Short extenders forcings – doing without preparations.

Moti Gitik*

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Abstract

We introduce certain morass type structures and apply them to blowing up powers of singular cardinals. As a bonus, a forcing for adding clubs with finite conditions to higher cardinals is obtained.

1 Introduction.

We would like to present a way of doing of short extenders forcings without forcing first with a preparation forcings of type $\mathcal{P}'$ of [3]. The main issue with short extenders forcings is to show that κ^{++} and cardinals above it are preserved in the final model. In [3] the preparation forcing (which added a structure with pistes) was used eventually to show κ^{++}-c.c. of the main forcing. A negative side of this preparation forcing is that it is only strategically closed which is not enough in order to preserve large cardinals like a supercompact. Actually it adds a version of the square principle which is incompatible with supercompacts [4].
Carmi Merimovich [9] used for the gap 3 a variation of Velleman's simplified morass [13] instead. κ^{++}-c.c. break down but he was able to show κ^{++}–properness instead. The forcing adding a simplified morass is directed closed enough in order to preserve supercompacts cardinals. Unfortunately generalizations (at least those that we considered) of Merimovich's idea of first adding a simplified morass and then to use a properness instead of a chain condition of the main forcing, run into server difficulties already for Gap 4.

Here we suggest an other way. The main forcing will be used directly over V without a preparation. Actually a simple version of the preparation forcing of [3] will be incorporated directly into the main forcing. Again as in [9] κ^{++}-c.c. will break down and we will show a properness instead.

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In this paper we will deal with a general situation - no bounds on a gap between a singular cardinal and its power. The situation where the gap between a singular cardinal κ and its power is bounded by $\kappa^{+\kappa^+} = \aleph_{\kappa^+}$ is considered in [6]. The arguments there are slightly easier, but not essentially.

The main instrument introduced here is called *structures with pistes*. It seems of an interest by its own. Beyond cardinal arithmetic applications stated above, it is applied in a further paper to a certain generalizations of Forcing Axioms to higher cardinals. Here we will apply it to adding clubs by finite conditions.

The paper is organized as follows.

In the first section we introduce a $\delta-$structure with pistes over η of the length θ. Basic properties of such structures like the intersection property, possible extensions etc. are studied here. A forcing with piste structures is introduced. Its properness is proved. An application to adding clubs is given at the end of this section.

The second section is devoted to a cardinal arithmetic application. We show how using structures of this type it is possible to blow up the power of a singular cardinal.

2 Structures with pistes–general setting.

Assume GCH.

The basic idea behind the structures defined below (2.2,$\delta-$structure with pistes over η of the length θ) is to stay as close as possible to an elementary chain of models. It cannot be literally a chain since models of different sizes are involved and not models of bigger cardinality can come before ones of a smaller. The first part (2.1) describes this "linear" part of conditions in the main forcing. It is called *a wide piste* and incorporates together elementary chains of models of different cardinalities. The main forcing, defined in 2.2, will be based on such wide pistes and involves an additional natural but non-linear component called splitting or reflection.

Definition 2.1 Let $\delta \leq \eta < \theta$ be regular cardinals.
A $(\theta, \eta, \delta)-$wide piste is a set $\langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle$ such that the following hold.[1]
Let us first specify sizes of models that are involved.

1. (Support) s is a closed set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:

   (a) $|s| < \delta$,

   (b) $\eta, \theta \in s$.
   Which means that the minimal and the maximal possible sizes are always present.

2. (Models) For every $\tau \in s$ and $A \in C^\tau$ the following holds:

   (a) $A \preccurlyeq \langle H(\theta^+), \in, \leq, \delta, \eta\rangle$,

   (b) $|A| = \tau$,

   (c) $A \supseteq \tau$,

   (d) $A \cap \tau^+$ is an ordinal,

   (e) elements of C^τ form a closed $\in -$chain with a largest element of a length $< \delta$,

   (f) if $X \in C^\tau \setminus C^{\tau lim}$ is a non-limit model (i.e. is not a union of elements of C^τ), then ${}^{\tau>}X \subseteq X$.

[1] The main application will be to the case when $\eta = \kappa^+$ for a cardinal κ which is an $\omega-$limit of strong enough (but not overlapping κ) cardinals. An other application is to forcing axioms, and for it we use $\delta = \eta = \omega$.

(g) if $X, Y \in C^\tau$ then $X \in Y$ iff $X \subsetneq Y$,

3. (Potentially limit points) Let $\tau \in s$.
$C^{\tau lim} \subseteq C^\tau$. We refer to its elements as *potentially limit points.*
The intuition behind is that once extending it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.
Let $X \in C^{\tau lim}$. Require the following:

(a) X is a successor point of C^τ.

(b) (Increasing union) There is an increasing continuous $\in$ −chain $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ [2] of elementary submodels of X such that

i. $\bigcup_{i<\mathrm{cof}(\sup(X\cap\theta^+))} X_i = X$,

ii. $|X_i| = \tau$,

iii. $X_i \supseteq \tau$,

iv. $X_i \in X$,

v. ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

(c) (Degree of closure of potentially limit point)
Either

i. ${}^{\tau>}X \subseteq X$
or

ii. $\mathrm{cof}(\sup(X \cap \theta^+)) = \xi$ for some $\xi \in s \cap \tau$ and then

A. ${}^{\xi>}X \subseteq X$,

B. there are $X_\theta \in C^{\theta lim}, X_\xi \in C^{\xi lim}$ such that $X \cap \theta^+ = \sup(X_\xi \cap \theta^+) = \sup(X \cap \theta^+)$ and there is a sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X\cap\theta^+))\rangle$ witnessing 3(b) which members belong to X_ξ.
Further the condition (9(b)) will imply that $X' \supseteq X \supseteq X''$. Eventually (once extending) for every regular $\mu, \tau \leq \mu \leq \theta$ there will be $X''' \in C^{\mu lim}, X \subseteq X''' \subseteq X'$.

Note that if $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ and $\langle X'_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ are two sequences which witness (3b) above, then the set $\{i < \mathrm{cof}(\sup(X\cap\theta^+)) \mid X_i = X'_i\}$ is closed and unbounded.

[2]This models need not be in C^τ, but rather allow to add in future extensions models below X

It is possible using the well ordering $\leq$ to define a canonical witnessing sequence $\langle X_i \mid i < \mathrm{cof}(X \cap \theta^+)\rangle$ for X.

Let first do this for X such that $\mathrm{cof}(X \cap \theta^+) = \tau$ (or for X_ξ of (3c(ii)(B)) above). Fix the well ordering $\langle x_\nu \mid \nu < \tau\rangle$. We proceed by induction. Once $i < \tau$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least elementary submodel of X such that

- $x_i \in X_{i+1}$,
- $X_i \in X_{i+1}$,
- $|X_i| = \tau$,
- $X_i \supseteq \tau$,
- ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

By (3b), it is possible to find such X_{i+1}.

Clearly $\bigcup_{i<\tau} X_i = X$.

Suppose now that $\mathrm{cof}(X \cap \theta^+) = \xi \in s \cap \tau$. Then let us use the canonical sequence $\langle X_{i\xi} \mid i < \xi = \mathrm{cof}(X \cap \theta^+)\rangle$ for X_ξ in order to define the canonical sequence $\langle X_i \mid i < \mathrm{cof}(X \cap \theta^+)\rangle$ for X.

Proceed by induction. Once $i < \tau$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least elementary submodel of $H(\theta)$ such that

- $X_{i+1} \in X_\xi$,
- $X_{i\xi} \in X_{i+1}$,
- $X_i \in X_{i+1}$,
- $|X_i| = \xi$,
- $X_i \supseteq \xi$,
- ${}^{\xi>}X_{i+1} \subseteq X_{i+1}$.

By (3c(ii)B), it is possible to find such X_{i+1} inside X_ξ.

Note that the existence of such canonical sequences implies that X itself is definable from X_ξ.

The next condition prevent unneeded appearances of small models between big ones.

4. If $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point and B_0 is its immediate predecessor, then there is no potentially limit point $A \in C^\tau$ with $\tau < \rho$ such that $B_0 \in A \in B_1$.

It is possible to require that no A at all, i.e. potentially limit or not, appears between B_0 and B_1. The requirement that B_1 is not a potentially limit point is important here. Once dealing with potentially limit points, we would like to allow reflections which may add small intermediate models.

Next condition is of a similar flavor, but deals with smallest models.

5. If $B \in C^\rho$, for some $\rho \in s$, is not a potentially limit point and it is the least element of C^ρ, then there is no potentially limit point $A \in C^\tau$ with $\tau > \rho$ such that $A \in B$[3].

   Both conditions 4 and 5 are desired to allow to add new models below potentially limit points which will be essential further for properness of the forcing.

   The next condition deals with with closure and is desired to prevent some pathological patterns.

6. Let $B \in C^\rho$, for some $\rho \in s$, be a non-limit point of C^ρ. If there are models $A \in \bigcup_{\xi \in s} C^\xi$ with $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then there is $A \in B \cap \bigcup_{\xi \in s} C^\xi$ such that

   (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$,

   (b) for every $A' \in \bigcup_{\xi \in s} C^\xi$ with $\sup(A' \cap \theta^+) < \sup(B \cap \theta^+)$, $\sup(A' \cap \theta^+) \leq \sup(A \cap \theta^+)$.

   Such A is the "real" immediate predecessor of B. Further, in the definition of the order, we will require that once B is not a potentially limit point, then no models E such that $A \in E \in B$ can be added.

   The purpose of the next two conditions is to allow to proceed down the pistes without interruptions at least before reaching a potentially limit point.

7. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^ρ, then $B' \in A$.

8. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is a limit point in C^ρ. Let $\langle B_\nu \mid \nu < \nu^* < \delta \rangle$ be $C^\rho \cap B$. Then a closed unbounded subsequence of $\langle B_\nu \mid \nu < \nu^* \rangle$ is in A.

9. (Linearity) If $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$, then

   (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ implies $A \in B$,

[3] If we drop the requirement $\tau > \rho$, then it may be impossible further to add models of sizes $> \eta$ once a potencially limit point of size η is around.

(b) $\sup(A\cap\theta^+)=\sup(B\cap\theta^+)$ implies $A\subseteq B$.

10. If $\tau,\rho\in s,\tau<\rho$, $A\in C^\tau, B\in C^\rho$, $\sup(A\cap\theta^+)>\sup(B\cap\theta^+)$ and $B\in A$, then for every $X\in\bigcup_{\mu\in s}C^\mu$, $\sup(X\cap\theta^+)=\sup(B\cap\theta^+)$ and $|X|\in A$ implies $X\in A$.

11. (Immediate successor restriction) Let $\tau,\rho\in s,\tau<\rho$, $A\in C^\tau, B\in C^{\rho lim}$,$\mathrm{cof}(\sup(B'\cap\theta^+))>\tau$ and $B\in A$. Suppose that there a model $B'\in B\cap C^\rho$ such that $\sup(B'\cap\theta^+)>\sup((A\cap B)\cap\theta^+)$, then the least such B' is a potentially limit model. I.e., if there is a model in C^ρ between $A\cap B$ and B, then the least such model is a potentially limit model.
It is designed to prevent the situation when there is $E\in A\cap C^\rho$ which has a non-potentially limit immediate successor E'' in B but not in A. Also it prevents a possibility that the least element Y of C^ρ is a non-potentially limit point which belongs to B is above $A\cap B$.
This condition is needed further for $\tau-$properness argument[4].

12. (Covering) If $\tau,\rho\in s,\tau<\rho$, $B\in C^\tau, D\in C^\rho$ and $\sup(B\cap\theta^+)>\sup(D\cap\theta^+)$, then there is $D^*\in B\cap C^{\rho^*}$ such that $D^*\supseteq D$[5], where $\rho^*=\min((B\setminus\rho)\cap Regular)$, i.e. the least regular cardinal in the interval $[\rho,\theta]$ which belongs to B. In particular, $\rho^*\in s$[6].

    The last condition describes a very particular way of covering and it is crucial for the properness arguments.

13. (Strong covering) Let $B\in C^\tau$, $D\in C^\rho,\rho>\tau$ and $\sup(D\cap\theta^+)<\sup(B\cap\theta^+)$. Then either

    (a) $D\in B$,
    or

    (b) $D\notin B$ and the least $D^*\in C^{\rho^*}\cap B, D^*\supset D$ is closed under $<\rho^*-$ sequence of its elements, where $\rho^*=\min((B\setminus\rho)\cap Regular)$. Then $B\cap D^*\subseteq D$ and

$$\{D'\in D^*\mid(|D'|=\rho^*)\wedge(\exists n<\omega)(\exists Z_{n-1}\in...\in Z_0\in B)$$

[4] In an earlier version of the paper, we defined a model $B_A:=\bigcup_{i\in A\cap\mathrm{cof}(\sup(B\cap\theta^+))}B_i$ (where $\langle B_i\mid i<\mathrm{cof}(\sup(B\cap\theta^+))\rangle$ is a chain which witnesses (3(b)) and added it to $C^{\rho lim}$. Having such B_A in $C^{\rho lim}$ implies impossibility of the situations above. Here we do without B_A and this simplifies the major arguments like intersection properties and properness. However getting a club that runs away from sets in V becomes a bit more complicated.

