Short extenders forcings – doing without preparations.

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Abstract

We introduce certain morass type structures and apply them to blowing up powers of singular cardinals. As a bonus, a forcing for adding clubs with finite conditions to higher cardinals is obtained.

1 Introduction.

We would like to present a way of doing short extenders forcings without forcing first with a preparation forcings of type $\mathcal{P}'$ of [6]. The main issue with short extenders forcings is to show that κ^{++} and cardinals above it are preserved in the final model. In [6] the preparation forcing (which added a structure with pistes) was used eventually to show κ^{++}-c.c. of the main forcing. A negative side of this preparation forcing is that it is only strategically closed which is not enough in order to preserve large cardinals like a supercompact. Actually it adds a version of the square principle which is incompatible with supercompacts [7].
Carmi Merimovich [12] used for the gap 3 a variation of Velleman's simplified morass [16] instead. κ^{++}-c.c. breaks down but he was able to show κ^{++}–properness instead. The forcing adding a simplified morass is directed closed enough in order to preserve supercompact cardinals. Unfortunately generalizations (at least those that we considered) of Merimovich's idea of first adding a simplified morass and then using a properness instead of a chain condition of the main forcing run into severe difficulties already for Gap 4.

Here we suggest an other way. The main forcing will be used directly over V without a preparation. Actually a simple version of the preparation forcing of [6] will be incorporated directly into the main forcing. Again as in [12] κ^{++}-c.c. will break down and we will show a properness instead.

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In this paper we will deal with a general situation - no bounds on a gap between a singular cardinal and its power. The situation where the gap between a singular cardinal κ and its power is bounded by $\kappa^{+\kappa^+} = \aleph_{\kappa^+}$ is considered in [9]. The arguments there are slightly easier, but not essentially.

The main instrument introduced here is called *structures with pistes*. It seems of an interest by its own. Beyond cardinal arithmetic applications stated above, it is applied in a further paper to a certain generalizations of Forcing Axioms to higher cardinals. Here we will apply it to adding clubs by finite conditions.

The paper is organized as follows.

In Section 2 we introduce a $\delta-$structure with pistes over η of the length θ. Basic properties of such structures like the intersection property, possible extensions etc. are studied here. A forcing with piste structures is introduced. Its properness is proved. An application to adding clubs is given at the end of this section.

Section 3 is devoted to a cardinal arithmetic application. We show how using structures of this type it is possible to blow up the power of a singular cardinal.

2 Structures with pistes–general setting.

Assume GCH.

The basic idea behind the structures defined below (Definition 2.3,$\delta-$structure with pistes over η of length θ) is to stay as close as possible to an elementary chain of models. It cannot be literally a chain since models of different sizes are involved and models of bigger cardinality can come before ones of a smaller. The first part (Definition 2.1) describes this "linear" part of conditions in the main forcing. It is called *a wide piste* and incorporates together elementary chains of models of different cardinalities. The main forcing, defined in Section 2.3, will be based on such wide pistes and involves an additional natural but non-linear component called splitting or reflection.

Definition 2.1 Let $\delta \leq \eta < \theta$ be regular cardinals.
A $(\theta, \eta, \delta)-$wide piste is a set $\langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle$ such that the following hold.[1]
Let us first specify sizes of models that are involved.

1. (Support) s is a set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:

   (a) $|s| < \delta$,

   (b) $\eta, \theta \in s$.
   Which means that the minimal and the maximal possible sizes are always present.

2. (Models) For every $\tau \in s$ and $A \in C^\tau$ the following holds:

   (a) $A \preccurlyeq \langle H(\theta^+), \in, \leq, \delta, \eta\rangle$, where $\leq$ is some fixed well ordering of $H(\theta^+)$,

   (b) $|A| = \tau$,

   (c) $A \supseteq \tau + 1$,

   (d) $A \cap \tau^+$ is an ordinal,

   (e) elements of C^τ form a closed $\in -$chain, of length $< \delta$, with a largest element ,

   (f) if $X \in C^\tau \setminus C^{\tau lim}$ is a non-limit model (i.e. is not a union of elements of C^τ), then ${}^{\tau >}X \subseteq X$.

   (g) if $X, Y \in C^\tau$ then $X \in Y$ iff $X \subsetneq Y$,

[1] The main application will be to the case when $\eta = \kappa^+$ for a cardinal κ which is an $\omega-$limit of sufficiently strong cardinals, but still without extenders which overlap κ. An other application is to forcing axioms, and for it we use $\delta = \eta = \omega$.

3. (Potentially limit points) Let $\tau \in s$.
$C^{\tau lim} \subseteq C^{\tau}$. We refer to its elements as *potentially limit points*.
The intuition behind this is that it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.
Let $X \in C^{\tau lim}$. Require the following:

   (a) X is a successor point of C^{τ}.

   (b) (Increasing union) There is an increasing continuous $\in -$chain $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+)) \rangle$ [2] of elementary submodels of X such that

      i. $\bigcup_{i<\mathrm{cof}(\sup(X\cap\theta^+))} X_i = X$,

      ii. $|X_i| = \tau$,

      iii. $X_i \supseteq \tau$,

      iv. $X_i \in X$,

      v. ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

   (c) (Degree of closure of potentially limit point)
   Either

      i. ${}^{\tau>}X \subseteq X$
      or

      ii. $\mathrm{cof}(\sup(X \cap \theta^+)) = \xi$ for some $\xi \in s \cap \tau$ and then there are $X_\xi \in C^{\xi lim}, X^\theta \in C^{\theta lim}$ such that

         A. $\sup(X_\xi \cap \theta^+) = \sup(X \cap \theta^+) = X_\theta \cap \theta^+$,

         B. $X_\xi \subseteq X \subseteq X_\theta$,

         C. $|X| = \tau \in X_\xi$,

         D. X is the least elementary submodel of X_θ which includes X_ξ and $\tau + 1$,

         E. there is a sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+)) \rangle$ witnessing 3(b) for X_θ whose members belong to X_ξ.

         F. For any $\mu \in s, \xi < \mu < \theta$, if $Y \in C^\mu$ is the least in C^μ such that $X_\xi \in Y$, then $X \in Y$ and Y is a potentially limit point.

      Note that if $\tau > \omega$, $\langle X_i \mid i < \mathrm{cof}(\sup(X\cap\theta^+)) \rangle$ and $\langle X'_i \mid i < \mathrm{cof}(\sup(X\cap\theta^+)) \rangle$ are two sequences which witness (3b) above, then the set $\{i < \mathrm{cof}(\sup(X \cap \theta^+)) \mid X_i = X'_i\}$ is closed and unbounded.

[2] These models need not be in C^τ, but rather allow to add in future extensions models below X

It is possible using the well ordering $\leq$ to define a canonical witnessing sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ for X.

Let us first do this for X such that $\mathrm{cof}(\sup(X \cap \theta^+)) = \tau$ (or for X_ξ of (3c(ii)(C)) above). Let $\langle x_\nu \mid \nu < \tau\rangle$ be an enumeration of X (defined using $\leq$). We proceed by induction. Once $i < \tau$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least elementary submodel of X such that

- $x_i \in X_{i+1}$,
- $X_i \in X_{i+1}$,
- $|X_i| = \tau$,
- $X_i \supseteq \tau$,
- ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

By (3b), it is possible to find such X_{i+1}.

Clearly $\bigcup_{i<\tau} X_i = X$.

Suppose now that $\mathrm{cof}(\sup(X \cap \theta^+)) = \xi \in s \cap \tau$. Then let us use the canonical sequence $\langle X_{i\xi} \mid i < \xi = \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ for X_ξ in order to define the canonical sequence $\langle X_i \mid i < \mathrm{cof}(\sup(X \cap \theta^+))\rangle$ for X.

Proceed by induction. If $i < \tau$ is a limit then set $X_i = \bigcup_{i'<i} X_{i'}$. Pick X_{i+1} to be the least (in the well order $\leq$ of $H(\theta^+)$) elementary submodel of $H(\theta)$ such that

- $X_{i+1} \in X_\xi$,
- $X_{i\xi} \in X_{i+1}$,
- $X_i \in X_{i+1}$,
- $|X_{i+1}| = \tau$,
- $X_i \supseteq \tau + 1$,
- ${}^{\tau>}X_{i+1} \subseteq X_{i+1}$.

By (3c(ii)B), it is possible to find such X_{i+1} inside X_ξ.

Note that the existence of such canonical sequences implies that X itself is definable from X_ξ and τ.

The next condition prevents unneeded appearances of small models between big ones.

4. If $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point and B_0 is its immediate predecessor, then there is no potentially limit point $A \in C^\tau$ with $\tau < \rho$ such that $B_0 \in A \in B_1$.

The requirement that B_1 is not a potentially limit point is important here. Once dealing with potentially limit points, we would like to allow reflections which may add small intermediate models.
However, small models which are non-potentially limit points are allowed.

5. Let $B_0, B_1 \in C^\rho$, for some $\rho \in s$, B_1 is not a potentially limit point, B_0 is its immediate predecessor and $A \in C^\tau \cap B_1$, with $\tau < \rho$. If $\sup(A \cap \theta^+) > \sup(B_0 \cap \theta^+)$, then $B_0 \in A$.

   The next condition is of a similar flavor, but deals with smallest models.

6. If $B \in C^\rho$, for some $\rho \in s$, is not a potentially limit point and it is the least element of C^ρ, then there is no potentially limit point $A \in C^\tau$ with $\tau > \rho$ such that $A \in B$[3].

   Both conditions 4 and 6 are designed to allow one to add new models below potentially limit points, which will be essential for properness of the forcing.

   The next condition deals with closure and is desired to prevent some pathological patterns.

7. Let $B \in C^\rho$, for some $\rho \in s$, be a non-limit point of C^ρ. If there are models $A \in \bigcup_{\xi \in s} C^\xi$ with $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then there is $A \in B \cap \bigcup_{\xi \in s} C^\xi$ such that

   (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$,

   (b) for every $A' \in \bigcup_{\xi \in s} C^\xi$ with $\sup(A' \cap \theta^+) < \sup(B \cap \theta^+)$, $\sup(A' \cap \theta^+) \leq \sup(A \cap \theta^+)$.

   Such A is the "real" immediate predecessor of B. Further, in the definition of the order, we will require that once B is not a potentially limit point, then no models E such that $A \in E \in B$ can be added.

   The purpose of the next four conditions is to allow to proceed down the pistes without interruptions at least before reaching a potentially limit point.

8. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^ρ. Then $B' \in A$.

9. Let $\tau, \rho, \rho^* \in s, \tau < \rho < \rho^*$, $A \in C^\tau, B \in C^{\rho^*}, D \in C^\rho$ and $B \in A$. Suppose that B is not a potentially limit point and B' is its immediate predecessor in C^ρ.
   Then $B' \in D \in B$ implies $D \in A$.

[3]If we drop the requirement $\tau > \rho$, then it may be impossible further to add models of sizes $> \eta$ once a potentially limit point of size η is around.

10. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is a limit point in C^ρ. Let $\langle B_\nu \mid \nu < \nu^* < \delta\rangle$ be $C^\rho \cap B$. Then a closed unbounded subsequence of $\langle B_\nu \mid \nu < \nu^*\rangle$ is in A.

11. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$ and $B \in A$. Suppose that B is a limit point in C^ρ. Let $D \in \bigcup_{\mu \in s} C^\mu$ be such that $\sup(D \cap \theta^+) = \sup(B \cap \theta^+)$. Then $|D| \in A$ implies that $D \in A$.

12. (Linearity) If $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^\rho$, then

    (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ implies $A \in B$,

    (b) $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$ implies $A \subseteq B$.

    The next condition will be used to ensure that the maximal models are linearly ordered by a combination of $\in, \subset$ −relations.

13. (Linearity at the top) Let $\tau, \rho \in s, \tau < \rho$, B be the maximal models of C^ρ and A be the maximal models of C^τ. Then

    (a) $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ implies $A \in B$,

    (b) $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$ implies $A \subset B$,

    (c) $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ implies $B \in A$.

    This condition is unneeded if $\theta < \aleph_{\eta^+}$, as will shown further in Lemma 2.2.

14. Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^{\rho lim}$, $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \in A$. Suppose that there exists $X \in \bigcup_{\mu \in s} C^\mu$, $\sup(X \cap \theta^+) = \sup(B \cap \theta^+)$ and $X \neq B$. Then $X_{\mathrm{cof}(\sup(X \cap \theta^+))}, X_\theta$ of Item 3c(ii) above, belong to A as well.

15. (Immediate successor restriction) Let $\tau, \rho \in s, \tau < \rho$, $A \in C^\tau, B \in C^{\rho lim}$, $\mathrm{cof}(\sup(B \cap \theta^+)) > \tau$ and $B \in A$. Suppose that there a model $B' \in B \cap C^\rho$ such that $\sup(B' \cap \theta^+) > \sup((A \cap B) \cap \theta^+)$. Then the least such B' is a potentially limit model. I.e., if there is a model in C^ρ between $A \cap B$ and B, then the least such model is a potentially limit model.
    The condition 15 is designed to prevent the situation when there is $E \in A \cap C^\rho$ which has a non-potentially limit immediate successor E'' in B but not in A. Also it prevents a possibility that the least element Y of C^ρ is a non-potentially limit point which

belongs to B and is above $A \cap B$.

The next condition is needed further for $\tau-$properness argument[4].

Next conditions deal with a possibility to cover a model $D \notin B$ such that $\sup(B \cap \theta^+) > \sup(D \cap \theta^+)$ by a model which belongs to B.

We deal separately with cases $\theta < \aleph_{\eta^+}$ and $\theta > \aleph_{\eta^+}$. It is not too hard to combine both cases together, but our imprecision is that considering first a simpler case $\theta < \aleph_{\eta^+}$ would better explain the intuition behind the general case. This seems to be true not only in this definition, but rather through the paper.

16. (Covering under $\theta < \aleph_{\eta^+}$) If $\tau, \rho \in s, \tau < \rho$, $B \in C^\tau, D \in C^\rho$ and $\sup(B \cap \theta^+) > \sup(D \cap \theta^+)$, then there is $D^* \in B \cap C^\rho$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$.[5] Note that the set $Z := \{\mu \leq \theta \mid \mu \text{ is a regular cardinal } \}$ belongs to B, by elementarity. Its cardinality is at most η, so $Z \subseteq B$.

    The next conditions describe a very particular way of covering and it is crucial for the properness arguments.

17. (Strong covering under $\theta < \aleph_{\eta^+}$) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau, \rho \in B$ and $\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Then either

    (a) $D \in B$,
    or

    (b) $D \notin B$ and the least $D^* \in C^\rho \cap B, D^* \supset D$ is closed under $< \rho-$ sequences of its elements.Then $B \cap D^* \subseteq D$ and

$$\{D' \in D^* \mid (|D'| = \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$

$$((\forall k < n)(|Z_k| < \rho)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D^6.$$

[4]In an earlier version of the paper, we defined a model $B_A := \bigcup_{i \in A \cap \text{cof}(\sup(B \cap \theta^+))} B_i$ (where $\langle B_i \mid i < \text{cof}(\sup(B \cap \theta^+))\rangle$ is a chain which witnesses (3(b)) and added it to $C^{\rho lim}$. Having such B_A in $C^{\rho lim}$ implies impossibility of the situations above. Here we do without B_A and this simplifies the major arguments like intersection properties and properness. However, getting a club that runs away from sets in V becomes a bit more complicated.

[5]Note that if $D \notin B$, then such D^* must be a potentially limit point by Items 8 and 10 above. Thus, it cannot be a successor non-potentially limit point, by Item 8, since its immediate predecessor $D^{*'}$ will be in B, and then, by minimality of D^*, $\sup(D^{*'} \cap \theta^+) < \sup(D \cap \theta^+)$, and so $D \supseteq D^{*'}$.
By Item 10, D^* cannot be a limit point of C^ρ.

[6]Note that GCH is assumed, so the cardinality of this set is less than ρ. Then it is in D^*, once D^* is closed under $< \rho-$sequences of its elements.

Or

(c) $D \notin B$ and the least $D^* \in C^\rho \cap B, D^* \supset D$ is not closed under $< \rho-$ sequence of its elements.
Let $\text{cof}(\sup(D^* \cap \theta^+)) = \xi$ for some $\xi \in s \cap \rho$ and let $E \in C^{\xi lim}$ such that $\sup(E \cap \theta^+) = \sup(D^* \cap \theta^+)$ (such E exists by Item 3c and $E \in B$ by Item 14, since $D^* \in B$).
Then either

i. $D \in E$, $B \cap D^* \subseteq D$ and

$$\{D' \in D^* \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$
$$((\forall k < n)(|Z_k| < \xi)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D.$$

Or

ii. $D \notin E$, $\sup(E \cap \theta^+) = \sup(D^* \cap \theta^+) > \sup(D \cap \theta^+)$,
and then, let $D^{**} \in C^\rho \cap E$ be the least such that $D^{**} \supset D$.If D^{**} is closed under $< \rho^{**}-$ sequence of its elements, then $B \cap D^* \subseteq D, E \cap D^{**} \subseteq D$ and

$$\{D' \in D^{**} \mid (|D'| \leq \rho \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)$$
$$((\forall k < n)(|Z_k| < \rho)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D$$

If D^{**} is not closed under $< \rho-$ sequence of its elements, then the process repeats itself going down below D^{**}. After finitely many steps we will either reach D or D will be above everything related to B. Let us state this formally.
So suppose that D^{**} is not closed under $< \rho-$ sequences of its elements.
Then are $n^* < \omega, \{\xi_n \mid n < n^*\} \subseteq s \setminus \rho + 1, \langle E_n \mid n < n^* \rangle, \langle D_n \mid n \leq n^* \rangle$ such that for every $n \leq n^*$ the following hold:

A. $D_0 = D^*$,

B. $E_0 = E$,

C. $D_n \in C^\rho$,

D. $D_n \supseteq D$,

E. $D_{n+1} \in D_n$,

F. $\mathrm{cof}(\sup(D_n \cap \theta^+)) = \xi_n$,

G. $E_n \in C^{\xi_n}$,

H. $E_{n+1} \in E_n$,

I. $\xi_{n+1} > \xi_n$,

J. $\sup(D_n \cap \theta^+) = \sup(E_n \cap \theta^+)$,

K. $D_{n+1} \in E_n$ is the least in $C^\rho \cap E_n$ with $D_{n+1} \supset D$,

L. $B \cap D_0 \subseteq D$,

M. $E_n \cap D_{n+1} \subseteq D$,

N. $\{D' \in D_{n+1} \mid (|D'| = \rho) \wedge (\exists m < \omega)(\exists Z_{m-1} \in ... \in Z_0 \in B)$
$((\forall k < m)(|Z_k| < \xi_n)) \wedge D' \in B \cup \bigcup_{k<m} Z_k))\} \in D$,

O. $D_{n^*} = D$ or, we have, $D \in D_{n^*}, {}^{\rho>}D_{n^*} \subseteq D_{n^*}$,

$$\{D' \in D_{n^*} \mid (|D'| = \rho) \wedge (\exists m < \omega)(\exists Z_{m-1} \in ... \in Z_0 \in B)$$

$$((\forall k < m)(|Z_k| < \rho)) \wedge D' \in B \cup \bigcup_{k<m} Z_k))\} \in D.$$

18. (An addition to the strong covering condition under $\theta < \aleph_{\eta^+}$) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau, \rho \in B$ and
$\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Suppose that there is $X \in C^\theta$ with $\sup(B \cap \theta^+) = X \cap \theta^+$.
Then either

 (a) $D \in B$,
 or

 (b) $D \notin B$ and (b),(c) of (17) hold with B replaced by any model $Y, B \subseteq Y \subseteq X$ of a regular cardinality $\mu \in B, \tau < \mu < \rho$ which is the smallest elementary submodel of X which includes B and $\mu + 1$. [7]

19. (Covering under $\theta \geq \aleph_{\eta^+}$.) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$ and
$\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Then either

 (a) $\rho \in B$ and then, there is $D^* \in C^\rho \cap B$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$ (and then, clearly, $B \cap D^* = B \cap D$),
 or

[7]Note that the total number of such Y's is $|B| = \tau$.

