Short Extenders Forcings II

Moti Gitik*

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Abstract

A model with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) = \omega_1 + 1$ is constructed, for countable set $\mathfrak{a}$ of regular cardinals.

1 Preliminary Settings

Let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an an increasing continuous sequence of singular cardinals of cofinality ω so that for each $\alpha < \omega_1$, if $\alpha = 0$ or α is a successor ordinal, then κ_α is a limit of an increasing sequence $\langle \kappa_{\alpha,n} \mid n < \omega \rangle$ of cardinals such that

(1) $\kappa_{\alpha,n}$ is strong up to a 2-Mahlo cardinal $< \kappa_{\alpha,n+1}$,

(2) $\kappa_{\alpha,0} > \kappa_{\alpha-1}$.

Fix a sequence $\langle g_\alpha \mid \alpha < \omega_1,\ \alpha = 0$ or it is a successor ordinal$\rangle$ of functions from ω to ω such that for every $\alpha, \beta, \alpha < \beta$ which are zero or successor ordinals below ω_1 the following holds

(a) $\langle g_\alpha(n) \mid n < \omega \rangle$ is increasing

(b) there is $m(\alpha, \beta) < \omega$ such that for every $n \geq m(\alpha, \beta)$

$$g_\alpha(n) \geq \sum_{m=0}^{n} g_\beta(m) \ .$$

(c) $g_\alpha(0) = 1$

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The easiest way is probably to force such a sequence.
Conditions are of the form

$\langle n, \{h_\alpha | \alpha \in I\}\rangle$, where $n < \omega, I$ is a finite subset of ω_1 and $h_\alpha : n \to \omega$.

The order is defined as follows:

$\langle n, \{h_\alpha | \alpha \in I\}\rangle \leq \langle m, \{t_\beta | \beta \in J\}\rangle$ iff $n \leq m, I \subseteq J$, for every $\alpha \leq \beta, \alpha, \beta \in I$, we have $t_\alpha | n = h_\alpha$ and if $n \leq k < m$ then require that $t_\alpha(k) \geq \sum_{0 \leq s \leq k} t_\beta(s)$.

It is possible to construct such a sequence in ZFC. Pick first a sequence $\langle h_\alpha | \alpha < \omega_1\rangle$ of functions from ω to ω such that

(1) $\langle h_\alpha(n) \mid n < \omega\rangle$ is non-decreasing and converges to infinity;

(2) if $\alpha < \beta$ then $h_\alpha > h_\beta$ mod finite.

Replace now each h_α by h'_α such that $h'_\alpha(n) = h_\alpha(n) + n + 1$.

Define $g_\alpha(n)$ to be $2^{(2^{\dots (2^{h'_\alpha(n)})\dots)}}$ where the number of powers is $h'_\alpha(n)$.

Let us argue that it is as required. Let $\alpha < \beta$. Pick $m(\alpha, \beta)$ to be such that for every $n \geq m(\alpha, \beta)$ we have $h'_\alpha(n) > h'_\beta(n)$.

Let $n \geq m(\alpha, \beta)$. Consider $\sum_{0 \leq s \leq n} g_\beta(s)$.

Then

$$\sum_{0 \leq s \leq n} g_\beta(s) \leq (n+1) \cdot g_\beta(n) \leq (g_\beta(n))^2 \leq 2^{g_\beta(n)} \leq g_\alpha(n).$$

In order to motivate the further construction let us consider first two simple (relatively) situation. The first will deal with only two cardinals κ_0 and κ_1, and the second with ω-many of them- $\langle \kappa_n \mid n < \omega\rangle$.

2 Two cardinals

We would like to blow up the powers (or pp) of both κ_0, κ_1 to κ_1^{++}. Organize this as follows. We have $\langle \kappa_{1,n} \mid n < \omega\rangle$. For each $n < \omega$ fix some regular $\kappa_{1,n,1}, \kappa_{1,n+1} > \kappa_{1,n,1} \geq \kappa_{1,n}^{+n+2}$. It will be the one connected to κ_1^{++} at the level n of κ_1. Denote by $\underset{\sim}{\rho}_{1,n,1}$ the canonical name of the indiscernible for $\kappa_{1,n,1}$, i.e. the cardinal corresponding to $\kappa_{1,n,1}$ in the one element Prikry forcing or more precisely in the short extenders forcing of [2].
So the sequence $\langle \rho_{1,n,1} \mid n < \omega\rangle$ will correspond to κ_1^{++} after the forcing.
Now turn to the 0–level. We have here $\langle \kappa_{0,n} \mid n < \omega\rangle$. Instead of a direct connection to κ_1^{++} let us arrange a connection to elements of the interval $[\kappa_0^+, \kappa_1)$ and then via $\langle \rho_{1,n,1} \mid n < \omega\rangle$ it will continue automatically further to κ_1^{++}.
Specify first an interval $[\kappa_{0,0}, \kappa_{0,0,1}]$ that will correspond to $[\kappa_0^+, \rho_{1,0,1}]$, for some regular large enough $\kappa_{0,0,1} < \kappa_{0,1}$, say a Mahlo or even a measurable (note that there are plenty of such

cardinals blow $\kappa_{0,1}$ since it is strong). Connection between them will be arranged and $\rho_{1,0,1}$ will correspond to $\kappa_{0,0,1}$. No more cardinals (i.e. those above $\rho_{0,0,1}$) will be connected to the 0–level of κ_0.

Turn to the next level of κ_0. We like to connect 0 and 1– levels of κ_1 to the 1–level of κ_0. In order to do this let us reserve two blocks of cardinals $[\kappa_{0,1}, \kappa_{0,0,0,1}]$ and $[\kappa_{0,0,1,0}, \kappa_{0,0,1,1}]$ such that $\kappa_{0,0,0,1} < \kappa_{0,0,1,0} < \kappa_{0,2}$ and $\kappa_{0,0,0,1}, \kappa_{0,0,1,0}, \kappa_{0,0,1,1}$ are large enough (again Mahlo or measurables). Now,the interval $[\kappa_0^+, \rho_{1,0,1}]$ will be connected to the first block $[\kappa_{0,1}, \kappa_{0,0,0,1}]$ and the interval $[\rho_{1,0,1}^+, \rho_{1,1,1}]$ to the second block $[\kappa_{0,0,1,0}, \kappa_{0,0,1,1}]$ with $\rho_{1,0,1}$ corresponding to $\kappa_{0,0,0,1}$ and $\rho_{1,1,1}$ to $\kappa_{0,0,1,1}$. No further cardinals from κ_1 will be connected to this level of κ_0.

Continue further in a similar fashion: connect the levels $0, 1, 2$ of κ_1 to the 2–level of κ_0 by specifying three blocks at this level, etc.

3 ω–many cardinals

We would like to blow up the powers (or pp) of all $\kappa_n, n < \omega$ to κ_ω^+. Organize this as follows. For each $k < \omega$, pick the first block for κ_k to be the interval $[\kappa_{k,0}, \kappa_{k,0,0,\omega}]$, where $\kappa_{k,0,0,\omega}$ is large enough cardinal below $\kappa_{k,1}$ which is a limit of an increasing sequence of large enough regular cardinals $\langle \kappa_{k,0,m,l} \mid m < g_k(0), l < \omega \rangle$ between $\kappa_{k,0}$ and $\kappa_{k,0,0,\omega+1}$, where $g_k : \omega \to \omega$ and each value $g_k(i)$ will be defined by induction at stage i. We set $g_k(0) = 1$.
Denote by $\rho_{k,0,0,l}$ the indiscernible (that will be forced further) for $\kappa_{k,0,0,l}$, for every $l \in \omega + 1$. Connect the interval $[\kappa_{m-1}^+, \rho_{m,0,0,\omega}]$ to $[\kappa_{k,0,0,m}, \kappa_{k,0,0,\omega}]$, for every $m, k < m - 1 < \omega$ so that κ_{m-1}^+ corresponds to $\kappa_{k,0,0,m}$, $\rho_{m,0,0,l}$ corresponds to $\kappa_{k,0,0,l}$, for each $l, m < l < \omega$ and $\rho_{m,0,0,\omega}$ corresponds to $\kappa_{k,0,0,\omega}$. The obvious commutativity is required.

Turn to the second levels of κ_k's. For each $k > 0$ we define $g_k(1) = 1$ and make no connections to m's above k. For $k = 0$ set $g_0(1) = 2$. Then at the level second level of κ_0, we reserve two blocks (instead of one) $[\kappa_{k,1}, \kappa_{k,1,0,\omega}]$ and $[\kappa_{k,1,1,0}, \kappa_{k,1,1,\omega}]$, where $\kappa_{k,1,0,\omega} < \kappa_{k,1,1,0} < \kappa_{1,2}$, $\kappa_{k,1,0,\omega}$ is a limit of increasing sequence of large enough regular cardinals $\langle \kappa_{k,1,0,l} \mid l < \omega \rangle$ above $\kappa_{k,1}$ and $\kappa_{k,1,1,\omega}$ is a limit of an increasing sequence of large enough regular cardinals $\langle \kappa_{k,1,1,l} \mid l < \omega \rangle$ above $\kappa_{k,1,0,\omega}$.
Then for each m,such that $k < m - 1 < \omega$ we connect the interval $[\kappa_{m-1}^+, \rho_{m,0,0,\omega}]$ with $[\kappa_{k,1,0,m}, \kappa_{k,1,0,\omega}]$ and $[\rho_{m,0,0,\omega}^{++}, \rho_{m,1,0,\omega}]$ with the second block starting from $\kappa_{k,1,1,m}$. Require for the first block that κ_{m-1}^+ corresponds to $\kappa_{k,1,0,m}$, $\rho_{m,0,0,l}$ corresponds to $\kappa_{k,1,0,l}$, for each $l, m < l < \omega$ and $\rho_{m,0,0,\omega}$ corresponds to $\kappa_{k,1,0,\omega}$. For the second block let us require that $\rho_{m,0,0,\omega}^{++}$ corresponds to $\kappa_{k,1,1,m}$, $\rho_{m,1,0,l}$ corresponds to $\kappa_{k,1,1,l}$, for each $l, m < l < \omega$ and $\rho_{m,1,0,\omega}$

corresponds to $\kappa_{k,1,1,\omega}$.

