PCF of a countable set can be uncountable*

Moti Gitik
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv 69978, Israel
gitik@tauex.tau.ac.il

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Abstract

A model with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) = \omega_1 + 1$ is constructed, for countable set $\mathfrak{a}$ of regular cardinals.

1 Preliminary Settings

Let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an an increasing continuous sequence of singular cardinals of cofinality ω so that for each successor ordinal $\alpha < \omega_1$, κ_α is a limit of an increasing sequence $\langle \kappa_{\alpha,n} \mid n < \omega \rangle$ of cardinals such that

(1) $\kappa_{\alpha,n}$ is strong up to a 2-Mahlo cardinal $< \kappa_{\alpha,n+1}$,

(2) $\kappa_{\alpha,0} > \kappa_{\alpha-1}$.

Fix a sequence $\langle g_\alpha \mid \alpha < \omega_1$ is a successor ordinal$\rangle$ of functions from ω to ω such that for every $\alpha, \beta, \alpha < \beta$ which are successor ordinals below ω_1 the following holds

(a) $\langle g_\alpha(n) \mid n < \omega \rangle$ is increasing

(b) there is $m(\alpha, \beta) < \omega$ such that for every $n \geq m(\alpha, \beta)$

$$g_\alpha(n) \geq \sum_{m=0}^{n} g_\beta(m) \ .$$

*It is a revised version of the paper "Short extenders forcings II".

The easiest way is probably to force such a sequence.

Conditions are of the form

$\langle n, \{h_\alpha | \alpha \in I\}\rangle$, where $n < \omega, I$ is a finite subset of ω_1 and $h_\alpha : n \to \omega$.

The order is defined as follows:

$\langle n, \{h_\alpha | \alpha \in I\}\rangle \leq \langle m, \{t_\beta | \beta \in J\}\rangle$ iff $n \leq m, I \subseteq J$, for every $\alpha \leq \beta, \alpha, \beta \in I$, we have $t_\alpha | n = h_\alpha$ and if $n \leq k < m$ then require that $t_\alpha(k) \geq \sum_{0 \leq s \leq k} t_\beta(s)$.

It is possible to construct such a sequence in ZFC. Pick first a sequence $\langle h_\alpha | \alpha < \omega_1\rangle$ of functions from ω to ω such that

(1) $\langle h_\alpha(n) \mid n < \omega\rangle$ is non-decreasing and converges to infinity;

(2) if $\alpha < \beta$ then $h_\alpha > h_\beta$ mod finite.

Replace now each h_α by h'_α such that $h'_\alpha(n) = h_\alpha(n) + n + 1$.

Define $g_\alpha(n)$ to be $2^{(2^{\dots (2^{h'_\alpha(n)})\dots)}}$ where the number of powers is $h'_\alpha(n)$.

Let us argue that it is as required. Let $\alpha < \beta$. Pick $m(\alpha, \beta)$ to be such that for every $n \geq m(\alpha, \beta)$ we have $h'_\alpha(n) > h'_\beta(n)$.

Let $n \geq m(\alpha, \beta)$. Consider $\sum_{0 \leq s \leq n} g_\beta(s)$.

Then

$$\sum_{0 \leq s \leq n} g_\beta(s) \leq (n+1) \cdot g_\beta(n) \leq (g_\beta(n))^2 \leq 2^{g_\beta(n)} \leq g_\alpha(n).$$

It would be more convenient further to have the following additional property:

For every odd ordinal $\alpha < \omega_1$, for every $n < \omega$,

$$g_\alpha(n) = \sum_{m=0}^{n} g_{\alpha+1}(m) \ .$$

It is straightforward to ensure this. Just deal with $\alpha, \alpha + 1$ simultaneously.