(b) $\rho \notin B$ and then, there is $D^* \in (\bigcup_{\mu\in s\setminus\rho+1} C^\mu)\cap B$ such that $D^* \supseteq D$ and $B\cap D^* \subseteq D$,

or

(c) $\rho \notin B$, there is no $D^* \in (\bigcup_{\mu\in s\setminus\rho+1} C^\mu)\cap B$ such that $D^* \supseteq D$ and $B\cap D^* \subseteq D$ and then, there is $D^* \in B$ such that $D^* \supseteq D$ and $B \cap D^* \subseteq D$.

We will elaborate this last possibility below.[8]

20. (Strong covering under $\theta \geq \aleph_{\eta^+}$.) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$ and $\sup(D\cap\theta^+) < \sup(B\cap\theta^+)$. Then there is $Z \in B \cap \bigcup_{\mu\in s} C^\mu$ such that $\sup(D\cap\theta^+) \leq \sup(Z\cap\theta^+)$. Let G be such Z with $\sup(Z\cap\theta^+)$ the smallest possible under $\in$ and the inclusion, i.e. for any other such model G', $G \in G'$ or $G \subseteq G'$.

Then ${}^{|G|>}G \subseteq G$ and either

(a) $G = D$, i.e. $D \in B$,

or

(b) $D \in G$, $|G| \geq |D| = \rho$, and then

i. $B \cap G \subseteq D$,

ii. $\{D' \in G \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)((\forall k < n)(|Z_k| < |G|)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D$.

Or

(c) $D \in G$, $|G| < |D| = \rho$, and then, there are H, D^* such that

i. $G \subseteq D^* \subseteq H$,

ii. $\sup(G\cap\theta^+) = H\cap\theta^+$,

iii. $H \in C^{\theta lim}$.

Note $H \in B$, since $G \in B$ and $\sup(G\cap\theta^+) = H\cap\theta^+$, by (14).

iv. $|D^*| = \min(B \cap On \setminus \rho)$,[9]

v. D^* is the smallest elementary submodel of H which includes G and $|D^*|$, i.e. the Skolem Hull of $G \cup |D^*|$.

Note that $D^* \in B$, since G, H and $|D^*|$ are there.

[8]Such possibility may occur already if $\eta = \delta = \omega$ and $\theta = \aleph_{\omega_1+1}$. Let B be a countable model. Then $B\cap\omega_1 < \omega_1$. Let D be a model of cardinality $\aleph_{\alpha+1}$ for some $\alpha, B\cap\omega_1 < \alpha < \omega_1$. So, a model $D^* \in B$ of a singular cardinality $\aleph_{\omega_1}$ may be needed in order to cover such D.

[9]Note that $|D^*|$ is a singular cardinal.

vi. $D \in D^*$.

This implies that $D \subseteq D^*$, as $|D| = \rho \in D^*$.

vii. $D^* \cap B \subseteq D$,

viii. $\{D' \in D^* \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)((\forall k < n)(|Z_k| < |G|)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D$.

Or

(d) $D \notin G$ (which implies $|G| < |D| = \rho$), and then, the above happens after finitely many steps, i.e. there are sequences
$\langle G_i \mid i < n \rangle, \langle H_i \mid i < n \rangle, \langle \rho_i \mid i < n \rangle, \langle D_i^* \mid i < n \rangle$such that

i. $D \in G_{n-1} \in G_{n-2} ... \in G_0 = G \in B$,

ii. for every $i < n$, $\sup(G_i \cap \theta^+) = H_i \cap \theta^+$,

iii. for every $i < n$, $G_i \in C^{|G_i| lim}, H_i \in C^{\theta lim}$,

iv. $|G_{i+1}| > |G_i|$, for every $i < n$,

v. $|D_0^*| = \min(B \cap On \setminus \rho)$,

vi. $|D_{i+1}^*| = \min(G_i \cap On \setminus \rho)$,

vii. D_i^* is the smallest elementary submodel of H_i which includes G_i and $|D_i^*|$, i.e. the Skolem Hull of $G_i \cup |D_i^*|$.

viii. $D \in D_0^*$.

This implies that $D \subseteq D_0^*$, as $|D| = \rho \in D_0^*$.

ix. $B \cap D_0^* \subseteq D$,

x. $D \in D_{i+1}^*$.

This implies that $D \subseteq D_{i+1}^*$, as $|D| = \rho \in D_{i+1}^*$.

xi. $G_i \cap D_{i+1}^* \subseteq D$,

xii. $\{D' \in \bigcup_{i<n} D_i^* \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)((\forall k < n)(|Z_k| < |G_{n-1}|)) \wedge D' \in B \cup \bigcup_{k<n} Z_k))\} \in D$.

21. (An addition to the strong covering condition under $\theta \geq \aleph_{\eta^+}$) Let $B \in C^\tau$, $D \in C^\rho, \rho > \tau$ and
$\sup(D \cap \theta^+) < \sup(B \cap \theta^+)$. Suppose that there is $X \in C^\theta$ with $\sup(B \cap \theta^+) = X \cap \theta^+$.
Then either

(a) $D \in B$,

or

(b) $D \notin B$ and then, (20) holds with B replaced by any model $Y, B \subseteq Y \subseteq X$ of a regular cardinality $\mu \in B, \tau < \mu < \rho$ which is the smallest elementary submodel of X which includes B and $\mu + 1$. [10]

It is probably worth to note that the main difference between the cases $\theta < \aleph_{\eta^+}$ and $\theta \geq \aleph_{\eta^+}$ is that in the later case models of singular cardinalities are used and we choose not to put them into C^μ's.

$\square$ of the definition.

Let us make now the following observation:

Lemma 2.2 *Let* $\langle\langle C^\tau, C^{\tau lim}\rangle \mid \tau \in s\rangle$ *be a* $(\theta, \eta, \delta)-$*wide piste. Let* $\tau, \rho \in s, \tau < \rho$ *and* A *and* B *be the maximal models in* C^τ *and* C^ρ *respectively. Suppose that if* $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$, *then* $\rho \in A$. *Then, without assuming 2.1(13), either* $B \in A$ *or* $A \subset B$. *Moreover,* $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$ *implies* $A \in B$ *and* $\sup(B \cap \theta^+) < \sup(A \cap \theta^+)$ *implies* $B \in A$.
So, the maximal models are linearly ordered by a combination of $\in$ *and* $\subset -$*relations.*

Proof. If $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then by Definition 2.1(12) $A \in B$.
If $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$, then by Definition 2.1(12) $A \subseteq B$.
Suppose that $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$.
Then $\rho \in A$, by the assumption. Apply Definition 2.1(16,19). Then there is $D^* \in A \cap C^\rho$ such that $D^* \supseteq B$. The maximality of B implies then that $B = D^*$. So, $B \in A$.
$\square$

Now we are ready to give the main definition.

Definition 2.3 Let $\delta \leq \eta < \theta$ be regular cardinals.
A $\delta-$*structure with pistes over* η *of length* θ is a set $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ such that the following hold.[11]
Let us first specify sizes of models that are involved.

1. (Support) s is a closed set of regular cardinals from the interval $[\eta, \theta]$ satisfying the following:

   (a) $|s| < \delta$,

[10] Note that the total number of such Y's is $|B| = \tau$.

[11] If $\delta = \omega$, then we call $\delta-$structure with pistes over η of the length θ just a finite structure with pistes over η of the length θ.

(b) $\eta, \theta \in s$.

Which means that the minimal and the maximal possible sizes are always present.

2. (Models) For every $\tau \in s$ the following holds:

   (a) $A^{0\tau} \preccurlyeq \langle H(\theta^+), \in, \leq, \delta, \eta \rangle$,

   (b) $|A^{0\tau}| = \tau$,

   (c) $A^{0\tau} \in A^{1\tau}$,

   (d) $A^{1\tau}$ is a set of less than δ elementary submodels of $A^{0\tau}$,

   (e) each element A of $A^{1\tau}$ has cardinality τ, $A \supseteq \tau + 1$ and $A \cap \tau^+$ is an ordinal.

3. (Potentially limit points) Let $\tau \in s$.
   $A^{1\tau lim} \subseteq A^{1\tau}$. We refer to its elements as *potentially limit points*.
   The intuition behind this is that once extending it will be possible to add new models unboundedly often below a potentially limit model, and this way it will be turned into a limit one.

4. (Piste function) The idea behind this is to provide a canonical way to move from a model in the structure to one below.
   Let $\tau \in s$.
   Then, $\mathrm{dom}(C^\tau) = A^{1\tau}$ and
   for every $B \in \mathrm{dom}(C^\tau)$, $C^\tau(B)$ is a closed chain of models in $A^{1\tau} \cap (B \cup \{B\})$ such that the following holds:

   (a) $B \in C^\tau(B)$,

   (b) if $X \in C^\tau(B)$, then $C^\tau(X) = \{Y \in C^\tau(B) \mid Y \in X \cup \{X\}\}$,

   (c) if B has immediate predecessors in $A^{1\tau}$, then one (and only one) of them is in $C^\tau(B)$,

5. (Wide piste) The set

$$\langle C^\tau(A^{0\tau}), C^\tau(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s \rangle$$

   is a $(\theta, \eta, \delta)-$wide piste.

   The next two condition describe the ways of splittings from wide pistes. This describes the structure of $A^{1\tau}$ and the way pistes allow one to move from one of its models to an other.

6. (Splitting points) Let $\tau \in s$. Let $X \in A^{1\tau}$ be a non-limit model (but possibly a potentially limit). Then either

   (a) X is minimal under $\in$ or equivalently under $\subsetneq$,

   or

   (b) X has a unique immediate predecessor in $A^{1\tau}$,

   or

   (c) (Splitting points of type 1) $\tau < \theta$, X has exactly two immediate predecessors X_0, X_1 in $A^{1\tau}$, none of X, X_0, X_1 is a limit or potentially limit point and X, X_0, X_1 form a Δ–*system triple* relatively to some $F_0, F_1 \in A^{1\tau^* lim}$, where $\tau^* \in s \setminus \tau + 1$[12], which means the following:

      i. $F_0 \subsetneq F_1$ and then $F_0 \in C^{\tau^*}(F_1)$, or $F_1 \subsetneq F_0$ and then $F_1 \in C^{\tau^*}(F_0)$,

      ii. ${}^{\tau^*>}F_0 \subseteq F_0$ and ${}^{\tau^*>}F_1 \subseteq F_1$,

      iii. $X_0 \in F_1$, if $F_0 \subsetneq F_1$ and $X_1 \in F_0$, if $F_1 \subsetneq F_0$,

      iv. $F_0 \in X_0$ and $F_1 \in X_1$,

      v. $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$,

      vi. ${}^{\tau>}X_0 \subseteq X_0$ and ${}^{\tau>}X_1 \subseteq X_1$,

      vii. the structures

      $$\langle X_0, \in, \langle X_0 \cap A^{1\rho}, X_0 \cap A^{1\rho lim}, (C^\rho \restriction X_0 \cap A^{1\rho}) \cap X_0 \mid \rho \in s \cap X_0 \rangle\rangle$$

      and

      $$\langle X_1, \in, \langle X_1 \cap A^{1\rho}, X_1 \cap A^{1\rho lim}, (C^\rho \restriction X_1 \cap A^{1\rho}) \cap X_1 \mid \rho \in s \cap X_1 \rangle\rangle$$

      are isomorphic over $X_0 \cap X_1$. Denote by π_{X_0, X_1} the corresponding isomorphism.

      viii. $X \in A^{0\tau*}$.

   Or

   (d) (Splitting points of type 2)[13] There is a singular cardinal $\lambda > \tau$ of cofinality $< \delta$[14] with $\lambda^+ \in s$, there are $\in$ –increasing sequences $\langle G_{0\xi} \mid \xi \in s \cap \lambda \rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda \rangle \in X$, $F_0, F_1 \in X \cap A^{1\lambda^+ lim}$ such that

[12] If there are only finitely many cardinals between η and θ, then we can take τ^* to be just τ^+.

[13] In previous versions of the paper models of singular cardinalities were allowed. This conditions corresponds to splitting for such models.

[14] So this type of splitting does not occur once $\delta = \omega$.

i. $F_0 \subsetneq F_1$ and then $F_0 \in C^{\lambda^+}(F_1)$, or $F_1 \subsetneq F_0$ and then $F_1 \in C^{\lambda^+}(F_0)$,

ii. ${}^{\lambda}F_0 \subseteq F_0$ and ${}^{\lambda}F_1 \subseteq F_1$,

iii. for every $\xi \in s \cap \lambda$, ${}^{\xi>}G_{0\xi} \subseteq G_{0\xi}$ and ${}^{\xi>}G_{1\xi} \subseteq G_{1\xi}$

iv. $G_{0\lambda} = \bigcup_{\xi \in s\cap\lambda} G_{0\xi}$ and $G_{1\lambda} = \bigcup_{\xi \in s\cap\lambda} G_{1\xi}$ are in X,

v. $G_{0\lambda} \in F_1$, if $F_0 \subsetneq F_1$ and $G_{1\lambda} \in F_0$, if $F_1 \subsetneq F_0$,

vi. $F_0 \in G_{0\lambda}$ and $F_1 \in G_{1\lambda}$,

vii. $G_{0\lambda} \cap G_{1\lambda} = G_{0\lambda} \cap F_0 = G_{1\lambda} \cap F_1$,

viii. the structures

$$\langle G_{0\lambda}, \in, \langle G_{0\lambda} \cap A^{1\rho}, G_{0\lambda} \cap A^{1\rho lim}, (C^{\rho} \restriction G_{0\lambda} \cap A^{1\rho}) \cap G_{0\lambda} \mid \rho \in s \cap G_{0\lambda} \rangle\rangle$$

and

$$\langle G_{1\lambda}, \in, \langle G_{1\lambda} \cap A^{1\rho}, G_{1\lambda} \cap A^{1\rho lim}, (C^{\rho} \restriction G_{1\lambda} \cap A^{1\rho}) \cap G_{1\lambda} \mid \rho \in s \cap G_{1\lambda} \rangle\rangle$$

are isomorphic over $G_{0\lambda} \cap G_{1\lambda}$. Denote by $\pi_{G_{0\lambda}G_{1\lambda}}$ the corresponding isomorphism.

ix. For every $\xi \in s \cap \lambda$, $\pi_{G_{0\lambda}G_{1\lambda}}(G_{0\xi}) = G_{1\xi}$.

x. $X_0 = G_{0\tau}$ and $X_1 = G_{1\tau}$.

xi. (Pistes go in the same direction) If $\xi \in s \cap \lambda, \xi \neq \tau$, $G_{0\xi}, G_{1\xi} \in A^{1\xi}$ and there is $G_{\xi} \in A^{1\xi}$ which is the immediate successor of $G_{0\xi}, G_{1\xi}$ in $A^{1\xi}$, then $G_{i\xi} \in C^{\mu}(G_{\xi}) \Leftrightarrow X_i \in C^{\tau}(X), i < 2$.

xii. X is not a limit or potentially limit point,

xiii. $X \in A^{0\lambda^+}$.

Or

(e) (Splitting points of type 3) There are $G, G_0, G_1 \in X \cap A^{1\mu}$ splitting points of types 1 or 2, for some $\mu \in s \setminus (\min(s \setminus \tau + 1) + 1)$, with witnessing models in X such that

i. $X_0 \in G_0$,

ii. $X_1 \in G_1$,

iii. $X_1 = \pi_{G_0G_1}[X_0]$.

iv. X is not a limit or potentially limit point,

v. $X \in A^{0\mu}$,

vi. (Pistes go in the same direction) $G_i \in C^\mu(G) \Leftrightarrow X_i \in C^\tau(X), i < 2$.

Further we will refer to such X, i.e. of types 1,2 or 3, as *splitting points.*

7. Let $\tau, \rho \in s$, $X \in A^{1\tau}, Y \in A^{1\rho}$. Suppose that X is a successor point, but not potentially limit point and $X \in Y$. Then all immediate predecessors of X are in Y, as well as the witnesses, i.e. F_0, F_1 if (6c) holds, $\langle G_{0\xi} \mid \xi \in s \cap \lambda \rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda \rangle, G_{0\lambda}, G_{1\lambda}$, F_0, F_1 if (6d) holds and G_0, G_1, G if (6e) holds.

8. Let $\tau \in s$. If $X \in A^{1\tau}$, $Y \in \bigcup_{\rho \in s} A^{1\rho}$ and $Y \in X$, then Y is *a piste reachable* from X, i.e. there is a finite sequence $\langle X(i) \mid i \leq n \rangle$ of elements of $A^{1\tau}$ which we call *piste leading to* Y such that

   (a) $X = X(0)$,

   (b) for every $i, 0 < i \leq n$, $X(i) \in C^\tau(X(i-1))$ or $X(i-1)$ has two immediate predecessors $X(i-1)_0, X(i-1)_1$ with $X(i-1)_0 \in C^\tau(X(i-1))$, $X(i) = X(i-1)_1$ and $Y \in X(i-1)_1 \setminus X(i-1)_0$ or $Y = X(i-1)_1$,

   (c) $Y = X(n)$, if $Y \in A^{1\tau}$ and if $Y \in A^{1\rho}$, for some $\rho \neq \tau$, then $Y \in X(n)$, $X(n)$ is a successor point and Y is not a member of any element of $X(n) \cap A^{1\tau}$.

The sequence $\langle X(i) \mid i \leq n \rangle$ is defined uniquely from X and Y.
In particular, every $Y \in A^{1\tau}$ is piste reachable from $A^{0\tau}$.
In order formulate further requirements, we will need to describe a simple process of changing the wide pistes. This leads to equivalent forcing conditions once the order will be defined.

