$'s are inside $\mathfrak{M}_{i\nu}$ by the choice of η and the item (2).

Let us turn to the assignment functions a of the level n (the isomorphisms function between the suitable structures) for κ^{++} and those of λ_n, and b of the level n for κ^{+3} and those of κ_n.
We require that each model A be in the domain of a is of the form $A' \cap \kappa^{++}$, for some $A' \in \mathrm{dom}(b)$. The rest of the requirements on a are as in [2].
Turn to b. Let A be in the domain of b. If A has cardinality κ^{++}, then $b(A)$ is a name of a model with type in $PT^{\kappa_n}_\nu$ depending on an indiscernible ν for the normal measure of $E^{\lambda_n}_n$. If A has cardinality κ^{+}, then $a(A) \cap \lambda_n^{+n+2}$ is an ordinal and $b(A)$ is a name of a model with type as those of $\mathfrak{N}_{i\nu}$, where ν is an indiscernible for the normal measure and i is the indiscernible for the measure $a(A) \cap \lambda_n^{+n+2}$ of $E^{\lambda_n}_n$. Again, the rest of the requirements are as in [2].

Lemma 1.1 *The forcing $\mathcal{P}$ is κ^+–proper (and even κ^+–strongly proper).*

Proof. Let $p \in \mathcal{P}$ and $M \prec H(\chi)$ with $|M| = \kappa^+$, ${}^{\kappa}M \subseteq M$, $p, \mathcal{P} \in M$. $a_n(M \cap \kappa^{++})$ is some $\alpha < \lambda_n^{+n+2}$. Run the corresponding argument of [2]. We will get finally some $\beta < \alpha$ that corresponds to the part of the extension which belongs to M. Now we will have that on a set of measure one $\beta^* \in \alpha^*$, where β^* denotes an indiscernible for β and α^* denotes an indiscernible for α. Then $\mathfrak{N}_{\beta^*\nu} \in \mathfrak{N}_{\alpha^*\nu}$, where ν is an indiscernible for the normal measure. Hence we have no problem in getting the needed type inside $b(M \cap \kappa^{+3})$.

$\square$

The argument of the next lemma is as those of [2], since models of big cardinality (κ_n^{+n+1}) are used here.

Lemma 1.2 *The forcing $\mathcal{P}$ is κ^{++}–proper (and even κ^{++}–strongly proper).*

2 Dropping cofinalities–gap 4.

We like to blow up the power of κ to κ^{+4} with drops in cofinalities.
Split into two cases according to places of drops.

2.1 κ^{+3} drops down to λ_n's.

We deal here with the case – $2^\kappa = \kappa^{+4}$ and the witnessing scale has points of cofinality κ^{+3} dropping down from κ_n's to smaller λ_n's.
The main difference (related to the dropping cofinality) here from the previous section is that there are two sizes κ^+ and κ^{++} of models witnessing the drop. Their images to κ_n's has sizes below λ_n. The issue of having enough types inside such models becomes a bit more delicate.

Fix $n < \omega$. Let $\lambda_n < \kappa_n, \eta < \kappa_n^{+n+2}$ be as above. The length of the extender $E_n^{\lambda_n}$ will be now λ_n^{+n+3} in order to accommodate three cardinals κ^+, κ^{++} and κ^{+3}. The assignment function a will act between κ^{+3} and λ_n^{+n+3}.

Define by induction for every $\nu < \lambda_n$ two $\in$–increasing continuous sequences $\langle \mathfrak{M}_{i\nu} \mid i < \nu^{+n+3}\rangle$, $\langle \mathfrak{N}_{i\nu} \mid i < \nu^{+n+3}\rangle$ and a sequence $\langle \mathfrak{S}_{x\nu} \mid x \in [\nu^{+n+3}]^{\leq \nu^{+n+1}}\rangle$ of elementary submodels of $H(\chi_n^{+\omega+1})$ such that