2 A basic description of the pcf- structure with ω_1–many cardinals

We would like to blow up the powers and pseudo-powers (pp) of all $\kappa_\alpha, \alpha < \omega_1$ to $\kappa^+_{\omega_1}$.[1]

The first tusk will be to arrange an appropriate pcf–structure that will be realized further. There are many possibilities for doing this. We will use one that seems to us to be among simplest. It requires some work since we allow only finitely many blocks at each level. Note

[1] $\mathrm{pp}(\lambda) = \sup\{\mathrm{cof}(\prod a/D \mid a \subseteq \lambda$ is a set of at most $\mathrm{cof}(\lambda)$ of regular cardinals, unbounded in λ and D an ultrafilter over a including all cobounded subsets of $a\}$.

that in view of [9] one cannot allow infinitely many blocks at least not under the large cardinals assumptions used here (below a strong or a little bit more).

Organize the things as follows.
Let $n < \omega$ and $1 \leq \alpha < \omega_1$ be a successor ordinal. We reserve at level n a splitting into $g_\alpha(n)$–blocks one above another:

$$\langle \kappa_{\alpha,n,m,i} \mid m < g_\alpha(n), i \leq \omega_1 \rangle,$$

so that

1. $\kappa_{\alpha,n} < \kappa_{\alpha,n,0,0}$,
2. $\kappa_{\alpha,n,m,i'} < \kappa_{\alpha,n,m,i}$, for every $m < g_\alpha(n), i' < i \leq \omega_1$,
3. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n,m+1,0}$, for every $m, m+1 < g_\alpha(n)$,
4. for every successor ordinal $i < \omega_1$ or if $i = 0$ let $\kappa_{\alpha,n,m,i}$ be large enough (say a Mahlo or even measurable),
5. for every limit $i, 0 < i \leq \omega_1$ let $\kappa_{\alpha,n,m,i} = \sup(\{\kappa_{\alpha,n,m,i'} \mid i' < i\})$,
6. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n+1}$, for every $m < g_\alpha(n)$.

For each successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$, we will fix a $(\kappa_{\alpha,n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1})$–extender $E_{\alpha n}$, i.e. an extender with the critical point $\kappa_{\alpha,n}$ which ultrapower contains $V_{\kappa_{\alpha,n,g_\alpha(n)-1,\omega_1}+2}$.

Let $\alpha < \omega_1$ be a successor ordinal.
We will refer further to $\kappa_{\alpha n}$'s (or, simplicity just to n's) as levels of κ_α (or, again, for simplicity just of α). In addition, if $n < \omega$ and $m < g_\alpha(n)$, then we refer to $\kappa_{\alpha,n,m,i}$'s ($i \leq \omega_1$) as members of the $m-$th block (of the level n of α).

Let us incorporate indiscernibles that will be generated by extender based forcings with $E_{\alpha n}$'s into the blocks as follows.[2] Denote as above the indiscernible for $\kappa_{\alpha,n,m,i}$ by $\rho_{\alpha,n,m,i}$.
$[\kappa^+_{\alpha-1}, \rho^+_{\alpha,0,0,\omega_1}]$ will be the first block of α of the level 0 (if $\alpha = 0$, then let it be $[\omega_1, \rho^+_{0,0,0,\omega_1}]$). Then for every $m < g_\alpha(0)$ let m–th block of α of the level 0 be $[\rho^{++}_{\alpha,0,m-1,\omega_1}, \rho^+_{\alpha,0,m,\omega_1}]$. The first block of the level 1 of α will be $[\rho^{++}_{\alpha,0,g_\alpha(0)-1,\omega_1}, \rho^+_{\alpha,1,0,\omega_1}]$. In general the first block of the level $n > 0$ of α will be $[\rho^{++}_{\alpha,n-1,g_\alpha(n-1)-1,\omega_1}, \rho^+_{\alpha,n,0,\omega_1}]$. The m-th block ($m > 0$) of the level $n > 0$ of α will be $[\rho^{++}_{\alpha,n,m-1,\omega_1}, \rho^+_{\alpha,n,m,\omega_1}]$.

[2] By indiscernibles here we mean members of a generic (one element) Prikry sequences produced by one-element Extender Based Prikry forcings with $E_{\alpha,n}$'s.

Special attention will be devoted to the very last blocks of each level,
i.e. to $[\rho^{++}_{\alpha,n,g_\alpha(n)-2,\omega_1}, \rho^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}]$.