At third levels of κ_k's let us do the following. For each $k > 1$ $g_k(2) = 1$ and make no connections to m's above k.
If $k = 1$, then set $g_1(2) = 2$ and proceed exactly as at the second levels with $k = 0$ replaced by $k = 1$.
If $k = 0$ then set $g_0(2) = 4$, reserve 4 blocks at the third level of κ_0 and arrange the connections to this blocks in the similar to that used for κ_0 above, covering three levels of κ_m's, for $m > k$.

At the forth levels we do a similar connection only stepping up by one, etc.

It is not hard under the same lines to generalize the above construction from ω–many cardinals to η–many for every countable η.
A structure suggest below in order to deal with to ω_1–many cardinals will require drops with infinite repetitions.

4 ω_1–many cardinals

We would like to blow up the powers (or pp) of all $\kappa_\alpha, \alpha < \omega_1$ to $\kappa^+_{\omega_1}$.
The first tusk will be to arrange a pcf–structure that will be realized. It requires some work since we allow only finitely many blocks at each level. Note that in view of [9] one cannot allow infinitely many blocks at least not under the large cardinals assumptions used here (below a strong or a little bit more).

Organize the things as follows.
Let $n < \omega$ and $1 \leq \alpha < \omega_1$ be a successor ordinal or $\alpha = 0$. We reserve at level n a splitting into $g_\alpha(n)$–blocks one above another:

$$\langle \kappa_{\alpha,n,m,i} \mid m < g_\alpha(n), i \leq \omega_1 \rangle,$$

so that

1. $\kappa_{\alpha,n} < \kappa_{\alpha,n,0,0}$,
2. $\kappa_{\alpha,n,m,i'} < \kappa_{\alpha,n,m,i}$, for every $m < g_\alpha(n), i' < i \leq \omega_1$,
3. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n,m+1,0}$, for every $m < g_\alpha(n)$,
4. for every successor ordinal $i < \omega_1$ or if $i = 0$ let $\kappa_{\alpha,n,m,i}$ be large enough (say a Mahlo or even measurable),

5. for every limit $i, 0 < i \leq \omega_1$ let $\kappa_{\alpha,n,m,i} = \sup(\{\kappa_{\alpha,n,m,i'} \mid i' < i\})$,

6. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n+1}$, for every $m < g_\alpha(n)$.

Further by $\alpha < \omega_1$ we will mean always a successor ordinal or 0.

Let us incorporate indiscernibles that will be generated by extender based forcings into the blocks as follows. Denote as above the indiscernible for $\kappa_{\alpha,n,m,i}$ by $\rho_{\alpha,n,m,i}$.
$[\kappa^+_{\alpha-1}, \rho^+_{\alpha,0,0,\omega_1}]$ will the first block of α of the level 0 (if $\alpha = 0$, then let it be $[\omega_1, \rho^+_{0,0,0,\omega_1}]$). Then for every $m < g_\alpha(0)$ let m–th block of α of the level 0 be $[\rho^{++}_{\alpha 0 m-1 \omega_1}, \rho^+_{\alpha,0,m,\omega_1}]$. The first block of the level 1 of α will be $[\rho^{++}_{\alpha 0 g_\alpha(0)-1\omega_1}, \rho^+_{\alpha,1,0,\omega_1}]$. In general the first block of the level $n > 0$ of α will be $[\rho^{++}_{\alpha n-1 g_\alpha(n-1)-1\omega_1}, \rho^+_{\alpha,n,0,\omega_1}]$. The m-th block ($m > 0$) of the level $n > 0$ of α will be $[\rho^{++}_{\alpha n m-1\omega_1}, \rho^+_{\alpha,n,m,\omega_1}]$.
Special attention will be devoted to the very last blocks of each level,
i.e. to $[\rho^{++}_{\alpha n g_\alpha(n)-2,\omega_1}, \rho^+_{\alpha,n,g_\alpha(n)-1,\omega_1}]$.

In the final (after the main forcing) model we will have the following structure. Every element of the set $\{\kappa^+_\beta \mid \alpha < \beta < \omega_1\}$ will be represented at all the levels up to level α. A countable set with uncountable pcf over α will be the set of indiscernibles

$$\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\} \ .$$

For every successor ordinal β, $\alpha < \beta < \omega_1$, each indiscernible $\rho^+_{\beta,n,m,\omega_1}$ $(n < \omega, m < g_\beta(n))$ will be in the pcf of this set. Thus, we will have the following:

$$\text{pcf}(\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\}) =$$

$$= \{\rho^+_{\beta,n,m,\omega_1} \mid \alpha < \beta < \omega_1, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_{\omega_1}\}.$$

Actually for each limit ordinal $\gamma, \alpha < \gamma \leq \omega_1$, the following will hold:

$$\text{pcf}(\{\rho^+_{\alpha,n,m,\gamma} \mid n < \omega, m < g_\alpha(n)\}) =$$

$$= \{\rho^+_{\beta,n,m,\gamma} \mid \alpha < \beta < \gamma, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_\gamma\}.$$

Note that for $\gamma < \omega_1$ the set on the right side of equality is countable.

Let us establish the first connection between the levels and blocks by induction.
Start with a connection of the level 1 to to the level 0.
Consider $m(0,1)$, i.e. the least $m < \omega$ such that for every $n \geq m$ we have

$$g_0(n) \geq \sum_{k=0}^{n} g_1(k) \ .$$

This is a place from which blocks of the second level fit nicely inside those of the first level. Let us arrange the connection as follows. Connect the all the blocks of the levels $n, n \leq m(0,1)$ of κ_1 to the blocks of the level $m(0,1)$ of κ_0 moving to the right as much as possible, i.e. if $r = g_0(m(0,1)) - \sum_{k=0}^{m(0,1)} g_1(k)$, then the first block of κ_1 is connected to the r–th block of the level $m(0,1)$ of κ_0, the second block of κ_1 is connected to $r+1$-th block of the level $m(0,1)$ of κ_0 etc., the last block of the level $m(0,1)$ of κ_1 will be connected to the last block of the level $m(0,1)$ of κ_0.

Let us deal now with a level $\alpha > 1$. Fix an enumeration $\langle \alpha_i \mid i < \omega \rangle$ of α (if $\alpha < \omega$, then the construction is the same). Connect blocks from $[\kappa_{\alpha-1}, \kappa_\alpha]$ (further refired as of α) to blocks from $[\kappa_{\alpha_0-1}, \kappa_{\alpha_0}]$ (further refired as of α_0) exactly as above (i.e. κ_1 and κ_0).
Let us deal now with α_1. We would like to have a tree order at least on the very last blocks of each level. Thus we would not allow a block of α to be connected to two unconnected blocks of α_0 and α_1. Split into two cases.
Case 1. $\alpha_1 > \alpha_0$. Let $l(\alpha_0, \alpha_1)$ be the first level where the connection between α_0 and α_1 starts. Then, by induction, $l(\alpha_0, \alpha_1) \geq m(\alpha_0, \alpha_1)$. Let $l(\alpha_0, \alpha)$ be the first level where the connection between α_0 and α starts. By the definition we have $l(\alpha_0, \alpha) = m(\alpha_0, \alpha)$. Consider $m(\alpha_1, \alpha)$. It is tempting to start the connection between α and α_1 with the levels $m(\alpha_1, \alpha)$, but we would like to avoid a situation when the last block of a level n of α is connected to last blocks of levels n of both α_0 and α_1, which are disconnected, i.e. the connection order is not a tree order. So set $l(\alpha_1, \alpha) = \max(l(\alpha_0, \alpha_1), m(\alpha_1, \alpha))$. Note that $m(\alpha_0, \alpha) = l(\alpha_0, \alpha) \leq l(\alpha_1, \alpha)$, since $l(\alpha_0, \alpha_1) \geq m(\alpha_0, \alpha_1)$.
Also note that there is a commutativity here, and for each $n \geq l(\alpha_1, \alpha)$, blocks of α of levels $\leq n$ are connected to the level n of α_1 and the levels $\leq n$ of α_1 are connected to the level n of α_0.
Case 2. $\alpha_1 < \alpha_0$.
The treatment is similar only now α_0 is connected to α_1. Set

$$l(\alpha_1, \alpha) = \max(l(\alpha_0, \alpha_1), l(\alpha_0, \alpha), m(\alpha_1, \alpha)).$$

Continue in the same fashion by induction.