Let $X \in A^{1\tau}$. We will define the $X-$wide piste. The definition will be by induction on number of turns (splits) needed in order to reach X by the piste from $A^{0\tau}$.
First, if $X \in C^\tau(A^{0\tau})$, then the $X-$wide piste is just $\langle C^\xi(A^{0\xi}), C^\xi(A^{0\xi}) \cap A^{1\xi lim} \mid \xi \in s \rangle$, i.e. the wide piste of the structure.
Second, if $X \notin C^\tau(A^{0\tau})$, but it is not a splitting point, then pick the least splitting point Y above X. Let Y_0, Y_1 be its immediate predecessors with $Y_0 \in C^\tau(Y)$. Then $X \in Y_i \cup \{Y_i\}$ for some $i < 2$. Set the $X-$wide piste to be the Y_i-wide piste.
So, in order to complete the definition, it remain to deal with the following principal case:
$X \in A^{1\tau}$ a splitting point of one of the types 1,2 or 3.

Let X_0, X_1 be its immediate predecessors with $X_0 \in C^\tau(X)$. Assume that $X-$wide piste $\langle C_X^\xi, C_X^{\xi lim} \mid \xi \in s\rangle$ for X is defined and assume that $C^\tau(X)$ is an initial segment of C_X^τ.
Let the X_0-wide piste be $\langle C_X^\xi, C_X^{\xi lim} \mid \xi \in s\rangle$.

Let us deal with type of splitting separately.
Case 1. X is a splitting point of type 1.
Define X_1-wide piste $\langle C_{X_1}^\xi, C_{X_1}^{\xi lim} \mid \xi \in s\rangle$ as follows:

- $C_{X_1}^\xi = C_X^\xi$, for every $\xi > \tau$.
  I.e. no changes for models of cardinality $> \tau$.
- $C_{X_1}^{\xi lim} = C_{X_1}^\xi \cap A^{1\xi lim}$, for every $\xi \in s$.
  Models that were potentially limit remain such and no new are added.
- $C_{X_1}^\tau = (C_X^\tau \setminus X) \cup C^\tau(X_1)$.
  Here we switched the piste from X_0 to X_1.
- $C_{X_1}^\xi = \{Z \in C_X^\xi \mid \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))\} \cup \{\pi_{X_0,X_1}(Z) \mid Z \in C_X^\xi \cap X_0\}$, for every $\xi \in s \cap \tau$.[15]

Note that such defined switch from X_0 to X_1 does not effect at all models of sizes above τ. Models of sizes $\leq \tau$ are effected only if they are contained in X_0 or in X_1.

If X is a splitting point of types 2 or 3, then we may need to turn some piste for models of cardinalities $> \tau$ into other directions, in order to satisfy the items 6(d)xi,6(e)vi above.
Proceed as follows.
Case 2. X is a splitting point of type 2.
Let $\lambda > \tau$, $\in -$increasing sequences $\langle G_{0\xi} \mid \xi \in s \cap \lambda\rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda\rangle \in X$, $F_0, F_1 \in X \cap A^{1\lambda^+ lim}$, $G_{0\lambda} = \bigcup_{\xi \in s\cap\lambda} G_{0\xi}, G_{1\lambda} = \bigcup_{\xi\in s\cap\lambda} G_{1\xi}$ be as in the item 6d.

Define X_1-wide piste $\langle C_{X_1}^\xi, C_{X_1}^{\xi lim} \mid \xi \in s\rangle$ as follows:

- $C_{X_1}^\xi = C_X^\xi$, for every $\xi > \lambda$.
  I.e. no changes for models of cardinality $> \lambda$.
- $C_{X_1}^{\xi lim} = C_{X_1}^\xi \cap A^{1\xi lim}$, for every $\xi \in s$.
  Models that were potentially limit remain such and no new are added.

[15]In particular, due to this, the next condition implies that for $\xi \in s \cap \tau$, if $Z \in C_X^\xi, \sup(Z \cap \theta^+) > \max(\sup(X_0 \cap \theta^+), \sup(X_1 \cap \theta^+))$, then $\{\pi_{X_0 X_1}(Z') \mid Z' \in C_X^\xi \cap X_0\} \subseteq Z$.

- $C^{\tau}_{X_1} = (C^{\tau}_X \setminus X) \cup C^{\tau}(X_1)$.
  Here we switched the piste from X_0 to X_1.
  Now, simultaneously, for every $\xi \in s \cap \lambda$, if $G_{0\xi}, G_{1\xi} \in A^{1\xi}$ and there is $G_\xi \in A^{1\xi}$ which is the immediate successor of $G_{0\xi}, G_{1\xi}$ in $A^{1\xi}$, then we switch pistes from $G_{0\xi}$ to $G_{1\xi}$ in $C^{\mu}(G_\xi)$, exactly as it is done above with X_0, X_1, X.

Case 3. X is a splitting point of type 3.
Let $G, G_0, G_1 \in X \cap A^{1\mu}$ be models which witness that X, X_0, X_1 are splitting points of type 3. Now using the induction on the rank of the models, we can assume that the G_1-wide piste is already defined.
Define the X_1-wide piste to be the G_1-wide piste.

Now we require the following:

9. Let $\tau \in s$ and $X \in A^{1\tau}$. Then $X-$wide piste is a wide piste, i.e. it satisfies 2.1.
   The problem is with the item (3c) of Definition 2.1 which, in general, is not preserved while splitting.

   Final conditions deal with largest models.

10. (Maximal models are above all the rest) For every $\tau \in s$ and $Z \in \bigcup_{\rho \in s} A^{1\rho}$, if $Z \notin A^{0\tau}$, then there is $\mu \in s$ such that $Z = A^{0\mu}$.

    Recall that by Lemma 2.2, maximal models $A^{0\tau}, \tau \in s$ are linearly ordered as top parts of the wide piste $\langle C^{\tau}(A^{0\tau}), C^{\tau}(A^{0\tau}) \cap A^{1\tau lim} \mid \tau \in s\rangle$.

This completes the definition of $\delta-$structure with pistes over η of the length θ.

2.1 Some properties of structures with pistes.

Let us turn now to the intersection property.
The intuition behind this is to replace an arbitrary intersection of models by an internal one.
We split the definitions below according to $\theta > \aleph_{\eta^+}$ and $\theta < \aleph_{\eta^+}$.

Definition 2.4 (Models of different sizes, $\theta > \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ.

Let $A \in A^{1\tau}, B \in A^{1\rho}$ and $\tau < \rho$.
By $ip(A, B)$ we mean the following:

1. $B \in A$,
   or

2. $A \subset B$,
   or

3. $B \notin A$, $A \not\subset B$ and then
   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \rho) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

Definition 2.5 (Models of the same size, $\theta > \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ.

Let $A, B \in A^{1\tau}$. By $ip(A, B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then
   - there are $\eta_1 < ... < \eta_m$ in $(s \setminus \tau) \cap A$ and $X_1 \in A^{1\eta_1} \cap A, ..., X_m \in A^{1\eta_m} \cap A$ such that $A \cap B = A \cap X_1 \cap ... \cap X_m$.

If both $ip(A, B)$ and $ip(B, A)$ hold, then we denote this by $ipb(A, B)$.

Definition 2.6 (Models of different sizes, $\theta < \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ.

Let $A \in A^{1\tau}, B \in A^{1\rho}$ and $\tau < \rho$.
By $ip(A, B)$ we mean the following:

1. $B \in A$,
   or

2. $A \subset B$,
   or

3. $B \notin A$, $A \not\subset B$ and then

   - there are $\eta_1 < ... < \eta_m$ in $A \setminus \rho$ and $X_1, ..., X_m \in A$ such that
     - $A \cap B = A \cap X_1 \cap ... \cap X_m$,
     - $|X_i| = \eta_i$, for every $i, 1 \leq i \leq m$,
     - if $\eta_i \in s$, then $X_i \in A^{1\eta_i}$, for every $i, 1 \leq i \leq m$,
     - if $\eta_i \notin s$, then there are $G_i \in A \cap A^{1|G_i|}, H_i \in A \cap A^{1\theta}$, $\sup(G_i \cap \theta^+) = H_i \cap \theta^+$ such that X_i is the smallest elementary submodel of H_i which includes $G_i \cup \eta_i$, for every $i, 1 \leq i \leq m$.

Definition 2.7 (Models of the same size, $\theta < \aleph_{\eta^+}$). Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ.

Let $A, B \in A^{1\tau}$. By $ip(A, B)$ we mean the following:

1. $A \subseteq B$,
   or

2. $B \subseteq A$,
   or

3. $A \not\subseteq B$, $B \not\subseteq A$ and then

   - there are $\eta_1 < ... < \eta_m$ in $A \setminus \rho$ and $X_1, ..., X_m \in A$ such that
     - $A \cap B = A \cap X_1 \cap ... \cap X_m$,
     - $|X_i| = \eta_i$, for every $i, 1 \leq i \leq m$,
     - if $\eta_i \in s$, then $X_i \in A^{1\eta_i}$, for every $i, 1 \leq i \leq m$,

– if $\eta_i \notin s$, then there are $G_i \in A \cap A^{1|G_i|}, H_i \in A \cap A^{1\theta}$, $\sup(G_i \cap \theta^+) = H_i \cap \theta^+$ such that X_i is the smallest elementary submodel of H_i which includes $G_i \cup \eta_i$, for every $i, 1 \leq i \leq m$.

If both $ip(A, B)$ and $ip(B, A)$ hold, then we denote this by $ipb(A, B)$.

Lemma 2.8 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ. Assume $A \in A^{1\tau}, B \in A^{1\rho}$, for some $\tau \leq \rho, \tau, \rho \in s$. Then $ip(A, B)$ and if $\tau = \rho$, then also $ipb(A, B)$.*

Proof. We will basically split the proof into two main cases: $\rho = \tau$ and $\rho \neq \tau$. However, the inductive assumption (on number of turns from the pistes) will be used simultaneously for both.

Case A. $\rho = \tau$.

So, $A, B \in A^{1\tau}$. Assume that $A \not\subseteq B$ and $B \not\subseteq A$. Consider the pistes leading from $A^{0\tau}$ to A and to B. Let X be their last common point. Then, by 2.3(8), X is a successor model.

Subcase A1. *X has a unique immediate predecessor.* Let X_0 be this immediate predecessor. Then, one of A or B is in X_0 and the other one is not. But, then it must be equal to X_0, which is impossible by our assumptions that $A \not\subseteq B$ and $B \not\subseteq A$.

Subcase A2. *X is a splitting point of type 1.*

Let X_0 and X_1 be the immediate predecessors of X. Let $F_0 \in X_0$ and $F_1 \in X_1$ witness that X, X_0, X_1 form a $\Delta-$system triple. Then $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$.
Assume that $A \in X_0 \cup \{X_0\}$ and $B \in X_1 \cup \{X_1\}$.
If $A = X_0$ and $B = X_1$, then $ipb(A, B)$ follows.
Suppose that $A \neq X_0$ or $B \neq X_1$. Say, $B \neq X_1$. Set $B' = \pi_{X_1,X_0}[B]$. Then $B' \in X_0$ and $B \cap X_0 = B' \cap F_0$. Hence,

$$A \cap B = A \cap B \cap X_0 = A \cap B' \cap F_0 = (A \cap B') \cap (A \cap F_0).$$

Now we apply induction to get $ip(A, B')$ and $ip(A, F_0)$.

Subcase A3. *X is a splitting point of types 2.*

Let X_0 and X_1 be the immediate predecessors of X.
Let $\langle G_{0\xi} \mid \xi \in s \cap \lambda\rangle, \langle G_{1\xi} \mid \xi \in s \cap \lambda\rangle, G_{0\lambda}, G_{1\lambda},\ F_0, F_1 \in X \cap A^{1\lambda^+ lim}$ be as in Definition 2.3(6d). Again, we have $X_0 \cap X_1 = X_0 \cap F_0 = X_1 \cap F_1$, since $G_{0\lambda} \cap G_{1\lambda} = G_{0\lambda} \cap F_0 = G_{1\lambda} \cap F_1$ and the isomorphism $\pi_{G_{0\lambda}G_{1\lambda}}$ moves $X_0 \subseteq G_{0\lambda}$ to $X_1 \subseteq G_{1\lambda}$.
Hence, it is possible to proceed as in the previous case.

Subcase A4. *X is a splitting point of types 3.*
The proof essentially the same as in **Subcase A3**.

Case B. $\rho > \tau$**.**

So, $A \in A^{1\tau}, B \in A^{1\rho}$. Assume that $A \not\subseteq B$ and $B \notin A$.
Suppose first that $A \notin A^{0\rho}$. Then Definition 2.3(10), $A = A^{0\tau}$ and if $B \notin A^{0\tau}$, then, again by Definition 2.3(10), $B = A^{0\rho}$. But any two maximal models on the wide piste of the structure are compatible by Lemma 2.2. Namely, if $\sup(A^{0\tau} \cap \theta^+) \leq \sup(A^{0\rho} \cap \theta^+)$, then $A^{0\tau} \subseteq A^{0\rho}$ by Definition 2.1(12). If $\sup(A^{0\tau} \cap \theta^+) > \sup(A^{0\rho} \cap \theta^+)$, then $A^{0\rho} \in A^{0\tau}$, by Lemma 2.2.

Suppose that $A \in A^{0\rho}$. Then $B \neq A^{0\rho}$, as $A \not\subseteq B$, and hence $A, B \in A^{0\rho}$.
By Definition 2.3(9) we can assume that A is on the wide piste of the structure. Consider the pistes leading from $A^{0\rho}$ to A and to B. Let $X \in C^\rho(A^{0\rho})$ be their last common point. The proof proceeds by induction on rank(X). Then, by Definition 2.3(8), X is a successor model.

Subcase B1. *X is a splitting point of type 1.*

The proof is essentially as in **Subcase A2** above.

Subcase B2. *X is a splitting point of type 3.*

Let X_0 and X_1 be the immediate predecessors of X. Let $G, G_0, G_1 \in X \cap A^{1\mu}$ be a corresponding $\Delta-$system triple, for some $\mu \in s \setminus \rho+1$. Also let $F_0 \in G_0 \cap X$ and $F_1 \in G_1 \cap X$ witness this, i.e. $G_0 \cap G_1 = G_0 \cap F_0 = G_1 \cap F_1$.
Assume that $A \in X_0$ and $B \in X_1 \cup \{X_1\}$.
Set $B' = \pi_{G_1,G_0}[B]$. Then $A \cap B = A \cap B' \cap F_0$. The induction applies to A and B', since $B' \subseteq X_0 \in X$. Also it applies to A and F_0, since $F_0 \in X$. Hence, $ip(A, B)$.

Subcase B3. *X is a splitting point of type 2.*

It is similar to the previous case.

Subcase B4. *X has a unique immediate predecessor.*

Let X_0 be this predecessor. Then either $B = X_0$ or $B \in X_0$.
Split into three cases according to the relation between A and X_0.

Subsubcase B4.1. $\sup(A \cap \theta^+) < \sup(X_0 \cap \theta^+)$.

Then $A \in X_0$, by Definition 2.1(12). But $B \in X_0$ as well, and we get a contradiction to the choice of X.

Subsubcase B4.2. $\sup(A \cap \theta^+) > \sup(X_0 \cap \theta^+)$.

Using Definition 2.3(9), we may assume that both A and B are on the wide piste of the structure. Just X_0 is there as the unique immediate predecessor of X. Switching pistes below X_0 would not effect the piste leading to A, since $\sup(A \cap \theta^+) > \sup(X_0 \cap \theta^+)$.

Apply Definition 2.1(17) or (20) to A and B. Let $Z \in A, Z \subseteq B$, $A \cap Z \subseteq B$ be the result. Then $A \cap B = A \cap Z$, and we are done.

Subsubcase B4.3. $\sup(A \cap \theta^+) = \sup(X_0 \cap \theta^+)$.

Then $A \subset X_0$. If $B = X_0$, then $A \cap B = A$ and we are done. So, $B \in X_0$. Then, $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$. Apply Definition 2.1(17) or (20) to A and B and continue as in **Subsubcase B4.2**.

□

Lemma 2.9 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ. Suppose that $\tau \in s$, $A \in A^{1\tau}$ is a potentially limit point and $A \cap A^{1\theta} \neq \emptyset$. Then there is $X \in A \cap A^{1\theta}$ which includes every element of $A \cap A^{1\theta}$.*

Proof. First note that $A^{1\theta}$ has no splitting points, since θ is the maximal cardinal involved. So, $A^{1\theta} = C^\theta(A^{0\theta})$ is a closed chain. A is a potentially limit point in $A^{1\tau}, \tau < \theta$, hence $\mathrm{cof}(A \cap \theta^+) = \tau \geq \eta \geq \delta$. Hence $A \cap A^{1\theta}$ is bounded in A. Set X to be the maximal element of $A \cap A^{1\theta}$.

□

Lemma 2.10 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ. Suppose that $\tau, \rho \in s, \sup(A^{0\tau} \cap \theta^+) \neq \sup(A^{0\rho} \cap \theta^+), \tau < \rho$, $A \in A^{1\tau}$ is a potentially limit point and $A \cap A^{1\rho} \neq \emptyset$. Then there is $X \in A \cap A^{1\rho}$ which includes every element of $A \cap A^{1\rho}$.*

Proof. If $A \notin A^{0\rho}$, then $A = A^{0\tau}$, by Definition 2.3(10). By Lemma 2.2, then $A^{0\rho} \in A = A^{0\tau}$. So $A^{0\rho}$ will be as required.

Assume that $A \in A^{0\rho}$. By Definition 2.3(9), we may assume that A is on the wide piste of the structure.

Let $Z \in C^\rho(A^{0\rho})$ be the least model which includes A. Consider its immediate predecessor Z' on the piste. It exists since, by the assumption of the lemma A is not a limit model, so Z cannot be a limit model, in addition, $A \cap A^{1\rho} \neq \emptyset$, and so the piste continues to elements of this intersection.

Note also, that by Definition 2.1(4), Z must be a potentially limit point, and as such, it cannot be a splitting point. So Z' is the unique immediate predecessor of Z.