[5] Note that the least such D^* must be a potentially limit point by 7, 8 above.

[6] Note that the set $Z:=\{\mu\leq\theta\mid\mu$ is a regular cardinal $\}$ belongs to B, by elementarity. If its cardinality is at most τ, then $Z\subseteq B$. So, in this case $\rho^*=\rho$.

$$((\forall k < n)(|Z_k| < \rho^*)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D^7.$$

Or

(c) $D \notin B$ and the least $D^* \in C^\rho \cap B, D^* \supset D$ is not closed under $< \rho-$ sequence of its elements.

Let $\mathrm{cof}(\sup(D^* \cap \theta^+)) = \xi$ for some $\xi \in s \cap \rho$ and let $E \in C^{\xi lim}$ such that $\sup(E \cap \theta^+) = \sup(D^* \cap \theta^+)$ (such E exists by 3c(b) and $E \in B$ by 10, since $D^* \in B$).

Then either

i. $D \in E$, $B \cap D^* \subseteq D$ and

$$\{D' \in D^* \mid (|D'| \leq \rho^*) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$

$$((\forall k < n)(|Z_k| < \xi)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D.$$

ii. $D \notin E$, and then, let be the least $D^{**} \in C^{\rho^{**}} \cap E$ with $D^{**} \supset D$, where $\rho^{**} = \min((E \setminus \rho) \cap Regular)$. If D^{**} is closed under $< \rho^{**}-$ sequence of its elements, then $B \cap D^* \subseteq D, E \cap D^{**} \subseteq D$ and

$$\{D' \in D^{**} \mid (|D'| \leq \rho^{**}) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$

$$((\forall k < n)(|Z_k| < \rho^{**})) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D$$

If D^{**} is not closed under $< \rho^{**}-$ sequence of its elements, then the process repeats itself going down below D^{**}. After finitely many steps we will either reach D or D will be above everything related to B. Let us state this formally. So suppose that D^{**} is not closed under $< \rho-$ sequence of its elements.

Then are $n^* < \omega, \{\xi_n \mid n \leq n^*\} \subseteq s \setminus \eta + 1, \langle E_n \mid n \leq n^* \rangle, \langle D_n \mid n \leq n^* \rangle$ such that for every $n \leq n^*$ the following hold:

A. $D_0 = D^*$,

B. $E_0 = E$,

C. $\rho_0 = \rho^*$,

D. $D_n \in C^{\rho_n}$,

[7] Note that GCH is assumed, so the cardinality of this set is less than ρ. Then it is in D^*, once D^* is closed under $< \rho-$sequences of its elements.

E. $D_n \supseteq D$,

F. $D_{n+1} \in D_n$,

G. $\mathrm{cof}(\sup(D_n \cap \theta^+)) = \xi_n$,

H. $E_n \in C^{\xi_n}$,

I. $\sup(D_n \cap \theta^+) = \sup(E_n \cap \theta^+)$,

J. $D_{n+1} \in E_n$ is the least in $C^{\rho_{n+1}} \cap E_n$ with $D_{n+1} \supset D$ and $\rho_{n+1} = \min((E_n \setminus \rho) \cap Regular)$.

K. $B \cap D_0 \subseteq D$,

L. $E_n \cap D_{n+1} \subseteq D$,

M. $\{D' \in D_{n+1} \mid (|D'| = \rho_{n+1}) \wedge (\exists m < \omega)(\exists Z_{m-1} \in ... \in Z_0 \in B)$ $((\forall k < m)(|Z_k| < \xi_n)) \wedge D' \in B \cup \bigcup_{k<m} Z_k))\} \in D$,

N. $D_{n^*} = D$ or, we have, $D \in D_{n^*}, {}^{\rho_{n^*}} D_{n^*} \subseteq D_{n^*}$,

$$\{D' \in D_{n^*} \mid (|D'| = \rho_{n^*}) \wedge (\exists m < \omega)(\exists Z_{m-1} \in ... \in Z_0 \in B)$$

$$((\forall k < m)(|Z_k| < \rho_{n^*})) \wedge D' \in B \cup \bigcup_{k<m} Z_k))\} \in D.$$

14. (An addition to the strong covering condition) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$ and $\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Suppose that there is $X \in C^\theta$ with $\sup(B \cap \theta^+) = X \cap \theta^+$. Then either

    (a) $D \in B$,
    or

    (b) $D \notin B$ and (b),(c) of (13) hold with B replaced by any model $Y, B \subseteq Y \subseteq X$ of a regular cardinality $\mu \in s, \tau < \mu < \rho$ which is definable in $\langle H(\theta^+), \in, \leq, \delta, \eta \rangle$ with parameters from the set $B \cup (\mu + 1) \cup \{B\}$[8].

Now we are ready to give the main definition.

Definition 2.2 Let $\delta \leq \eta < \theta$ be regular cardinals.
$\delta-$structure with pistes over η of the length θ is a set $\langle \langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau \rangle \mid \tau \in s \rangle$ such

[8]Note that the total number of such Y's for a fixed regular $\mu \in s, \tau < \mu < \rho$ is $|B| = \tau$. Hence, there are less than ρ possibilities for Y's. Also, note that the model X is definable from B, as it was observed above in (3)

that the following hold.[9]

Let us first specify sizes of models that are involved.

1. (Support) s is a closed set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:

   (a) $|s| < \delta$,

   (b) $\eta, \theta \in s$.
   Which means that the minimal and the maximal possible sizes are always present.

2. (Models) For every $\tau \in s$ the following holds:

   (a) $A^{0\tau} \preccurlyeq \langle H(\theta^+), \in, \leq, \delta, \eta \rangle$,

   (b) $|A^{0\tau}| = \tau$,

   (c) $A^{0\tau} \in A^{1\tau}$,

   (d) $A^{1\tau}$ is a set of less than δ elementary submodels of $A^{0\tau}$,

   (e) each element A of $A^{1\tau}$ has cardinality τ, $A \supseteq \tau$ and $A \cap \tau^+$ is an ordinal and it is above the number of cardinals in the interval $[\eta, \theta]$.

3. (Potentially limit points) Let $\tau \in s$.
   $A^{1\tau lim} \subseteq A^{1\tau}$. We refer to its elements as *potentially limit points*.
   The intuition behind is that once extending it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.

4. (Piste function) The idea behind is to provide a canonical way to move from a model in the structure to one below.
   Let $\tau \in s$.
   Then, $\mathrm{dom}(C^\tau) = A^{1\tau}$ and
   for every $B \in \mathrm{dom}(C^\tau)$, $C^\tau(B)$ is a closed chain of models in $A^{1\tau} \cap (B \cup \{B\})$ such that the following holds:

   (a) $B \in C^\tau(B)$,

   (b) if $X \in C^\tau(B)$, then $C^\tau(X) = \{Y \in C^\tau(B) \mid Y \in X \cup \{X\}\}$,

[9] If $\delta = \omega$, then we call $\delta-$structure with pistes over η of the length θ just a finite structure with pistes over η of the length θ.

(c) if B has immediate predecessors in $A^{1\tau}$, then one (and only one) of them is in $C^\tau(B)$,

5. (Wide piste) The set
$$\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s\rangle$$
is a $(\theta, \eta, \delta)-$wide piste.

Next two condition describe the ways of splittings from wide pistes. This describes the structure of $A^{1\tau}$ and the way pistes allow to move from one of its models to an other.

6. (Splitting points) Let $\tau \in s$. Let $X \in A^{1\tau}$ be a non-limit model (but possibly a potentially limit), then either

   (a) X is a minimal under $\in$ or equivalently under $\supsetneq$,
   or

   (b) X has a unique immediate predecessor in $A^{1\tau}$,
   or

   (c) X has exactly two immediate predecessors X_0, X_1 in $A^{1\tau}$, non of X, X_0, X_1 is a limit or potentially limit points and X, X_0, X_1 form a $\Delta-$*system triple* relatively to some $F_0, F_1 \in A^{1\tau^* lim}$, for some $\tau^* \in s \setminus \tau + 1$[10], which means the following:

      i. $F_0 \subsetneq F_1$ and then $F_0 \in C^{\tau^*}(F_1)$, or $F_1 \subsetneq F_0$ and then $F_1 \in C^{\tau^*}(F_0)$,

      ii. ${}^{\tau^*>}F_0 \subseteq F_0$ and ${}^{\tau^*>}F_1 \subseteq F_1$,

      iii. $X_0 \in F_1$ (or $X_1 \in F_0$),

      iv. $F_0 \in X_0$ and $F_1 \in X_1$,

      v. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

      vi. ${}^{\tau>}X_0 \subseteq X_0$ and ${}^{\tau>}X_1 \subseteq X_1$,

      vii. the structures
      $$\langle X_0, \in, \langle X_0 \cap A^{1\rho}, X_0 \cap A^{1\rho lim}, (C^\rho \restriction X_0 \cap A^{1\rho}) \cap X_0 \mid \rho \in s \cap X_0\rangle\rangle$$
      and
      $$\langle X_1, \in, \langle X_1 \cap A^{1\rho}, X_1 \cap A^{1\rho lim}, (C^\rho \restriction X_1 \cap A^{1\rho}) \cap X_1 \mid \rho \in s \cap X_1\rangle\rangle$$
      are isomorphic over $X_0 \cap X_1$. Denote by π_{X_0, X_1} the corresponding isomorphism.

[10] If there are only finitely many cardinals between η and θ, then we can take τ^* to be just τ^+.

viii. $X \in A^{0\tau *}$.

Further we will refer to such X as *a splitting point.*

Or

(d) (Splitting points of higher order) There are $G, G_0, G_1 \in X \cap A^{1\mu}$, for some $\mu \in s \setminus \min(s \setminus \tau + 1)$, which form a Δ-system triple with witnessing models in X such that

i. $X_0 \in G_0$,

ii. $X_1 \in G_1$,

iii. $X_1 = \pi_{G_0 G_1}[X_0]$.

iv. X is not a limit or potentially limit point,

v. $X \in A^{0\mu}$,

vi. (Pistes go in the same direction) $G_i \in C^\mu(G) \Leftrightarrow X_i \in C^\tau(X), i < 2$.

Further we will refer to such X as *a splitting point of higher order.*

7. Let $\tau, \rho \in s$, $X \in A^{1\tau}, Y \in A^{1\rho}$. Suppose that X is a successor point, but not potentially limit point and $X \in Y$. Then all immediate predecessors of X are in Y, as well as the witnesses, i.e. F_0, F_1 if (6c) holds and G_0, G_1, G if (6d) holds.

8. Let $\tau \in s$. If $X \in A^{1\tau}$, $Y \in \bigcup_{\rho \in s} A^{1\rho}$ and $Y \in X$, then Y is *a piste reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n \rangle$ of elements of $A^{1\tau}$ which we call *a piste leading to* Y such that

   (a) $X = X(0)$,

   (b) for every $i, 0 < i \leq n$, $X(i) \in C^\tau(X(i-1))$ or $X(i-1)$ has two immediate successors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^\tau(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$ or $Y = X(i-1)_1$,

   (c) $Y = X(n)$, if $Y \in A^{1\tau}$ and if $Y \in A^{1\rho}$, for some $\rho \neq \tau$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\tau}$.

In particular, every $Y \in A^{1\tau}$ is piste reachable from $A^{0\tau}$.

In order formulate further requirement, we will need to describe a simple process of changing the wide pistes. This leads to equivalent forcing conditions once the order will be defined.

Let $X \in A^{1\tau}$. We will define $X-$wide piste. The definition will be by induction on number of turns (splits) needed in order to reach X by the piste from $A^{0\tau}$.

First, if $X \in C^\tau(A^{0\tau})$, then $X-$wide piste is just $\langle C^\xi(A^{0\xi}), C^\xi(A^{0\xi}) \cap A^{1\xi lim} \mid \xi \in s\rangle$, i.e. the wide piste of the structure.

Second, if $X \not\in C^\tau(A^{0\tau})$, but it is not a splitting point, then pick the least splitting point Y above X. Let Y_0, Y_1 be its immediate predecessors with $Y_0 \in C^\tau(Y)$. Then $X \in Y_i \cup \{Y_i\}$ for some $i < 2$. Set $X-$wide piste to be the Y_i-wide piste.

So, in order to complete the definition, it remain to deal with the following principle case:

$X \in A^{1\tau}$ a splitting point with witnesses $F_0, F_1 \in C^{\tau^*}(A^{0\tau^*})$. Let X_0, X_1 be its immediate predecessors with $X_0 \in C^\tau(X)$. Assume that $X-$wide piste $\langle C^\xi_X, C^{\xi lim}_X \mid \xi \in s\rangle$ for X is defined and assume that $C^\tau(X)$ is an initial segment of C^τ_X.

Let the X_0-wide piste be $\langle C^\xi_X, C^{\xi lim}_X \mid \xi \in s\rangle$.