In the final (after the main forcing) model we will have the following structure:

1. every element of the set $\{\kappa^+_\beta \mid \alpha < \beta < \omega_1\}$ will be represented at each $\alpha' \leq \alpha$;

2. the set of indiscernibles

$$\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\}$$

will be a countable set with uncountable pcf over α;

3. for every successor ordinal β, $\alpha < \beta < \omega_1$, each indiscernible $\rho^+_{\beta,n,m,\omega_1}$ $(n < \omega, m < g_\beta(n))$ will be in the pcf of this set. Thus, we will have the following:

$$\mathrm{pcf}(\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\}) \supseteq$$

$$\{\rho^+_{\beta,n,m,\omega_1} \mid \alpha < \beta < \omega_1, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_{\omega_1}\};$$

4. for each limit ordinal $\gamma, \alpha < \gamma \leq \omega_1$, the following will hold:

$$\mathrm{pcf}(\{\rho^+_{\alpha,n,m,\gamma} \mid n < \omega, m < g_\alpha(n)\}) \supseteq$$

$$\{\rho^+_{\beta,n,m,\gamma} \mid \alpha < \beta < \gamma, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_\gamma\}.$$

Note that for $\gamma < \omega_1$ the set on the right side is countable.

3 The connection

Let us establish the connection between the levels and blocks.
Let $\alpha < \beta < \omega_1$. Define the connection of levels and blocks of κ_β to the levels and blocks of κ_α.
Consider $m(\alpha, \beta)$, i.e. the least $m < \omega$ such that for every $n \geq m$ we have

$$g_\alpha(n) \geq \sum_{k=0}^{n} g_\beta(k) \ .$$

This is a place from which blocks of the $\beta-$level fit nicely inside those of the $\alpha-$level.
Let us arrange the connection as follows. Connect all the blocks of the levels $n, n \leq m(\alpha, \beta)$ of κ_β to the blocks of the level $m(\alpha, \beta)$ of κ_α (or in short - of α) moving to the right as much

as possible, i.e. if $r = g_\alpha(m(\alpha,\beta)) - \sum_{k=0}^{m(\alpha,\beta)} g_\beta(k)$, then the first block of κ_β (in short - β) is connected to the r–th block of the level $m(\alpha,\beta)$ of κ_α, the second block of κ_β is connected to $r+1$-th block of the level $m(\alpha,\beta)$ of κ_α etc., the last block of the level $m(\alpha,\beta)$ of κ_β will be connected to the last block of the level $m(\alpha,\beta)$ of κ_α.
For every $s, m(\alpha,\beta) \leq s < \omega$, we continue to connect blocks of all the levels $s' \leq s$ of β to the s block of α in the same fashion, moving to the right as much as possible.

Let us refine the above and introduce $\ell(\alpha,\beta)$ as a replacement of $m(\alpha,\beta)$ used above.

Let $\alpha < \omega_1$ be a limit ordinal define the connection inside the interval $(\kappa_\alpha, \kappa_{\alpha+\omega})$.

Set $\ell(\alpha+2,\alpha+3) = m(\alpha+2,\alpha+3)$. Next, let $\ell(\alpha+2k,\alpha+2k+1) = \max(k, m(\alpha+2k,\alpha+2k+1))$, $k < \omega$. The connection of $\alpha+2k+1$ to smaller levels goes via the connection of $\alpha+2k$ to smaller levels.

Proceed by induction. Let $0 < \beta < \omega_1$ be a limit ordinal. Define the connection of $\beta+1$. We chose in advance a cofinal in β sequence $\langle \beta_i \mid i < \omega \rangle$ consisting of even non-limit ordinals and which is a Cohen generic. Define an increasing cofinal subsequence $\langle \beta_{i_k} \mid k < \omega \rangle$ of it.
Connect the first level of β to β_{i_0}, where $i_0 \geq 0$ is the least such that the number of blocks of the level 0 of β_{i_0} fits exactly with the number of blocks of the first level of β. Such i_0 exists by the genericity of the sequence $\langle \beta_i \mid i < \omega \rangle$.
Define β_{i_1}. If there is no limit $\gamma < \beta$ with $\beta_{i_0} = \gamma_{i_0}$, then we connect the first + second levels of β to β_{i_1}, where $i_1 > i_0$ is the least such that the number of blocks of the second level of β_{i_1} fits exactly with the number of blocks of the first two levels of β. Such i' exists by the genericity.[3]
If there are limit γ's below β with $\beta_{i_0} = \gamma_{i_0}$, then we pick $i_1 > i_0$ to be the least such that

1. the number of blocks of the second level of β_{i_1} fits exactly with the number of blocks of the first two levels of β,
   and

2. β_{i_1} connects with $\beta_{i_0}+1$ at the same place as γ_{i_1}.