Now, both A and Z' are on the wide piste, $\tau < \rho$ and $A \not\subseteq Z'$. Hence, by Definition 2.1(12), $\sup(A \cap \theta^+) > \sup(Z' \cap \theta^+)$. Apply now Definition 2.1(16) or (19) to A and Z' (note that $|Z'| = \rho \in A$). So, there will be $X \in A \cap C^\rho(A^{0\rho})$ such that $X \supseteq Z'$. But then $Z' = X$ and

we are done, since if $Y \in A \cap C^\rho(A^{0\rho})$, then $Y \in Z \supset A$. Hence $Y \in Z' \cup \{Z'\}$.
□

Lemma 2.11 *Let $\langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be a $\delta-$structure with pistes over η of the length θ. Suppose that $\tau \in s$, $A \in A^{1\tau}$ and $A \cap A^{1\tau} \neq \emptyset$. If A is a potentially limit point then there is $X \in A \cap A^{1\tau}$ which includes every element of $A \cap A^{1\tau}$.*

Proof. Just by Definition 2.3(6), A has a unique immediate predecessor. It will be as desired.
□

Note that if A is a splitting point, then the lemma is not true anymore.
Also, if one likes to find the largest model of a small cardinality inside a larger one, then it should not be true in general (however any $\delta-$structure with pistes over η of the length θ can be extended to one that satisfies this). Thus, for example reflect in an increasing order $\omega-$many models of size η into a fixed potentially limit model A of size η^+. There will be no maximal model of cardinality η inside A. But an additional reflection will produce such.

2.2 Forcing with structures with pistes.

Definition 2.12 Define $\mathcal{P}_{\theta\eta\delta}$ to be the set of all
$\delta-$structures with pistes over η of the length θ.
Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle \in \mathcal{P}_{\theta\eta\delta}$.
Denote further $A^{0\tau}$ by $A^{0\tau}(p), A^{1\tau}$ by $A^{1\tau}(p)$, $A^{1\tau lim}$ by $A^{1\tau lim}(p), C^\tau$ by $C^\tau(p)$ and s by $s(p)$.
Call s the support of p.

Let us define a partial order on $\mathcal{P}_{\theta\eta\delta}$ as follows.

Definition 2.13 Let
$p_0 = \langle\langle A_0^{0\tau}, A_0^{1\tau}, A_0^{1\tau lim}, C_0^\tau\rangle \mid \tau \in s_0\rangle$, $p_1 = \langle\langle A_1^{0\tau}, A_1^{1\tau}, A_1^{1\tau lim}, C_1^\tau\rangle \mid \tau \in s_1\rangle$ be two elements of $\mathcal{P}_{\theta\eta\delta}$.
Set $p_0 \leq p_1$ (p_1 extends p_0) iff

1. $s_0 \subseteq s_1$,

2. $A_0^{1\tau} \subseteq A_1^{1\tau}$, for every $\tau \in s_0$,

3. let $A \in A_0^{1\tau}$, then $A \in A_0^{1\tau lim}$ iff $A \in A_1^{1\tau lim}$.
   The next item deals with a property called switching in [6]. It allows to change piste directions.

4. For every $A \in A_0^{1\tau}$, $C_0^\tau(A) \subseteq C_1^\tau(A)$,
   or
   there are finitely many places where pistes change their directions, i.e. there are splitting points $B(0), ..., B(k) \in A_0^{1\tau}$ with $B(j)', B(j)''$ the immediate predecessors of $B(j)$ ($j \leq k$) such that

   (a) $B(j)' \in C_0^\tau(B(j))$,

   (b) $B(j)'' \in C_1^\tau(B(j))$.

5. If $A \in A_0^{1\tau}$ is a splitting point in p_0, then it remains such in p_1 with the same immediate predecessors.

6. Let $B \in A_0^{1\tau}$ be a successor point, not in $A_0^{1\tau\,\mathrm{lim}}$ and with a unique immediate predecessor. Consider the wide piste that runs via B (in p_0). Let A be as in 2.1(7). Then there is no model E in p_1 such that $A \in E \in B$.

   This requirement guarantees intervals without models, even after extending a condition.

By 2.13(6), potentially limit points are the only places where non-end-extensions can be made.

Next two lemmas will insure that generic clubs produced by $\mathcal{P}_{\theta\eta\delta}$ run away from old sets.

Lemma 2.14 *Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be an element of $\mathcal{P}_{\theta\eta\delta}$. Let $X \in A^{1\rho lim}$, for some $\rho \in s$.*
Assume that if $\mathrm{cof}(\sup(X \cap \theta^+)) < \rho$, then $\rho \in B$, where $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ is the model with $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$ (exists by 2.1(3)(c)(ii))).
Suppose that for every $t \in X$ there is $D \preceq X$ such that

1. *$D \in X$,*

2. *$t \in D$,*

3. *$|D| = \rho$,*

4. *$D \supseteq \rho$*

5. *${}^{\rho>}D \subseteq D$,*

6. *D is a union of a chain of its elementary submodels which are members of D and satisfy items 1-5* [16].

Then for every $\beta < \sup(X \cap \theta^+)$ *there is* T *of size* ρ *with* $\sup(T \cap \theta^+) > \beta, T \in X$ *such that adding* T *as a potentially limit point and reflecting it through* $\Delta-$*system type triples*[17] *gives an extension of* p.
Moreover, if $\operatorname{cof}(\sup(X \cap \theta^+)) < \rho$, *then we can pick such* T *in* B.

Proof. Let $\tau = \operatorname{cof}(\sup(X \cap \theta^+))$. We deal with the case $\tau < \rho$. The case $\tau = \rho$ is similar and a bit simpler.
By Definition 2.1(3(c(ii)), then $\tau \in s$ and there are $B \in A^{1\tau lim}$ and $X_\theta \in A^{1\theta lim}$ such that $\sup(B \cap \theta^+) = \sup(X \cap \theta^+) = X_\theta \cap \theta^+$ and $B \subseteq X \subseteq X_\theta$. In addition, $\operatorname{cof}(X \cap \rho^+) = \tau$, by Definition 2.1(3(b)).
Let $\langle X_i \mid i < \tau \rangle$ be the canonical sequence of models of 2.1(3(b)) which members are in B. It exists since $\rho \in B$. We have ${}^{\rho>}X_{i+1} \subseteq X_{i+1}$, for every $i < \tau$. By the assumption, we can assume that for every $i < \tau$, X_{i+1} is a union of a chain of its elementary submodels which satisfy items 1,3-5 above.
Pick now T to be one of X_{i+1}, such that

1. $\sup(T \cap \theta^+) > \beta$,

2. for every model E which appears in p and belongs to X, require that $E \in T$,

3. for every model Z which appears in p, has cardinality $< \rho$ and $\sup(Z \cap \theta^+) > \sup(X \cap \theta^+)$, we require that $Z \cap X \in T$[18].
   The next item is added in order to satisfy 2.1(18), (21).

4. For every pair of models Z, Y in p such that

   (a) $Z \subseteq Y$,

   (b) $\sup(Z \cap \theta^+) = \sup(Y \cap \theta^+) > \sup(B \cap \theta^+) = \sup(X \cap \theta^+)$,

   (c) $|Y| = \theta$,

[16] The issue here is to satisfy 2.1(3(b)).

[17] By "reflecting reflecting through $\Delta-$system type triples", we mean that whenever $Z \in \bigcup_{\zeta \in s} A^{1\zeta}$ is a splitting point with immediate predecessors Z_0, Z_1 and $T \in Z_0$, then the image $\pi_{Z_0 Z_1}(T)$ of T under the isomorphism $\pi_{Z_0 Z_1}$ is added together with T.

[18] Recall that ${}^{\rho>}X \subseteq X$ by 2.1(3(c(i))). So, $Z \cap X \in X$, and hence, $Z \cap X \in X_i$ for every large enough $i < \rho$.

(d) $|Z| = \zeta = \mathrm{cof}(\sup(Z \cap \theta^+))$, for some $\zeta < \tau$,

we require that $R \cap X \in T$, for every $R, Z \subseteq R \subseteq Y$ of regular cardinality $\mu \in Z, \zeta < \mu < \rho$ which is the smallest elementary submodel of Y which includes Z and $\mu + 1$.[19].

5. Let D, D_0, D_1 from p form a $\Delta-$system triple, i.e. D is a splitting point and D_0, D_1 are its immediate predecessors. Suppose that $D_0 \supseteq B$.[20] Require the following analogs of the above conditions (2),(3),(4):

   - for every model E which appears in p and belongs to $\pi_{D_0,D_1}(X)$, [21] require that $E \in \pi_{D_0,D_1}(T)$,[22,23]
   - for every model Z which appears in p, has cardinality $< \rho$ and $\sup(Z \cap \theta^+) > \sup(\pi_{D_0,D_1}(X) \cap \theta^+)$, we require that $Z \cap \pi_{D_0,D_1}(X) \in \pi_{D_0,D_1}(T)$.
   - For every pair of models Z, Y in p such that

     (a) $Z \subseteq Y$,

     (b) $\sup(Z \cap \theta^+) = \sup(Y \cap \theta^+) > \sup(B \cap \theta^+) = \sup(X \cap \theta^+)$,

     (c) $|Y| = \theta$,

     (d) $|Z| = \zeta = \mathrm{cof}(\sup(Z \cap \theta^+))$, for some $\zeta < \tau$,

     we require that $R \cap \pi_{D_0,D_1}(X) \in \pi_{D_0,D_1}(T)$, for every $R, Z \subseteq R \subseteq Y$ of regular cardinality $\mu \in Z, \zeta < \mu < \rho$ which is the smallest elementary submodel of Y which includes Z and $\mu + 1$.

Let us argue that T is as desired.

We need check that adding T as a potentially limit point and reflecting it through $\Delta-$system type triples gives an extension of p. The only thing that may go wrong here is that T (or its images through $\Delta-$system type triples) interferes badly with models of p.

First note that if D is one of the models of p and $\sup(D \cap \theta^+) < \sup(X \cap \theta^+)$, then $\sup(D \cap \theta^+) < \sup(T \cap \theta^+)$.

It follows by (2) for models which are in X.

[19]Note that the total number of such R's is $|Z| = \zeta$. Hence, there are less than τ many possibilities for R's.

[20]Note that then $B \subsetneq D_0$, and then $B \in D_0$, since by Definition 2.3(6(c)), D_0 is not a potentially limit point.

[21]This makes sense, since $B \in D_0$ implies $X_\theta \in D_0$, and, by Definition 2.1(3(c(D))), X is the smallest elementary submodel of X_θ which includes B and $\rho + 1$.

[22]Note that π_{D_0,D_1} does not move ρ. Also note that E need not have a pre-image under π_{D_0,D_1}.

[23]$T \in B \subseteq D_0$, hence $T \in \mathrm{dom}(\pi_{D_0,D_1})$.

So, suppose that $D \not\in X$. Changing the wide piste of p if necessary and using (5) above, we can assume that both X and D are on the same wide piste. Apply Definition 2.1(16) or (20) to X and D. Let $D^* \in X \cap C^{|D^*|}(A^{0|D^*|})$ be the witnessing model. But then by (2) above, we have $D^* \in T$.
The requirement (3) ensures the items (17), (20) of Definition 2.1 and (4) ensures the items (18), (21) of Definition 2.1

Let us argue now that adding T does not case any harm once moving through $\Delta-$system type triples. Let D, D_0, D_1 from p of a $\Delta-$system triple, i.e. D is a splitting point and D_0, D_1 are its immediate predecessors.
Note that by (3) above $T \not\in Z$, for any Z in p of cardinality $< \rho$. So, let us assume that $|D| \geq \rho$.
If $X \not\subseteq D_0$ and $X \not\subseteq D_1$, then then let us argue that T not in the domain of π_{D_0,D_1} and π_{D_1,D_0}, and so, does not move. For this apply the intersection property $ip(X, D_0)$. Then

$$X \cap D_0 = X \cap T_1 \cap ... \cap T_n,$$

for some $T_1, ..., T_n \in X$. Hence, by (2) above, $T_1, ..., T_n \in T$. So T cannot be in $X \cap D_0$.

Now, if $X \subseteq D_0$, then the condition (5) above provides the desired conclusion.
□

Lemma 2.15 *Let $p = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau\rangle \mid \tau \in s\rangle$ be an element of $\mathcal{P}_{\theta\eta\delta}$. Let $X \in A^{1\rho lim}$, for some $\rho \in s$.*
Assume that if $\mathrm{cof}(\sup(X \cap \theta^+)) < \rho$, then $\rho \in B$, where $B \in A^{1\,\mathrm{cof}(\sup(X\cap\theta^+))lim}$ is the model with $\sup(B \cap \theta^+) = \sup(X \cap \theta^+)$ (exists by 2.1(3)(c)(ii)).
Suppose that for every $t \in X$ there is $D \preceq X$ such that

1. *$D \in X$,*

2. *$t \in D$,*

3. *$|D| = \rho$,*

4. *$D \supseteq \rho$*

5. *${}^{\rho>}D \subseteq D$,*

6. *D is a union of a chain of its elementary submodels which satisfy items 1-5.*

Let $\beta < \sup(X \cap \theta^+)$ and T be a potentially limit point of size ρ with $\sup(T \cap \theta^+) > \beta, T \in X$ added by the previous lemma 2.14. Then for every $\gamma, \sup(T \cap \theta^+) < \gamma < \sup(X \cap \theta^+)$ there is T' of size ρ with $\sup(T' \cap \theta^+) > \gamma, T' \in X$ such that adding T' as a non-potentially limit point and reflecting it through $\Delta-$system type triples gives an extension of the previous condition.
Moreover, if $\mathrm{cof}(\sup(X \cap \theta^+)) < \rho$, then we can pick such T' in B.

Proof. The proof repeats that of 2.14. The purpose of first adding T and only then T' is to satisfy Item (15) of Definition 2.1. Thus we add first a potentially limit point T above everything relevant, then we are free to add above it a non-potentially limit point T'.
$\square$

We turn now to properness of $\mathcal{P}_{\theta\eta\delta}$.

Recall the following basic definition due to S. Shelah [15]:

Definition 2.16 Let $\mu \geq \omega$ be a regular cardinal and P a forcing notion. P is called $\mu-$proper iff for every $p \in P$ and $M \prec H(\lambda)$ (for large enough λ) with $|M| = \mu, {}^{\mu>}M \subseteq M$, $P, p \in M$ there is $p' \geq_P p$ such that for every dense open $D \subseteq P, D \in M$, $p' \Vdash$ "$D \cap \underset{\sim}{G} \cap M \neq \emptyset$." Such p' is called (M, P)-generic.

The following is obvious:

Lemma 2.17 *If P is μ-proper, then it preserves μ^+.*

Lemma 2.18 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\eta-$proper*

Proof. Let $p \in \mathcal{P}_{\theta\eta\delta}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ regular large enough such that

1. $|\mathfrak{M}| = \eta$,
2. $\mathfrak{M} \supseteq \eta$,
3. $\mathcal{P}_{\theta\eta\delta}, p \in \mathfrak{M}$,
4. ${}^{\eta>}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\theta^+)$.
Clearly, M satisfies 2.1(3(b)). Moreover, using the elementarity of $\mathfrak{M}$, for every $x \in M$ there will be $Z \in M$ such that

- $Z \preceq H(\theta^+)$,
- $|Z| = \theta$,
- $Z \supseteq \theta$,
- ${}^{\theta>}Z \subseteq Z$,
- $x \in Z$.

This allows to find a chain of models $\langle N_i \mid i < \eta\rangle$ of size θ which members are in M, witnesses 2.1(3(b)) for $N := \bigcup_{i<\eta} N_i$ and $N \supseteq M$.

Extend p by adding M as a new $A^{0\eta}$ and N as a new $A^{0\theta}$. Require them to be potentially limit points. Denote the result by $p^\frown\{M, N\}$.
We claim that $p^\frown\{M, N\}$ is $(\mathcal{P}_{\theta\eta\delta}, \mathfrak{M})-$generic. So, let $p' \geq p^\frown\{M, N\}$ and $\tilde{D} \in \mathfrak{M}$ be a dense open subset of $\mathcal{P}_{\theta\eta\delta}$. It is enough to find $q \in \mathfrak{M} \cap D$ which is compatible with p'.
Let

$$p' = \langle\langle A^{0\tau}(p'), A^{1\tau}(p'), A^{1\tau lim}(p'), C^{\tau}(p')\rangle \mid \tau \in s(p')\rangle.$$

By Lemmas 2.9,2.10,2.11, for every $\tau \in s(p')$ there will be a maximal model in $A^{1\tau}(p') \cap M$ (once non-empty). They all are on the wide piste which runs through M. Just the wide piste that runs through M runs through N and all other relevant for maximality models, since $|M| = \eta$ is the smallest possible cardinality of the model and pistes of models of different sizes go in the same directions by Definition 2.3(9,6d(xi), 6e(vi)).
Let us argue that they are linearly ordered by $\in, \subseteq$. Thus, let $\tau, \rho \in s(p'), \tau < \rho$, $A \in A^{1\tau}(p') \cap M, B \in A^{1\rho}(p') \cap M$ be such maximal models. If $\sup(A \cap \theta^+) < \sup(B \cap \theta^+)$, then, by Definition 2.1(12(a)), $A \in B$. If $\sup(A \cap \theta^+) = \sup(B \cap \theta^+)$, then, by Definition 2.1(12(b)), $A \subseteq B$. Suppose that $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$.
If $\rho \in A$, for example if $\theta < \aleph_{\eta^+}$, then, by Definition 2.1(16), there is $B^* \in A, |B^*| = \rho$ which is on the same wide piste and $B^* \supseteq B$. If $B^* \in M$, then by maximality of B this may occur only when $B^* = B$. If $B^* \notin M$, then, by Definition 2.1(16), there is $B^{**} \in M, B^{**} \supseteq B^*$ of cardinality ρ, since $\rho \in M$. The maximality of B implies then that $B^{**} = B$, and hence, $B = B^*$ as well. So, we are done.
If $\rho \in A$, for some A, B as above, then let us extend p' by adding to it an $\in -$increasing sequence of models in M which will be maximal.

In general, i.e. without assuming $\theta < \aleph_{\eta^+}$, we just add such $\in -$increasing sequence of models from M.

Set

$$q' = \langle\langle \max(A^{1\tau}(p') \cap M), A^{1\tau}(p') \cap M, A^{1\tau lim}(p') \cap M, C^{\tau}(p') \restriction A^{1\tau}(p') \cap M\rangle \mid \tau \in s(q')\rangle,$$

where $s(q') = s(p') \cap M$.

It is routine to check that $q' \in \mathcal{P}_{\theta\eta\delta}$ and $q' \leq p'$. Also, $q' \in \mathfrak{M}$, since ${}^{\eta>}\mathfrak{M} \subseteq \mathfrak{M}$.

Now let $q = \langle\langle A^{0\tau}(q), A^{1\tau}(q), A^{1\tau lim}(q), C^{\tau}(q)\rangle \mid \tau \in s(q)\rangle$ be an extension of q' in $\mathfrak{M}$ which belongs to $\tilde{D}$.

We claim that p' and q are compatible. Namely, set $s = s(q) \cup s(p')$. Let $A^{0\tau} = A^{0\tau}(p')$, for every $\tau \in s(p')$. Let $\langle \tau_i \mid i < i^*\rangle$ be an increasing (or just any one to one) enumeration of $s \setminus s(p')$. Pick $\in -$increasing sequence of models $\langle A_i \mid i < i^*\rangle$ such that for every $i < i^*$ the following hold:

1. $p', q \in A_i$,
2. $|A_i| = \tau_i$,
3. A_i satisfies 2.3(2).