Define X_1-wide piste $\langle C^\xi_{X_1}, C^{\xi lim}_{X_1} \mid \xi \in s\rangle$ as follows:

- $C^\xi_{X_1} = C^\xi_X$, for every $\xi \geq \tau^*$.
  I.e. no changes for models of cardinality $\geq \tau^*$.
- $C^{\xi lim}_{X_1} = C^\xi_{X_1} \cap A^{1\xi lim}$, for every $\xi \in s$.
  Models that were potentially limit remain such and no new are added.
- $C^\tau_{X_1} = (C^\tau_X \setminus X) \cup C^\tau(X_1)$.
  Here we switched the piste from X_0 to X_1.
- $C^\xi_{X_1} = \{Z \in C^\xi_X \mid \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))\} \cup \{\pi_{X_0,X_1}(Z) \mid Z \in C^\xi_X \cap X_0\}$, for every $\xi \in s \cap \tau^*$[11].

Now we require the following:

9. Let $\tau \in s$ and $X \in A^{1\tau}$. Then $X-$wide piste is a wide piste, i.e. it satisfies 2.1.
   The problem is with (3c) of 5 which, in general, is not preserved while splitting.

   Final conditions deal with largest models.

10. (Maximal models are above all the rest) For every $\tau \in s$ and $Z \in \bigcup_{\rho \in s} A^{1\rho}$, if $Z \not\in A^{0\tau}$, then there is $\mu \in s$ such that $Z = A^{0\mu}$.

    Recall that by 5, maximal models $A^{0\tau}, \tau \in s$ are linearly ordered as top parts of the wide piste $\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s\rangle$.

[11] In particular, due to this, the next condition implies that for $\xi \in s \cap \tau^*$, if $Z \in C^\xi_X, \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))$, then $\{\pi_{X_0X_1}(Z') \mid Z' \in C^\xi_X \cap X_0\} \subseteq Z$.

This completes the definition of $\delta-$structure with pistes over η of the length θ.

2.1 Some properties of structures with pistes.

Let us turn now to the intersection property.
The intuition behind is to replace an arbitrary intersection of models by an internal one.

Definition 2.3 (Models of different sizes). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ.

Let $A \in A^{1\tau}, B \in A^{1\rho}$ and $\tau < \rho$.
By $ip(A, B)$ we mean the following:

1. $B \in A$,
   or
2. $A \subset B$,
   or
3. $B \notin A$, $A \not\subset B$ and then
   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \rho) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

Definition 2.4 (Models of a same size). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ.

Let $A, B \in A^{1\tau}$. By $ip(A, B)$ we mean the following:

1. $A \subseteq B$,
   or
2. $B \subseteq A$,
   or
3. $A \not\subseteq B$, $B \not\subseteq A$ and then
   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \tau) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

If both $ip(A, B)$ and $ip(B, A)$ hold, then we denote this by $ipb(A, B)$.

Lemma 2.5 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a δ structure with pistes over η of the length θ. Assume $A \in A^{1\tau}, B \in A^{1\rho}$, for some $\tau \leq \rho, \tau, \rho \in s$. Then $ip(A, B)$ and if $\tau = \rho$, then also $ipb(A, B)$.*

Proof. We will basically split the proof into two main cases: $\rho = \tau$ and $\rho \neq \tau$. However, the inductive assumption will be used simultaneously for both.

Case A. $\rho = \tau$**.**

So, $A, B \in A^{1\tau}$. Assume that $A \not\subseteq B$ and $B \not\subseteq A$. Consider the pistes leading from $A^{0\tau}$ to A and to B. Let X be their last common point. Then, by 2.2(8), X is a successor model.

Subcase A1. *X has a unique immediate predecessor.* Let X_0 be this immediate predecessor. Then, one of A or B is in X_0 and the other one is not. But, then it must be equal to X_0, which is impossible by our assumptions that $A \not\subseteq B$ and $B \not\subseteq A$.

Subcase A2. *X is a splitting point.*

Let X_0 and X_1 be the immediate predecessors of X. Let $F_0 \in X_0$ and $F_1 \in X_1$ witness that X, X_0, X_1 form a $\Delta-$system triple. Then $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$.
Assume that $A \in X_0 \cup \{X_0\}$ and $B \in X_1 \cup \{X_1\}$.
If $A = X_0$ and $B = X_1$, then $ipb(A, B)$ follows.
Suppose that $A \neq X_0$ or $B \neq X_1$. Say, $B \neq X_1$. Set $B' = \pi_{X_1,X_0}[B]$. Then $B' \in X_0$ and $B \cap X_0 = B' \cap F_0$. Hence,

$$A \cap B = A \cap B \cap X_0 = A \cap B' \cap F_0 = (A \cap B') \cap (A \cap F_0).$$

Now we apply induction to get $ip(A, B')$ and $ip(A, F_0)$.

Subcase A3. *X is a splitting point of higher order.*

The proof essentially the same as in **Subcase A2**.

Case B. $\rho > \tau$**.**

So, $A \in A^{1\tau}, B \in A^{1\rho}$. Assume that $A \not\subseteq B$ and $B \notin A$.
Suppose first that $A \notin A^{0\rho}$. Then 2.2(10), $A = A^{0\tau}$ and if $B \notin A^{0\tau}$, then, again by 2.2(10), $B = A^{0\rho}$. But any two maximal models on the wide piste of the structure are compatible as follows from 5. Thus, if $\sup(A^{0\tau} \cap \theta^+) \leq \sup(A^{0\rho} \cap \theta^+)$, then $A^{0\tau} \subseteq A^{0\rho}$ by 5(9). If $\sup(A^{0\tau} \cap \theta^+) > \sup(A^{0\rho} \cap \theta^+)$, then $A^{0\rho} \in A^{0\tau}$, by 5(12).

Suppose that $A \in A^{0\rho}$. Then $B \neq A^{0\rho}$, as $A \not\subseteq B$, and hence $A, B \in A^{0\rho}$.
By 2.2(9) we can assume that A is on the wide piste of the structure. Consider the pistes leading from $A^{0\rho}$ to A and to B. Let $X \in C^\rho(A^{0\rho}$ be their last common point. The proof proceeds by induction on $\mathrm{rank}(X)$. Then, by 2.2(8), X is a successor model.

Subcase B1. *X is a splitting point.*

The proof is essentially as in **Subcase A2** above.

Subcase B2. *X is a splitting point of higher order.*

Let X_0 and X_1 be the immediate predecessors of X. Let $G, G_0, G_1 \in X \cap A^{1\mu}$ be a corresponding $\Delta-$system triple, for some $\mu \in s \setminus \rho+1$. Also let $F_0 \in G_0 \cap X$ and $F_1 \in G_1 \cap X$

witness this, i.e. $G_0 \cap G_1 = G_0 \cap F_0 = G_1 \cap F_1$.
Assume that $A \in X_0$ and $B \in X_1 \cup \{X_1\}$.
Set $B' = \pi_{G_1,G_0}[B]$. Then $A \cap B = A \cap B' \cap F_0$. The induction applies to A and B', since $B' \subseteq X_0 \in X$. Also it applies to A and F_0, since $F_0 \in X$. Hence, $ip(A, B)$.

Subcase B3. *X has a unique immediate predecessor.*

Let X_0 be this predecessor. Then either $B = X_0$ or $B \in X_0$.
Split into three cases according to the relation between A and X_0.

Subsubcase B3.1. $\sup(A \cap \theta^+) < \sup(X_0 \cap \theta^+)$.

Then $A \in X_0$, by 2.1(9). But $B \in X_0$ as well, and we get a contradiction to the choice of X.

Subsubcase B3.2. $\sup(A \cap \theta^+) > \sup(X_0 \cap \theta^+)$.

Using 2.2(9), we may assume that both A and B are on the wide piste of the structure.

Apply 2.1(13) to A and B. Let $Z \in A \cap C^{\rho^*}(A^{1\rho^*})$ be a least model with $B \subseteq Z$, where $\rho^* = \min((A \setminus \rho) \cap Regular)$. Then $B \in Z$, as $B \notin A$ and $B \supseteq A \cap Z$, by 2.1(13). Hence, $A \cap B = A \cap Z$ and we are done.

Subsubcase B3.3. $\sup(A \cap \theta^+) = \sup(X_0 \cap \theta^+)$.

Then $A \subset X_0$. If $B = X_0$, then $A \cap B = A$ and we are done. So, $B \in X_0$. Then, $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$. Apply 2.1(13) to A and B and continue as in **Subsubcase B3.2**.

□

Lemma 2.6 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ. Suppose that $\tau, \rho \in s, \tau < \rho$, $A \in A^{1\tau}$ and $A \cap A^{1\rho} \neq \emptyset$. Then there is $X \in A \cap A^{1\rho}$ which includes every element of $A \cap A^{1\rho}$.*

Proof. If $A \notin A^{0\rho}$, then $A = A^{0\tau}$, by 2.2(10), and again, by 2.2(10), then $A^{0\rho} \in A = A^{0\tau}$. So $A^{0\rho}$ will be as required.
Assume that $A \in A^{0\rho}$. By 2.2(5), we may assume that A is on the wide piste of the structure. Let $Z \in C^\rho(A^{0\rho})$ be the least model which includes A. Consider its immediate predecessor Z' on the piste. It exists since, by the assumption of the lemma $A \cap A^{1\rho} \neq \emptyset$, and so the piste continues to elements of this intersection.
Now, both A and Z' are on the wide piste, $\tau < \rho$ and $A \not\subseteq Z'$. Hence, by 2.1(9), $\sup(A \cap \theta^+) > \sup(Z' \cap \theta^+)$. Apply now 2.1(12) to A and Z'. So, there will be $X \in A \cap C^\rho(A^{0\rho})$ such that $X \supseteq Z'$. But then $Z' = X$ and we are done.
□

Lemma 2.7 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ. Suppose that $\tau \in s$, $A \in A^{1\tau}$ and $A \cap A^{1\tau} \neq \emptyset$. If A is a potentially limit point then there is $X \in A \cap A^{1\tau}$ which includes every element of $A \cap A^{1\tau}$.*

Proof. Just by 2.2(6), A has a unique immediate predecessor. It will be as desired.
□

Note that if A is a splitting point or a splitting point of higher order then the lemma is not true anymore.
Also, if one likes to find the largest model of a small cardinality inside a larger one, then it should not be true in general (however any $\delta-$structure with pistes over η of the length θ can be extended to one that satisfies this). Thus, for example reflect in an increasing order $\omega-$many models of size η into a fixed potentially limit model A of size η^+. There will be no maximal model of cardinality η inside A. But an additional reflection will produce such.

2.2 Forcing with structures with pistes.

Definition 2.8 Define $\mathcal{P}_{\theta\eta\delta}$ to be the set of all
$\delta-$structures with pistes over η of the length θ.
Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle \in \mathcal{P}_{\theta\eta\delta}$.
Denote further $A^{0\tau}$ by $A^{0\tau}(p)$, $A^{1\tau}$ by $A^{0\tau}(p)$, $A^{1\tau lim}$ by $A^{1\tau lim}(p)$, C^{τ} by $C^{\tau}(p)$ and s by $s(p)$.
Call s the support of p.

Let us define a partial order on $\mathcal{P}_{\theta\eta\delta}$ as follows.

Definition 2.9 Let
$p_0 = \langle\langle A_0^{0\tau}, A_0^{1\tau}, A_0^{1\tau lim}, C_0^{\tau}\rangle \mid \tau \in s_0\rangle$, $p_1 = \langle\langle A_1^{0\tau}, A_1^{1\tau}, A_1^{1\tau lim}, C_1^{\tau}\rangle \mid \tau \in s_1\rangle$ be two elements of $\mathcal{P}_{\theta\eta\delta}$.
Set $p_0 \leq p_1$ (p_1 extends p_0) iff

1. $s_0 \subseteq s_1$,
2. $A_0^{1\tau} \subseteq A_1^{1\tau}$, for every $\tau \in s_0$,
3. let $A \in A_0^{1\tau}$, then $A \in A_0^{1\tau lim}$ iff $A \in A_1^{1\tau lim}$.
   The next item deals with a property called switching in [3]. It allows to change piste directions.

4. For every $A \in A_0^{1\tau}$, $C_0^\tau(A) \subseteq C_1^\tau(A)$,
   or
   there are finitely many splitting (or generalized splitting) points $B(0), ..., B(k) \in A_0^{1\tau}$ with $B(j)', B(j)''$ the immediate predecessors of $B(j)$ $(j \leq k)$ such that
   (a) $B(j)' \in C_0^\tau(B(j))$,
   (b) $B(j)'' \in C_1^\tau(B(j))$.
5. If $A \in A_0^{1\tau}$ is a splitting point or a splitting point of higher order in p_0, then it remains such in p_1 with the same immediate predecessors.
6. Let $B \in A_0^{1\tau}$ be a successor point, not in $A_0^{1\tau\,\mathrm{lim}}$ and with a unique immediate predecessor. Consider the wide piste that runs via B (in p_0). Let A be as in 2.1(6). Then there is no model E in p_1 such that $A \in E \in B$.

   This requirement guaranties intervals without models, even after extending a condition.

By 2.9(6), potentially limit points are the only places where not end-extensions can be made.

Next two lemmas will insure that generic clubs produced by $\mathcal{P}_{\theta\eta\delta}$ run away from old sets.