1. $|\mathfrak{M}_{i\nu}| = \kappa_n^{+n+1}$,
2. $\mathfrak{M}_{i\nu} \cap \kappa_n^{+n+2}$ is an ordinal above η of cofinality ν^{+n+3},
3. $|\mathfrak{N}_{i\nu}| = \nu^{+n+2}$,
4. $\mathfrak{N}_{i\nu} \cap \nu^{+n+3}$ is an ordinal,
5. $|\mathfrak{S}_{x\nu}| = \nu^{+n+1}$,

6. $\mathfrak{S}_{x\nu} \cap \nu^{+n+2}$ is an ordinal,

7. $\mathfrak{M}_{i\nu} \in \mathfrak{N}_{i\nu}$, if $i = 0$ or i is a successor ordinal,

8. $\langle \mathfrak{M}_{j\nu} \mid j \leq i \rangle, \langle \mathfrak{N}_{j\nu} \mid j \leq i \rangle \in \mathfrak{M}_{i+1\nu}$,

9. $\langle \mathfrak{M}_{j\nu} \mid j \leq i \rangle, \langle \mathfrak{N}_{j\nu} \mid j \leq i \rangle \in \mathfrak{N}_{i+1\nu}$,

10. for each $x \in [\mathfrak{N}_{i+1\nu} \cap \nu^{+n+3}]^{\leq \nu^{+n+1}}, \mathfrak{S}_{x\nu} \in \mathfrak{N}_{i+1\nu}$,

11. ${}^{\nu^{+n+2}}\mathfrak{M}_{i\nu} \subseteq \mathfrak{M}_{i\nu}$, if $i = 0$ or i is a successor ordinal,

12. ${}^{\nu^{+n+1}}\mathfrak{N}_{i\nu} \subseteq \mathfrak{N}_{i\nu}$, if $i = 0$ or i is a successor ordinal,

13. if $i, \mathfrak{N}_{i\nu} \cap \nu^{+n+3} \in x$, then $\mathfrak{M}_{i\nu}, \mathfrak{N}_{i\nu} \in \mathfrak{S}_{x\nu}$,

14. if $y \in x$, then $\mathfrak{S}_{y\nu} \in \mathfrak{S}_{x\nu}$,

15. if $\nu < \nu'$, then $\langle \mathfrak{M}_{i\nu} \mid i < \nu^{+n+3} \rangle, \langle \mathfrak{N}_{i\nu} \mid i < \nu^{+n+3} \rangle \in \mathfrak{M}_{0\nu'} \cap \mathfrak{N}_{0\nu'} \cap \mathfrak{S}_{\emptyset\nu'}$.

The set of permitted types will be the set of all types of models $\mathfrak{M}_{i\nu}$, with parameters ordinals bigger than κ_n^{++} types of models $\mathfrak{N}_{i\nu}, \mathfrak{S}_{x\nu}$ with parameters ordinals in ν^{+n+1} and ν^{+n} respectively. Formally set

$$PT_\nu^{\kappa_n} = \{tp_m(\mathfrak{M}_{i\nu}) \mid i < \nu^{+n+3}, 2 < m < \omega\}, PT_\nu^{\lambda_n,2} = \{tp_m(\mathfrak{N}_{i\nu}) \mid i < \nu^{+n+3}, 2 < m < \omega\},$$

$$PT_\nu^{\lambda_n,1} = \{tp_m(\mathfrak{S}_{x\nu}) \mid x \in [\nu^{+n+3}]^{\leq \nu^{+n+1}}, 2 < m < \omega\}, PT_\nu = PT_\nu^{\kappa_n} \cup PT_\nu^{\lambda_n,1} \cup PT_\nu^{\lambda_n,2}.$$