Set $A^{0\tau_i} = A_i$.

Finally let for every $\tau \in s$,

$$A^{1\tau} = \{A^{0\tau}\} \cup A^{1\tau}(p') \cup A^{1\tau}(q) \cup \{B \mid$$

$$\exists(D, D_0, D_1) \quad \Delta - \text{system triple in } p' \text{ with } M \in \mathrm{dom}(\pi_{D_0,D_1})$$

$$\text{and there is a model } A \text{ in } q \text{ which does not appear in } p' \text{ such that } B = \pi_{D_0,D_1}(A)\}\cup$$

$$\{B \mid \exists(D, D_0, D_1) \quad \Delta - \text{system triple in } q \text{ but not in } p' \text{ and there is a model}$$

$$A \in \mathrm{dom}(\pi_{D_0,D_1}) \text{ in } p' \text{ which does not appear in } q \text{ such that } B = \pi_{D_0,D_1}(A)\}.$$

Intuitively, we just put together models (of same cardinalities) of p' and q and add to them the images of new models (those in q and not in p') under isomorphisms of models of p' with M inside.

Define $A^{0\tau lim}$ and $C^{\tau}(\tau \in s)$ in the obvious fashion now.

Set

$$p^* = \langle\langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^{\tau}\rangle \mid \tau \in s\rangle.$$

We claim that $p^* \in \mathcal{P}_{\theta\eta\delta}$ and then, by 2.13 and the definition of p^*, $p^* \geq p', q$.

Deal with the wide piste of p^*. Let us show that it satisfies Definition 2.1. The main issue will be to check the covering conditions of Definition 2.1. Other conditions hold trivially.

Let $D \in A^{0\rho}(p')$ be on the wide piste of p' for some $\rho \in s \setminus \eta + 1$. If $\sup(D \cap \theta^+) \geq \sup(M \cap \theta^+)$, then $M \subseteq D$ by Definition 2.1(12). Hence every new model (i.e. one in q and not in p') is in D.

Note it is impossible to have a situation when $M \in D$, D has an immediate predecessor with sup below $\sup(M \cap \theta^+)$ and D is not a potentially limit point. It follows by Definition 2.1(4), since M is a potentially limit point.

Also such D cannot be both minimal in $C^\rho(A^{0\rho}(p'))$ and not potentially limit, by Definition 2.1(6), since N appears in p'. Actually, this the only reason of picking $\mathfrak{N}$ and adding N to p.

Assume that $\sup(D \cap \theta^+) < \sup(M \cap \theta^+)$. If D is in M, then D is in q', and hence new models are fine with D since they are in $q \geq q'$. Assume that $D \notin M$. Let $|D| = \rho$. Then $\rho > \eta$. Let A be on the wide piste of q. If A appears in p', then we are done. Suppose otherwise, i.e. A is a new model.

Case 1. $\sup(A \cap \theta^+) < \sup(D \cap \theta^+)$.

Let D^* be the least model in M which includes D and $M \cap D^* \subseteq D$. It exists and belongs to $C^\rho(A^{0\rho})$, if $\theta < \aleph_{\eta^+}$, by (16,17) of Definition 2.1, and if $\theta \geq \aleph_{\eta^+}$, then such D^* exists by (19) of Definition 2.1.

Subcase 1.1 $A \in D^*$.

Then $M \cap D^* \subseteq D$ implies $A \in D$.

Subcase 1.2 $A \notin D^*$.

Suppose first that $\theta < \aleph_{\eta^+}$. Then, necessarily, $|A| > \rho = |D^*| = |D|$. Both A, D^* are in q, hence there is the least model $A^* \in C^{|A|}(A^{0|A|})(q)$ in D^* above A.

Then $A^* \in D$, as $M \cap D^* \subseteq D$, and so, A, D satisfy Definition 2.1(16). Also they satisfy Definition 2.1(17), since A, D^* satisfy it as members of q and $D \subseteq D^*$.

Suppose now that $\theta \geq \aleph_{\eta^+}$.

If $\rho \in M$, then by (19(a)) of Definition 2.1, $D^* \in C^\rho(A^{0\rho})$ and $|A| > \rho = |D^*| = |D|$.

Both A, D^* are in q, hence, by (19) of Definition 2.1, there is the least model $A^* \in D^*$ such that $D^* \cap A^* \subseteq A$.

Then $A^* \in D$, as $M \cap D^* \subseteq D$, and so, A, D satisfy Definition 2.1(19). If $|A| \in D \subseteq D^*$, then $|A^*| = |A|$, and A, D will satisfy Definition 2.1(20) as well.

Suppose that $|A| \notin D$. Then $|A| \notin D^*$, since $M \cap D^* \subseteq D$. Apply (20) of Definition 2.1 to D^* and A.

Suppose, for example, that (c) of (20) of Definition 2.1 holds. I.e. there are G, H, A^* in D^* such that

1. $A \in G$,

2. $|G| < |A|$,

3. $G \subseteq A^* \subseteq H$,

4. $\sup(G \cap \theta^+) = H \cap \theta^+$,

5. $G \in C^{|G|}(A^{0|G|})(q), H \in C^{\theta}(A^{0\theta})(q)$,

6. $|A^*| = \min(D^* \cap On \setminus |A|)$,

7. A^* is the smallest elementary submodel of H which includes G and $|A^*|$, i.e. the Skolem Hull of $G \cup |A^*|$.

8. $A \in A^*$, $A^* \supseteq A$,

9. $A^* \cap D^* \subseteq A$,

10. $\{A' \in A^* \mid (|A'| \leq |A|) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in D^*)((\forall k < n)(|Z_k| < |G|)) \wedge A' \in D^* \cup \bigcup_{k<n} Z_k))\} \in A$.

But G, H, A^* in M as $q \in M$, then they are in D, as well, since $M \cap D^* \subseteq D$. So, G, H, A^* witness (20) of Definition 2.1 for D and A.
The other possibilities ((a),(b), (d) of (20) of Definition 2.1) are similar.

The condition (21) of Definition 2.1 for D and A holds, since any relevant Y (as in (b) of (21)), is in D^*, and so, $\min(D^* \cap On \setminus |A|) \leq \min(Y \cap On \setminus |A|)$, once $|Y| < |A|$. Hence, $\min(D^* \cap On \setminus |A|) = \min(Y \cap On \setminus |A|)$, as $\min(D^* \cap On \setminus |A|) = \min(D \cap On \setminus |A|) \geq \min(Y \cap On \setminus |A|)$. So, G, H, A^* will work for Y, in this case. If $|Y| \geq |A|$, then $|Y| \geq |A^*|$, since $|Y| \in D$. Recall that $A^* \in D \subseteq Y$ and $A \in A^*$. So, $A^* \subseteq Y$ and $A \in Y$.

Assume now that $\rho \notin M$. Apply (20) of Definition 2.1 to M and D (in p'). Let $G \in M \cap \bigcup_{\mu \in s(p')} C^{\mu}(A^{0\mu}(p'))$ be the smallest possible such that $\sup(D \cap \theta^+) < \sup(G \cap \theta^+)$. Assume first that (b) of (20) of Definition 2.1 holds. Then $D \in G, |G| \geq |D| = \rho$ and

1. $M \cap G \subseteq D$,

2. $\{D' \in G \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)((\forall k < n)(|Z_k| < |G|)) \wedge D' \in M \cup \bigcup_{k<n} Z_k))\} \in D$.

If $A \in G$ then $A \in D$, since $M \cap G \subseteq D$. In particular, if $|A| \leq |G|$, then $A \in G$, since both are in q, and, so $A \in D$.
Suppose $A \notin G$. Assume first that $|A| \in G$. By (19) of Definition 2.1 applied to A and G inside q, there is $A^* \in G \cap C^{|A|}(A^{0|A|})$ such that $A^* \supseteq A$ and $G \cap A^* \subseteq A$. We have this A^* in q, and hence in M. So, $A^* \in G \cap M \subseteq D$.
Recall that $G \supseteq D$. Hence, A^* witnesses the covering property (19(a)) of Definition 2.1 for D and A, as well. The properties (20,21) of Definition 2.1 follow.
Suppose now that that $|A| \notin G$. Apply then (20) of Definition 2.1 to A and G inside q. The witnesses of this property will serve as witnesses for A and D, as well.

Assume now that (c) of (20) of Definition 2.1 holds instead of (b). Then $D \in G$, $|G| < |D| = \rho$ and there are H, D^* such that

1. $G \subseteq D^* \subseteq H$,
2. $\sup(G \cap \theta^+) = H \cap \theta^+$,
3. $G \in C^{|G|}(A^{0|G|})(p'), H \in C^{\theta}(A^{0\theta})(p')$,
4. $|D^*| = \min(M \cap On \setminus \rho)$,
5. D^* is the smallest elementary submodel of H which includes G and $|D^*|$, i.e. the Skolem Hull of $G \cup |D^*|$.
6. $D \in D^*$, $D^* \supseteq D$,
7. $D^* \cap M \subseteq D$,
8. $\{D' \in D^* \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in M)((\forall k < n)(|Z_k| < |G|)) \wedge D' \in M \cup \bigcup_{k<n} Z_k))\} \in D$.

If $A \in D^*$ then $A \in D$, since $M \cap D^* \subseteq D$. In particular, if $|A| \leq |G|$, then $A \in G$, since both are in q, and, so $A \in D^*$, and then $A \in D$.
Suppose $A \notin D^*$. Assume first that $|A| \in G$. Use (19) of Definition 2.1, to find the least model $A^* \in G \cap C^{|A|}(A^{0|A|}(q))$ such that $A^* \supseteq A$ and $G \cap A^* \subseteq A$. Then $A^* \in M \cap D^* \subseteq D$ and it will witness the covering properties for A and D.
Assume now that $|A| \notin G$. Apply then (20) of Definition 2.1 to A and G inside q.
Let $G' \in G \cap \bigcup_{\mu \in s(q)} C^{\mu}(A^{0\mu}(q)$ be the smallest possible such that $\sup(A \cap \theta^+) < \sup(G' \cap \theta^+)$. Then (b) or (c) or (d) of (20) of Definition 2.1 holds. It is not hard to see, using the arguments similar to those above, that witnessing models in this cases will be good for D as well.

Case 2. $\sup(A \cap \theta^+) > \sup(D \cap \theta^+)$.
Suppose first that $\theta < \aleph_{\eta^+}$. Apply (16) of Definition 2.1 to M and D inside p'. Let D^* be the least model in M above D in $C^\rho(A^{0\rho}(p'))$. Then $M \cap D^* \subseteq D$.
Clearly, $\sup(A \cap \theta^+) \geq \sup(D^* \cap \theta^+)$.
Both A and D^* are in q, so $|A| \geq \rho$ implies, by 2.1(12), that $A \supseteq D^* \supseteq D$.
Suppose that $|A| < \rho$. If $D^* \in A$, then, by 2.1(17), D^* will be the least in $A \cap C^{\rho^*}(A^{0\rho^*})$ above D. Suppose that $D^* \notin A$. Then, since both A and D^* in q, there will be the least $D^{**} \in A \cap C^\rho(A^{0\rho})(q)$ above D^*, and this D^{**} will satisfy 2.1(16) for A, D. The condition 2.1(17) holds for A, D since it is true for A, D^* and for M, D .

Suppose now that $\theta > \aleph_{\eta^+}$. Apply (20) of Definition 2.1 to M and D inside p'. Let $G \in M \cap \bigcup_{\mu \in s(p')} C^\mu(A^{0\mu}(p'))$ be the smallest possible such that $\sup(D \cap \theta^+) < \sup(G \cap \theta^+)$. If (a) of (20) of Definition 2.1 holds, then $G = D$, and so, $D \in M$, but we assumed that it is not the case.
Suppose that (b) of (20) of Definition 2.1 holds. Then $D \in G, |G| \geq |D| = \rho$ and

1. $M \cap G \subseteq D$,
2. $\{D' \in G \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in B)((\forall k < n)(|Z_k| < |G|)) \wedge D' \in M \cup \bigcup_{k<n} Z_k))\} \in D$.

The minimality of G implies that $\sup(A \cap \theta^+) \geq \sup(G \cap \theta^+)$. Also, by (20) of Definition 2.1, we have ${}^{|G|>}G \subseteq G$, so $\sup(A \cap \theta^+) = \sup(G \cap \theta^+)$ implies $|A| \geq |G|$ and $A \supseteq G$. Hence, $D \in A$ in this case.
Suppose that $\sup(A \cap \theta^+) > \sup(G \cap \theta^+)$. If $|A| \geq |G|$, then $A \supseteq G$, and so again, $D \in A$. Suppose that $|A| < |G|$. Apply (20) of Definition 2.1 again. This time to A and G inside q. We will obtain $G^* \in A$ which covers G and $G^* \cap A = G \cap A$.
Let us argue that then $G^* \cap A \subseteq D$. It is enough to deal with ordinals. So, let $\alpha < \theta^+$ be in $G^* \cap A$. Then, clearly, $a = \{\alpha\} \in G^* \cap A = G \cap A$. Recall that G satisfies (20) of Definition 2.1 for M and D. Then, by (b)(ii) or (c)(vii) or (d)(x) of this condition and the fact that $A \in M$,$|A| < |G|$, we will have $a \in D$. So, the witnesses of (20) of Definition 2.1 for A and G will be fine for A and D.

Suppose now that (c) of (20) of Definition 2.1 holds (for M and D). Then $D \in G$, $|G| < |D| = \rho$ and there are H, D^* such that

1. $G \subseteq D^* \subseteq H$,
2. $\sup(G \cap \theta^+) = H \cap \theta^+$,

3. $G \in C^{|G|}(A^{0|G|})(p'), H \in C^{\theta}(A^{0\theta})(p')$,

4. $|D^*| = \min(M \cap On \setminus \rho)$,

5. D^* is the smallest elementary submodel of H which includes G and $|D^*|$, i.e. the Skolem Hull of $G \cup |D^*|$.

6. $D \in D^*$, $D^* \supseteq D$,

7. $D^* \cap M \subseteq D$,

8. $\{D' \in D^* \mid (|D'| \leq \rho) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in M)((\forall k < n)(|Z_k| < |G|)) \wedge D' \in M \cup \bigcup_{k<n} Z_k))\} \in D$.

The minimality of G implies that $\sup(A \cap \theta^+) \geq \sup(G \cap \theta^+)$. Also, by (20) of Definition 2.1, we have ${}^{|G|>}G \subseteq G$, so $\sup(A \cap \theta^+) = \sup(G \cap \theta^+)$ implies $|A| \geq |G|$ and $A \supseteq G$. Hence, $D \in A$ in this case.
Suppose that $\sup(A \cap \theta^+) > \sup(G \cap \theta^+)$. If $|A| \geq |G|$, then $A \supseteq G$, and so again, $D \in A$. Suppose that $|A| < |G|$.
Assume first that $G \in A$. Then $H \in A$, and if $\rho \in A$, then also $D^* \in A$.
Let $\rho_A = \min(A \cap On \setminus \rho)$ and let D_A be the Skolem Hull of $G \cup \rho_A$. If $D^* \in A$, then $D_A = D^*$, since $A \in M$, and so $\rho_A \geq |D^*|$. If $D^* \notin A$, then, clearly, $\rho_A = |D_A| > |D^*|$, again since $A \in M$. In both cases we have $D_A \supset D^*$ and $A \cap D_A = A \cap D^*$. But then $A \cap D_A \subseteq D$, by (20)(c)(vii) of Definition 2.1 for M and D. So, G, H, D_A work for A and D.

Suppose now that $G \notin A$.
Apply (20) of Definition 2.1 to A and G inside q. Consider a typical case - (20)(c) of Definition 2.1. Other cases are simpler ((b)) or similar (d)). Let $G', H', D' \in A$ be the witnessing models. So, the following hold:

1. ${}^{|G'|>}G' \subseteq G'$,

2. $G \in G'$,

3. $|G'| < |G|$

4. $G' \subseteq D' \subseteq H'$,

5. $\sup(G' \cap \theta^+) = H' \cap \theta^+$,

6. $G' \in C^{|G'|}(A^{0|G'|})(q), H \in C^{\theta}(A^{0\theta})(q)$,

7. $|D'| = \min(A \cap On \setminus |G|)$,

8. D' is the smallest elementary submodel of H' which includes G' and $|D'|$, i.e. the Skolem Hull of $G' \cup |D'|$.

9. $G \in D'$, $D' \supseteq G$,

10. $D' \cap A \subseteq G$,

11. $\{D'' \in D' \mid (|D''| \leq |D'|) \wedge (\exists n < \omega)(\exists Z_{n-1} \in ... \in Z_0 \in A)((\forall k < n)(|Z_k| < |G'|)) \wedge D'' \in A \cup \bigcup_{k<n} Z_k))\} \in G$.

Let $\rho' = \min(A \cap On \setminus \rho)$. Note that $\rho' \geq |D'|$.
Set D'' to be the Skolem Hull of $G' \cup \rho'$ in H'. Then $D'' \supseteq D'$.

Claim. $A \cap D'' = A \cap D^*$.

Proof. Note that each member of $A \cap D''$ is an application of a Skolem function on elements of $(A \cap G') \cup (A \cap \rho')$,as follows from following simple general fact:

Fact. *Assume that Z, Y are elementary submodels of $H(\chi)$. Suppose that $\chi > \zeta > |Z| > |Y|, Z, \zeta \in Y$. Then $Y \cap cl(Z \cup \zeta) = cl((Z \cap Y) \cup (Y \cap \zeta))$, where $cl(X)$ denotes the Skolem Hull of X.*

Proof. Clearly, $Y \cap cl(Z \cup \zeta) \supseteq cl((Z \cap Y) \cup (Y \cap \zeta))$, since $Y = cl(Y) \supseteq cl((Z \cap Y) \cup (Y \cap \zeta))$ and $cl(Z \cup \zeta) \supseteq cl((Z \cap Y) \cup (Y \cap \zeta))$.
Let us show that $Y \cap cl(Z \cup \zeta) \subseteq cl((Z \cap Y) \cup (Y \cap \zeta))$.
So, let $\mu \in Y \cap cl(Z \cup \zeta)$. Then there is a Skolem function h, an element $g \in Z$ and an ordinal $\beta < \zeta$ such that $\mu = h(g, \beta)$. Then, by the elementarity, since $G, \rho, \mu \in A$, $A \models \exists x \in G \exists \alpha < \rho(\mu = h(x, \alpha))$. Hence, there are $g' \in Y \cap Z, \beta' \in Y \cap \zeta$ such that $\mu = h(g', \beta')$. So, $\mu \in cl((Z \cap Y) \cup (Z \cap \zeta))$.
$\square$ of the fact.