Continue further in the same fashion and define i_k for every $k < \omega$.

Now for $\beta+k, 1 < k < \omega$, connect $\beta+k$ down via $\beta+1$.

We will use the defined connection of such β's to $\alpha+2k+1$ in order to keep information about its connection to $\alpha+2k$, in the definition of the forcing once a non-direct extension was made. This way higher levels may be involved. For example, if $\beta_0 = \alpha+2$ and $\beta_1 = \alpha+4$.

[3]This allows us to eliminate a manual connection used in the previous version, which simplifies the presentation.

The established connection connects any two $\alpha < \beta < \omega_1$. Moreover every block n of a level k of α is connected for $\aleph_1-$many β to some block n' of a level k' of β.
Let us denote by *connect*(α, β) the set of such $((n, k), (n', k'))$.

Let us describe now an additional connection which will be used in the definition of the main forcing for dropping in cofinalities.
Suppose that α is an even successor ordinal and suppose that for some limit $\beta > \alpha$, β is connected to α, i.e. $\beta_{i_n} = \alpha$.
Consider $\beta_{i_{n+1}}$ and the place of its connection to $\alpha + 1$. Suppose that it is connected to the level $m, n + 1 \leq m < \omega$ and the first n levels of $\beta + 1$ correspond here to an interval of blocks of the level m of $\alpha + 1$ which starts with a block k. Consider the block $k - 1$ of the level m of $\alpha + 1$ and all the blocks of levels of $\alpha + 1$ below $m-$th level, if exist. There is no room for them at the level n of α, and so, we drop below α to $\alpha - 1$. Let us use the already established connection between $\alpha - 1$ and $\alpha + 1$. So, (ℓ, m) corresponds to $(\ell - k, n)$ of α, for every $\ell, k \leq \ell < g_\alpha(n) + k$, and (ℓ, s) corresponds to (t, h) of $\alpha - 1$, for every $\ell, \ell < k, s = m$ and $s < m, \ell < g_{\alpha+1}(s)$, where (t, h) is such that $((\ell, s), (t, h)) \in$ *connect*$(\alpha - 1, \alpha + 1)$.
Let *dconnect*(α) be the set which consists of all such pairs.
Note that the choice of particular β for α is not important, since the defined connections for all of them behave the same.

4 The preparation forcing.

We would like to use a generic set for the forcing $\mathcal{P}'$ of Chapter 3 (Preserving Strong Cardinals) of [6] in order to supply models for the main forcing defined further. Some degree of strongness of $\kappa_{\alpha,n}$ will be needed as well, for every successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$.

Two ways were described in Chapter 3 of [6]. Either can be applied for our purpose.

The first one is as follows.

Assume that for some regular cardinal θ the following set is stationary:

$$S = \{\nu < \theta \mid \nu \text{ is a superstrong with the target } \theta (\text{i.e. there is } i : V \to M, crit(i) = \nu$$

$$i(\nu) = \theta \text{ and } M \supseteq V_\theta)\}.$$

Return to the definition of κ_γ's and $\kappa_{\gamma,k}$'s. Let us choose them by induction such that all $\kappa_{\gamma,k}$'s are from S. Suppose that $\langle \kappa_{\gamma,k} \mid k < \omega \rangle$ is defined. Then $\kappa_\gamma = \bigcup_{k<\omega} \kappa_{\gamma,k}$. Let $\tilde{\kappa}_\gamma$ be the next element of S. Pick $\kappa_{\gamma+1,0}$ to be an element of S above $\tilde{\kappa}_\gamma$.