Lemma 2.10 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ *be an element of* $\mathcal{P}_{\theta\eta\delta}$. *Let* $X \in A^{1\rho lim}$, *for some* $\rho \in s$.
Assume that if $\mathrm{cof}(\sup(X \cap \theta^+)) < \rho$, *then* $\rho \in B$, *where* $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ *is the model with* $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$ *(exists by 2.1(3)(c)B))*[12].
Suppose that for every $t \in X$ *there is* $D \preceq X$ *such that*

1. $D \in X$,

2. $t \in D$,

3. $|D| = \rho$,

4. $D \supseteq \rho$

5. ${}^{\rho>}D \subseteq D$,

6. D *is a union of a chain of its elementary submodels which satisfy items 1-5* [13].

[12]We will see further that it is possible to remove this assumption at least in interesting cases.
[13]The issue here is to satisfy 2.1(3(b)).

Then for every $\beta < \sup(X \cap \theta^+)$ there is T of size ρ with $\sup(T \cap \theta^+) > \beta, T \in X$ such that adding T as a potentially limit point and reflecting it through $\Delta-$system type triples gives an extension of p.

Proof. Let $\tau := \mathrm{cof}(\sup(X \cap \theta^+))$. We deal with the case $\tau < \rho$. The case $\tau = \rho$ is similar and a bit simpler.
By 2.1(3(c(ii)), then $\tau \in s$ and there is $B \in A^{1\tau lim}$ such that $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$ and $B \subseteq X$.
Let $\langle X_i \mid i < \tau \rangle$ be the canonical sequence of models of 2.1(3(b),(c)ii) which members are in B. We have ${}^{\rho>}X_{i+1} \subseteq X_{i+1}$, for every $i < \tau$. By the assumption, we can assume that for every $i < \eta$, X_{i+1} is a union of a chain of its elementary submodels which satisfy items 1,3-5 above.
Pick now T to be one of X_{i+1}, such that

1. $\sup(T \cap \theta^+) > \beta$,
2. for every model E which appears in p and belongs to X, require that $E \in T$,
3. for every model Z which appears in p, has cardinality $< \tau$ and $\sup(Z \cap \theta^+) > \sup(B \cap \theta^+) = \sup(X \cap \theta^+)$, we require that $Z \cap X \in T$[14].
   The next item is added in order to satisfy 2.1(14).
4. For every models Z, Y in p such that
   (a) $\sup(Z \cap \theta^+) = \sup(Y \cap \theta^+) > \sup(B \cap \theta^+) = \sup(X \cap \theta^+)$,
   (b) $|Y| = \theta$,
   (c) $|Z| = \zeta$, for some $\zeta < \tau$,

   we require that $R \cap X \in T$, for every R of regular cardinality $\mu, \zeta < \mu < \tau$ which is definable in $\langle H(\theta^+), \in, \leq, \delta, \eta \rangle$ with parameters from the set $Z \cup (\mu + 1) \cup \{Z, Y\}$[15].
5. Let D, D_0, D_1 from p of a $\Delta-$system triple, i.e. D is a splitting point and D_0, D_1 are its immediate predecessors. Suppose that $D_0 \supseteq B$[16]. Require the following analogs of the above conditions (2),(3),(4):

[14]Recall that ${}^{\tau>}X \subseteq X$ by 2.1(3(c)). So, $Z \cap X \in X$, and hence, $Z \cap X \in X_i$ for every large enough $i < \tau$.
[15]Note that the total number of such R's for a fixed regular $\mu, \zeta < \mu < \tau$ is $|Z| = \zeta$. Hence, there are less than τ possibilities for R's.
[16]Note that then $B \subsetneq D_0$, since by 2.2(6(c)), D_0 is not a potentially limit point.

- for every model E which appears in p and belongs to $\pi_{D_0,D_1}(X)$, require that $E \in \pi_{D_0,D_1}(T)$[17],
- for every model Z which appears in p, has cardinality $< \tau$ and $\sup(Z \cap \theta^+) > \sup(\pi_{D_0,D_1}(B) \cap \theta^+)$, we require that $Z \cap \pi_{D_0,D_1}(X) \in \pi_{D_0,D_1}(T)$.
- For every models Z, Y in p such that
  (a) $\sup(Z \cap \theta^+) = \sup(Y \cap \theta^+) > \sup(B \cap \theta^+) = \sup(X \cap \theta^+)$,
  (b) $|Y| = \theta$,
  (c) $|Z| = \zeta$, for some $\zeta < \tau$,

  we require that $R \cap \pi_{D_0,D_1}(X) \in \pi_{D_0,D_1}(T)$, for every R of regular cardinality $\mu, \zeta < \mu < \tau$ which is definable in $\langle H(\theta^+), \in, \leq, \delta, \eta \rangle$ with parameters from the set $Z \cup (\mu + 1) \cup \{Z, Y\}$

Let us argue that T is as desired.
First note that if D is one of models of p and $\sup(D \cap \theta^+) < \sup(X \cap \theta^+)$, then $\sup(D \cap \theta^+) < \sup(T \cap \theta^+)$.
It follows by (2) for models which are in X.
So, suppose that $D \notin X$. Changing the wide piste of p if necessary, we can assume that both X and D are on the same wide piste. Apply 2.1(12) to X and D. There is $D^* \in X \cap C^{|D|}(A^{0|D|}), D^* \supseteq D$. But then by (2) above we have $D^* \in T$.
The requirement (3) insures 2.1(13).

Let us argue now that adding T does not case any harm once moving through $\Delta-$system type triples. Let D, D_0, D_1 from p of a $\Delta-$system triple, i.e. D is a splitting point and D_0, D_1 are its immediate predecessors.
Note that by (3) above $T \notin Z$, for any Z in p of cardinality $< \tau$. So, let us assume that $|D| \geq \tau$.
If $B \not\subseteq D_0$ and $B \not\subseteq D_1$, then then let us argue that T not in the domain of π_{D_0,D_1} and π_{D_1,D_0}, and so, does not move. For this apply the intersection property $ip(B, D_0)$ (or similar $ip(B, D_0)$). Then

$$B \cap D_0 = B \cap T_1 \cap ... \cap T_n,$$

for some $T_1, ..., T_n \in B$. But $B \subseteq X$, and hence, by (2) above, $T_1, ..., T_n \in T$. So T cannot be in $B \cap D_0$.

Now, if $B \subseteq D_0$, then the condition (4) above provides the desired conclusion.
□

[17] Note that π_{D_0,D_1} does not move τ. Also note that E need not have a pre-image under π_{D_0,D_1}.

Lemma 2.11 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ *be an element of* $\mathcal{P}_{\theta\eta\delta}$*. Let* $X \in A^{1\rho lim}$*, for some* $\rho \in s$.
Assume that if $\mathrm{cof}(\sup(X \cap \theta^+)) < \rho$*, then* $\rho \in B$*, where* $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ *is the model with* $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$ *(exists by 2.1(3)(c)B)).*
Suppose that for every $t \in X$ *there is* $D \preceq X$ *such that*

1. $D \in X$,
2. $t \in D$,
3. $|D| = \rho$,
4. $D \supseteq \rho$
5. ${}^{\rho>}D \subseteq D$,
6. D *is a union of a chain of its elementary submodels which satisfy items 1-5.*

Let $\beta < \sup(X \cap \theta^+)$ *and* T *be a potentially limit point of size* ρ *with* $\sup(T \cap \theta^+) > \beta, T \in X$ *added by the previous lemma 2.10. Then for every* $\gamma, \sup(T \cap \theta^+) < \gamma < \sup(X \cap \theta^+)$ *there is* T' *of size* ρ *with* $\sup(T' \cap \theta^+) > \gamma, T' \in X$ *such that adding* T' *as a non-potentially limit point and reflecting it through* $\Delta-$*system type triples gives an extension of the previous condition.*

Proof. The proof repeats those of 2.10. The purpose of first adding T and only then T' is to satisfy 2.1(11). Thus we add first a potentially limit point T above everything relevant, then we are free to add above it a non-potentially limit point T'.
□

Lemma 2.12 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ *be an element of* $\mathcal{P}_{\theta\eta\delta}$*. Let* $X \in A^{1\rho lim}$*, for some* $\rho \in s$.
Assume that $\mathrm{cof}(\sup(X \cap \theta^+)) = \tau < \rho$*,* $\rho \notin B$*, for* $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ *such that* $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$*. Let* $Y \in A^{1\theta lim}$ *with* $Y \cap \theta^+ = \sup(X \cap \theta^+)$ *(it exists by 2.1(3(c)B)). Suppose that for every* $t \in Y$ *there is* $D \preceq Y$ *such that*

1. $D \in Y$,
2. $t \in D$,
3. $|D| = \theta$,

4. $D \supseteq \theta$

5. ${}^{\theta>}D \subseteq D$*,*

6. D *is a union of a chain of its elementary submodels which satisfy items 1-5.*

Then for every $\beta < \sup(X \cap \theta^+)$ *there is* T *of size* ρ *with* $\sup(T \cap \theta^+) > \beta, T \in X$ *such that adding* T *as a potentially limit point and reflecting it through* $\Delta-$*system type triples gives an extension of* p*.*

Proof. Apply first Lemma 2.10 to Y twice in order to obtain $T_0 \in T_1$ closed under $< \theta-$sequences which satisfy its conclusion. Then $T_0, T_1 \in X$, since $T_0, T_1 \in B$ and $B \subseteq X$. Now it is not hard to find $D \in T_1$ such that

1. $D \in X$,
2. $T_0 \in X \in T_1$
3. $t \in D$,
4. $|D| = \rho$,
5. $D \supseteq \rho$
6. ${}^{\rho>}D \subseteq D$,
7. D is a union of a chain of its elementary submodels which satisfy items 1-5

and T is higher enough relatively to models of p of cardinality $< \rho$ (B among them). Add such T.
□

We turn now to properness of $\mathcal{P}_{\theta\eta\delta}$.

Recall the following basic definition due to S. Shelah [12]:

Definition 2.13 Let $\mu \geq \omega$ be a regular cardinal and P a forcing notion. P is called $\mu-$proper iff for every $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \mu, {}^{\mu>}M \subseteq M$, $P, p \in M$ there is $p' \geq_P p$ such that for every dense open $D \subseteq P, D \in M$, $p' \Vdash$ "$D \cap \underset{\sim}{G} \cap M \neq \emptyset$." Such p' is called (M, P)-generic.

The following is obvious:

Lemma 2.14 *If P is μ-proper, then it preserves μ^+.*

Lemma 2.15 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\eta-$proper*

Proof. Let $p \in \mathcal{P}_{\theta\eta\delta}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ regular large enough such that such that

1. $|\mathfrak{M}| = \eta$,
2. $\mathfrak{M} \supseteq \eta$,
3. $\mathcal{P}_{\theta\eta\delta}, p \in \mathfrak{M}$,
4. ${}^{\eta>}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\theta^+)$.
Clearly, M satisfies 2.1(3(b)). Moreover, using the elementarity of $\mathfrak{M}$, for every $x \in M$ there will be $Z \in M$ such that

- $Z \preceq H(\theta^+)$,
- $|Z| = \theta$,
- $Z \supseteq \theta$,
- ${}^{\theta>}Z \subseteq Z$,
- $x \in Z$.

This allows to find a chain of models $\langle N_i \mid i < \eta \rangle$ of size θ which members are in M, witnesses 2.1(3(b)) for $N := \bigcup_{i<\eta} N_i$ and $N \supseteq M$.

Extend p by adding M as a new $A^{0\eta}$ and N as a new $A^{0\theta}$. Require them to be a potentially limit points. Denote the result by $p^\frown\{M, N\}$.
We claim that $p^\frown\{M, N\}$ is $(\mathcal{P}_{\theta\eta\delta}, \mathfrak{M})-$generic. So, let $p' \geq p^\frown\{M, N\}$ and $D \in M$ be a dense open subset of $\mathcal{P}_{\theta\eta\delta}$. It is enough to find $q \in \mathfrak{M} \cap D$ which is compatible with p'.
Let

$$p' = \langle \langle A^{0\tau}(p'), A^{1\tau}(p'), A^{1\tau lim}(p'), C^{\tau}(p') \rangle \mid \tau \in s(p') \rangle.$$

By 2.6,2.7, for every $\tau \in s$ there will be the maximal model in $A^{1\tau}(p') \cap M$ (once non-empty). They all are on the wide piste which runs through M. Just the wide piste that runs through M runs trough N and all other relevant for maximality models, since $|M| = \eta$

is the smallest possible cardinality of the model and pistes of models of different sizes go in the same directions by 2.2(9, 6d(vi)).

Let us argue that they are linear ordered by $\in, \subseteq$. Thus, let $\tau, \rho \in s(p'), \tau < \rho$, $A \in A^{1\tau}(p') \cap M, B \in A^{1\rho}(p') \cap M$ be such maximal models. If $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then, by 2.1(9(a)), $A \in B$. If $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$, then, by 2.1(9(b)), $A \subseteq B$. Suppose that $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$, then then, by 2.1(12), there is $B^* \in A, |B^*| = \rho$ which is on the same wide piste and $B^* \supseteq B$. By maximality of B this may occur only when $B^* = B$. So, we are done.