Let us called the established connection *automatic connection.* Last blocks ordered by this connection form a tree order by the construction.
It will be shown below (in Lemma 4.4) that there is no ω_1–branches. Note that we required that $g_\alpha(0) = 1$ for all α's, and hence first levels fit nicely one with an other. However, the automatic connection is defined so that if $\alpha < \alpha' < \omega_1$ and for some $n < \omega$ n-th levels

of α and α' are connected, then for every $m, n \leq m < \omega$, m-th levels of α and α' are automatically connected as well. Hence the ability to connect first levels does not imply that they will be actually connected by the automatic connection.

Let $\alpha < \omega_1, n < \omega$ and $m < g_\alpha(n)$. Set

$$a_\alpha(n,m) = \{(\alpha', n', m') \mid \alpha' < \alpha, \text{ the block } m' \text{ of } n' \text{ of } \alpha'$$

$$\text{is connected automatically to those of } m \text{ of } n \text{ of } \alpha\}.$$

Lemma 4.1 *Let $\alpha < \omega_1, n_1, n_2 < \omega$, $m_1 < g_\alpha(n_1), m_2 < g_\alpha(n_2)$ and ($n_1 \neq n_2$ or $n_1 = n_2$ but $m_1 \neq m_2$). Then $a_\alpha(n_1, m_1) \cap a_\alpha(n_2, m_2) = \emptyset$.*

Proof. Let $\langle \alpha_k \mid k < \omega \rangle$ be the enumeration of α which was used in the definition of the automatic connection. Clearly, the connection to α_0 cannot map different blocks of α to a same block. Consider α_1. If $\alpha_1 < \alpha_0$, then be start the connection from α to α_1 not before the level $m(\alpha_0, \alpha_1)$. For any further $\beta \leq \alpha_1$, and any level $n < \omega$ of β to which both α_0 and α_1 are connected, we must to have $m(\alpha_0, \beta) \leq n$ and $m(\alpha_1, \beta) \leq n$. Then all the blocks of α up to (and including) the level n are connected with the blocks of the level n of α_1 starting with the most right block of the level n of α_1, and then the blocks of α_1 up to and including the level n of α_1 are connected to the level n of β again starting with the most right block of the level n of β. So we have a kind of commutativity there. Hence no collisions occur over a level n of β.
The same argument works if $\alpha_1 > \alpha_0$. Just replace α_0 by α_1 above.

Suppose now that $k > 0$ and $\{\alpha_0, ..., \alpha_k\}$ provide the empty intersection. Let us argue that adding α_{k+1} does not change this. Split the set $\{\alpha_0, ..., \alpha_k\}$ into two sets $\{\alpha_{i_0}, ..., \alpha_{i_s}\}$, $\{\alpha_{j_0}, ..., \alpha_{j_r}\}$ such that the members of the first are below α_{k+1} and the members of the second one are above it. Consider $\beta \leq \alpha_{k+1}$ and a level n of β where a potential intersection can occur once α is connected to α_{k+1}. Let $\{\alpha_{l_0}, ..., \alpha_{l_t}\}$ be the subset of $\{\alpha_{i_0}, ..., \alpha_{i_s}\}$ which consists of all the elements $\geq \beta$.
Then $n \geq m(\alpha_{k+1}, \alpha_{j_q})$ and $n \geq m(\alpha_{i_p}, \alpha_{k+1})$, for every $q \leq r, p \leq s$. Also we can assume that $n \geq m(\beta, \alpha_{k+1})$. Otherwise there is no connection from α_{k+1} to the level n of β. There must be some $\alpha^* \in \{\alpha_{l_0}, ..., \alpha_{l_t}\} \cup \{\alpha_{j_0}, ..., \alpha_{j_r}\}$ with $m(\beta, \alpha^*) \leq n$, since otherwise only α_{k+1} will be connected to the level n of β and then the intersection will not have elements there. We deal now with $\alpha^*, \alpha_{k+1}, \beta$ and n exactly as above.
$\square$

Note that many blocks remain unconnected. If no further connection will be made, then the following will occur. Unconnected blocks of an $\alpha < \omega_1$ will correspond to κ_α^+. By [8] we will have here always $\max(\mathrm{pcf}(\kappa_\alpha^+ \mid \alpha < \beta)) = \kappa_\beta^+$, for every $\beta < \omega_1$, due to the initial large cardinal assumptions. So, eventually there will be $\beta < \omega_1$ such that all blocks of all $\alpha < \beta$ will correspond to κ_β^+. It is clearly bad for our purpose.

We would like to extend the automatic connection such that for every α, if ρ and η are the last members of different blocks for α (it does not matter if levels are the same or not), then $\mathfrak{b}_{\rho^+} \neq \mathfrak{b}_{\eta^+}$. A problematic for us situation is once a connection was established in a way that for some $\alpha < \omega_1$ there are two different blocks for α that are connected to same blocks for unboundedly many levels below α. A problem will be then with a chain condition over α. Note that by Localization Property (see [12] or [1]) once pcf of a countable set is uncountable, there will be countable sets which correspond to cardinals much above their sup. Our construction uses only finitely many blocks at each level. If the connection is not built properly, then some countable set of blocks that should be connected with $\aleph_1$–many may turn to be connected with a single block of some $\alpha < \omega_1$ which will spoil everything.

Let us do the following. We force using a c.c.c. forcing a new connection based on the automatic connection.

Definition 4.2 Let Q be a set consisting of all pairs of finite functions q, ρ such that

1. $\rho : [\omega_1]^2 \to \omega \setminus \{0\}$ is a partial finite function,
2. $l(\alpha, \beta) \leq \rho(\alpha, \beta) < \omega$, for every $\alpha < \beta$ in the domain of ρ.
   Intuitively, $\rho(\alpha, \beta)$ will specify the place from which the automatic connection between α and β will step into the play.
3. $\mathrm{dom}(q) \subseteq \omega_1 \times (\omega \times \omega)$,
4. for every $(\alpha, n, m) \in \mathrm{dom}(q)$, $q(\alpha, n, m)$ is a finite subset of $\alpha \times \omega \times \omega$ such that
   (a) if $\langle \beta, r, s \rangle \in q(\alpha, n, m)$, then $s < g_\beta(r)$.
   This will mean that s-th block of the level r of β is connected to m-th block of the level n of α.
   (b) $(\beta, \alpha) \in \mathrm{dom}(\rho)$ iff $\alpha > \beta$ and $\alpha \in \mathrm{dom}(\mathrm{dom}(q))$ and there are $n, m, r, s < \omega$ such that $(\beta, s, r) \in q(\alpha, n, m)$,
   (c) if $(\alpha, n, m) \in \mathrm{dom}(q)$, then $q(\alpha, n, m) \neq \emptyset$,

(d) if $(\beta, r, s) \in q(\alpha, n, m)$, $r \geq \rho(\beta, \alpha)$ and (β, r, s) is automatically connected to (α, n, m), then $(\beta, r, s) \notin q(\alpha, n', m')$ whenever $(n', m') \neq (n, m)$,

(e) if $(\beta, r, s) \in q(\alpha, n, m)$, then for every $s' < g_\beta(r)$ there are $n', m' < \omega$ such that $(\beta, r, s') \in q(\alpha, n', m')$,

(f) if $(\beta, r, s) \in q(\alpha, n, m)$ and the connection is not automatic, then $n > r$,

(g) if $(\beta, r, s) \in q(\alpha, n, m)$ and $(\alpha, n, m) \in q(\alpha', n', m')$, then $(\beta, r, s) \in q(\alpha', n', m')$,

Let us define the order on Q.

Definition 4.3 Let $\langle q_1, \rho_1\rangle, \langle q_2, \rho_2\rangle \in Q$. Set $\langle q_1, \rho_1\rangle \geq \langle q_2, \rho_2\rangle$ iff

1. $\rho_1 \supseteq \rho_2$,
2. $\text{dom}(q_1) \supseteq \text{dom}(q_2)$,
3. for every $\langle \alpha, n, m\rangle \in \text{dom}(q_2)$ we have $q_2(\alpha, n, m) = q_1(\alpha, n, m)$.

Let us give a bit more intuition behind the definition of Q and explain the reason of adding ρ instead of just using the function l of the automatic connection.
The point is to prevent a situation like this: let $\gamma < \beta < \alpha$, $\alpha, \gamma \in \text{dom}(\text{dom}(q))$, $\beta \notin \text{dom}(\text{dom}(q))$ and we like to add it, for some $q \in Q$. Suppose that $l(\gamma, \alpha) = n < l(\beta, \alpha)$ and the level n of γ is connected automatically in q to all the blocks of α up to and including the level n. We need to add β. In order to do this the level n of γ should be connected to β. Then, due to the commutativity, the established connection is continued to α to the level n or below. One may try to use blocks of β of the level n and below for this purpose, but the total number of such blocks may be less than the number of blocks of the level n of γ, i.e. of $g_\gamma(n)$. So some non connected automatically to α blocks of higher levels of β should be used. There may be no such blocks at all or even if there are still this may conflict with automatic connections of bigger than α ordinals in the domain of q.
Once we have ρ, it is possible just to "fix" the automatic connection setting $\rho(\beta, \alpha)$ (i.e. the point from which the automatic connection starts actually to work) higher enough.