Let us turn to the assignment functions a of the level n (the isomorphisms function between the suitable structures) for κ^{+3} and those of λ_n, and b of the level n for κ^{+4} and those of κ_n.
We require that each model A be in the domain of a is of the form $A' \cap \kappa^{+3}$, for some $A' \in \mathrm{dom}(b)$. The rest of the requirements on a are as in [2].
Turn to b. Let A be in the domain of b. If A has cardinality κ^{+3}, then $b(A)$ is a name of a model with type in $PT_\nu^{\kappa_n}$ depending on an indiscernible ν for the normal measure of $E_n^{\lambda_n}$.
If A has cardinality κ^{++}, then $a(A) \cap \lambda_n^{+n+3}$ is an ordinal and $b(A)$ is a name of a model with type as those of $\mathfrak{N}_{i\nu}$, where ν is an indiscernible for the normal measure and i is the indiscernible for the measure $a(A) \cap \lambda_n^{+n+3}$ of $E_n^{\lambda_n}$. The rest of the requirements are as in [2].
If A has cardinality κ^+, then $a(A) \cap \lambda_n^{+n+3}$ is a set of cardinality λ_n^{+n+1} and $b(A)$ is a name of a model with type as those of $\mathfrak{S}_{x\nu}$, where ν is an indiscernible for the normal measure and $x \in [\nu^{+n+3}]^{\leq \nu^{+n+1}}$ is the indiscernible for the measure $a(A) \cap \lambda_n^{+n+3}$ of $E_n^{\lambda_n}$. Again, the rest of the requirements are as in [2].

2.2 κ^{+3} does not drop down to λ_n's.

We deal here with the case – $2^\kappa = \kappa^{+4}$ and the witnessing scale has points of cofinality κ^{++} dropping down from κ_n's to smaller λ_n's, but those of cofinality κ^{+3} do not drop down. Here only models of the size κ^+ will witness the drop. Their images to κ_n's will have sizes below λ_n.

Fix $n < \omega$. Let $\lambda_n < \kappa_n, \eta < \kappa_n^{+n+2}$ be as above. The length of the extender $E_n^{\lambda_n}$ will be now λ_n^{+n+2} and of $E_n^{\kappa_n}$ will be κ_n^{+n+3}. The assignment function a will act between κ^{++} and λ_n^{+n+2}.

Define by induction for every $\nu < \lambda_n$ two $\in$–increasing continuous sequences $\langle \mathfrak{M}_{i\nu} \mid i < \nu^{+n+2}\rangle$, $\langle \mathfrak{B}_{i\nu} \mid i < \nu^{+n+2}\rangle$ and a sequence $\langle \mathfrak{N}_{i\nu} \mid i < \nu^{+n+2}\rangle$ of elementary submodels of $H(\chi_n^{+\omega+1})$ such that

1. $|\mathfrak{M}_{i\nu}| = \kappa_n^{+n+3}$,
2. $\mathfrak{M}_{i\nu} \cap \kappa_n^{+n+3}$ is an ordinal above η,
3. $|\mathfrak{B}_{i\nu}| = \kappa_n^{+n+2}$,
4. $\mathfrak{B}_{i\nu} \cap \kappa_n^{+n+2}$ is an ordinal above η of cofinality ν^{+n+2},
5. $|\mathfrak{N}_{i\nu}| = \nu^{+n+1}$,
6. $\mathfrak{N}_{i\nu} \cap \nu^{+n+2}$ is an ordinal,
7. $\mathfrak{M}_{i\nu} \in \mathfrak{B}_{i\nu} \in \mathfrak{N}_{i\nu}$, if $i = 0$ or i is a successor ordinal,
8. $\langle \mathfrak{M}_{j\nu} \mid j \leq i\rangle, \langle \mathfrak{B}_{j\nu} \mid j \leq i\rangle, \langle \mathfrak{N}_{j\nu} \mid j \leq i\rangle \in \mathfrak{M}_{i+1\nu}$,
9. $\langle \mathfrak{M}_{j\nu} \mid j \leq i\rangle, \langle \mathfrak{B}_{j\nu} \mid j \leq i\rangle, \langle \mathfrak{N}_{j\nu} \mid j \leq i\rangle \in \mathfrak{B}_{i+1\nu}$,
10. $\langle \mathfrak{M}_{j\nu} \mid j \leq i\rangle, \langle \mathfrak{B}_{j\nu} \mid j \leq i\rangle, \langle \mathfrak{N}_{j\nu} \mid j \leq i\rangle \in \mathfrak{N}_{i+1\nu}$,
11. for each $x \in [\mathfrak{N}_{i+1\nu} \cap \nu^{+n+3}]^{\leq \nu^{+n+1}}$, $\mathfrak{S}_{x\nu} \in \mathfrak{N}_{i+1\nu}$,
12. ${}^{\nu^{+n+2}}\mathfrak{B}_{i\nu} \subseteq \mathfrak{B}_{i\nu}$, if $i = 0$ or i is a successor ordinal,
13. ${}^{\nu^{+n+1}}\mathfrak{N}_{i\nu} \subseteq \mathfrak{N}_{i\nu}$, if $i = 0$ or i is a successor ordinal,
14. ${}^{\kappa^{+n+1}}\mathfrak{M}_{i\nu} \subseteq \mathfrak{M}_{i\nu}$, if $i = 0$ or i is a successor ordinal,
15. if $\nu < \nu'$, then $\langle \mathfrak{M}_{i\nu} \mid i < \nu^{+n+3}\rangle, \langle \mathfrak{N}_{i\nu} \mid i < \nu^{+n+3}\rangle \in \mathfrak{M}_{0\nu'} \cap \mathfrak{B}_{0\nu'} \cap \mathfrak{N}_{0\nu'}$.