Force with $\mathcal{P}'(\theta)$ with a smallest size of models say $\aleph_8$. Then, by Lemma 3.0.23 of Chapter 3 (Preserving Strong Cardinals) of [6], each $\kappa_{\alpha,n}$ will remain $\tilde{\kappa}_\alpha$–strong (and even $\kappa^+_{\omega_1}$–strong). Moreover, $\mathcal{P}'(\tilde{\kappa}_\alpha)$ is a nice subforcing of $\mathcal{P}'(\theta)$ by Lemma 3.0.18 of Chapter 3 (Preserving Strong Cardinals) of [6], since $V_{\tilde{\kappa}_\alpha} \preceq V_\theta$ due to the choice of $\tilde{\kappa}_\alpha$ in S.

An other way, which uses initial assumptions below $\mathbf{0}^\P$, is as follows.

Let θ be a 2-Mahlo cardinal and $\kappa < \theta$ be a strong up to θ cardinal. Pick $\delta, \kappa < \delta < \theta$ a Mahlo cardinal such that $V_\delta \prec_{\Sigma_1} V_\theta$. By Lemma 3.0.15 of Chapter 3 (Preserving Strong Cardinals) of [6] or just directly, there will unboundedly many cardinals $\eta < \kappa$ with $\delta_\eta < \kappa$ such that the function $\eta \mapsto \delta_\eta$ represents δ and $V_{\delta_\eta} \prec_{\Sigma_1} V_\theta$. Then, by Lemma 3.0.18 of Chapter 3 of [6], $\mathcal{P}'(\delta_\eta)$ is a nice subforcing of $\mathcal{P}'(\theta)$.

Denote by S the set of all such η's.

Force now with $\mathcal{P}'(\theta)$. Let G' be a generic. By Lemma 3.0.24 of Chapter 3 of [6], embeddings wich witness δ-strongness of κ for large enough δ's below θ extend in $V[G']$. Then, below κ in $V[G']$, we will have unboundedly many η's which are strong up to δ_η for which $V_{\delta_\eta}[G' \cap V_{\delta_\eta}] \prec_{\Sigma_1} V_\theta[G']$, since every $\eta \in S$ is like this.

We define now by induction $\kappa_{\gamma,k}$'s, κ_γ's and $\tilde{\kappa}_\gamma$'s using such η's and δ_η's.

Let η_0 be the first element of S. Define $\tilde{\kappa}_0$ be δ_{η_0}. Set κ_{00} to be the least element of S above $\tilde{\kappa}_0$. Let κ_{01} to be the least element of S above $\delta_{\kappa_{00}}$. Continue by induction. Suppose that $n < \omega$ and $\kappa_{0n} \in S$ is defined. Let then κ_{0n+1} to be the least element of S above $\delta_{\kappa_{0n}}$.

Set $\kappa_0 = \bigcup_{k<\omega} \kappa_{0k}$.
Continue to $\gamma > 0$ in a similar fashion.
Thus, if $\gamma, 0 < \gamma < \omega_1$ is a limit ordinal and $\langle \kappa_{\gamma'} \mid \gamma' < \gamma \rangle$ is defined, then set $\kappa_\gamma = \bigcup_{\gamma'<\gamma} \kappa_{\gamma'}$.
Suppose now that κ_γ is defined. Define $\langle \kappa_{\gamma+1,k} \mid k < \omega \rangle$ and $\kappa_{\gamma+1}$.
Let $\tilde{\kappa}_\gamma = \delta_\eta$ for the least $\eta > \kappa_\gamma, \eta \in S$. Pick $\kappa_{\gamma+1,0}$ to be the first $\eta \in S$ above $\tilde{\kappa}_\gamma$ and $\kappa_{\gamma+1,1}$ to be the first $\eta \in S$ above $\delta_{\kappa_{\gamma+1,0}}$, etc. Finally, set $\kappa_{\gamma+1} = \bigcup_{k<\omega} \kappa_{\gamma+1,k}$.

5 Suitable and suitable generic structures.

Suitable structures and suitable generic structures are defined similar to those in Sections 1.2 or 2.4 of [6].