Lemma 4.4 *Q satisfies c.c.c.*

Proof. Suppose that $\langle\langle q_i, \rho_i\rangle \mid i < \omega_1\rangle$ is a sequence of ω_1 elements of Q. Let us concentrate on q_i's. Set $b_i = \text{dom}(\text{dom}(q_i)) \cup \text{dom}\,\text{dom}(\text{rng}(q_i)$ (i.e. the finite sequence of ordinals of dom(q_i) and of its range). Form a Δ-system. Suppose that $\langle b_i \mid i < \omega_1\rangle$ is already a Δ-system and let b^* be its kernel. By shrinking more if necessary, we can assume that the following holds:

- for every $i < j < \omega_1$, $\sup(b_i) < \min(b_j \setminus b^*)$,
- for every $i < j < \omega_1$, q_i and q_j are isomorphic over $\omega \cup \sup(b^*)$.

Now, let $i < j < \omega_1$. Then $\langle q_i \cup q_j, \rho_i \cup \rho_j\rangle$ will be a condition in Q stronger than both $\langle q_i, \rho_i\rangle$ and $\langle q_j, \rho_j\rangle$.
□

Let G be a generic subset of Q. It naturally defines a connection between blocks. Namely we connect s-th block of a level r of β with m-th block of a level n of α iff for some $(q, \rho) \in G$, $\langle \beta, r, s\rangle \in q(\alpha, n, m)$. Let us call further the part of this connection that is not the automatic connection by *manual connection* .
Denote for $\alpha, n < \omega, m < g_\alpha(m), \alpha_1 < \alpha_2 < \omega_1$,

$$connect(\alpha, n, m) = \{\langle \beta, n_1, m_1\rangle \mid \exists (q, \rho) \in G \quad \langle \beta, n_1, m_1\rangle \in q(\alpha, n, m)\}, \text{ or } \langle \beta, n_1, m_1\rangle$$

$$\text{is automatically connected to } \langle a, n, m\rangle \text{ and } \rho(\beta, \alpha) \leq n_1\},$$

$$connect(\alpha_1, \alpha_2) = \{(n_1, m_1), (n_2, m_2)) \mid \langle \alpha_1, n_1, m_1\rangle \in connect(\alpha_2, n_2, m_2)\},$$

$$aconnect(\alpha_1, \alpha_2) = \{(n_1, m_1), (n_2, m_2)) \in connect(\alpha_1, \alpha_2) \mid \langle \alpha_1, n_1, m_1\rangle, \langle \alpha_2, n_2, m_2\rangle$$

$$\text{are automatically connected and for some } (q, \rho) \in G \text{ we have } \rho(\alpha_1, \alpha_2) \leq n_1\}.$$

$$mconnect(\alpha_1, \alpha_2) = connect(\alpha_1, \alpha_2) \setminus aconnect(\alpha_1, \alpha_2).$$

Let us refer further to elements of $mconnect(\alpha_1, \alpha_2)$ connected by the *manual connection* .

Lemma 4.5 *Suppose that $\langle \beta, r, s\rangle$ is a block of β and $\alpha > \beta$. Then for some $n, m < \omega$ we have $\langle \beta, r, s\rangle \in connect(\alpha, n, m)$.*

Proof. Let $\langle q, \rho\rangle \in Q$. We will construct a stronger condition $\langle q^*, \rho^*\rangle$ with $\langle \alpha, n, m\rangle \in$ dom(q^*) and $\langle \beta, r, s\rangle \in q^*(\alpha, n, m)$, or $\rho^*(\beta, \alpha) \leq r$ and $\langle \beta, r, s\rangle$ is automatically connected with $\langle \alpha, n, m\rangle$, for some $n, m < \omega$.
If $\langle \beta, r, s\rangle$ is automatically connected with $\langle \alpha, n, m\rangle$, for some $n, m < \omega$ and $(\beta, \alpha) \notin$ dom(ρ) or $(\beta, \alpha) \in$ dom(ρ), $\rho(\beta, \alpha) \leq r$, then just set $\rho^*(\beta, \alpha) = r$ or $\rho^*(\beta, \alpha) = \rho(\beta, \alpha)$, if defined and we are done.

Suppose now that the above is not the case. So $r < \rho(\beta, \alpha)$ or $\rho(\beta, \alpha)$ is undefined. In the later case define it just to take any value above r. Pick $n < \omega$ to be big enough such that (α, n) does not appears in q. Extend q to q^* by adding $\langle \alpha, n, g_\alpha(n) - 1 \rangle$ to its domain. Set $q^*(\alpha, n, g_\alpha(n) - 1) = \langle \beta, r, s \rangle$.

□

Lemma 4.6 *For every $\langle \beta, r, s \rangle, \langle \beta, r', s' \rangle$ with $\beta < \omega_1, r, r' < \omega, r \neq r', s < g_\beta(r), s' < g_\beta(r')$ there are $\alpha < \omega_1$ (or even $\alpha < \beta + \omega$),$n < \omega, m < g_\alpha(n)$ such that $\langle \beta, r, s \rangle, \langle \beta, r', s' \rangle \in$ connect(α, n, m).*

Proof. This follows by the density argument.

Assume that $r < r'$.

Let $\langle q, \rho \rangle \in Q$. We will construct a stronger condition $\langle q^*, \rho^* \rangle$ as follows. First let us pick $\alpha > \beta$ which does not appear in q. If the automatic connection between β and α starts at a level $\leq r'$ and the block s' is connected by it, then just set $\rho^*(\beta, \alpha) = r'$. Let $n = r'$ and m be the block of the level n of α which corresponds to the block s' of the level r' of β. Extend q to q^* by adding $\langle \alpha, n, m \rangle$ to its domain. Set $q^*(\alpha, n, m) = \langle \beta, r, s \rangle$.

Suppose now that the automatic connection between β and α starts at a level $> r'$ or it starts at a level $\leq r'$, but the block s' is too low and remains unconnected by it.

Set then $\rho^*(\beta, \alpha)$ to be the place where automatic connection between β and α starts. Pick a level n above it. Extend q to q^* by adding $\langle \alpha, n, g_\alpha(n) - 1 \rangle$ to its domain. Set $q^*(\alpha, n, g_\alpha(n) - 1) = \{\langle \beta, r, s \rangle, \langle \beta, r', s' \rangle\}$.

□

Lemma 4.7 *For every $\alpha < \omega_1$, $n, n' < \omega$ and $m < g_\alpha(n), m' < g_\alpha(n')$. connect$(\alpha, n, m) \cap$ connect(α, n', m') is bounded in α, unless $n = n'$ and $m = m'$.*

Proof. Note that the automatic connection has this property (even we have disjoint sets by 4.1). The additions made (if at all) are finite.

□

In order to realize the defined above connection there is a need in dropping cofinalities technics. Thus, for example, for some α the very first block of α may be connected (by the manual connection) to the last block of a level $n > 0$ of $\alpha + 1$. So in order to accommodate all the blocks of levels $\leq n$ of $\alpha + 1$ on and below the very first block of α there is a need to drop down below α. Note that on $\alpha - 1$ there is enough places to which such blocks are connected automatically, just starting with a higher enough level of $\alpha - 1$.

In this respect $\alpha = 0$ should be treated separately, since $\alpha - 1$ does not exist and so no place to drop. Let us just assume that all blocks of 0 are connected to blocks of 1 automatically. This can be achieved easily by changing g_0, g_1 a bit in order to fit together nicely. In addition do not allow to use blocks of 0 in the forcing Q above.

5 The preparation forcing.

We would like to use a generic set for the forcing $\mathcal{P}'$ of Chapter 3 (Preserving Strong Cardinals) of [6] in order to supply models for the main forcing defined further. Some degree of strongness of $\kappa_{\alpha,n}$ will be needed as well, for every successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$.
Two ways were described in Chapter 3 of [6]. Either can be applied for our purpose.
The first one is as follows.
Assume that for some regular cardinal θ the following set is stationary:

$$S = \{\nu < \theta \mid \nu \text{ is a superstrong with the target } \theta (\text{i.e. there is } i : V \to M, crit(i) = \nu$$

$$i(\nu) = \theta \text{ and } M \supseteq V_\theta)\}.$$

Return to the definition of κ_γ's and $\kappa_{\gamma,k}$'s. Let us choose them by induction such that all $\kappa_{\gamma,k}$'s are from S. Suppose that $\langle \kappa_{\gamma,k} \mid k < \omega \rangle$ is defined. Then $\kappa_\gamma = \bigcup_{k<\omega} \kappa_{\gamma,k}$. Let $\tilde{\kappa}_\gamma$ be the next element of S. Pick $\kappa_{\gamma+1,0}$ to an element of S above $\tilde{\kappa}_\gamma$.
Force with $\mathcal{P}'(\theta)$ with a smallest size of models say $\aleph_8$. Then, by Lemma 3.0.23 of Chapter 3 (Preserving Strong Cardinals) of [6], each $\kappa_{\alpha,n}$ will remain $\tilde{\kappa}_\alpha$–strong (and even $\kappa^+_{\omega_1}$–strong). Moreover, $\mathcal{P}'(\tilde{\kappa}_\alpha)$ is a nice subforcing of $\mathcal{P}'(\theta)$ by Lemma 3.0.18 of Chapter 3 (Preserving Strong Cardinals) of [6], since $V_{\tilde{\kappa}_\alpha} \preceq V_\theta$ due to the choice of $\tilde{\kappa}_\alpha$ in S.