Set

$$q' = \langle \langle \max(A^{1\tau}(p') \cap M), A^{1\tau}(p') \cap M, A^{1\tau lim}(p') \cap M, C^\tau(p') \restriction A^{1\tau}(p') \cap M \rangle \mid \tau \in s(p') \rangle.$$

It is routine to check that $q' \in \mathcal{P}_{\theta\eta\delta}$ and $q' \leq p'$. Also, $q' \in \mathfrak{M}$, since ${}^{\eta>}\mathfrak{M} \subseteq \mathfrak{M}$.

Now let $q = \langle \langle A^{0\tau}(q), A^{1\tau}(q), A^{1\tau lim}(q), C^\tau(q) \rangle \mid \tau \in s(q) \rangle$ be an extension of q' in $\mathfrak{M}$ which belongs to D.

We claim that p' and q are compatible. Namely, set $s = s(q)$. Let $A^{0\tau} = A^{0\tau}(p')$, for every $\tau \in s(p')$. Let $\langle \tau_i \mid i < i^* \rangle$ be an increasing (or just any one to one) enumeration of $s \setminus s(p')$. Pick $\in -$increasing sequence of models $\langle A_i \mid i < i^* \rangle$ such that for every $i < i^*$ the following hold:

1. $p', q \in A_i$,
2. $|A_i| = \tau_i$,
3. A_i satisfies 2.2(2).

Set $A^{0\tau_i} = A_i$.

Finally let for every $\tau \in s$,

$$A^{1\tau} = \{A^{0\tau}\} \cup A^{1\tau}(p') \cup A^{1\tau}(q) \cup \{B \mid$$

$$\exists (D, D_0, D_1) \quad \Delta - \text{system triple in } p' \text{ with } M \in \mathrm{dom}(\pi_{D_0,D_1})$$

$$\text{and there is a model } A \text{ in } q \text{ which does not appear in } p' \text{ such that } B = \pi_{D_0,D_1}(A)\} \cup$$

$$\{B \mid \exists (D, D_0, D_1) \quad \Delta - \text{system triple in } q \text{ but not in } p' \text{ and there is a model}$$

$$A \in \mathrm{dom}(\pi_{D_0,D_1}) \text{ in } p' \text{ which does not appear in } q \text{ such that } B = \pi_{D_0,D_1}(A)\}.$$

Intuitively, we just put together models (of same cardinalities) of p' and q and to them the images of new models (those in q and not in p') under isomorphisms of models of p' with

M inside.
Define $A^{0\tau lim}$ and $C^\tau(\tau \in s)$ in the obvious fashion now.

Set

$$p^* = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle.$$

We claim that $p^* \in \mathcal{P}_{\theta\eta\delta}$ and then, by 2.9 and the definition of p^*, $p^* \geq p', q$.

Deal with the wide piste of p^*. The main issue is to show that it satisfies 2.1.
Let $D \in A^{0\rho}(p')$ be on the wide piste of p' for some $\rho \in s\backslash\eta+1$. If $\sup(D\cap\theta^+) \geq \sup(M\cap\theta^+)$, then $M \subseteq D$ by 2.1(9). Hence every new model (i.e. one in q and not in p') is in D.
Note it is impossible to have a situation when $M \in D$, D has an immediate predecessor with sup below $\sup(M \cap \theta^+)$ (or by 2.1(12), equivalently in M) and D is not a potentially limit point. It follows by 2.1(4), since M is a potentially limit point.
Also such D cannot be both minimal in $C^\rho(A^{0\rho}(p'))$ and not potentially limit, by 2.1(5), since N appears in p'. Actually, this the only reason of picking $\mathfrak{N}$ and adding N to p.

Assume that $\sup(D \cap \theta^+) < \sup(M \cap \theta^+)$. If D is in M, then D is in q', and hence new models are fine with D since they are in $q \geq q'$. Assume that $D \notin M$. Let $|D| = \rho$. Then $\rho > \eta$. Let A be on the wide piste of q. If A appears in p', then we are done. Suppose otherwise, i.e. A is a new model.

Case 1. $\sup(A \cap \theta^+) < \sup(D \cap \theta^+)$.
Let D^* be the least model in M above D in $C^\rho(A^{0\rho})$.

Subcase 1.1 $A \in D^*$.
Suppose first that $\mathrm{cof}(D^* \cap \theta^+) = \rho$. By 2.1(3(b)) and elementarity of M there is a sequence $\langle D^*_i \mid i < \mathrm{cof}(D^* \cap \theta^+)\rangle \in M$ which witnesses 2.1(3(b)). Then , for some $i^* < \mathrm{cof}(D^* \cap \theta^+) = \rho, i^* \in M$, we have $A \in D^*_{i^*}$. Now, by 2.1(13(b)), $D^*_{i^*} \in D$, and so, $D^*_{i^*} \subseteq D$. Hence $A \in D$.
If $\mathrm{cof}(D^* \cap \theta^+) = \xi$, for some $\xi \in s \cap \rho$, then the argument is similar only using 2.1(13(c)).

Subcase 1.2 $A \notin D^*$.
Then necessary, $|A| > \rho$. Both A, D^* are in q, hence there is the least model $A^* \in C^{|A|}(A^{0|A|})(q)$ in D^* above A. Apply the argument of Subcase 1.1 to A^* and D^*. Then $A^* \in D$ and so A, D satisfy 2.1(12). Also they satisfy 2.1(13), since A, D^* satisfy it as members of q and $D \subseteq D^*$.

Case 2. $\sup(A \cap \theta^+) > \sup(D \cap \theta^+)$.
Let D^* be the least model in M above D in $C^\rho(A^{0\rho})$. Then $\sup(A \cap \theta^+) > \sup(D^* \cap \theta^+)$, since otherwise by 2.1(13) (applied to M and D^*) we will have $\sup(A \cap \theta^+) < \sup(D \cap \theta^+)$.
Both A and D^* are in q, so $|A| \geq \rho$ implies, by 2.1(9), that $A \supseteq D^* \supseteq D$.
Suppose that $|A| < \rho$. If $D^* \in A$, then, by 2.1(13), D^* will be the least in $A \cap C^\rho(A^{0\rho})$

above D. Suppose that $D^* \not\in A$. Then, since both A and D^* in q, there will be the least $D^{**} \in A \cap C^\rho(A^{0\rho})(q)$ above D^*, and this D^{**} will satisfy 2.1(12) for A, D. The condition 2.1(13) holds for A, D since it is true for A, D^* and for M, D .

Case 3. $\sup(A \cap \theta^+) = \sup(D \cap \theta^+)$.
Let D^* be the least model in M above D in $C^\rho(A^{0\rho})$. By 2.1(13) applied to M, D and 2.1(3(b)), we derive a contradiction.

Finally us deal with a situation where D is the wide piste of p' and so of p^*.

Case A. D is not on the wide piste and the first splitting on the piste from $A^{0\rho}$ to D is above M.
Then we just consider the image of M under such splitting and proceed with it as before.

Case B. D is not on the wide piste and the first splitting on the piste from $A^{0\rho}$ to D is below M (according to sup of the models).
Change inside p' the wide piste in order to put D on it. Such change will preserve M on the wide piste since the relevant splittings are below M. Now, both D and M will be on the (new) wide piste of p' and p^*. Proceed as before.

The above shows that p^* satisfies 2.2(5, 9). The rest of 2.2, as well as $p^* \geq p', q$ follows easily from the definition of p^* using 2.1(11).
$\square$

Our next tusk will be to show that the forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$. Let us first prove three technical lemmas that allow to add new models at places of specific type.

Lemma 2.16 *Let $p = \langle \langle C^\tau, C^{\tau lim} \rangle \mid \tau \in s \rangle$ be a wide piste and B, D are models of p such that $|B| = \tau$, for some $\tau \in s \cap \theta$, $|D| = \theta$ and $\sup(B \cap \theta^+) = D \cap \theta^+$.*
Let $\rho \in (\tau, \theta)$ be a regular cardinal. Suppose that for every $A \in \bigcup_{\mu<\rho} C^\mu$, if sup$(A \cap \theta^+) >$ $\sup(B \cap \theta^+)$, then $B \in A$. Then a model of cardinality ρ can be added to p between B and D such that the result remains a wide piste.

Proof. Let B, D be as in the statement of the lemma and $\rho \in (\tau, \theta)$ a regular cardinal. Suppose that there is no model of size ρ between B and D inside p. Without loss of generality we can assume that $\rho \in s$. Just otherwise extend p by adding the largest (under $\in$) model of cardinality ρ making it potentially limit one.
Let E be the least elementary submodel of D such that

- $|E| = \rho$,

- $E \supseteq B$,
- $E \supseteq \rho$,
- ${}^{\mathrm{cof}(\sup(E\cap\theta^+))>}E \subseteq E$.

Add E to $C^{\rho lim}$. Let us check that the result is a wide piste.

Once a model A appears in p and there is $H \in A$ in p with $\sup(H \cap \theta^+) = \sup(E \cap \theta^+)$, then by 2.1(10) also $B, D \in A$, and then by elementarity, $E \in A$.

First note that if $A \in C^\xi$, for some $\xi \in s \setminus \rho$, then either $E \in A$ or $A \in E$. Thus if $\sup(A\cap\theta^+) > \sup(E\cap\theta^+)$, then $\sup(A\cap\theta^+) > \sup(B\cap\theta^+)$, since $\sup(E\cap\theta^+) = \sup(B\cap\theta^+)$. Then, by 2.1(9), $B \in A$. Hence, by 2.1(10), $D \in A$, as $\sup(B \cap \theta^+) = D \cap \theta^+$. But E is definable from B, D, so $E \in A$.
Assume now that $\sup(A\cap\theta^+) < \sup(E\cap\theta^+)$. If $A \in B$, then we are done, since $B \subseteq E$. So, suppose that $A \notin B$. Then, by 2.1(12), there is $A^* \in B \cap C^\rho(A^{0\rho})$ such that $A^* \supseteq A$. But then, $A \in A^*$, and by above $A^* \in E$ and $|A^*| = \rho$, so $A^* \subseteq E$. Hence $A \in E$.

Let now $A \in C^\xi$, for some $\xi \in s \cap \rho$. If $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$, then, by the assumption of the lemma, $B \in A$. Then, as above, we conclude that $E \in A$.

Suppose now that $\sup(A \cap \theta^+) < \sup(E \cap \theta^+)$.
If $|A| = \xi \leq \rho$, then $A \in B$ will trivially imply that $A \in E$. If $A \notin B$, then by 2.1(9), $\xi > \tau$. Apply then 2.1(12) to B and A and find least possible $A^* \in C^\xi \cap B$ which contains A. But then $A \in A^* \in B \subseteq E$ and $|E| = \rho \geq |A^*| = \xi$, hence $A^* \subseteq E$, and so , $A \in E$.

Consider now the remaining case: $\xi > \rho$ and $A \notin E$. It follows that $A \notin B$. Apply 2.1(12) to B and A and find least possible $A^* \in C^\xi \cap B$ which contains A. Clearly it witnesses 2.1(12) for E and A as well, but we would like to show that 2.1(13) holds for E and A. Note that E is definable in $\langle H(\theta^+), \in, \leq, \delta, \eta\rangle$ using $\{B, D, \rho\}$ as parameters. So, 2.1(14) applies to E and A which implies 2.1(13) for them[18].
Finally, 2.1(14) holds for E and A due to definability of E and 2.1(14) for B and A.

The rest of the conditions follow easily.

□

Lemma 2.17 *Let* $p = \langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle$ *be a wide piste and* B, D *are models of* p *such that* $|B| = \tau$, *for some* $\tau \in s \cap \theta$, $|D| = \theta$ *and* $\sup(B \cap \theta^+) = D \cap \theta^+$.
Let $\rho \in (\tau, \theta)$ *be a regular cardinal. Then a model of cardinality* ρ *can be added to* p *between* B *and* D *such that the result remains a wide piste.*

[18]Here is the only placed where we use 2.1(14).

Proof. We just continue the argument of the previous lemma (2.16) from the point where the appeal to the assumption "for every $A \in \bigcup_{\mu<\rho}$, if $sup(A \cap \theta^+) > \sup(B \cap \theta^+)$, then $B \in A$" was made. So, assume that $A \in C^\xi$, for some $\xi \in s \cap \rho$,$\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \notin A$.
Assume that such A was picked to be the least possible (under $\in -$relation) with $\sup(A \cap \theta^+) > \sup(E \cap \theta^+)$ and $E \notin A$. If there is A_1 so that $\sup(A_1 \cap \theta^+) > \sup(B^* \cap \theta^+)$ and $B^* \notin A_1$, then either $\sup(A_1 \cap \theta^+) < \sup(A \cap \theta^+)$ and, then $|A_1| > |A|$ (due to minimality of A), or $\sup(A_1 \cap \theta^+) > \sup(A \cap \theta^+)$ and, then $|A_1| < |A|$ (since $|A_1| \geq |A|$ will imply $A_1 \supseteq A$, by 2.1(9)). Let us show that the former possibility is impossible.
Claim. *It is impossible to have* $\sup(A_1 \cap \theta^+) < \sup(A \cap \theta^+)$.
Proof. Suppose otherwise. Then, by minimality of of A, $|A_1| > |A|$ and $B \in A_1$. If $A_1 \in A$, then 2.1(13) for A, B provides the desired contradiction. If $A_1 \notin A$, then then apply 2.1(12) to A and A_1 and find $A_1^* \in A \cap C^{|A_1|}(A^{0|A_1})$ such that $A_1 \subseteq A_1^*$. Now we derive a contradiction replacing A_1 with A_1^*.
□ of the claim.