The set of permitted types will be the set of all types of models $\mathfrak{M}_{i\nu}$, $\mathfrak{B}_{i\nu}$, with parameters ordinals bigger than κ_n^{++} types of models $\mathfrak{N}_{i\nu}$ with parameters ordinals in ν^{+n}. Formally set

$$PT_\nu^{\lambda_n} = \{tp_m(\mathfrak{N}_{i\nu}) \mid i < \nu^{+n+2}, 2 < m < \omega\}, PT_\nu^{\kappa_n,2} = \{tp_m(\mathfrak{M}_{i\nu}) \mid i < \nu^{+n+2}, 2 < m < \omega\},$$

$$PT_\nu^{\kappa_n,1} = \{tp_m(\mathfrak{B}_{x\nu}) \mid x \in [\nu^{+n+2}]^{\leq \nu^{+n+1}}, 2 < m < \omega\}, PT_\nu = PT_\nu^{\lambda_n} \cup PT_\nu^{\kappa_n,1} \cup PT_\nu^{\kappa_n,2}.$$

Let us turn to the assignment functions a of the level n (the isomorphisms function between the suitable structures) for κ^{++} and those of λ_n, and b of the level n for κ^{+3}, κ^{+4} and those of κ_n.

We require that each model A be in the domain of a is of the form $A' \cap \kappa^{++}$, for some $A' \in \mathrm{dom}(b)$. The rest of the requirements on a are as in [2].

Turn to b. Let A be in the domain of b. If A has cardinality κ^{+3}, then $b(A)$ is a name of a model with type in $PT_\nu^{\kappa_n,2}$.

If A has cardinality κ^{++}, then $b(A)$ is a name of a model with type in $PT_\nu^{\kappa_n,1}$ depending on an indiscernible ν for the normal measure of $E_n^{\lambda_n}$.

If A has cardinality κ^+, then $a(A) \cap \lambda_n^{+n+2}$ is an ordinal and $b(A)$ is a name of a model with type as those of $\mathfrak{N}_{i\nu}$, where ν is an indiscernible for the normal measure and i is the indiscernible for the measure $a(A) \cap \lambda_n^{+n+2}$ of $E_n^{\lambda_n}$.

The rest of the requirements are as in [2].

3 General case.

The treatment is similar to those used in Gap 4 case. We are free to choose a point of splitting between cardinals that go to λ_n's and to κ_n's as it was done in 2.1, 2.2.

References

[1] M. Gitik, Short extenders forcings I, http://www.math.tau.ac.il/~gitik/short%20extenders%20forcings%201.pdf

[2] M. Gitik, Short extenders forcings–doing without preparations.