Let us briefly address main components of the preparation forcing $\mathcal{P}'$ ($\mathcal{P}'(\theta)$) used here. A typical member of $\mathcal{P}'$ is of the form $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$.

- s is a closed set of cardinals from the interval $[\aleph_8, \theta]$ having the Easton support. For every $\tau \in s$ the following holds:
- $A^{1\tau}$ is a set of cardinality at most τ consisting of elementary submodels of $H(\theta)$ of size τ, and $A^{0\tau}$ is its largest element under both $\in, \subseteq$.
- C^τ (pistes) is function with domain $A^{1\tau}$ which attach to every $X \in A^{1\tau}$ an increasing continuous sequence of models in $(X \cap A^{1\tau}) \cup \{X\}$ with X being the maximal element.
- The basic property here is that every $B \in A^{1\tau}$ can be reached in finitely many steps from the top model $A^{0\tau}$ going down by pistes of C^τ.
- For every $\tau \in s$, $C^\tau(A^{0\tau})$ is called $\tau-$*central line* and $\langle C^\tau(A^{0\tau} \mid \tau \in s \rangle$ is called *central line (or the main piste)* of the condition $\langle \langle A^{0\tau}, A^{1\tau}, C^\tau \rangle \mid \tau \in s \rangle$.

It is allowed to change directions of pistes (this is called switching) of elements of $\mathcal{P}'$ in the obvious sense, i.e. at splitting points we can choose a direction which is different from one given by the central line. Such process will create a new central line. This way equivalent conditions (in the forcing sense) are obtained.
The order (pre-order) on $\mathcal{P}'$ is defined by combining switchings with end-extensions.

The notions of *a suitable and suitable generic structures* (from SEF I) are used in the main forcing.
The idea is to code elements of $\mathcal{P}'$ as a single structure (i.e. not three sorted as appears in

the definition of $\mathcal{P}'$) and then to deal with isomorphisms of such structures over different cardinals. Suitable structures are such codes. Let us recall the definition.

Definition 5.1 A structure $\mathfrak{X} = \langle X \cup \{X\}, E, C \ \in, \subseteq \ \rangle$, where $E \subseteq [X \cup \{X\}]^2$ and $C \subseteq [X \cup \{X\}]^3$ is called *suitable structure* iff there is
$p(\mathfrak{X}) = \langle \langle A^{0\tau}(\mathfrak{X}), A^{1\tau}(\mathfrak{X}), C^{\tau}(\mathfrak{X}) \rangle \mid \tau \in s(\mathfrak{X}) \rangle \in \mathcal{P}'$ such that

1. $X = A^{0\kappa^+}(\mathfrak{X})$,
2. $s(\mathfrak{X}) \in X$,
3. $s(\mathfrak{X}) \subseteq X$,
4. $\langle a, b \rangle \in E$ iff $a \in s(\mathfrak{X})$ and $b \in A^{1a}(\mathfrak{X})$,
5. $\langle a, b, d \rangle \in C$ iff $a \in s(\mathfrak{X}), b \in A^{1a}(\mathfrak{X})$ and $d \in C^a(\mathfrak{X})(b)$.

We will use further *suitable structures over β of level n*, where $\beta = 0$ or is a successor ordinal $< \omega_1$ and $n < \omega$. The definition is the same only $\mathcal{P}'$ is replaced by $\mathcal{P}' \cap V_{\delta_{\kappa_{\beta,n}}}$.

Let $G(\mathcal{P}')$ be a generic subset of $\mathcal{P}'$.
A suitable generic structure is basically a substructure (not necessarily elementary) of the suitable structure generated by an element of $G(\mathcal{P}')$. It can and, typically, would have a smaller cardinality, which is archived by omitting some models from the pistes.

Let us state the main properties.