An other way, which uses initial assumptions below $\mathbf{0}^\P$, is as follows.
Let θ be a 2-Mahlo cardinal and $\kappa < \theta$ be a strong up to θ cardinal. Pick $\delta, \kappa < \delta < \theta$ a Mahlo cardinal such that $V_\delta \prec_{\Sigma_1} V_\theta$. By Lemma 3.0.15 of Chapter 3 (Preserving Strong Cardinals) of [6] or just directly, there will unboundedly many cardinals $\eta < \kappa$ with $\delta_\eta < \kappa$ such that the function $\eta \mapsto \delta_\eta$ represents δ and $V_{\delta_\eta} \prec_{\Sigma_1} V_\theta$. Then, by Lemma 3.0.18 of Chapter 3 of [6], $\mathcal{P}'(\delta_\eta)$ is a nice subforcing of $\mathcal{P}'(\theta)$.
Denote by S the set of all such η's.
Force now with $\mathcal{P}'(\theta)$. Let G' be a generic. By Lemma 3.0.24 of Chapter 3 of [6], embeddings wich witness δ-strongness of κ for large enough δ's below θ extend in $V[G']$. Then, below κ in $V[G']$, we will have unboundedly many η's which are strong up to δ_η for which $V_{\delta_\eta}[G' \cap V_{\delta_\eta}] \prec_{\Sigma_1} V_\theta[G']$, since every $\eta \in S$ is like this.
Now we define by induction $\kappa_{\gamma,k}$'s, κ_γ's and $\tilde{\kappa}_\gamma$'s using this η's and δ_η's.
Thus, suppose that $\langle \kappa_{\gamma,k} \mid k < \omega \rangle$ is defined. Then $\kappa_\gamma = \bigcup_{k<\omega} \kappa_{\gamma,k}$. Let $\tilde{\kappa}_\gamma = \delta_\eta$ for some such $\eta > \kappa_\gamma$. Pick $\kappa_{\gamma+1,0}$ to be the first $\eta \in S$ above $\tilde{\kappa}_\gamma$ and $\kappa_{\gamma+1,1}$ to be the first $\eta \in S$ above $\delta_{\kappa_{\gamma+1,0}}$, etc.

6 Types of Models

Force with $\mathcal{P}'$. Let $G' \subseteq \mathcal{P}'$ be a generic subset. Work in $V[G']$. For each successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$ let us fix a $(\kappa_{\alpha,n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1})$– extender $E_{\alpha n}$, i.e. an extender with the critical point $\kappa_{\alpha,n}$ which ultrapower contains $V_{\kappa_{\alpha,n,g_\alpha(n)-1,\omega_1}+2}$.
Also, using GCH (we assume GCH in V and then it will holds in $V[G']$ as well), fix an enumeration $\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle$ of $[\kappa_{\alpha n}]^{<\kappa_{\alpha n}}$ so that for every successor cardinal $\delta < \kappa_{\alpha n}$ the initial segment $\langle x_\gamma \mid \gamma < \delta\rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_{\alpha n}(\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle) = \langle x_\gamma \mid \gamma < j_{\alpha n}(\kappa_{\alpha n})\rangle$, where $j_{\alpha n}$ is a canonical embedding of $E_{\alpha n}$. Then $\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle$ will enumerate $[\kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}]^{\leq \kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}}$.

For every $k < \omega$, we consider a structure

$$\mathfrak{A}_{\alpha,n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, E_{\alpha n}, \kappa_{\alpha n}, \kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}, \langle \kappa_{\alpha,n,m,i} \mid m < g_\alpha(n), i \leq \omega_1\rangle, \chi,$$

$$\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle, G', \theta, \langle \kappa_{\beta m} \mid \beta < \omega_1 \text{ is a successor ordinal or zero }, m < \omega\rangle,$$

$0, 1, \ldots, \xi, \ldots \mid \xi < \kappa^{+k}_{\alpha n}\rangle$
in an appropriate language which we denote $\mathcal{L}_{\alpha,n,k}$, with a large enough regular cardinal χ. Note that we have G' inside, so suitable structures may be chosen inside G' or $G' \cap \mathcal{P}'(\kappa_{\alpha,n})$.

Let $\mathcal{L}'_{\alpha,n,k}$ be the expansion of $\mathcal{L}_{\alpha,n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$ let $\mathfrak{A}_{\alpha,n,k,a}$ be the expansion of $\mathfrak{A}_{\alpha,n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. Denote by $tp_{\alpha,n,k}(b)$ the $\mathcal{L}_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k}$. Further we identify it with the ordinal coding it and refer to it as the k-type of b. Let $tp_{\alpha,n,k}(a,b)$ be a the $\mathcal{L}'_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [2].

Now, repeating the usual arguments we obtain the following:

Lemma 6.1 *(a)* $|\{tp_{\alpha,n,k}(b) \mid b \in H(\chi^{+k})\}| = \kappa^{+k+1}_{\alpha n}$

(b) $|\{tp_{\alpha,n,\kappa}(a,b) \mid a, b \in H(\chi^{+k})\}| = \kappa^{+k+1}_{\alpha n}$

Lemma 6.2 *Let $A \prec \mathfrak{A}_{\alpha,n,k+1}$ and $|A| \geq \kappa^{+k+1}_{\alpha n}$. Then the following holds:*

(a) for every $a, b \in H(\chi^{+k})$ there $c, d \in A \cap H(\chi^{+k})$ with $tp_{\alpha,n,k}(a,b) = tp_{\alpha,n,k}(c,d)$

(b) for every $a \in A$ and $b \in H(\chi^{+k})$ there is $d \in A \cap H(\chi^{+k})$ so that $tp_{\alpha,n,k}(a \cap H(\chi^{+k}), b)=tp_{\alpha,n,k}(a \cap H(\chi^{+k}), d)$.

Lemma 6.3 *Suppose that $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_{\alpha n}^{+k+1}$. Let τ be a cardinal in the interval $[\kappa_{\alpha n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}]$ those $k+1$-type is realized unboundedly often below $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. Then there are $\tau' < \tau$ and $A' \prec A \cap H(\chi^{+k})$ such that $\tau', A' \in A$ and $\langle \tau', A' \rangle$ and $\langle \tau, A \cap H(\chi^{+k}) \rangle$ realize the same $tp_{\alpha,n,k}$. Moreover, if $|A| \in A$, then we can find such A' of cardinality $|A|$.*

Lemma 6.4 *Suppose that $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_{\alpha n}^{+k+1}$, $B \prec \mathfrak{A}_{\alpha,n,k}$, and $C \in \mathcal{P}(B) \cap A \cap H(\chi^{+k})$. Then there is D so that*

(a) $D \in A$

(b) $C \subseteq D$

(c) $D \prec A \cap H(\chi^{+k}) \prec H(\chi^{+k})$.

(d) $tp_{\alpha,n,k}(C, B) = tp_{\alpha,n,k}(C, D)$.

7 The Main Forcing.

Suitable structures and suitable generic structures are defined similar to those in Sections 1.2 or 2.4 of [6].

We would like to define the main forcing $\mathcal{P}$. Let us split the definition into ω–many steps. First we define pure conditions $\mathcal{P}_0$, at the next step $\mathcal{P}_1$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_0$, then $\mathcal{P}_2$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_1$, etc. Finally $\mathcal{P}$ will be $\bigcup_{n<\omega} \mathcal{P}_n$.

Definition 7.1 The set $\mathcal{P}_0$ consists of all sequences

$$\langle p_\alpha \mid \alpha < \omega_1 \text{ and } (\alpha = 0 \text{ or } \alpha \text{ is a successor ordinal }) \rangle$$

such that $p_\alpha = \langle p_{\alpha\beta} \mid \alpha < \beta < \omega_1 \text{ is a successor ordinal } \rangle$, and for all $n < \omega, \alpha < \beta < \omega_1$ is a successor ordinal ,
$p_{\alpha\beta} = \langle p_{\alpha\beta x} \mid x \in connect(\alpha, \beta) \rangle$, where for every $x \in connect(\alpha, \beta)$,
$p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha\beta x}, f_{\alpha\beta x} \rangle$ is such that:

1. (Automatic connection) If $x \in aconnect(\alpha, \beta)$, $x = ((n_1, k_1), (n_2, k_2))$, for some $k_1, k_2, n_1, n_2 < \omega$, then

(a) $A_{\alpha\beta x} = A_{\alpha n_2}$, i.e. it does not depend on β, x, but rather on on level n_2 of α (and α itself).
It is a set of measure one for a measure of the extender $E_{\kappa_\alpha n_2}$.

(b) $a_{\alpha\beta x} = a_{\alpha\beta n_2}$, i.e. it depends on α, β and n_2 only.

(c) $\text{dom}(a_{\alpha\beta n_2})$ is a $(\prod_{k \leq n_2} A_{\beta k})-$name of a generic suitable structure of size $< \kappa_{\alpha n_2}$.

(d) $\text{rng}(a_{\alpha\beta n_2})$ is a suitable structure over level n_2 of α.