Consider now the later possibility.
Replace now A,B by A_1, B^*. We will reach, after finitely many steps, models $B^{**} \in C^\tau(A^{0\tau})$ and $D^{**} \in C^\theta$ with $\sup(B^{**} \cap \theta^+) = D^{**} \cap \theta^+$ such that for every A' in p, with $\sup(A' \cap \theta^+) > \sup(B^{**} \cap \theta^+)$, we have $B^{**}, D^{**} \in A'$. Apply the previous lemma 2.16 to B^{**}, D^{**} and add $E^{**}, B^{**} \subseteq E^{**} \subseteq D^{**}$. Then continue, go down and apply 2.16 again and again until finally B and D will be reached.
□

Lemma 2.18 *Let* $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}_{\theta\eta\delta}$ *and* B, D *are models on the wide piste of* p *such that* $|B| = \tau$*, for some* $\tau \in s \cap \theta$*,* $|D| = \theta$ *and* $\sup(B \cap \theta^+) = D \cap \theta^+$*. Then for every regular cardinal* $\rho \in (\tau, \theta)$ *a model of cardinality* ρ *can be added to* p *between* B *and* D*.*

Proof. Let B, D be as in the statement of the lemma and $\rho \in (\tau, \theta)$ a regular cardinal. Suppose that there is no model of size ρ between B and D inside p. Without loss of generality we can assume that $\rho \in s$. Just otherwise extend p by adding the largest (under $\in$) model of cardinality ρ making it potentially limit one.
Let E be the least elementary submodel of D such that

- $|E| = \rho$,

- $E \supseteq B$,
- $E \supseteq \rho$,
- ${}^{\mathrm{cof}(\sup(E\cap\theta^+))>}E \subseteq E$.

Now we would like to add E to p. However in order to so more models probably need to be added. Namely we proceed as follows:

- add E to the wide piste of p,
- add to the wide piste of p models that are needed to be added by 2.17 together with E,
- add all their images under $\Delta-$system triples isomorphisms to $A^{1\rho lim}$,
- change the wide piste and add to new one models that may be needed by 2.17.

The result will be as desired.

□

Lemma 2.19 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$.*

Proof. Let τ be a regular cardinal in the interval $[\eta, \theta]$. We would like to show that $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper. If $\tau = \eta$, then this follows by the previous lemma (2.15). Suppose that $\tau > \eta$. Let $p \in \mathcal{P}_{\theta\eta\delta}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ regular large enough such that

1. $|\mathfrak{M}| = \tau$,
2. $\mathfrak{M} \supseteq \tau$,
3. $\mathcal{P}_{\theta\eta\delta}, p \in \mathfrak{M}$,
4. ${}^{\tau>}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\theta^+)$.
Clearly, M satisfies 2.1(3(b)). Moreover, using the elementarity of $\mathfrak{M}$, for every $x \in M$ there will be $Z \in M$ such that

- $Z \preceq H(\theta^+)$,

- $|Z| = \theta$,
- $Z \supseteq \theta$,
- ${}^{\theta>}Z \subseteq Z$,
- $x \in Z$.

This allows to find a chain $\langle N_i \mid i < \eta \rangle$ of models of size θ which members are in M, witnesses 2.1(3(b)) for $N := \bigcup_{i<\eta} N_i$ and $N \supseteq M$.

Extend p by adding M as a new $A^{0\eta}$ and N as a new $A^{0\theta}$. Require them to be a potentially limit points. Denote the result by $p^\frown\{M, N\}$.
We claim that $p^\frown\{M, N\}$ is $(\mathcal{P}_{\theta\eta\delta}, \mathfrak{M})-$generic. So, let $p' \geq p^\frown M$ and $D \in M$ be a dense open subset of $\mathcal{P}_{\theta\eta\delta}$.
Extending p' more if necessary, we can assume, without loss of generality, that $p' \in D$.
Extend p' further, by applying repeatedly Lemma 2.18, in order to achieve the following:

- for every $\xi \in s(p')$, there is a model B on the wide piste of p' of cardinality ξ such that $M \subseteq B \subseteq N$.

In particular, $\sup(M \cap \theta^+) = \sup(B \cap \theta^+) = N \cap \theta^+$. Denote such B by M_ξ.
Let us denote such extension of p' still by p'.

Pick now $A \preceq H(\theta^+)$ which satisfies the following:

- $|A| = \eta$,
- $A \supseteq \eta$,
- $A \cap \eta^+$ is an ordinal,
- ${}^{\eta>}A \subseteq A$,
- $p' \in A$.
  In particular every model of p' belongs to A.

Extend p' to p'' by adding A as new largest model of cardinality η, i.e. $p'' = p'^\frown A$.
Let us reflect $A = A^{0\eta}(p'')$ down to $\mathfrak{M}$ over over $A^{0\eta}(p'') \cap M$, i.e. we pick some $A' \in M$ and q which realizes the same k-type (for some $k < \omega$ sufficiently big) over $A^{0\eta}(p'') \cap M$ as

$A^{0\eta}(p'')$ and p''. Do this in a rich enough language which includes D as well. [19]
In particular $q \in D \cap M$.

Let us argue that q is compatible with p''.
Set $s = s(q) = s(p'')$. Let $\langle \tau_i \mid i < i^* \rangle$ be an increasing enumeration of s. Pick $\in -$increasing sequence of models $\langle A_i \mid i < i^* \rangle$ such that for every $i < i^*$ the following hold:

1. $p'', q \in A_i$,
2. $|A_i| = \tau_i$,
3. A_i satisfies 2.2(2).

Set $A^{0\tau_i} = A_i$.
Finally let for every $\tau \in s$,

$$A^{1\tau} = \{A^{0\tau}\} \cup A^{1\tau}(p'') \cup A^{1\tau}(q).$$

Define $A^{0\tau lim}$ and $C^\tau(\tau \in s)$ in the obvious fashion now, but do not make $A^{0\xi}, \xi \in s$ potentially limit.

Set

$$p^* = \langle \langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau \rangle \mid \tau \in s \rangle.$$

Then, in p^*, the triple $(A^{0\eta}, A^{0\eta}(p''), A^{0\eta}(q))$ will form a Δ–system triple relatively to M and to the model which corresponds to M under the reflection.

Let check that the wide piste of p^* satisfies 2.1. Suppose that it goes through $C^{0\xi}(A^{0\xi})(p'')$, for each $\xi \in s \cap \tau$, i.e. via the part before the reflection.
Let $B \in C^\rho(A^{0\rho})$ be above M (i.e. $\sup(B \cap \theta^+) > \sup(M \cap \theta^+)$). If B is $A^{0\rho}$, then $p'', q \in B$, and so, every model which is below M is in B.
Suppose that $B \neq A^{0\rho}$. Then B is in p''.

Case 1. $\rho \in s \setminus \tau$.
By 2.1(9(a)), for p'', we have $M \in B$. Hence, all models added by reflection are in Z as well. In addition, by 2.1(9(b)), for p'', we have $N \in B$. So, by 2.1(5), Z cannot be minimal in $C^\rho(A^{0\rho})(p'')$. In addition, the least B on $C^\rho(A^{0\rho})(p'')$ which contains M should be a potentially limit point. So, adding new models of size ρ below M is legitimate.

[19] We follow here a suggestion by Carmi Merimovich to include D into the language which simplifies the original argument considerably.

Case 2. $\rho \in s \cap \tau$.
Then B is among models of p'' that reflect down to M.
Suppose now that $D \in C^{\xi}(A^{0\zeta})$ is below B. Assume that D does not appear in p''. Then D is below M and is the image of a model of p'' under the reflection.
If D is on the wide piste of p^*, then $\xi \geq \tau$. Then there is a model D_ξ on the wide piste of p' of cardinality ξ such that $M \subseteq D_\xi \subseteq N$. Clearly, $D \subseteq D_\xi$ and $D \in D_\xi$.
So, B, D satisfy 2.1(12).

Let $B \in C^{\rho}(A^{0\rho})$ be below M (i.e. $\sup(B \cap \theta^+) < \sup(M \cap \theta^+)$). If $\rho \leq \tau$, then $B \in M$ either by 2.1(9), if B appears in p' or by the reflection otherwise.
Suppose that $\rho > \tau$. If $B \notin M$, then B in p' and there is $B^* \in C^{\rho}(A^{0\rho}) \cap M$ the least such above B, by 2.1(12) for p'. Let $\tilde{B}$ be the image of B under the reflection. Then $\tilde{B} \in C^{\rho}(A^{0\rho}) \cap M$ and also $\tilde{B} \in B^*$, since B satisfies this. Then by 2.1(13) (for p') we must to have $\tilde{B} \in B$. Note that by 2.1(11) (for p', M, B^*), $\tilde{B}$ can be added since the least element of $B^* \cap C^{\rho}(A^{0\rho})$ which is above $B^* \cap M$ is a potentially limit point.

Let us turn to 2.2. The only non-trivial thing to check here is what happens once we change the wide piste to the one that replaces the part of p^* that was reflected by its reflection, according to 2.2(9) .
So, suppose that such switching between the reflecting part and its reflection was made. We need to argue that the result still satisfies 2.1. The issue is the covering. Namely the conditions (12),(13) and (14) of 2.1.
Start with (12).
Let $\xi, \rho \in s, \xi < \rho$, $B \in C^{\xi}(A^{0\xi}), D \in C^{\rho}(A^{0\rho})$ and $\sup(B \cap \theta^+) > \sup(D \cap \theta^+))$. The principle case is when $\tau \leq \xi, \rho$, B is above M and D is below M. Just all models but the maximal ones of cardinalities below τ are below M on the new piste under the consideration.

Recall now that by the choice of p', there is a model $M_\rho, M \subseteq M_\rho \subseteq N$ in $C^{\rho}(p')(A^{0\rho}(p'))$. Then $D \subseteq M_\rho$, since $D \in M$. Apply 2.1(12) to B, M_ρ. It is possible since both are on the wide piste of p'. The model obtained witnesses 2.1(12) for B and D.

Deal now with the requirements (13),(14).
Note that once B of cardinality $\xi \geq \tau$ is above M and both are on the wide piste of p', we have $B \supseteq M$, by 2.1(9). But, D is below M implies $D \in M$, since the reflection made is into M. So, $D \in B$ and we are done.
□

The next lemma is straightforward.

Lemma 2.20 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $< \delta-$strategically closed.*

Proof. Use the strategy to switch each time back to the same (extended) wide piste. Take unions along the wide piste at limit stages. Note that 2.1(12, 13,14) will hold with such limit models, since non-limit ones are closed at least under $<\eta-$sequences and in particular, once including members of an increasing sequence (which length is less than $\delta \leq \eta$) will have the limit inside. Also definable parts (relevant for 2.1(13,14)), will have cofinality $\geq \eta$, and so cannot break down with this new limit models.
$\square$

Combining together Lemmas 2.15,2.19, 2.20, we obtain the following:

Theorem 2.21 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ preserves all cardinals $\leq \delta$ and all cardinals $> \eta$.*
In particular, if $\delta = \eta$, then all cardinals are preserved.

We are not going to force with $\mathcal{P}_{\theta\eta\delta}$ in the cardinal arithmetic applications, but rather to use its members as domains of conditions of a further forcing. However, the forcing with it is of an interest. Thus, for example, $\mathcal{P}_{\theta,\omega,\omega}$, where $\theta < \aleph_{\omega_1}$ regular, may be of an interest on its own since the forcing with it will add a club subset to θ by finite conditions which runs away from every countable set in the ground model.

Let $G \subseteq \mathcal{P}_{\theta,\eta,\delta}$ be a generic. Set

$$C = \{X \cap \theta^+ \mid \exists p \in G(X \in A^{1\theta lim}(p) \wedge \mathrm{cof}(X \cap \theta^+) = \theta)\}$$

and let

$$C' = \{\alpha < \theta^+ \mid \alpha \text{ is a limit of points in } C\}.$$

Lemma 2.22 *Let $\alpha \in C'$ be of cofinality θ, then $\alpha \in C$.*

Proof. Suppose otherwise. Let $p \in \mathcal{P}_{\theta\eta\delta}$ and $\alpha < \theta^+$ be of cofinality θ and

$$p \Vdash \alpha \notin \underset{\sim}{C} \text{ and } \underset{\sim}{C} \text{ is unbounded in } \alpha.$$

Split into two cases.

Case 1. *There is no model A in p with* $\sup(A \cap \theta^+) = \alpha$.