Now, $A \cap G = A \cap D'$ and $A \cap \rho' = A \cap \rho$. We have $A \cap D^* = A \cap cl(G \cup \rho) \supseteq cl((A \cap G) \cup (A \cap \rho))$. Also, $cl((A \cap G) \cup (A \cap \rho)) = cl((A \cap D') \cup (A \cap \rho'))$ and $cl((A \cap D') \cup (A \cap \rho')) = cl((A \cap cl(G' \cup |D'|)) \cup (A \cap \rho'))$. By the fact above, then $cl((A \cap cl(G' \cup |D'|)) \cup (A \cap \rho')) = cl((A \cap G') \cup (A \cap \rho')) = A \cap D''$. So, we need to show that $A \cap cl(G \cup \rho) \subseteq cl((A \cap G) \cup (A \cap \rho))$. Let $\mu \in A \cap cl(G \cup \rho)$. Then there is a Skolem function h, an element $g \in G$ and an ordinal $\beta < \rho$ such that $\mu = h(g, \beta)$. We have $D' \supseteq G$ and $\rho < \rho'$. Both D' and ρ' are in A. Then, by the elementarity, $A \models \exists x \in D' \exists \alpha < \rho'(\mu = h(x, \alpha))$. Hence, there are $g' \in A \cap D', \beta' \in A \cap \rho'$ such that $\mu = h(g', \beta')$. But, $A \cap D' \subseteq G$ and $A \cap \rho' \subseteq \rho$, so $g' \in G$ and $\beta' < \rho$. Hence,

$\mu \in cl((G \cap A) \cup (A \cap \rho))$, and we are done.

$\square$ of the claim.

So, G', H', D'' witness (20, 21) of Definition 2.1 for A and D, since $A \in M$ and $|A| < |G|$.

Case 3. $\sup(A \cap \theta^+) = \sup(D \cap \theta^+)$.

Then by Definition 2.1(12(b)), we will have $A \subseteq D$, if $|A| \leq \rho$ and $A \supseteq D$ otherwise.

Finally us deal with a situation where D is not on the wide piste of p' and so of p^*.

Case A. D is not on the wide piste and the first splitting on the piste from $A^{0\rho}$ to D is above M.

Then we just consider the image of M under such splitting and proceed with it as before.

Case B. D is not on the wide piste and the first splitting on the piste from $A^{0\rho}$ to D is below M (according to sup of the models).

Change inside p' the wide piste in order to put D on it. Such change will preserve M on the wide piste since the relevant splittings are below M. Now, both D and M will be on the (new) wide piste of p' and p^*. Proceed as before.

The above shows that p^* satisfies 2.3(5, 9). The rest of 2.3, as well as $p^* \geq p', q$ follows easily from the definition of p^* using 2.1(15).

$\square$

Our next task will be to show that the forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$. Let us first prove three technical lemmas that allow to add new models at places of specific type.

Lemma 2.19 *Let $p = \langle \langle C^\tau, C^{\tau lim} \rangle \mid \tau \in s \rangle$ be a wide piste and B, D are models of p such that $|B| = \tau$, for some regular $\tau \in s \cap \theta$, $|D| = \theta$ and $\sup(B \cap \theta^+) = D \cap \theta^+$.*
Let $\rho \in (\tau, \theta) \cap B$ be a regular cardinal.
Assume that $\rho \in A$, for every model $A \in \bigcup_{\mu \in s} C^\mu$ such that
$\sup(A \cap \theta^+) \geq \sup(B \cap \theta^+)$. [24]
Suppose that for every $A \in \bigcup_{\mu \in s \cap \tau} C^\mu$, if $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$, then $B \in A$. Then a model of cardinality ρ can be added to p between B and D such that the result remains a wide piste.

Proof. Suppose that there is no model of size ρ between B and D inside p. Without loss of generality we can assume that $\rho \in s$. Just otherwise extend p by adding the largest (under $\in$) model of cardinality ρ making it a potentially limit one.

Let E be the least elementary submodel of D such that

[24]This holds automatically once $\theta < \aleph_{\eta^+}$.

- $|E| = \rho$,
- $E \supseteq B$,
- $E \supseteq \rho + 1$,

So, E is the Skolem Hull of $B \cup \rho + 1$ in D.

Add E to $C^{\rho lim}$. Let us check that the result is a wide piste.

If a model A appears in p and there is $H \in A$ in p with $\sup(H \cap \theta^+) = \sup(E \cap \theta^+)$, then by 2.1(14) also $B, D \in A$, and then by elementarity, $E \in A$.

Let $A \in C^\xi$, for some $\xi \in s$.

If $\sup(A \cap \theta^+) > \sup(E \cap \theta^+)$, then $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$, since $\sup(E \cap \theta^+) = \sup(B \cap \theta^+)$. Then, by 2.1(12), once $\xi \geq \tau$ and the assumption of the lemma, once $\xi < \tau$, we have $B \in A$. Hence, by 2.1(14), $D \in A$, as $\sup(B \cap \theta^+) = D \cap \theta^+$. But E is definable from B, D and ρ ($\rho \in B, A$), so $E \in A$.

Assume now that $\sup(A \cap \theta^+) < \sup(E \cap \theta^+)$.

Suppose first that $\xi \leq \rho$. If $A \in B$, then we are done, since $B \subseteq E$. Note that it the case once $\xi \leq \tau$. Suppose that $A \notin B$. Assume first that $\xi \in B$. Then, by 2.1(16), there is $A^* \in B \cap C^\xi(A^{0\xi})$ such that $A^* \supseteq A$. But then, $A \in A^*$, and by above $A^* \in E$. But $\xi \leq \rho$, so $A^* \subseteq E$. Hence $A \in E$.

Consider now the remaining case: $\xi > \rho$ and $A \notin E$. It follows that $A \notin B$. Apply Definition 2.1(16) or (19) to B and A and find least possible $A^* \in C^\xi \cap B$ which contains A, if $\xi \in B$.

Clearly, it witnesses 2.1(16) or (19) for E and A as well, but we would like to show that 2.1(17) holds for E and A and to remove the assumption that $\xi \in B$. Note that E is definable in $\langle H(\theta^+), \in, \leq, \delta, \eta \rangle$ using $\{B, D, \rho + 1\}$ as parameters. So, 2.1(18) or (21) applies to E and A which implies 2.1(17) for them[25].

Finally, 2.1(18) holds for E and A due to definability of E and 2.1(18) for B and A.

The rest of the conditions follow easily.

□

Lemma 2.20 *Let $p = \langle \langle C^\tau, C^{\tau lim} \rangle \mid \tau \in s \rangle$ be a wide piste and B, D are models of p such that $|B| = \tau$, for some regular $\tau \in s \cap \theta$, $|D| = \theta$ and $\sup(B \cap \theta^+) = D \cap \theta^+$.*
Let $\rho \in (\tau, \theta) \cap B$ be a regular cardinal. Then a model of cardinality ρ can be added to p between B and D such that the result remains a wide piste.

[25] Here is the only placed where we use 2.1(18).

Proof. Suppose first that $\theta < \aleph_{\eta^+}$. We just continue the argument of the previous lemma (2.19) from the point where the appeal to the assumption "for every $A \in \bigcup_{\mu \in s \cap \tau} C^\mu$, if $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$, then $B \in A$" was made. So, assume that $A \in C^\xi$, for some $\xi \in s \cap \rho$, $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \notin A$.
Assume that such A was picked to be the least possible (under $\in -$relation) with $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \notin A$.
Apply (16,17) of Definition 2.1 to A,B and find smallest $B^* \in C^\tau \cap A, B^* \supseteq B$. If there is A_1 so that $\sup(A_1 \cap \theta^+) > \sup(B^* \cap \theta^+)$ and $B^* \notin A_1$, then either $\sup(A_1 \cap \theta^+) < \sup(A \cap \theta^+)$ and, then $|A_1| > |A|$ (due to minimality of A), or $\sup(A_1 \cap \theta^+) > \sup(A \cap \theta^+)$ and, then $|A_1| < |A|$ (since $|A_1| \geq |A|$ will imply $A_1 \supseteq A$, by Definition 2.1(12)). Let us show that the former possibility is impossible.
Claim. *It is impossible to have* $\sup(A_1 \cap \theta^+) < \sup(A \cap \theta^+)$.
Proof. Suppose otherwise. Then, by minimality of A, $|A_1| > |A|$ and $B \in A_1$. If $A_1 \in A$, then 2.1(17) for A, B provides the desired contradiction, since $B \in B^*, B \in A_1 \in A, |A_1| < |B^*|$ (since $\sup(A_1 \cap \theta^+) > \sup(B^* \cap \theta^+)$ and $B^* \notin A_1$). If $A_1 \notin A$, then then apply Definition 2.1(16) to A and A_1 and find $A_1^* \in A \cap C^{|A_1|}$ such that $A_1 \subseteq A_1^*$. Now we derive a contradiction replacing A_1 with A_1^*.
$\square$ of the claim.

Note that $D \notin A$, since that by Definition 2.1, B will be in A, which is not the case. Let $D^* \in A \cap C^\theta$ be the minimal cover of D. Then it is impossible to have $D^* \cap \theta^+ > \sup(B^* \cap \theta^+)$, since then $B^* \in D^*$, but by Definition 2.1(17), $A \cap D^* \subseteq D$ and $B^* \notin D$.
Also, it is impossible to have $D^* \cap \theta^+ < \sup(B^* \cap \theta^+)$, since then consider the maximal element D' of C^θ which such that $D' \cap \theta^+ < \sup(B^* \cap \theta^+)$. By Definition 2.1(17) applied to both A and B^*, then $D' \in A \cap B^*$. But, $A \cap B^* \subseteq B$, by Definition 2.1(17), and $B^* \notin B$. Hence, the only possibility is $D^* \cap \theta^+ = \sup(B^* \cap \theta^+)$.

Consider now the later possibility.
Replace A,B by A_1, B^*.

After finitely many steps we will reach models $B^{**} \in C^\tau(A^{0\tau})$ and $D^{**} \in C^\theta$ with $\sup(B^{**} \cap \theta^+) = D^{**} \cap \theta^+$ such that for every A' in p, with $\sup(A' \cap \theta^+) > \sup(B^{**} \cap \theta^+)$, we have $B^{**}, D^{**} \in A'$. Apply the previous lemma 2.19 to B^{**}, D^{**} and add $E^{**}, B^{**} \subseteq E^{**} \subseteq D^{**}$. Then continue, go down and apply 2.19 again and again until finally B and D will be reached.

Suppose now that $\theta \geq \aleph_{\eta^+}$.
Assume that there is $A \in C^\xi$, for some $\xi \in s \cap \rho$, $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \notin A$.

Suppose first that $\tau = |B| \in A$. Note that $\tau \notin A$ implies $B \notin A$. Proceed as in the case $\theta < \aleph_{\eta^+}$.
Assume that such A was picked to be the least possible (under $\in$ −relation) with $\sup(A \cap \theta^+) > \sup(B \cap \theta^+)$ and $B \notin A$.
Apply (20) of Definition 2.1 to A,B. Let $B^* \in C^{|B^*|} \cap A$ witnesses it.
Suppose first that (b) or (c) of (20) of Definition 2.1 hold. Then $B \in B^*$.
Deal with (b) of (20) of Definition 2.1 first. Then $|B^*| \geq |B| = \tau$. If there is A_1 so that $\sup(A_1 \cap \theta^+) > \sup(B^* \cap \theta^+)$ and $B^* \notin A_1$, then either $\sup(A_1 \cap \theta^+) < \sup(A \cap \theta^+)$ and, then $|A_1| > |A|$ (due to minimality of A), or $\sup(A_1 \cap \theta^+) > \sup(A \cap \theta^+)$ and, then $|A_1| < |A|$ (since $|A_1| \geq |A|$ will imply $A_1 \supseteq A$, by Definition 2.1(12)).
Claim. *It is impossible to have* $\sup(A_1 \cap \theta^+) < \sup(A \cap \theta^+)$.
Proof. Suppose otherwise. Then, by minimality of A, $|A_1| > |A|$ and $B \in A_1$. If $A_1 \in A$, then (20)(b)(ii) of Definition 2.1 for A, B provides the desired contradiction, since $B \in B^*, B \in A_1 \in A, |A_1| < |B^*|$ (since $\sup(A_1 \cap \theta^+) > \sup(B^* \cap \theta^+)$ and $B^* \notin A_1$). If $A_1 \notin A$, then then apply Definition 2.1(20) to A and A_1 and find $A_1^* \in A \cap C^{|A_1^*|}$. If (b) or (c) of (20) of Definition 2.1 hold, then $A_1 \in A_1^*$.
If (b) holds, then $|A_1| \geq |A_1^*|, A_1 \subseteq A_1^*$, in particular $B \in A_1^*$. If $B^* \in A_1^*$, then $B^* \in A \cap A_1^* \subseteq A_1$, but $B^* \notin A_1$. So, $B^* \notin A_1^*$. Now we derive a contradiction replacing A_1 with A_1^*.
If (c) holds, then we have $|A_1^*| < |A_1| < |B^*|$ and $B \in A_1 \in A_1^* \in A$. But such situation is impossible by (b(ii)) of (20) of Definition 2.1 for A, B, B^*.
If (d) holds, then we have $|A_1^*| = |G_0| < ... < |G_{n-1}| < |A_1| < |B^*|$ and $B \in A_1 \in G_{n-1} \in ... \in G_0 = A_1^* \in A$, for finite sequence $\langle G_k \mid k < n \rangle$ of models which witness (d). But, again, such situation is impossible by (b(ii)) of (20) of Definition 2.1 for A, B, B^*.
□ of the claim.

If $\sup(A_1 \cap \theta^+) > \sup(A \cap \theta^+)$, then we proceed as in the case $\theta > \aleph_{\eta^+}$ above.

Deal now with (c) of (20) of Definition 2.1. Then $|B^*| < |B| = \tau$, but the further arguments repeat those of (b) above only in the claim instead of (b(ii)) of (20) of Definition 2.1, (c(viii)) should be used.

The treatment of the possibility (d) of (20) of Definition 2.1 is similar to (b,c).
□

Lemma 2.21 *Let* $p = \langle \langle A^{0\tau}, A^{1\tau}, A^{1\tau lim}, C^\tau \rangle \mid \tau \in s \rangle \in \mathcal{P}_{\theta\eta\delta}$ *and* B, D *are models on the wide piste of* p *such that* $|B| = \tau$, *for some regular* $\tau \in s \cap \theta$, $|D| = \theta$ *and* $\sup(B \cap \theta^+) = D \cap \theta^+$. *Then for every regular cardinal* $\rho \in (\tau, \theta) \cap B$ *a model of cardinality* ρ *can be added to* p

between B and D.

Proof. Let B, D be as in the statement of the lemma and $\rho \in (\tau, \theta) \cap B$ be a regular cardinal. Suppose that there is no model of size ρ between B and D inside p. Without loss of generality we can assume that $\rho \in s$. Just otherwise extend p by adding the largest (under $\in$) model of cardinality ρ making it potentially limit one.
Let E be the least elementary submodel of D such that

- $|E| = \rho$,
- $E \supseteq B$,
- $E \supseteq \rho + 1$,

Now we would like to add E to p. However in order to do so more models probably need to be added. Namely we proceed as follows:

- add E to the wide piste of p,
- add to the wide piste of p models that are needed to be added by 2.20 together with E,
- add all their images under $\Delta-$system triples isomorphisms to $A^{1\rho lim}$,
- change the wide piste and add to new one models that may be needed by 2.20.

The result will be as desired.

□

Lemma 2.22 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper for every regular $\tau, \eta \leq \tau \leq \theta$.*

Proof. Let τ be a regular cardinal in the interval $[\eta, \theta]$. We would like to show that $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $\tau-$proper. If $\tau = \eta$, then this follows by the previous lemma (2.18). Suppose that $\tau > \eta$. Let $p \in \mathcal{P}_{\theta\eta\delta}$. Pick $\mathfrak{M}$ to be an elementary submodel of $H(\chi)$ for some χ regular large enough such that

1. $|\mathfrak{M}| = \tau$,
2. $\mathfrak{M} \supseteq \tau$,
3. $\mathcal{P}_{\theta\eta\delta}, p \in \mathfrak{M}$,

4. ${}^{\tau>}\mathfrak{M} \subseteq \mathfrak{M}$.

Set $M = \mathfrak{M} \cap H(\theta^+)$.
Clearly, M satisfies 2.1(3(b)). Moreover, using the elementarity of $\mathfrak{M}$, for every $x \in M$ there will be $Z \in M$ such that

- $Z \preceq H(\theta^+)$,
- $|Z| = \theta$,
- $Z \supseteq \theta$,
- ${}^{\theta>}Z \subseteq Z$,
- $x \in Z$.

This allows to find a chain $\langle N_i \mid i < \tau\rangle$ of models of size θ which members are in M, witnesses (3(b)) of Definition 2.1 for $N := \bigcup_{i<\tau} N_i$ and $N \supseteq M$.

Extend p by adding M as a new $A^{0\tau}$ and N as a new $A^{0\theta}$. Require them to be potentially limit points. Denote the result by $p^\frown\{M, N\}$.
We claim that $p^\frown\{M, N\}$ is $(\mathcal{P}_{\theta\eta\delta}, \mathfrak{M})-$generic. So, let $p' \geq p^\frown\{M, N\}$ and $\bar{D} \in \mathfrak{M}$ be a dense open subset of $\mathcal{P}_{\theta\eta\delta}$.
Extending p' more if necessary, we can assume, without loss of generality, that $p' \in \bar{D}$. Extend p' further, if necessary, in order to insure that the following holds:

- maximal models of p' increase according their cardinalities,
- if τ is a successor of a regular cardinal, then its predecessor is in $s(p')$,
- if τ is an inaccessible cardinal, then $\tau \cap s(p')$ has a maximal element and it is regular,
- if τ is a successor of a singular cardinal τ^- of cofinality $\geq \delta$, then $\tau^- \cap s(p')$ has a maximal element and it is regular.

Let η^* denote the predecessor of τ, if τ is a successor of a regular cardinal or of a singular cardinal of cofinality $< \delta$. If τ is an inaccessible cardinal, then let $\eta^* = \max(\tau \cap s(p'))$. If τ is a successor of a singular cardinal τ^- of cofinality $\geq \delta$, then let $\eta^* = \max(\tau^- \cap s(p'))$.

Extend p' further, by applying repeatedly Lemma 2.21, in order to achieve the following:

- for every $\xi \in s(p') \cap (\tau, \theta) \cap M \cap Regular$, there is a model B on the wide piste of p' of cardinality ξ such that $M \subseteq B \subseteq N$.

In particular, $\sup(M \cap \theta^+) = \sup(B \cap \theta^+) = N \cap \theta^+$. Denote such B by M_ξ. Let us denote such extension of p' still by the same letter p'.