(e) For each $k \leq n_2$ and $\eta \in A_{\beta k}$ let us denote by $\rho_{\beta k}$ the projection of η to the normal measure of the extender $E_{\beta k}$.
For each $m < g_\beta(k)$ and $\gamma \leq \omega_1$ let $\rho_{\beta k m \gamma}$ be $\pi_{coor(A_{\beta k})\kappa_{\beta k m \gamma}}(\eta)$, i.e. the indiscernible which corresponds to $\kappa_{\beta k m \gamma}$, where $coor(A_{\beta k})$ is the coordinate of $E_{\kappa_\beta k}$ to which $A_{\beta k}$ belongs.
We require that for each
$\langle \eta_0, ..., \eta_{n_2} \rangle \in \prod_{k \leq n_2} A_{\beta k}$, for every $k \leq n_2, m < g_\beta(k)$ and $\beta < \gamma \leq \omega_1$,
$a_{\alpha\beta n_2}[\langle \eta_0, ..., \eta_{n_2} \rangle]$ (i.e. the interpretation of $a_{\alpha\beta n_2}$ according to $\langle \eta_0, ..., \eta_{n_2} \rangle$) is the isomorphism between $\text{dom}(a_{\alpha\beta n_2})[\langle \eta_0, ..., \eta_{n_2} \rangle]$ and $\text{rng}(a_{\alpha\beta n_2})$ which maps models of sizes $\rho_{\beta k m \gamma}$ and $(\rho_{\beta k m \gamma})^+$ to models over the level n_2 of α of cardinalities $\kappa_{\alpha n_2 m^* \gamma}$ and $(\kappa_{\alpha n_2 m^* \gamma})^+$ respectively, where $m^* = (g_\alpha(n_2) - \sum_{s=k}^{n_2} g_\beta(s)) + m$ (i.e. we start as far right as possible).
This means, in particular, that once a non-direct extension was made at the level n_2 of α, then $\rho_{\beta k m \gamma}$ and $(\rho_{\beta k m \gamma})^+$ will correspond to $\rho_{\alpha n_2 m^* \gamma}$ and $(\rho_{\alpha n_2 m^* \gamma})^+$ respectively.
Models of sizes from the interval $((\rho_{\beta k m-1 \omega_1})^+, \rho_{\beta k m \beta+1})$ will be connected with models of sizes in the interval $(\kappa_{\alpha n_2 m^* 0}, \kappa_{\alpha n_2 m^* \beta+1})$, if $m > 0$.
If $m = 0$ and $k > 0$, then models of sizes from $((\rho_{\beta k-1 g_\beta(k-1)-1 \omega_1})^+, \kappa_{\beta k-1}) \cup [\kappa_{\beta k-1}, \rho_{\beta k 0 \beta+1})$ will be connected with $(\kappa_{\alpha n_2 m^* 0}, \kappa_{\alpha n_2 m^* \beta+1})$.
If $m = 0$ and $k = 0$, then $(\kappa_{\beta-1}, \rho_{\beta 0 0 \beta+1})$ will be connected with
$(\kappa_{\alpha n_2 m^* \beta}, \kappa_{\alpha n_2 m^* \beta+1})$.
The difference is that there are two types - one having pre-images from higher β' (i.e. from $\beta' > \beta$) and an other that do not have them.

(f) $f_{\alpha\beta x} = f_{\alpha\beta n_2}$ is a $(\prod_{k \leq n_2} A_{\beta k})-$name of a partial function from κ_{β, n_2} to κ_{α, n_2} of cardinality at most $\kappa_{\beta-1}$.

2. (Manual connection) $x \in mconnect(\alpha, \beta)$, $x = ((n_1, k_1), (n_2, k_2))$, for some $k_1, k_2, n_1, n_2 < \omega$.

The cardinals corresponding is similar to the case of the automatic connection. Note that $m-$connection connects to a single level and the rest drops down. Describe such droppings.
First we assume (arrange) that for every limit $\alpha > 0$,

$$g_{\alpha+1}(0) = g_{\alpha+2}(0) \text{ and } \forall n > 0(g_{\alpha+1}(n) = \sum_{k \leq n} g_{\alpha+2}(k)),$$

i.e. the blocks of $\alpha + 1$ and $\alpha + 2$ fit precisely one to another.
Do the same 0 and 1, i.e.

$$g_0(0) = g_1(0) \text{ and } \forall n > 0(g_0(n) = \sum_{k \leq n} g_1(k)).$$

We require that all connections to $\alpha + 1$ or to 0 from above go via $\alpha + 2$ or via 1, respectively, and the only connections between $\alpha+2$ and $\alpha+1$, 1 and 0, are automatic with all blocks of $\alpha + 1$ (or of 0) are connected to blocks of $\alpha + 2$ (or, respectively, to 1) by $a-$connections.
This way $\alpha + 1$ (or 0) will be used for dropping from $\alpha + 2$ (or from 1).
Also, if $\alpha < \alpha'$ are both limit, then connections from $\alpha' + k', 0 < k' < \omega$ to $\alpha + k, 0 < k < \omega$ are only $a-$connections. I.e. $m-$connections applied only between $\alpha + s, \alpha + s', 0 < s, s' < \omega$ with a same limit $\alpha < \omega_1$.

Describe now manual connections droppings and state commutativity requirements.

Suppose that two blocks of β are connected to the same block over α. In particular, one (at least one) (from higher level) must be then $m-$connected.
Let (α, n, m) be connected with (β, r, s) and (β, r', s'), where $\beta > \alpha, r' > r$.
It may be the case that $r \leq n$, and then necessary (β, r', s') is a part of $a-$connection of the level n of β to the level n of α.
By genericity of connections, there will be $\gamma, \beta < \gamma < \beta + \omega$,$t, u < \omega$ such that (γ, u, t) is $a-$connected to (β, r', s') and (β, r, s) is $m-$connected to it.
We require the obvious commutativity here.

Similar we treat the situation in which $\beta > \alpha + \omega$ is connected ($a-$connection, since $\beta > \alpha + \omega$) to (α, n), but its connection to $\alpha + 1$ starts above level n. Thus, by genericity of connections, there will be some (r, s)(actually, there will be infinitely many such r's) so that $(\alpha, n, g_\alpha(n) - 1)$ is connected manually to $(\alpha + 1, r, s)$ and also, (β, n) is $a-$connected to it. Require commutativity here as well.
This way, in particular, we will not loose information on connection of $(\beta, n), (\alpha, n)$

once a non-direct extension made over $\alpha+1$.

Let us describe dropping in cofinality that occurs here.

Suppose that α is a non-limit ordinal, $\beta > \alpha+1$ and we have the following:

(β, n') connected to levels n and n' of $\alpha+1$ (say $n' > n$), wth $a-$connection to $(\alpha+1, n')$.

Both $(\alpha+1, n), (\alpha+1, n')$ are connected to a same level at α.

It may be the case that level n of α is $a-$connected with the level n of α, but it is possible that this does not occurs and then a block (α, k, l) is $m-$connected to blocks $(\alpha+1, n, m)$ and $(\alpha+1, n', m')$. Consider the least block $s' < g_{\alpha+1}(n')$ which is connected to α (i.e. to $k-$th level of α). Then below it the drop to $\alpha-1$ occurs.

If the connection from α to the $n-$th level of $\alpha+1$ is not automatic, then there will be $s < g_{\alpha+1}(n)$ the last with connection to α (i.e. to $k-$th level of α). So, again, drop will occur here to $\alpha-1$. We have n' big enough so the connections to $(\alpha+1, n')$, $(\alpha+1, n)$ which are from $n'-$level of β cover all blocks including s, s' (counting down from above). So, commutativity requirements apply to all relevant blocks before those that drop to $\alpha-1$.

Let n' drops to some $\tilde{n}' \geq n'$ (over $\alpha-1$) and n to $\tilde{n} \geq n$, where $\tilde{n}', \tilde{n}$ depend on places where the $a-$connection between β and $\alpha-1$ starts to work. Also $\tilde{n}' > \tilde{n}$. This implies that the corresponding assignment functions ($\underset{\sim}{b}$'s) will have domains of different cardinalities ($< \kappa_{\alpha-1\tilde{n}'}$ and $< \kappa_{\alpha-1\tilde{n}}$). Repeat Sec. 6 [6] and split into intervals over central pistes of suitable structures.

3. (Commutativity of connections) Let β, γ be successor ordinals, $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$. Assume that $k_\alpha-$th block of $n_\alpha-$th level of α is connected to $k_\beta-$th block of a level n_β of β and to $k_\gamma-$th block of a level n_γ of γ. Suppose that in addition that $k_\beta-$th block of a level n_β of β and $k_\gamma-$th block of a level n_γ of γ are connected.

   Then for each $Z \in \mathrm{dom}(a_{\alpha\gamma((n_\alpha,k_\alpha),(n_\gamma,k_\gamma))})$ we have $Z \in \mathrm{dom}(a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,kn_\gamma))})$ and

$$a_{\alpha\gamma((n_\alpha,k_\alpha),(n_\gamma,k_\gamma))}(Z) = a_{\alpha\beta((n_\alpha,k_\alpha),(n_\beta,k_\beta))}(\underset{\sim}{a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}}(Z)),$$

   where $\underset{\sim}{a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}}(Z)$ is a name of the indiscernible corresponding to $a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}(Z)$.