Pick then B to be the least model on the wide piste of p with $\sup(B \cap \theta^+) > \alpha$. Let $A \in B$ be given by 2.1(6). Then, $\sup(A \cap \theta^+) < \alpha$.

If B is not a potentially limit point, then, by 2.9(6), no models can be added between A and B, which contradicts unboundedness of C in α.

If B is a potentially limit point, then we would like to use 2.10 and 2.11 to add a potentially limit point T and a non potentially limit model T' such that

1. $A \in T \in T' \in B$,
2. $T, T' \in C^{|B|}(B)$,
3. $\sup(T \cap \theta^+) < \alpha < \sup(T' \cap \theta^+)$.

This will provide a contradiction, since no element of C then will be inside the interval $(\sup(T \cap \theta^+), \sup(T' \cap \theta^+))$.

By the assumption $\mathrm{cof}(\alpha) = \theta$, $|p| < \delta$ and

$$p \Vdash \underset{\sim}{C} \text{ is unbounded in } \alpha,$$

we can find such T and then by 2.11 also T'.

Case 2. *There is a model B in p with* $\sup(B \cap \theta^+) = \alpha$.

B is then in $A^{1\theta}$, since $\mathrm{cof}(\alpha) = \theta$. Hence $\alpha \in C$. Contradiction.

□

Lemma 2.23 *Let $\alpha \in C'$ be of cofinality $< \delta$, then $\alpha \in C$.*

Proof. Suppose otherwise. Let $p \in \mathcal{P}_{\theta\eta\delta}$ and $\alpha < \theta^+$ be of cofinality $< \eta$ and

$$p \Vdash \alpha \notin \underset{\sim}{C} \text{ and } \underset{\sim}{C} \text{ is unbounded in } \alpha.$$

Split into two cases.

Case 1. $p \Vdash$*(There is no model A in an element of $\underset{\sim}{G}$ with* $\sup(A \cap \theta^+) = \alpha$*).*

We use the assumption

$$p \Vdash \underset{\sim}{C} \text{ is unbounded in } \alpha$$

and construct an increasing sequence of extensions of p of the length $\mathrm{cof}(\alpha)$ having an upper bound which decide elements of C below α. Let q be such an upper bound. Then by 2.1(2(e)), there will be $A \in A^{1\theta}(q)$ with $A \cap \theta^+ = \alpha$. Contradiction.

Case 2. $p \not\Vdash$*(There is no model A in an element of $\underset{\sim}{G}$ with* $\sup(A \cap \theta^+) = \alpha$*).*

Extend p to some p' such that

$p' \Vdash$(There is a model A in an element of $\underset{\sim}{G}$ with $\sup(A \cap \theta^+) = \alpha$).

Let $p'' \geq p'$ decides such A. If $|A| = \theta$, then we are done. Suppose that $|A| < \theta$.

Without loss of generality assume that A is on the wide piste of p''. Being A not limit or not potentially limit contradicts to unboundedness of C in α.

Subcase 2.1. *A is a potentially limit point.*

Let $r \geq p''$. We extend it to some q which has a model D of cardinality θ with $D \cap \theta^+ = \alpha$. Proceed as follows.

Let $\langle A_i \mid i < \mathrm{cof}(\sup(A \cap \theta^+))\rangle$ be the sequence which witnesses 2.1(3(b)) for A.

Claim. *For every $i < \mathrm{cof}(\sup(A \cap \theta^+))$ there is $D \in A$ such that*

1. $|D| = \theta$,

2. $D \supseteq \theta$,

3. $D \preceq H(\theta^+)$,

4. ${}^{\theta>}D \subseteq D$,

5. $A_i \in D$.

Proof. By the assumptions on α and on p, there is $t \geq r$ which has a model D' satisfying the conditions of the claim with $D' \cap \theta^+ < \sup(A \cap \theta^+)$. However such D' may not be in A. If this is the case, then apply 2.1(12) to A and D' (in t). So we will get a model $D^* \in A$ which contains D' and satisfies all conditions of the claim, but may be not (4). Apply elementarity then:

$$H(\theta^+) \models \exists D \in D^* \text{ which satisfies (1),(2),(4),(5) and } D \preceq D^*.$$

Hence there will be such D in B as well.
□ of the claim.

Take now the least $\in -$increasing continuous chain of models $\langle D_i \mid i < \mathrm{cof}(\sup(A \cap \theta^+))\rangle$ with members in A, satisfy (1)-(3) of the claim and successor models satisfy in addition also (4),(5).
Set $D = \bigcup_{i<\mathrm{cof}(\sup(A\cap\theta^+))} D_i$. Now it is possible to add this D to t as a potentially limit point and reflecting under $\Delta-$systems, if necessary, we obtain a desired strengthening q of t.

Subcase 2.2. *B is a limit point.*
Then, we can use the arguments of Subcase 2.1 and to construct an increasing continuous chain of extensions of p' with the upper bound q which has a limit model $D \in A^{1\theta}$, $D \cap \theta^+ = \alpha$.
□

Lemma 2.24 *Let $x \in V$ be a subset of θ^+ of cardinality δ. Then $x \not\subseteq C'$.*

Proof. Let $x \in V$ be a set of cardinality δ and $p \in \mathcal{P}_{\theta\eta\delta}$. Without loss of generality assume that $\sup(x)$ is a limit ordinal of cofinality δ. Set $\nu = \sup(x)$.
If

$$p \Vdash (\underset{\sim}{C}' \text{ is bounded in } \nu),$$

then we are done. So, suppose that

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Split models of p into two groups. Set

$$H_0 = \{A \mid A \text{ appears in } p \text{ and } \sup(A \cap \nu) < \nu\}$$

and

$$H_1 = \{A \mid A \text{ appears in } p \text{ and } \sup(A \cap \nu) = \nu\}.$$

Note that since the total number of models in p is less than δ and δ is a regular, we have $\bigcup_{A \in H_0} \sup(A \cap \nu) < \nu$. Also, if $\delta < \eta$, then $\nu \in A$, for every $A \in H_1$.
Let $\nu^* = \bigcup_{A \in H_0} \sup(A \cap \nu)$.
Take now any $B \in H_1$.

Claim. *For every $\beta, \nu^* < \beta < \nu$, there is $D \in B$ such that*

1. $|D| = \theta$,
2. $D \subseteq \theta$,
3. ${}^{\theta>}D \subseteq D$,
4. $\beta < D \cap \theta^+ < \nu$,
5. D satisfies 2.1(3(b)).

Proof. Assume that $\beta \in B$, since $B \in H_1$ just replacing it by a bigger ordinal if necessary. We have

$$p \Vdash (\underset{\sim}{C}' \text{ is bounded in } \nu).$$

Let $p \in G$. So in $V[G]$, there is $D' \in C$ such that $\beta < D' \cap \theta^+ < \nu$. Then there is an extension q of p with D' inside. If $D' \in B$, then we are done. Suppose otherwise. Apply 2.1(13) to B, D' for q. Then there will be $D'' \in B \cap A^{1\theta lim}$ which includes D' and $D' \cap \theta^+ < D'' \cap \theta^+ < \nu$, since $B \in H_1$.
Now,

$$H(\theta^+) \models \exists D \in D''(\beta \in D \text{ and it satisfies conditions (1)-(5) of the claim}).$$

Just D' witnesses this. By elementarity then

$$B \models \exists D \in D''(\beta \in D \text{ and it satisfies conditions (1)-(5) of the claim}).$$

$\square$ of the claim.

Suppose that the following holds in $V[G]$:

(*) There is an elementary $\in -$chain $\langle D_i \mid i < \delta \rangle$ of elementary submodels of $\langle H(\theta^+), \in, \leq, \delta, \eta \rangle$ such that

- for every $A \in H_1$ on the wide piste of p, $\{D_i \mid i < \delta\} \subseteq A$,
- each D_i satisfies the items (1),(2),(4)(without β),(5) of the claim,
- $\{D_i \cap \theta^+ \mid i < \delta\}$ is unbounded in ν.

Without loss of generality we can assume that $\langle D_i \mid i < \delta \rangle$ is a closed chain.
Recall that

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Hence for every $i < \delta$ there is the least $i', i < i' < \delta$ such that

$$D_{i'+1} \models \exists D \preceq D_{i'}(D \text{ satisfies the items (1)-(5) of the claim, with (4)without } \beta \wedge D_i \in D).$$

Pick the least such D and make it the new D_{i+1}.
Let p' be an extension of p which adds a model X in C above ν^*, but below ν. Change, if necessary, our sequence $\langle D_i \mid i < \delta \rangle$ by removing an initial segment such that $X \in D_1$.
Now we pick some $\xi \in x, D_1 \cap \theta^+ < \xi < D_{i^*+1} \cap \theta^+$, for some $i^* < \delta$. Next, add D_1, D_{i^*+1} to p, D_1 as a potentially limit point and D_2 as a non-limit and non-potentially limit point. The requirement 2.1(11) will hold, since D_1 is a potentially limit point above X (this for models in H_0) and D_1, D_{i^*+1} are in every model of H_1 on the wide piste. Reflect them through all relevant splittings. Let q be the result. Then

$$q \Vdash x \not\subseteq \underset{\sim}{C},$$

since nothing can be added between D_1 and D_{i^*+1}.

Let us now argue that (*) holds.
Split into two cases.

Case 1. $\delta > \omega$.
Work in $V[G]$ with $p \in G$. Let $\langle E_i \mid i < \delta \rangle$ be an increasing sequence of members of C unbounded in ν with $E_i \cap \theta^+ < \nu$.
Let $A \in H_1$ be a non-limit model on the main piste of p and $i < \delta$. Pick an extension $p_i \in G$ of p such that E_i appears in p_i. Then, by 2.1(12), there is $E_i^A \in A \cap C^\theta(A^{0\theta})(p_i)$ the least which contains E_i. By 2.1(13), $E_i^A \cap \theta^+ < \nu$, since $A \in H_1$.
Consider $\{E_i^A \mid i < \delta\}$. Adding limit models if necessary we can assume that it is a closed chain. A is a non-limit, hence it is closed under less than $\delta-$sequence of its elements, so, $\{E_i^A \mid i < \delta\} \subseteq A$. Set

$$Y^A := \{E_i^A \cap \theta^+ \mid i < \delta\}.$$

Consider

$$Y := \bigcap\{Y^A \mid A \in H_1 \text{ non-limit and on the wide piste of } p\}.$$

Then Y is an intersection of fewer than δ clubs, and hence is a club in ν. Now, $Y \subseteq A$, for every $A \in H_1$ non-limit and on the wide piste of p. If B is a limit model on the wide piste of p and $B \in H_1$, then B is an increasing union of less than δ non-limit models from the wide piste. Then the final segment of them is in H_1, and hence contains Y. So, $Y \subseteq B$.
Finally note that if E, E' two models which appear in $A^{1\theta}(r)$, for some r, and $E \cap \theta^+ = E' \cap \theta^+$, then $E = E'$, by 2.1(2). So, take any $A \in H_1$ non-limit and on the wide piste of p, consider the sequence

$$\langle E_i^A \mid i < \delta, E_i^A \cap \theta^+ \in Y\rangle.$$

It will witness (*).

Case 2. $\delta = \omega$.
Work in V. Conditions are finite now.
Pick now the least A on the wide piste of p of cardinality η in H_1. It easy to insure that such A exists just by extending p, if necessary.
Apply the claim to A and construct an elementary chain $\langle D_i \mid i < \delta\rangle$ such that

- $\{D_i \mid i < \delta\} \subseteq A$,
- for every $i < \delta$, D_{i+1} satisfies the conditions (1)-(5) of the claim.

If $\{D_i \mid i < \delta\} \subseteq B$, for every $B \in H_1$ on the wide piste, then we are done.
Suppose now that there is $B \in H_1$ on the wide piste which does not contain $\{D_i \mid i < \delta\}$. Then, necessary, $\theta > |B| > \eta$ and $\sup(B \cap \theta^+) < \sup(A \cap \theta^+)$ by 2.1(9).
Let A_1 be such B of the least possible cardinality and the least such of this cardinality. Denote $\eta_1 = |A_1|$.
If $A_1 \in A$, then set $A_1^* = A_1$. Otherwise, use 2.1(13). There will $A_1^* \in A \cap C^{\eta_1}(A^{0\eta_1})(p)$, $A_1^* \supseteq A_1$ with $A \cap A_1 = A \cap A_1^*$. Let $\langle D_i^1 \mid i < \delta\rangle$ be the sequence defined inside A but applying the claim to A_1^* and picking the least possible model there. So, $\{D_i^1 \mid i < \delta\} \subseteq A \cap A_1^* = A \cap A_1$.
If $\{D_i^1 \mid i < \delta\} \subseteq B$, for every $B \in H_1$ on the wide piste, then we are done.
Suppose that there is $B \in H_1$ on the wide piste which does not contain $\{D_i^1 \mid i < \delta\}$. Then, necessary, $\theta > |B| > \eta_1$ and $\sup(B \cap \theta^+) < \sup(A_1 \cap \theta^+)$ by 2.1(9).
Let A_2 be such B of the least possible cardinality and the least such of this cardinality. Denote $\eta_2 = |A_2|$.