Pick now $A \preceq H(\theta^+)$ which satisfies the following:

1. $|A| = \eta^*$,

2. $A \supseteq \eta^* + 1$,

3. $A \cap \eta^{*+}$ is an ordinal,

4. ${}^{\mathrm{cof}(\eta^*)>}A \subseteq A$,

5. $p' \in A$.
   In particular every model of p' belongs to A.

Extend p' to p'' by adding A as new largest model of cardinality η^*, i.e. $p'' = p'^\frown A$, if η^* is a regular cardinal. If η^* is a singular cardinal (and, then by its definition, $\mathrm{cof}(\eta^*) < \delta$), then we add an increasing under inclusion and cardinality sequence of models with limit A instead, which correspond to a splitting point of type 2) Namely, fix an increasing sequence of regular cardinals $\langle \eta_\alpha \mid \alpha < \mathrm{cof}(\eta^*) \rangle$ cofinal in η^* and with $\eta_0 \geq \eta$. Pick a sequence of elementary submodels $\langle K_\alpha \mid \alpha < \mathrm{cof}(\eta^*) \rangle$ of $H(\theta^+)$ such that for every $\alpha < \mathrm{cof}(\eta^*)$ the following holds:

1. $|K_\alpha| = \eta_\alpha$,

2. $K_\alpha \supseteq \eta_\alpha + 1$,

3. $K_\alpha \cap \eta_\alpha$ is an ordinal,

4. ${}^{\eta_\alpha >}K_\alpha \subseteq K_\alpha$,

5. $p' \in K_0$.

6. $\alpha < \beta$ implies $K_\alpha \in K_\beta$, and so, $K_\alpha \subseteq K_\beta$,

7. $A = \bigcup_{\alpha < \mathrm{cof}(\eta^*)} K_\alpha$.

Add $\langle K_\alpha \mid \alpha < \mathrm{cof}(\eta^*) \rangle$ to p'. Denote the result by p''. So, $K_\alpha = A^{0\eta_\alpha}(p'')$, for every $\alpha < \mathrm{cof}(\eta^*) < \delta$.

Let us reflect A down to $\mathfrak{M}$ over over $A \cap M$, i.e. we pick some $A' \in M$ and q which realizes the same k-type (for some $k < \omega$ sufficiently big) over $A \cap M$ as A and p''.[26] Do this reflection in a rich enough language which includes $\bar{D}$ as well. [27]
In particular $q \in \bar{D} \cap M$.

Extend p'' further, by applying repeatedly Lemma 2.21, in order to achieve the following:

- for every regular cardinal $\xi \in (s(p'') \cup s(q)) \cap M$, there is a model B on the wide piste of p'' of cardinality ξ such that $M \subseteq B \subseteq N$.

In particular, $\sup(M \cap \theta^+) = \sup(B \cap \theta^+) = N \cap \theta^+$. Denote such B by M_ξ. Note that $q \in M$, so $s(q) \in M$ and $s(q) \subseteq M$. Also, if $\theta < \aleph_{\eta^+}$, then $s(q) = s(p'')$. However, if $\theta > \aleph_{\eta^+}$, then the reflection process may add cardinals to $s(p'')$. All of this "new" cardinals come from M.
Let us denote such extension of p'' still by p''.

Let us argue that q is compatible with p''.
Set $s = s(q) \cup s(p'')$. Let $\langle \xi_i \mid i < i^* \rangle$ be an increasing enumeration of s. Pick $\in -$increasing sequence of models $\langle A_i \mid i < i^* \rangle$ such that for every $i < i^*$ the following hold:

1. $p'', q \in A_i$,
2. $|A_i| = \tau_i$,
3. A_i satisfies 2.3(2).

Set $A^{0\xi_i} = A_i$.
Finally let for every $\xi \in s$,

$$A^{1\xi} = \{A^{0\xi}\} \cup A^{1\xi}(p'') \cup A^{1\xi}(q).$$

Define $A^{0\xi lim}$ and $C^\xi (\xi \in s)$ in the obvious fashion now, but do not make $A^{0\xi}, \xi \in s$ potentially limit.

Set

$$p^* = \langle \langle A^{0\xi}, A^{1\xi}, A^{1\xi lim}, C^\xi \rangle \mid \xi \in s \rangle.$$

[26]The meaning is that A' satisfies the same formulas in $\mathfrak{M} \cap H(\theta^{+k})$ as A does with parameters from $A \cap M$ in $H(\theta^{+k})$.
[27]We follow here a suggestion by Carmi Merimovich to include $\bar{D}$ into the language which simplifies the original argument considerably.

Then, in p^*, the triple $(A^{0\eta^*}, A^{0\eta^*}(p''), A^{0\eta^*}(q))$ will form a Δ–system triple relatively to M and to the model which corresponds to M under the reflection, provided that η^* is a regular cardinal.
If η^* is a singular cardinal, then we have here a splitting point of type 2.

Let check that the wide piste of p^* satisfies Definition 2.1. Suppose that it goes through $C^{0\xi}(A^{0\xi})(p'')$, for each $\xi \in s \cap \tau$, i.e. via the part before the reflection.
Let $B \in C^\rho(A^{0\rho})$ be above M (i.e. $\sup(B \cap \theta^+) > \sup(M \cap \theta^+)$). If B is $A^{0\rho}$, then $p'', q \in B$, and so, every model which is below M is in B.
Suppose that $B \neq A^{0\rho}$. Then B is in p''.

Case 1. $\rho \in s \setminus \tau$.
By Definition 2.1(12(a)), for p'', we have $M \in B$. Hence, all models added by reflection are in B as well. In addition, by Definition 2.1(12(b)), for p'', we have $N \in B$. So, by Definition 2.1(6), B cannot be minimal in $C^\rho(A^{0\rho})(p'')$. In addition, the least B on $C^\rho(A^{0\rho})(p'')$ which contains M should be a potentially limit point. So, adding new models of size ρ below M is legitimate.

Case 2. $\rho \in s \cap \tau$.
Then B is among models of p'' that reflect down to M.
Suppose now that $E \in C^\xi(A^{0\xi})$ is below B. Assume that E does not appear in p''. Then E is below M and it is the image of a model of p'' under the reflection.
If E is on the wide piste of p^*, then $\xi \geq \tau$. Then there is a model E_ξ on the wide piste of p'' of cardinality ξ such that $M \subseteq E_\xi \subseteq N$. Clearly, $E \subseteq E_\xi$ and $E \in E_\xi$.
So, B, E satisfy Definition 2.1(16).

Let $B \in C^\rho(A^{0\rho})$ be below M (i.e. $\sup(B \cap \theta^+) < \sup(M \cap \theta^+)$). If $\rho \leq \tau$, then $B \in M$ either by Definition 2.1(12), if B appears in p' or by the reflection otherwise.
Suppose that $\rho > \tau$. If $B \notin M$, then B in p'.
Suppose first that $\theta < \aleph_{\eta^+}$. Then there is $B^* \in C^\rho(A^{0\rho}) \cap M$ the least such above B,by Definition 2.1(16) for p'. Let $\tilde{B}$ be the image of B under the reflection. Then $\tilde{B} \in C^\rho(A^{0\rho}) \cap M$. Also $\tilde{B} \in B^*$, since B satisfies this. Then, by Definition 2.1(17) (for p') we must to have $\tilde{B} \in B$. Note that by Definition 2.1(15) (for p', M, B^*), $\tilde{B}$ can be added since the least element of $B^* \cap C^\rho(A^{0\rho})$ which is above $B^* \cap M$ is a potentially limit point.

Suppose now that $\theta > \aleph_{\eta^+}$. Apply (20) of Definition 2.1 to M and B inside p'. Let $G \in M$ be the model which witnesses this. Let $\tilde{B}$ be the image of B under the reflection. Then $\tilde{B} \in C^{\rho'}(A^{0\rho'}) \cap M$, where ρ' is the image of ρ under the reflection. Also $\tilde{B} \in G$, if $B \in G$, i.e. if (b) or (c) of (20) hold.

Then, by Definition 2.1(20(b)(ii) or (c)(viii)) (for p') we must to have $\tilde{B} \in B$.
Suppose that (d) of Definition 2.1(20) holds, then instead of a single G, we will have a finite sequence of models $\langle G_i \mid i < n \rangle$ such that $B \in G_{n-1} \in ... \in G_0 = G \in M$. Let $\tilde{B}, \tilde{G}_{n-1}, ..., \tilde{G}_1$ be the images of $B, G_{n-1}, ..., G_1$ under the reflection. Then, $\tilde{B} \in \tilde{G}_{n-1} \in ... \in \tilde{G}_1 \in G \in M$. By (d)(xii) of Definition 2.1(20), then $\tilde{B} \in \tilde{G}_{n-1} \in ... \in \tilde{G}_1 \in B$.

Let us turn to Definition 2.3. The only non-trivial thing to check here is what happens once we change the wide piste to the one that replaces the part of p^* that was reflected by its reflection, according to 2.3(9) .
So, suppose that such switching between the reflecting part and its reflection was made. We need to argue that the result still satisfies Definition 2.1. The issue is the covering. Namely, the conditions (16)- (21) of Definition 2.1.
Let $\xi, \rho \in s, \xi < \rho$, $B \in C^{\xi}(A^{0\xi}), R \in C^{\rho}(A^{0\rho})$ and $\sup(B \cap \theta^+) > \sup(R \cap \theta^+))$. The principle case is when $\tau \leq \xi, \rho$, B is above M and R is below M. Just all models but the maximal ones of cardinalities below τ are below M on the new piste under the consideration, since the reflection made was into M.

Note that if B of cardinality $\xi \geq \tau$ is above M and both are on the wide piste of p'', then we have $B \supseteq M$, by Definition 2.1(12). But, R is below M implies $R \in M$, since the reflection made is into M. So, $R \in B$ and we are done.
□

The next lemma is straightforward.

Lemma 2.23 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ is $< \delta-$strategically closed.*

Proof. Use the strategy to switch each time back to the same (extended) wide piste. Take unions along the wide piste at limit stages. Note that 2.1(16, 17,18) will hold with such limit models, since non-limit ones are closed at least under $< \eta-$sequences and in particular, once including members of an increasing sequence (which length is less than $\delta \leq \eta$) will have the limit inside. Also definable parts (relevant for 2.1(17,18)), will have cofinality $\geq \eta$, and so cannot break down with this new limit models.
□

Combining together Lemmas 2.18,2.22, 2.23, we obtain the following:

Theorem 2.24 *The forcing notion $\langle \mathcal{P}_{\theta\eta\delta}, \leq \rangle$ preserves all cardinals $\leq \delta$ and all cardinals $> \eta$.*
In particular, if $\delta = \eta$, then all cardinals are preserved.

We are not going to force with $\mathcal{P}_{\theta\eta\delta}$ in the cardinal arithmetic applications, but rather to use its members as domains of conditions of a further forcing. However, the forcing with it is of an interest. Thus, for example, $\mathcal{P}_{\theta\omega\omega}$, where $\theta > \omega$ is a regular cardinal, may be of an interest on its own since the forcing with it will add a closed unbounded subset to θ^+ by finite conditions which runs away from every countable set in the ground model.

Let $G \subseteq \mathcal{P}_{\theta\eta\delta}$ be a generic. Set

$$C = \{X \cap \theta^+ \mid \exists p \in G(X \in A^{1\theta lim}(p) \wedge \mathrm{cof}(X \cap \theta^+) = \theta)\}$$

and let

$$C' = \{\alpha < \theta^+ \mid \alpha \text{ is a limit of points in } C\}.$$[28]

Lemma 2.25 *Let $\alpha \in C'$ be of cofinality θ, then $\alpha \in C$.*

Proof. Suppose otherwise. Let $p \in \mathcal{P}_{\theta\eta\delta}$ and $\alpha < \theta^+$ be of cofinality θ and

$$p \Vdash \alpha \notin \underset{\sim}{C} \text{ and } \underset{\sim}{C} \text{ is unbounded in } \alpha.$$

Split into two cases.

Case 1. *There is no model A in p with* $\sup(A \cap \theta^+) = \alpha$.

Pick then B to be the least model on the wide piste of p with $\sup(B \cap \theta^+) > \alpha$. Let $A \in B$ be given by 2.1(7). Then, $\sup(A \cap \theta^+) < \alpha$.

If B is not a potentially limit point, then, by 2.13(6), no models can be added between A and B, which contradicts unboundedness of C in α.

If B is a potentially limit point, then we would like to use 2.14 and 2.15 to add a potentially limit point T and a non potentially limit model T' such that

1. $A \in T \in T' \in B$,
2. $T, T' \in C^{|B|}(B)$,
3. $\sup(T \cap \theta^+) < \alpha < \sup(T' \cap \theta^+)$.

This will provide a contradiction, since no element of C then will be inside the interval $(\sup(T \cap \theta^+), \sup(T' \cap \theta^+))$.

[28]Note that the set C is not closed. Thus, densely often there are conditions $p \in \mathcal{P}_{\theta\eta\delta}$ with models A, B inside such that both are on the wide piste of p and are potentially limit, $|A| = \eta, |B| = \theta$, $B \in A$, ${}^{\eta>}A \subseteq A$, ${}^{\theta>}B \subseteq B$ and $\sup(A \cap B)$ is is a limit of models of size θ (clearly most of them not in p, due to $|p| < \delta$) that may be added to p.

By the assumption $\mathrm{cof}(\alpha) = \theta$, $|p| < \delta$ and

$$p \Vdash \underset{\sim}{C} \text{ is unbounded in } \alpha,$$

we can find such T and then by 2.15 also T'.

Case 2. *There is a model B in p with $\sup(B \cap \theta^+) = \alpha$.*
B is then in $A^{1\theta}$, since $\mathrm{cof}(\alpha) = \theta$. Hence $\alpha \in C$. Contradiction.
□

Lemma 2.26 *Let $\alpha \in C'$ be of cofinality $< \delta$, then $\alpha \in C$.*

Proof. Let $p \in \mathcal{P}_{\theta\eta\delta}$ and $\alpha < \theta^+$ be of cofinality $< \delta$ and

$$p \Vdash \underset{\sim}{C} \text{ is unbounded in } \alpha.$$

We construct, using a strategic closure of the forcing, an increasing sequence of extensions of p of the length $\mathrm{cof}(\alpha) < \delta$ having an upper bound which decide unboundedly many elements of C below α. Let q be such an upper bound. Then by 2.1(2(e)), there will be $A \in A^{1\theta}(q)$ with $A \cap \theta^+ = \alpha$.
□

Lemma 2.27 *Let $\alpha \in C'$ be such that $\delta \leq \mathrm{cof}(\alpha) < \eta$. Then $\alpha \notin C$.*

Proof. Let us argue that it is impossible to have a model A in a condition $p \in \mathcal{P}_{\theta\eta\delta}$ such that $\sup(A \cap \theta^+) = \alpha$. Suppose for a moment that this is the case. Without loss of generality assume that A is on the wide piste of p. A cannot be a limit point since $\sup(A \cap \theta^+) = \alpha$, $\delta \leq \mathrm{cof}(\alpha) < \eta$. Also, A cannot be a successor non-potentially limit model, since this will contradict to the unboundedness of C in α. So, A must be a potentially limit point. But then either ${}^{|A|>}A \subset A$, which rules out the possibility that $\sup(A \cap \theta^+) = \alpha$, $\mathrm{cof}(\alpha) < \eta$, or by Definition 2.1 (3(c)(i,ii)) there are models A_η, A_θ in p, such that $A_\eta \subseteq A \subseteq A_\theta$, $\sup(A_\eta \cap \theta^+) = \sup(A \cap \theta^+) = \sup(A_\theta \cap \theta^+) = \alpha$ and ${}^{\eta>}A_\eta \subset A_\eta$, which is again impossible, since $\mathrm{cof}(\alpha) < \eta$. Contradiction.
□.

Lemma 2.28 *Let $x \in V$ be a subset of θ^+ of cardinality δ. Then $x \not\subseteq C'$.*

Proof. Let $x \in V$ be a set of cardinality δ and $p \in \mathcal{P}_{\theta\eta\delta}$. Without loss of generality assume that $\sup(x)$ is a limit ordinal of cofinality δ. Set $\nu = \sup(x)$.

If

$$p \Vdash (\underset{\sim}{C}' \text{ is bounded in } \nu),$$

then we are done. So, suppose that

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Split models of p into two groups. Set

$$H_0 = \{A \mid A \text{ appears in } p \text{ and } \sup(A \cap \nu) < \nu\}$$

and

$$H_1 = \{A \mid A \text{ appears in } p \text{ and } \sup(A \cap \nu) = \nu\}.$$

Note that since the total number of models in p is less than δ and δ is a regular, we have $\bigcup_{A \in H_0} \sup(A \cap \nu) < \nu$. Also, if $\delta < \eta$, then $\nu \in A$, for every $A \in H_1$.
Let $\nu^* = \bigcup_{A \in H_0} \sup(A \cap \nu)$.
Take now any $B \in H_1$.

Claim. *For every $\beta, \nu^* < \beta < \nu$, there is $D \in B$ such that*

1. $|D| = \theta$,
2. $D \subseteq \theta$,
3. ${}^{\theta>}D \subseteq D$,
4. $\beta < D \cap \theta^+ < \nu$,
5. D satisfies 2.1(3(b)).

Proof. Assume that $\beta \in B$, since $B \in H_1$ just replacing it by a bigger ordinal if necessary. We have

$$p \Vdash (\underset{\sim}{C}' \text{ is bounded in } \nu).$$

Let $p \in G$. So in $V[G]$, there is $D' \in C$ such that $\beta < D' \cap \theta^+ < \nu$. Then there is an extension q of p with D' inside. If $D' \in B$, then we are done. Suppose otherwise. Apply 2.1(17) to B, D' for q. Then there will be $D'' \in B \cap A^{1\theta lim}$ which includes D' and $D' \cap \theta^+ < D'' \cap \theta^+ < \nu$, since $B \in H_1$.
Now,

$$H(\theta^+) \models \exists D \in D''(\beta \in D \text{ and it satisfies conditions (1)-(5) of the claim}).$$

Just D' witnesses this. By elementarity then

$$B \models \exists D \in D''(\beta \in D \text{ and it satisfies conditions (1)-(5) of the claim}).$$

$\square$ of the claim.

Suppose that the following holds in $V[G]$:

(*) There is an elementary $\in -$chain $\langle D_i \mid i < \delta \rangle$ of elementary submodels of $\langle H(\theta^+), \in, \leq, \delta, \eta \rangle$ such that

- for every $A \in H_1$,$\{D_i \mid i < \delta\} \subseteq A$,
- each D_i satisfies the items (1),(2),(4)(without β),(5) of the claim,
- $\{D_i \cap \theta^+ \mid i < \delta\}$ is unbounded in ν.