Definition 7.2 (One element extension.)
Suppose $p = \langle p_\alpha \mid \alpha < \omega_1 \text{ and } (\alpha = 0 \text{ or } \alpha \text{ is a successor ordinal })\rangle \in \mathcal{P}_0$, $\alpha < \omega_1$ be zero or a successor ordinal, $\beta, \alpha < \beta < \omega_1$ a successor ordinal and $x = ((n_\alpha, m_\alpha), (n_\beta, m_\beta)) \in$ *connect*(α, β). Let $p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha,x}, f_{\alpha\beta x}\rangle$ and $\eta \in A_{\alpha,x}$.

Assume that $n_\alpha = 0$. In general, if $n_\alpha \neq 0$, then taking a non-direct extension over the level n_α we would like simultaneously to make a non-direct extension at each level $n < n_\alpha$ over α. Define $p^\frown\eta$, the one element non direct extension of p by η, to be $q = \langle q_\xi \mid \xi < \omega_1$ and $(\xi = 0$ or ξ is a successor ordinal $)\rangle$ so that

1. for every $\xi, \zeta, \alpha < \xi < \zeta < \omega_1, y \in connect(\xi\zeta), \quad p_{\xi\zeta y} = q_{\xi\zeta y}$,
2. for every $y \in connect(\alpha, \gamma)$ with the level on α bigger than n_α we have $p_{\alpha\beta y} = q_{\alpha\beta y}$.
3. for every successor ordinal $\gamma, \alpha < \gamma < \omega_1, \quad q_{\alpha\gamma y} = f_{\alpha\gamma y} \cup \{\langle \tau, \pi^{E_{\kappa_\alpha, n_\alpha}}_{mc(\alpha,n), a_{\alpha\gamma y}(\tau)}(\eta)\rangle \mid \tau \in \mathrm{dom}(a_{\alpha\gamma y})\}$, where $y \in connect(\alpha, \gamma)$ and the level of y over α is n_α as those of x.
4. Let $\alpha', \tau, \alpha' > \alpha > \tau$, be successor ordinals or zero. Then connections $a_{\tau\alpha' y}$ of p will split now in q into connections from α' to α followed by a connection from α to τ. Namely, let $\langle \tau, r, s\rangle$ be connected with $\langle \alpha', n', m'\rangle$. For each (n, m) such that $((n, m), (n', m')) \in aconnect(\alpha', \alpha)$ and $\langle \tau, r, s\rangle \in connect(\alpha, n, m)$ (the are such n, m by Lemma 4.6) split $a_{(\alpha',n',m'),(\gamma,r,s)}$ into $a_{(\alpha',n',m'),(\alpha,n,m)}$ followed by $a_{(\gamma,r,s),(\alpha,n,m)}$.
5. For each level $n' < n_\alpha$ of α, the same things occur, i.e. 2-4 above hold with (n_α, m_α) replaced by (n', k'), where k' is any block of the level n'.
6. Each connection which drops in cofinality below the block of η, i.e. below the level n_α of α, we freeze such drops and deal only with drops to cofinalities above η in a fashion used in Section 6 of [6] for same purpose.

Definition 7.3 Set $\mathcal{P}_1$ to be the set all $p^\frown\eta$ as in Definition 7.2. Proceed by induction. For each $n < \omega$, once $\mathcal{P}_n$ is defined, define $\mathcal{P}_{n+1}$ to be the set of all $p^\frown\eta$, where $p \in \mathcal{P}_n$. Finally set $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}_n$.

Definition 7.4 Let $p, q \in \mathcal{P}$.

1. We say that p is a direct extension of q and denote this by $p \geq^* q$ iff p is obtained from q by extending $a_{\alpha\beta x}, f_{\alpha\beta x}$'s and by shrinking the sets of measures one probably by passing to bigger measure first.
2. The forcing order $\geq$ is defined as follows:
   $p \geq q$ iff there are $q_1, ..., q_n \in \mathcal{P}, \eta_1, ..., \eta_n$ such that
   (a) $q \leq^* q_1$,

(b) for every $k, 1 \leq k \leq n, \quad q_k{}^\frown\eta_k \in \mathcal{P}$,

(c) for every $k, 1 \leq k < n, \quad q_k{}^\frown\eta_k \leq^* q_{k+1}$,

(d) $q_n{}^\frown\eta_n \leq^* p$.

For each $\alpha < \omega_1$. $\mathcal{P}$ splits into $(\mathcal{P}\backslash\kappa_\alpha) * \mathcal{P} \restriction \kappa_{\alpha+1}$, where $\mathcal{P}\backslash\kappa_\alpha$ is the part of $\mathcal{P}$ is defined as $\mathcal{P}$ but with $\kappa_{\alpha+1}$ replacing κ_0, i.e. everything is above κ_α and the first cardinal we deal with is $\kappa_{\alpha+1,0}$. $\mathcal{P} \restriction \kappa_{\alpha+1}$ is defined in $V[G']^{\mathcal{P}\backslash\kappa_\alpha}$ as $\mathcal{P}$ was defined in $V[G']$, but cutting everything at $\kappa_{\alpha+1}$, where $G' = G(\mathcal{P}')$ is a generic subset of the preparation forcing $\mathcal{P}'$.

Let us prove now the Prikry condition.

Lemma 7.5 *$\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.*

Proof. Work in $V[G(\mathcal{P}')]$. Let σ be a statement of the forcing language and $p \in \mathcal{P}$. Suppose for simplicity that $p \in \mathcal{P}_0$.
We peak an elementary chain of elementary submodels of H_χ (for χ big enough)

$$\langle M(\kappa_{\alpha n}, \xi) \mid \alpha < \omega_1,\ 0 \text{ or non-limit ordinal }, n < \omega, \xi \leq \kappa_{\alpha n} \rangle$$

such that

1. $p, \sigma \in M(\kappa_{00}, 0)$,

2. $|M(\kappa_{\alpha n}, \xi)| = \kappa_{\alpha n}$,

3. if ξ is a limit ordinal then $M(\kappa_{\alpha n}, \xi) = \bigcup_{\xi' < \xi} M(\kappa_{\alpha n}, \xi')$,

4. $\langle M(\kappa_{\alpha n}, \xi') \mid \xi' < \xi \rangle \in M(\kappa_{\alpha n}, \xi)$, for every successor ξ,

5. $\langle M(\kappa_{\alpha n}, \xi) \mid \xi \leq \kappa_{\alpha n} \rangle \in M(\kappa_{\alpha n+1}, 0)$,

6. $\langle M(\kappa_{\alpha n}, \xi) \in G(\mathcal{P}')$.

7. Let $M(\kappa_{\omega_1}) = \bigcup_{\alpha < \omega_1, n < \omega} M(\kappa_{\alpha n}, \kappa_{\alpha n})$.
   Then

   (a) $M(\kappa_{\omega_1}) \in G(\mathcal{P}')$,

   (b) each model $M(\kappa_{\alpha n}, \xi)$ is on the main piste of $M(\kappa_{\omega_1})$.

Proceed by induction. Suppose we got to level n of some α. Denote by X the corresponding set of measure one of the condition q built (i.e. $A_{\alpha n}$ of it). Continue by induction on members of X. We use here models $\langle M(\kappa_{\alpha n}, \xi) \mid \xi \leq \kappa_{\alpha n} \rangle$. Thus, if $\nu \in X$, then work inside $\langle M(\kappa_{\alpha n}, \nu + 1)$. We ask if there is an extension of $q(\nu)^\frown \nu$ (where $q(\nu)$ was formed on the previous stage) which decides σ and is a direct extension above α, n. If so, then pick such extension and add $\langle M(\kappa_{\alpha n}, \nu + 1)$ to be the largest model. Otherwise, we make no change. Non-direct parts below α, n will be stabilized once all ν's in X are considered. More precisely, we stabilize each $\langle \tau_1^\nu, ..., \tau_s^\nu \rangle$ that is below ν and a direct extension of $q(\nu)^\frown \langle \tau_1^\nu, ..., \tau_s^\nu \rangle^\frown \nu$ decides σ. Isomorphisms between structures (a's) and Cohen functions (f's) below are dealt as names depending on ν's.
Being of models $M(\kappa_{\beta m})$'s on the main piste of $M(\kappa_{\omega_1})$ allows freely to take unions.
□

Lemma 7.6 *$\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets to κ_0.*

Proof. Let $p \in \mathcal{P}$, $\underset{\sim}{z}$ be a $\mathcal{P}-$name and $p \Vdash \underset{\sim}{z}$ is a bounded subset of κ_0. Extending p if necessary we can assume that $p \Vdash \underset{\sim}{z} \subseteq \kappa_{0m}$, for some $m < \omega$. Extend p further, if necessary, and assume that non-direct extensions were made in it at every level $n \leq m$ of 0.
Pick an elementary submodel $M \preceq H\chi$ of cardinality κ_0^+ such that

1. $p, \underset{\sim}{z} \in M$,
2. $M \in G(\mathcal{P}')$,
3. there is an increasing continuous sequence $\langle M_\xi \mid \xi < \kappa_{0m} \rangle$ of elementary submodels of M such that
   (a) $p, \underset{\sim}{z} \in M_0$,
   (b) $\langle M_\xi \mid \xi < \kappa_{0m} \rangle$ on the piste of M of models of size κ_0^+,
   (c) $M = \bigcup_{\xi < \kappa_{0m}} M_\xi$.