Set $A_2^* = A_2$, if $A_2 \in A$. Otherwise, use 2.1(13). There will $A_2^* \in A \cap C^{\eta_1}(A^{0\eta_1})(p), A_2^* \supseteq A_2$ with $A \cap A_2 = A \cap A_2^*$.
If $A_2^* \in A_1^*$, then let $\langle D_i^2 \mid i < \delta \rangle$ be the sequence defined inside A but applying the claim to A_2^* and picking the least possible model there. So, $\{D_i^2 \mid i < \delta\} \subseteq A \cap A_1^* \cap A_2^* = A \cap A_1 \cap A_2$.
Suppose that $A_2^* \notin A_1^*$. If $A_2^* \supseteq A_1^*$, then $\langle D_i^1 \mid i < \delta \rangle$ will do for $A \cap A_1 \cap A_2$, which is not the case. Hence $A_2^* \not\supseteq A_1^*$. Then $\sup(A_2^* \cap \theta^+) < \sup(A_1^* \cap \theta^+)$, since $\eta_1 < \eta_2$. Apply 2.1(13)to A_2^* and A_1^*. So there is $A_{12} \in A_1^* \cap C^{\eta_2}(A^{0\eta_2})(p), A_{12} \supseteq A_2^*$ with $A_1^* \cap A_{12} = A_1^* \cap A_2^*$. If $A_{12} \in A$, then we look for a sequence inside it. If $A_{12} \notin A$, then apply 2.1(13) to A_{12} and A and get $A_{12}^* \in A \cap C^{\eta_2}(A^{0\eta_2})(p), A_{12}^* \supseteq A_{12}$. The process necessarily terminates after finitely many steps, since $\delta = \omega$, and hence, p is finite.
Suppose for simplicity that already $A_{12}^* \in A_1^*$.
Let then $\langle D_i^3 \mid i < \delta \rangle$ be the sequence defined inside A but applying the claim to A_{12}^* and picking the least possible model there. So,

$$\{D_i^3 \mid i < \delta\} \subseteq A \cap A_1^* \cap A_{12}^* = (A \cap A_{12}^*) \cap A_1^* = A \cap A_{12} \cap A_1^* = A \cap (A_{12} \cap A_1^*) =$$

$$A \cap A_2^* \cap A_1^* = A \cap A_2 \cap A_1^* = (A \cap A_1^*) \cap A_2 = A \cap A_1 \cap A_2.$$

If $\{D_i^3 \mid i < \delta\} \subseteq B$, for every $B \in H_1$ on the wide piste, then we are done. Otherwise there is $B \in H_1$ on the wide piste which does not contain $\{D_i^3 \mid i < \delta\}$. Then, necessary, $\theta > |B| > \eta_2$ and $\sup(B \cap \theta^+) < \sup(A_2 \cap \theta^+)$ by 2.1(9). So, again we are going down. After finitely many steps the desired sequence will be reached.
□

In particular, taking $\delta = \eta = \omega$, we obtain the following generalization of corresponding results by J. Baumgartner [1], S. Friedman [2] and by W. Mitchell [10] to higher cardinals[20]:

Corollary 2.25 *The forcing $\mathcal{P}_{\theta\omega\omega}$ is cardinals preserving forcing adding a club in θ^+ using finite conditions.*

Remark 2.26 Given a stationary subset S of θ^+ such that for every $\alpha < \theta^+$ if $\mathrm{cof}(\alpha) < \theta$, then $\alpha \in S$. It is easy to modify the forcing $\mathcal{P}_{\theta\eta\delta}$ such that it will add a club through S. Only require that for every X of cardinality θ in a condition we have $X \cap \theta^+ \in S$.

3 Suitable structures.

We reorganize here the structures with pistes of the previous section in order to allow isomorphisms of them over different cardinals.

[20]Note that Magidor and Radin forcings ([7],[11]) also add clubs by finite conditions.

Definition 3.1 Let $\delta < \kappa < \theta$ be cardinals and δ, θ is a regular. A structure $\mathfrak{X} = \langle X, E, E^{lim}, C, S, \in, \subseteq \rangle$, where $E \subseteq [X]^2$ and $C \subseteq [X]^3$ is called a $\delta-$suitable structure with pistes over κ of the length θ iff there is a δ structure with pistes over κ^+ of the length θ $p(\mathfrak{X}) = \langle\langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), A^{1\tau lim}(\mathfrak{X}), C^\tau(\mathfrak{X})\rangle \mid \tau \in s(\mathfrak{X})\rangle$ such that

1. $X = A^{0\eta}(\mathfrak{X}) \cup \{A^{0\eta}(\mathfrak{X})\}$, where $\eta \in s(\mathfrak{X})$ is such that for every $\tau \in s(\mathfrak{X})$ we have then $A^{0\tau}(\mathfrak{X}) \in X$ or $A^{0\tau}(\mathfrak{X}) \subseteq X$,
2. $S = s(\mathfrak{X})$,
3. $\langle a, b\rangle \in E$ iff $a \in S$ and $b \in A^{1a}(\mathfrak{X})$,
4. $\langle a, b\rangle \in E^{lim}$ iff $a \in S$ and $b \in A^{1alim}(\mathfrak{X})$,
5. $\langle a, b, d\rangle \in C$ iff $a \in S, b \in A^{1a}(\mathfrak{X})$ and $d \in C^a(\mathfrak{X})(b)$.

Let us refer to $\mathfrak{X}$ for shortness as a a $\delta-$suitable structure once κ, θ are fixed.

Note that $p(\mathfrak{X})$ is uniquely defined from $\mathfrak{X}$. Also, it is easy to define a δ-suitable structure from $p \in \mathcal{P}_{\kappa^+\theta\delta}$.

Definition 3.2 Let $\mathfrak{X}, \mathfrak{Y}$ be δ-suitable structures. Set $\mathfrak{X} \leq \mathfrak{Y}$ iff $p(\mathfrak{X}) \leq p(\mathfrak{Y})$.

3.1 Forcing conditions.

Let κ be a limit of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega\rangle$ with each κ_n being strong up to the least Mahlo cardinal λ_n above κ_n as witnessed by an extender E_n.

For every $n < \omega$ define Q_{n0}.

Definition 3.3 Let Q_{n0} be the set of the triples $\langle a, A, f\rangle$ so that:

1. f is a partial function from θ^+ to κ_n of cardinality at most κ,
2. a is an isomorphism between a κ_n–suitable structure $\mathfrak{X}$ over κ of the length θ and a κ_n–suitable structure $\mathfrak{X}'$over κ_n^{+n} of the length λ_n such that
   (a) X' is above every model which appears in $(\bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')) \setminus \{X'\}$, in the order $\leq_{E_n}$, (or actually after codding X' by an ordinal),

(b) if $t \in \bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')$,then for some $k, 2 < k < \omega$, $t \prec H(\chi^{+k})$, with χ big enough fixed in advance.

Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image t of a model from X will be elementary submodel of $H(\chi^{+k})$.

The way to compare such models $t_1 \prec H(\chi^{+k_1})$, $t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:

move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{nX'}$,

4. for every ordinals α, β, γ which code models in $\bigcup_{\tau \in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')$, we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \text{ implies}$$

$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho)),$$

for every $\rho \in \pi''_{X'\alpha} A$.

Definition 3.4 Let $\langle a, A, f\rangle, \langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $\mathrm{dom}(a) \geq \mathrm{dom}(b)$,
2. $\mathrm{ran}(a) \geq \mathrm{ran}(b)$,
3. $a \supseteq b$,
4. $f \supseteq g$,
5. $\pi^{E_n}_{\max(\mathrm{ran}(a)),\max(\mathrm{ran}(b))}$ "$A \subseteq B$.

Definition 3.5 Q_{n1} consists of all partial functions $f : \theta \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

Definition 3.6 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq^*_n = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f\rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown\nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{ran}(a)),a(\alpha)}(\nu) \mid \alpha \in A^{1\theta}(\mathrm{dom}(a)) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $A^{1\theta}(\mathrm{dom}(a))$ are used to produce non direct extensions, the rest of models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq_n^* q$ or $p \in Q_{n1}, q = \langle b, B, g\rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown \nu$.

Definition 3.7 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega\rangle$ so that

1. for every $n < \omega, \quad p_n \in Q_n$,
2. there is $\ell(p) < \omega$ such that
   (a) for every $n < \ell(p), \quad p_n \in Q_{n1}$,
   (b) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n\rangle \in Q_{n0}$,
   (c) if $\ell(p) \leq n \leq m$, then $\mathrm{dom}(a_n) \leq \mathrm{dom}(a_m)$,
   (d) if $\ell(p) \leq n \leq m$, then $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$.
3. For every $n \geq m \geq \ell(p), \quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,
4. for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite.] (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).
5. For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.
6. There is a κ-structure with pistes $\mathfrak{p}$ over κ such that
   (a) $\mathfrak{p} \geq \mathrm{dom}(a_n)$, for every $n, \ell(p) \leq n < \omega$,
   (b) if a model A appears in $\mathfrak{p}$, then A appears in $\mathrm{dom}(a_n)$ for some $n, \ell(p) \leq n < \omega$ (and then in a final segment of them),
   (c) $\max(\mathrm{dom}(a_n)) = \max(\mathfrak{p})$ (actually this follows from the previous condition).

Note that $\mathfrak{p}$ of 3.7(6) is uniquely determined by p. Let us refer to it further as the κ-structure with pistes over κ of p.

Lemma 3.8 *$\langle Q_{n0}, \leq_{n0}\rangle$ is $< \kappa_n$-strategically closed.*

Lemma 3.9 *$\langle \mathcal{P}, \leq^*\rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 3.10 *$\langle \mathcal{P}, \leq^*\rangle$ satisfies the Prikry condition.*

Lemma 3.11 *Let $p \in \mathcal{P}$ and $\alpha < \theta^+$, then there are $q \geq^* p$ and $\beta, \alpha < \beta < \theta^+$ such that $\beta = M \cap \theta^+$, for some M which appears in q.*

Proof. Pick some $M \prec H(\theta^+)$ of size θ which is above the maximal model of $\mathfrak{p}$ (say $\mathfrak{p} \in M$) and such that $M \cap \theta^+ > \alpha$. Add it to p. Let q be the resulting condition. Then it is as desired.
$\square$

The next lemma follows now:

Lemma 3.12 *Let G be a generic subset of $\langle \mathcal{P}, \leq \rangle$. Then in $V[G]$ there are $\mathrm{cof}((\theta^+)^V)$–many ω–sequences of ordinals below κ.*

Define $\to$ on $\mathcal{P}$ as in [3].

κ^{++}–c.c. and even θ^+–c.c. break down here for the forcing $\langle \mathcal{P}, \to \rangle$.
Following C. Merimovich [9] we replace them by properness.

3.2 Properness.

We will turn now to the properness of the forcing. The proofs repeat almost completely those of Lemmas 2.15,2.19

Lemma 3.13 *$\langle \mathcal{P}, \to \rangle$ is κ^+-proper.*

Lemma 3.14 *$\langle \mathcal{P}, \to \rangle$ is η-proper, for every regular $\eta, \kappa^+ < \eta \leq \theta$.*

The proofs repeat almost completely those of Lemmas 2.15,2.19. The only additional ingredient is to put new models that were added below κ in the process of extension of conditions inside old ones. As usual, in [3], we use $\longleftrightarrow$ for this purpose and pass to equivalent models.

Finally, combining together Lemmas 3.9, 3.10, 3.12, 3.13, 3.14, we obtain the following:

Theorem 3.15 *Let G be a generic subset of $\langle \mathcal{P}, \to \rangle$. Then $V[G]$ is cofinalities preserving extension of V in which $2^\kappa = \kappa^\omega = \theta^+$.*

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References

[1] J. Baumgartner,

[2] S. Friedman,

[3] M. Gitik, Short extenders forcings I,
http://www.math.tau.ac.il/∼gitik/short%20extenders%20forcings%201.pdf

[4] M. Gitik, Remarks on non-closure of the preparation forcing and off-piste versions of it,
http://www.math.tau.ac.il/∼gitik/closed.pdf

[5] M. Gitik and S. Unger, Adding many ω–sequences to a singular cardinal, Notes from Appalachian Set Theory workshop April 3, 2010,
http://www.math.tau.ac.il/∼gitik/GitikAST.pdf

[6] M. Gitik, Short extenders forcings – doing without preparations.Bounded gaps.
http://www.math.tau.ac.il/∼gitik

[7] M. Magidor, Changing co

nality of cardinals, Fundamenta Mathematicae, 99(1):61-71, 1978.

[8] C. Merimovich, The short extenders gap two forcing is of Prikry type, Arch. Math. Logic, 48(2009),737-747.

[9] C. Merimovich, The short extenders gap three forcing using a morass, Arch. Math. Logic,50(2011),115-135.

[10] W. Mitchell,

[11] L. Radin, Adding closed co

nal sequences to large cardinals, Annals of Mathematical Logic, 22(3):243-261, 1982.

[12] S. Shelah, Proper and Improper forcing, Springer 1998.

[13] D. Velleman, Simplified morasses, JSL 49(1),257-271(1984)