Without loss of generality we can assume that $\langle D_i \mid i < \delta \rangle$ is a closed chain. Recall that

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Hence for every $i < \delta$ there is the least $i', i < i' < \delta$ such that

$$D_{i'+1} \models \exists D \preceq D_{i'}(D \text{ satisfies the items (1)-(5) of the claim, with (4)without } \beta \wedge D_i \in D).$$

Pick the least such D and make it the new D_{i+1}.
Let p' be an extension of p which adds a model X in C above ν^*, but below ν. Change, if necessary, our sequence $\langle D_i \mid i < \delta \rangle$ by removing an initial segment such that $X \in D_1$.
Now we pick some $\xi \in x$, $D_1 \cap \theta^+ < \xi < D_{i^*+1} \cap \theta^+$, for some $i^* < \delta$. Next, add D_1, D_{i^*+1} to p, D_1 as a potentially limit point and D_{i^*+1} as a non-limit and non-potentially limit point. The requirement 2.1(15) will hold, since D_1 is a potentially limit point above X (this for models in H_0) and D_1, D_{i^*+1} are in every model of H_1 on the wide piste. Reflect them through all relevant splittings. Let q be the result. Then

$$q \Vdash x \not\subseteq \underset{\sim}{C},$$

since nothing can be added between D_1 and D_{i^*+1}.

Let us now argue that (*) holds.
Split into three cases.

Case 1. $\eta > \delta$.
Then $\nu \in A$, for every $A \in H_1$, since if A is a non-limit, then it is closed under $< \eta-$sequences

of its elements. If A is a limit model, then it is a union of less than δ non-limit models. So the regularity of δ implies that a final segment of them is in H_1. But then ν belongs to each non-limit model of this final segment, and hence it belongs to A as well.
Pick the least cofinal in ν sequence $\langle \nu_i \mid i < \delta \rangle$. Then $\langle \nu_i \mid i < \delta \rangle \in A$ and $\{\nu_i \mid i < \delta\} \subseteq A$ for every $A \in H_1$ by elementarity.
Now it is easy to construct $\langle D_i \mid i < \delta \rangle$ which satisfies (*). Just proceed by induction and apply the claim with β there replaced by ν_i's.

Case 2. $\delta > \omega$.
Work in $V[G]$ with $p \in G$. Let $\langle E_i \mid i < \delta \rangle$ be an increasing sequence of members of C unbounded in ν with $E_i \cap \theta^+ < \nu$.
Let $A \in H_1$ be a non-limit model of p and $i < \delta$. Pick an extension $p_i \in G$ of p such that E_i appears in p_i and A is on the wide piste of p_i. Then, by Definition 2.1(16)or (20), there is $E_i^A \in A \cap C^\theta(A^{0\theta})(p_i)$ the least which contains E_i (recall that θ is in every model). By Definition 2.1(17)or (20), $E_i^A \cap \theta^+ < \nu$, since $A \in H_1$.
Consider $\{E_i^A \mid i < \delta\}$. Adding limit models if necessary we can assume that it is a closed chain. A is a non-limit, hence it is closed under less than $\delta-$sequence of its elements, so, $\{E_i^A \mid i < \delta\} \subseteq A$. Set

$$Y^A := \{E_i^A \cap \theta^+ \mid i < \delta\}.$$

Consider

$$Y := \bigcap \{Y^A \mid A \in H_1 \text{ non-limit}\}.$$

Then Y is an intersection of fewer than δ clubs, and hence is a club in ν. Now, $Y \subseteq A$, for every $A \in H_1$ non-limit . If B is a limit model of p and $B \in H_1$, then B is an increasing union of less than δ non-limit models. Then, the final segment of them is in H_1, and hence contains Y. So, $Y \subseteq B$.
Finally note that if E, E' two models which appear in $A^{1\theta}(r)$, for some r, and $E \cap \theta^+ = E' \cap \theta^+$, then $E = E'$, by Definition 2.1(2). So, take any $A \in H_1$ non-limit , consider the sequence

$$\langle E_i^A \mid i < \delta, E_i^A \cap \theta^+ \in Y \rangle.$$

It will witness (*).

Case 3. $\delta = \omega = \eta$.
Work in V. Conditions are finite now.
We have

$$p \Vdash (\underset{\sim}{C}' \text{ is unbounded in } \nu).$$

Let $\beta < \nu$.
Extend p to some p' such that there is $D' \in A^{1\theta}(p'), \beta < D' \cap \theta^+ < \nu$.
Suppose that there is $A \in H_1$ such that $D' \notin A$. Replace p' by an equivalent condition, if necessary, to ensure that A is on the wide piste. Then by Definition 2.1(16)or (20), there is $D^A \in\in A \cap C^\theta(A^{0\theta})(p')$ the least which contains D' (recall that θ is in every model). By Definition 2.1(17)or (20), $D^A \cap \theta^+ < \nu$, since $A \in H_1$.
Now, if there is $B \in H_1$ such that $D^A \notin B$, then we repeat the process with D', A replaced by D^A, B. Remember that p' is finite, so after finitely many steps there will be $D^* \in C^\theta(A^{0\theta})(p'), D' \subseteq D^*$ such that $D^* \in Z$, for every $Z \in H_1$.
The ordinal $\beta < \nu$ was arbitrary, so there is a cofinal in ν sequence $\langle \nu_n \mid n < \omega \rangle$ such that $\{\nu_n \mid n < \omega\} \subseteq A$, for every $A \in H_1$.
Now, we proceed as in Case 1 and use $\langle \nu_n \mid n < \omega \rangle$ in order to define $\langle D_n \mid n < \omega \rangle$ which satisfies (*).
$\square$

In particular, taking $\delta = \eta = \omega$, we obtain the following generalization of corresponding results by U. Abraham [1], J. Baumgartner, S. Friedman [2] and by W. Mitchell [13] to higher cardinals[29]:

Corollary 2.29 *The forcing $\mathcal{P}_{\theta\omega\omega}$ is cardinals preserving forcing adding a club in θ^+ which avoids old countable sets using finite conditions.*

Remark 2.30 Given a stationary subset S of θ^+ such that for every $\alpha < \theta^+$ if $\mathrm{cof}(\alpha) < \theta$, then $\alpha \in S$, it is easy to modify the forcing $\mathcal{P}_{\theta\eta\delta}$ such that it will add a club through S. Only require that for every X of cardinality θ in a condition we have $X \cap \theta^+ \in S$.

[29]Note that Magidor and Radin forcings ([10],[14]) also add clubs by finite conditions.

3 Suitable structures.

We reorganize here the structures with pistes of the previous section in order to allow isomorphisms of them over different cardinals.

Definition 3.1 Let $\delta < \kappa < \theta$ be cardinals, with δ and θ regular. A structure $\mathfrak{X} = \langle X, E, E^{lim}, C, S, \in, \subseteq \rangle$, where $E \subseteq [X]^2$ and $C \subseteq [X]^3$ is called a $\delta-$suitable structure with pistes over κ of the length θ iff there is a δ structure with pistes over κ^+ of the length θ $p(\mathfrak{X}) = \langle \langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), A^{1\tau lim}(\mathfrak{X}), C^{\tau}(\mathfrak{X}) \rangle \mid \tau \in s(\mathfrak{X}) \rangle$ such that

1. $X = A^{0\eta}(\mathfrak{X}) \cup \{A^{0\eta}(\mathfrak{X})\}$, where $\eta \in s(\mathfrak{X})$ is such that for every $\tau \in s(\mathfrak{X})$ we have then $A^{0\tau}(\mathfrak{X}) \in X$ or $A^{0\tau}(\mathfrak{X}) \subseteq X$,
2. $S = s(\mathfrak{X})$,
3. $\langle a, b \rangle \in E$ iff $a \in S$ and $b \in A^{1a}(\mathfrak{X})$,
4. $\langle a, b \rangle \in E^{lim}$ iff $a \in S$ and $b \in A^{1alim}(\mathfrak{X})$,
5. $\langle a, b, d \rangle \in C$ iff $a \in S, b \in A^{1a}(\mathfrak{X})$ and $d \in C^{a}(\mathfrak{X})(b)$.

Let us refer to $\mathfrak{X}$ for shortness as a $\delta-$suitable structure if κ, θ are fixed.

Note that $p(\mathfrak{X})$ is uniquely defined from $\mathfrak{X}$. Also, it is easy to define a δ-suitable structure from $p \in \mathcal{P}_{\theta\kappa^+\delta}$.

Definition 3.2 Let $\mathfrak{X}, \mathfrak{Y}$ be δ-suitable structures. Set $\mathfrak{X} \leq \mathfrak{Y}$ iff $p(\mathfrak{X}) \leq p(\mathfrak{Y})$.

3.1 Forcing conditions.

Let κ be a limit of an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega \rangle$ with each κ_n being strong up to the least Mahlo cardinal λ_n above κ_n as witnessed by an extender E_n.
Let $\theta > \kappa$ be a regular cardinal.

The definitions below follow closely the corresponding definitions starting with [4], Chapter 2, then [3],[8], and finally [6], Chapter 2.4.

For every $n < \omega$ define Q_{n0}.

Definition 3.3 Let Q_{n0} be the set of the triples $\langle a, A, f \rangle$ so that:

1. f is a partial function from θ^+ to κ_n of cardinality at most κ,

2. a is an isomorphism between a κ_n–suitable structure $\mathfrak{X}$ over κ of the length θ and a κ_n–suitable structure $\mathfrak{X}'$over κ_n^{+n} of the length λ_n such that

   (a) X' is above every model which appears in $(\bigcup_{\tau\in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')) \setminus \{X'\}$, in the order $\leq_{E_n}$, (or actually after coding X' by an ordinal),[30]

   (b) if $t \in \bigcup_{\tau\in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')$,then for some $k, 2 < k < \omega$, $t \prec H(\chi^{+k})$, with χ big enough fixed in advance.
   Further passing from Q_{n0} to $\mathcal{P}$ we will require that for every $k < \omega$ for all but finitely many n's the n-th image t of a model from X will be elementary submodel of $H(\chi^{+k})$.
   The way to compare such models $t_1 \prec H(\chi^{+k_1})$, $t_2 \prec H(\chi^{+k_2})$, when $k_1 \neq k_2$, say $k_1 < k_2$, will be as follows:
   move to $H(\chi^{+k_1})$, i.e. compare t_1 with $t_2 \cap H(\chi^{+k_1})$.

3. $A \in E_{nX'}$,

4. for every ordinals α, β, γ which code models in $\bigcup_{\tau\in s(\mathfrak{X}')} A^{1\tau}(\mathfrak{X}')$, we have

$$\alpha \geq_{E_n} \beta \geq_{E_n} \gamma \text{ implies}$$

$$\pi^{E_n}_{\alpha\gamma}(\rho) = \pi^{E_n}_{\beta\gamma}(\pi^{E_n}_{\alpha\beta}(\rho)),$$

   for every $\rho \in \pi''_{X'\alpha}A$.

Definition 3.4 Let $\langle a, A, f\rangle, \langle b, B, g\rangle$ be in Q_{n0}. Set $\langle a, A, f\rangle \geq_{n0} \langle b, B, g\rangle$ iff

1. $\mathrm{dom}(a) \geq \mathrm{dom}(b)$ in the order of suitable structures (Definition 3.2),

2. $\mathrm{ran}(a) \geq \mathrm{ran}(b)$ in the order of suitable structures (Definition 3.2),

3. $a \supseteq b$,

4. $f \supseteq g$,

5. $\pi^{E_n}_{\max(\mathrm{ran}(a)),\max(\mathrm{ran}(b))}$“$A \subseteq B$.

Definition 3.5 Q_{n1} consists of all partial functions $f : \theta \to \kappa_n$ with $|f| \leq \kappa$. If $f, g \in Q_{n1}$, then set $f \geq_{n1} g$ iff $f \supseteq g$.

[30] $\leq_{E_n}$ is the order of the extender E_n, see [4], Chapter 2.

Definition 3.6 Define $Q_n = Q_{n0} \cup Q_{n1}$ and $\leq_n^* = \leq_{n0} \cup \leq_{n1}$.
Let $p = \langle a, A, f \rangle \in Q_{n0}$ and $\nu \in A$. Set

$$p^\frown \nu = f \cup \{\langle \alpha, \pi_{\max(\mathrm{ran}(a)),a(\alpha)}(\nu)\rangle \mid \alpha \in A^{1\theta}(\mathrm{dom}(a)) \setminus \mathrm{dom}(f)\}.$$

Note that here a contributes only the values for α's in $\mathrm{dom}(a) \setminus \mathrm{dom}(f)$ and the values on common α's come from f. Also only the ordinals in $A^{1\theta}(\mathrm{dom}(a))$ are used to produce non direct extensions, the rest of models disappear.

Now, if $p, q \in Q_n$, then we set $p \geq_n q$ iff either $p \geq_n^* q$ or $p \in Q_{n1}, q = \langle b, B, g \rangle \in Q_{n0}$ and for some $\nu \in B$, $p \geq_{n1} q^\frown \nu$.

Definition 3.7 The set $\mathcal{P}$ consists of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

1. for every $n < \omega$, $\quad p_n \in Q_n$,
2. there is $\ell(p) < \omega$ such that
   (a) for every $n < \ell(p)$, $\quad p_n \in Q_{n1}$,
   (b) for every $n \geq \ell(p)$, we have $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$,
   (c) if $\ell(p) \leq n \leq m$, then $\mathrm{dom}(a_n) \leq \mathrm{dom}(a_m)$,
   (d) if $\ell(p) \leq n \leq m$, then $\max(\mathrm{dom}(a_n)) = \max(\mathrm{dom}(a_m))$.
3. For every $n \geq m \geq \ell(p)$, $\quad \mathrm{dom}(a_m) \subseteq \mathrm{dom}(a_n)$,
4. for every n, $\ell(p) \leq n < \omega$, and $X \in \mathrm{dom}(a_n)$ we have that for each $k < \omega$ the set $\{m < \omega \mid \neg(a_m(X) \cap H(\chi^{+k}) \prec H(\chi^{+k}))\}$ is finite. (Alternatively require only that $a_m(X) \subseteq \lambda_m$ but there is $\widetilde{X} \prec H(\chi^{+k}))$ such that $a_m(X) = \widetilde{X} \cap \lambda_m$. It is possible to define being k-good this way as well).[31]
5. For every $n \geq \ell(p)$ and $\alpha \in \mathrm{dom}(f_n)$ there is $m, n \leq m < \omega$ such that $\alpha \in \mathrm{dom}(a_m) \setminus \mathrm{dom}(f_m)$.
6. There is a κ-structure with pistes $\mathfrak{p}$ over κ such that
   (a) $\mathfrak{p} \geq \mathrm{dom}(a_n)$, for every $n, \ell(p) \leq n < \omega$,
   (b) if a model A appears in $\mathfrak{p}$, then A appears in $\mathrm{dom}(a_n)$ for some $n, \ell(p) \leq n < \omega$ (and then in a final segment of them),

[31]See [3] Definition 2.8, where this idea appears for the first time.

(c) $\max(\mathrm{dom}(a_n)) = \max(\mathfrak{p})$ (actually this follows from the previous condition).

Note that $\mathfrak{p}$ of 3.7(6) is uniquely determined by p. Let us refer to it further as the κ-structure with pistes over κ of p.

The forcing order $\leq$ and the direct extension orders $\leq^*$ are defined on $\mathcal{P}$ as follows:

Definition 3.8 Let $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle$ be in $\mathcal{P}$. We define $p \geq q (p \geq^* q)$ iff for every $n < \omega$, $p_n \geq_n q_n (p_n \geq^*_n q_n)$.

The proofs of the next lemmas repeat those of corresponding lemmas in [3],[5],[6] Chapter 1.

Lemma 3.9 *$\langle Q_{n0}, \leq_{n0} \rangle$ is $< \kappa_n$-strategically closed.*

Lemma 3.10 *$\langle \mathcal{P}, \leq^* \rangle$ does not add new sequences of ordinals of the length $< \kappa_0$.*

Lemma 3.11 *$\langle \mathcal{P}, \leq^* \rangle$ satisfies the Prikry condition.*

Lemma 3.12 *Let $p \in \mathcal{P}$ and $\alpha < \theta^+$, then there are $q \geq^* p$ and $\beta, \alpha < \beta < \theta^+$ such that $\beta = M \cap \theta^+$, for some M which appears in q.*

Proof. Pick some $M \prec H(\theta^+)$ of size θ which is above the maximal model of $\mathfrak{p}$ (say $\mathfrak{p} \in M$) and such that $M \cap \theta^+ > \alpha$. Add it to p. Let q be the resulting condition. Then it is as desired.

□

The next lemma follows now:

Lemma 3.13 *Let G be a generic subset of $\langle \mathcal{P}, \leq \rangle$. Then in $V[G]$ there are* $\mathrm{cof}((\theta^+)^V)$*–many ω–sequences of ordinals below κ.*

Define $\rightarrow$ and $\leftrightarrow$ on $\mathcal{P}$ as in [3] or [5].

κ^{++}–c.c. and even θ^+–c.c. break down here for the forcing $\langle \mathcal{P}, \rightarrow \rangle$.
Following C. Merimovich [12] we replace them by properness.

3.2 Properness.

We will turn now to the properness of the forcing. The proofs repeat almost completely those of Lemmas 2.18,2.22

Lemma 3.14 *$\langle \mathcal{P}, \rightarrow \rangle$ is κ^+-proper.*

Lemma 3.15 *$\langle \mathcal{P}, \rightarrow \rangle$ is η-proper, for every regular $\eta, \kappa^+ < \eta \leq \theta$.*

The proofs repeat almost completely those of Lemmas 2.18,2.22. The only additional ingredient is to put new models that were added below κ in the process of extension of conditions inside old ones. As usual, in [6], we use $\longleftrightarrow$ for this purpose and pass to equivalent models.

Namely, in the proof of Lemma 2.18, we shrink a condition to a model $\mathfrak{M}$ and then inside $\mathfrak{M}$ an extension of it in the dense open set was taken. Here we will need in addition to reflect the images of such "new" models into the images M_n of $\mathfrak{M} \cap H(\theta^+)$ under the isomorphisms a_n between structures over κ and those over κ_n. It is possible for every $n < \omega$ large enough due to (4) of Definition 3.7. We refer, for example, to [3] Lemma 3.19 or [6] Lemma 1.2.21 for such reflection arguments.

In the proof of Lemma 2.22 the reflection was made over κ into the model $\mathfrak{M}$. Here we will need in addition to reflect into the images M_n of $\mathfrak{M} \cap H(\theta^+)$. Again, it is possible for every $n < \omega$ large enough due to (4) of Definition 3.7.

Finally, combining together Lemmas 3.10, 3.11, 3.13, 3.14, 3.15, we obtain the following:

Theorem 3.16 *Let G be a generic subset of $\langle \mathcal{P}, \rightarrow \rangle$. Then $V[G]$ is cofinalities preserving extension of V in which $2^\kappa = \kappa^\omega = \theta^+$.*

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