Now, we use the previous lemma 7.5 and build by induction a $\leq^* -$increasing sequence $\langle p(\xi) \mid \xi < \kappa_{0m} \rangle$ of extensions of p such that $p(\xi) \in M_{\xi+1}$ and $p(\xi) \| \xi \in \underset{\sim}{z}$.
We have enough closure to run the process and eventually the upper bound of $\langle p(\xi) \mid \xi < \kappa_{0m} \rangle$ will decide $\underset{\sim}{z}$ completely.
□

Similar argument gives the following:

Lemma 7.7 *For every $\alpha < \omega_1$, $\langle \mathcal{P} \setminus \kappa_\alpha, \leq \rangle$ does not add new bounded subsets to $\kappa_{\alpha+1}$.*

Define now $\longleftrightarrow$ and $\longrightarrow$.

Definition 7.8 Let $p, q \in \mathcal{P}$. Set $p \longleftrightarrow q$ iff there is $\alpha < \omega_1$ such that

1. $p \setminus \kappa_\alpha = q \setminus \kappa_\alpha$,
2. for every $k < \omega$, for all but finitely many $\beta \leq \alpha$, for all but finitely many $n < \omega$ the following hold:
   (a) if no non-direct extension was made at the level n of β in p_β and q_β, then $0_{\mathcal{P}\setminus\kappa_\beta} \Vdash_{\mathcal{P}\setminus\kappa_\beta}$ over the level n of β the following hold in p_β and q_β:
      i. f's, A's and $\mathrm{dom}(a)$'s are the same,
      ii. $\mathrm{rng}(a)$'s realize the same k-type;
   (b) if a non-direct extension was made at the level n of β in one of p_β or q_β, then it was made in another as well, and they are equal.

   This means basically that $p \upharpoonright \kappa_{\alpha+1} \longleftrightarrow_{\mathcal{P}\upharpoonright\kappa_{\alpha+1}} q \upharpoonright \kappa_{\alpha+1}$,
   where $\longleftrightarrow_{\mathcal{P}\upharpoonright\kappa_{\alpha+1}}$ states that for each $k < \omega$ all but finitely many coordinates realize the same k-type.

Now we define $\longrightarrow$ in the usual fashion.

Definition 7.9 Let $p, q \in \mathcal{P}$. Set $p \longrightarrow q$ iff there is a sequence of conditions $\langle r_k \mid k < m < \omega \rangle$ so that

(1) $r_0 = p$

(2) $r_{m-1} = q$

(3) for every $k < m - 1$,
$r_k \leq r_{k+1}$ or $r_k \longleftrightarrow r_{k+1}$.

Lemma 7.10 *Let $\alpha < \omega_1$. Then, in $V^{\mathcal{P}' * \mathcal{P}\setminus\kappa_\alpha}$, the forcing $\langle \mathcal{P} \upharpoonright \kappa_{\alpha+1}, \longrightarrow \rangle$ satisfies κ_α^{++}-c.c.*

Proof. Suppose otherwise. Assume that

$$0_{\mathcal{P}\setminus\kappa_\alpha} \Vdash_{\mathcal{P}\setminus\kappa_\alpha} \{\underset{\sim}{p}_\xi \mid \xi < \kappa_\alpha^{++}\} \subseteq \mathcal{P} \upharpoonright \kappa_{\alpha+1} \text{ is an antichain .}$$

Force over $V[G(\mathcal{P}')]$ (not over $V[G(\mathcal{P}')][G(\mathcal{P} \setminus \kappa_\alpha)]$!) with the obvious forcing (i.e. initial segments) which produces a κ_α^{++}-chain of members of $G(\mathcal{P}')$ of size κ_α^+. This forcing does not add new sequences of length $\leq \kappa_\alpha^+$.
Pick an elementary submodel $M \preceq H_\chi$ of such generic extension which is a union of an elementary chain $\langle M_\xi \mid \xi < \kappa_\alpha^{++}\rangle$ of its elementary submodels of size κ_α^+ which are in $G(\mathcal{P}')$, and such that for every $\xi < \kappa_\alpha^{++}$,
$\langle M_{\xi'} \mid \xi' < \xi\rangle \in M_\xi$ and is on the central piste of M_ξ.
Now we proceed by induction. On stage ξ decide $\underset{\sim}{p}_\xi$ inside M_ξ and add M_ξ as a largest model. The rest of the proof follows completely the lines of the analogues arguments for short extenders forcings (see, for example, Sec 1 of [6]). Eventually, we will have $\xi < \rho < \kappa_\alpha^{++}$ and a condition in $\mathcal{P} \setminus \kappa_\alpha$ which forces compatibility of $\underset{\sim}{p}_\xi$ and $\underset{\sim}{p}_\rho$.
□

Lemma 7.11 *The forcing $\langle \mathcal{P}, \longrightarrow \rangle$ over $V[G']$ preserves all the cardinals (and every cofinality).*

Proof. Let η be a cardinal in $V[G']$. We show by induction on $\alpha < \omega_1$ that if $\eta \leq \kappa_\alpha$ then it is preserved in the generic extension. Clearly, it is enough to deal only with regular η's. Hence, we need to consider only the following situation:

$$\kappa_\alpha < \eta < \kappa_{\alpha+1},$$

for some $\alpha < \omega_1$. Split the forcing $\mathcal{P}$ into $\mathcal{P}\setminus\kappa_\alpha$ followed by $\mathcal{P} \restriction \kappa_{\alpha+1}$. By Lemma **??**, $\mathcal{P}\setminus\kappa_\alpha$ does not add new bounded subsets to $\kappa_{\alpha+1}$ (namely, this lemma together with the Prikry condition imply that no new subsets are added to $\kappa_{\alpha+1,0}$, but taking non-direct extensions over $\kappa_{\alpha+1,n}$'s it is easy to push this up to $\kappa_{\alpha+1}$). By Lemma 7.10 the forcing $\mathcal{P} \restriction \kappa_{\alpha+1}$ preserves all the cardinals above κ_α^+. So, the only case that remains is $\eta = \kappa_\alpha^+$. But it is not problematic, since we have here the successor of the singular cardinal and the usual arguments apply.
□

Lemma 7.12 *For every $\alpha < \omega_1$, α non-accumulation point (i.e. $\alpha = 0$ or α non-limit ordinal) the following hold in $V^{\mathcal{P}' * \langle \mathcal{P}, \longrightarrow\rangle}$:*

$$\mathrm{pcf}(\{(\rho_{\alpha n m \omega_1})^+ \mid n < \omega, m < g_\alpha(n)\}) \setminus \kappa_\alpha =$$

$$\{(\rho_{\beta r s \omega_1})^+ \mid \alpha < \beta < \omega_1 \textit{ is a successor ordinal}, r < \omega, s < g_\beta(r)\} \cup \{\kappa_{\omega_1}^+\},$$

moreover, for every limit $\gamma, \alpha < \gamma < \omega_1$,

$$\mathrm{pcf}(\{(\rho_{\alpha n m \gamma})^+ \mid n < \omega, m < g_\alpha(n)\}) \setminus \kappa_\alpha =$$

$$\{(\rho_{\beta r s \gamma})^+ \mid \alpha < \beta < \gamma \textit{ is a successor ordinal}, r < \omega, s < g_\beta(r)\} \cup \{\kappa_\gamma^+\},$$

where $\rho_{\delta t u \xi}$ *denotes the indiscernible for* $\kappa_{\delta t u \xi}$.

Proof. The proof is by induction on β using the assignment functions (a's) of the conditions and that $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.
□

8 Concluding remarks.

The construction of the previous section gives a countable set of regular cardinals $\mathfrak{a}$ with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) = \omega_1 + 1$. It is natural to try to get a bigger order type. The present methods allow to obtain $\omega_1 \cdot \alpha + 1$, for every $\alpha < \omega_1$. Just repeat the construction α– many times (one above another). However it is unclear how to get to $\omega_1 \cdot \omega_1 + 1$ and beyond.

Question 1. Is it possible to increase $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a}))$ beyond $\omega_1 \cdot \omega_1$, for a countable set of regular cardinals $\mathfrak{a}$?

We think that it may be possible under same lines, but using more elaborated techniques, to get any successor order type $< \omega_2$.

Shelah Weak Hypothesis states that the set

$$\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$$

is at most countable. The construction of the previous section provides a counterexample, but very restricted one. The cardinality and even the order type there is ω_1. So the following question is natural:

Question 2. Is it possible to increase $\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$ beyond ω_1, for a cardinal κ?
Note that no upper bound on cardinality of $\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$ is known.
Going further beyond ω_1, in view of results of [7] and [9] will require some completely new ideas. The same once one likes to have a set $\{\eta \mid \eta < \kappa, \mathrm{cof}(\eta) > \omega, \mathrm{pp}(\eta) > \kappa\}$ infinite, for some κ.

Question 3. How to move everything down, in particular is it possible to get down to $\aleph_\omega$?

It is possible to add collapses to the present construction, but only very inessential ones. By [8], the supercompact Prikry forcing looks be needed in order to collapse successors of singular cardinals, but this complicates the matters largely. It is unclear how to combine this forcing with sort extenders forcings in a productive way.

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