A suitable structure $\mathfrak{X} = \langle X, E, C \in, \subseteq \rangle$ is called *suitable generic structure* iff there is $\langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \rangle \in G(\mathcal{P}')$ such that

- $\langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \setminus \{\mu\} \rangle \in A^{0\mu}$, where $\mu \in s$ is a regular cardinal and it is the least size of models of our particular interest. Typically, μ is a successor of a singular cardinal which power blows up and the forcing used for this purpose satisfies μ^+−c.c. For example, in our particular setting, once we would like to show that κ_α^{++} is preserved, for some $\alpha < \omega_1$, $\mu = \kappa_\alpha^+$ will be taken.
- $\mathfrak{X}$ is a substructure (not necessarily elementary) of the suitable structure generated by $\langle \langle A^{0\tau}, A^{1\tau}, C^{\tau} \rangle \mid \tau \in s \rangle$, i.e.
  $\langle A^{0\mu} \cup \{A^{0\mu}\}, \{\langle \tau, B \rangle \mid \tau \in s, B \in A^{1\tau}\}, \{\langle \tau, B, D \rangle \mid \tau \in s, B \in A^{1\tau}, D \in C^{\tau}(B)\},$
- $X \in C^{\mu}(A^{0\mu})$,

- $p(\mathfrak{X})$ (the decoding of $\mathfrak{X}$) and $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$ agree about the pistes to members of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$. In other words we require that all the elements of pistes in $\langle\langle A^{0\tau}, A^{1\tau}, C^{\tau}\rangle \mid \tau \in s\rangle$ to elements of $X \cap \bigcup\{A^{1\tau} \mid \tau \in s\}$ are in X.
  The idea here is to reduce the cardinality of the structure still keeping all the essential information.

6 Types of Models

Force with $\mathcal{P}'$. Let $G' \subseteq \mathcal{P}'$ be a generic subset. Work in $V[G']$. For each successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$ let us fix a $(\kappa_{\alpha,n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1})$– extender $E_{\alpha n}$, i.e. an extender with the critical point $\kappa_{\alpha,n}$ which ultrapower contains $V_{\kappa_{\alpha,n,g_\alpha(n)-1,\omega_1}+2}$.
Also, using GCH (we assume GCH in V and then it will holds in $V[G']$ as well), fix an enumeration $\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle$ of $[\kappa_{\alpha n}]^{<\kappa_{\alpha n}}$ so that for every successor cardinal $\delta < \kappa_{\alpha n}$ the initial segment $\langle x_\gamma \mid \gamma < \delta\rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_{\alpha n}(\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle) = \langle x_\gamma \mid \gamma < j_{\alpha n}(\kappa_{\alpha n})\rangle$, where $j_{\alpha n}$ is a canonical embedding of $E_{\alpha n}$. Then $\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle$ will enumerate $[\kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}]^{\leq\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}}$.

For every $k < \omega$, we consider a structure

$$\mathfrak{A}_{\alpha,n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, \chi, E_{\alpha n}, \langle \kappa_\beta \mid \beta < \omega_1\rangle,$$

$$\langle \kappa_{\beta s} \mid \beta < \omega_1 \text{ is a successor ordinal or zero }, s < \omega\rangle,$$

$$\langle \kappa_{\beta,s,r,i} \mid \beta < \omega_1 \text{ is a successor ordinal or zero }, s < \omega, r < g_\beta(s), i \leq \omega_1\rangle,$$

$$\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle, G', \theta, 0, 1, \ldots, \xi, \ldots \mid \xi < \kappa^{+k}_{\alpha n}\rangle$$

in an appropriate language which we denote $\mathcal{L}_{\alpha,n,k}$, with a large enough regular cardinal χ. Note that we have G' inside, so suitable structures may be chosen inside G' or $G' \cap \mathcal{P}'(\kappa_{\alpha,n})$.

Let $\mathcal{L}'_{\alpha,n,k}$ be the expansion of $\mathcal{L}_{\alpha,n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$ let $\mathfrak{A}_{\alpha,n,k,a}$ be the expansion of $\mathfrak{A}_{\alpha,n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. Denote by $tp_{\alpha,n,k}(b)$ the $\mathcal{L}_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k}$. Further we identify it with the ordinal coding it and refer to it as the k-type of b. Let $tp_{\alpha,n,k}(a, b)$ be a the $\mathcal{L}'_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [2].

Now, repeating the usual arguments we obtain the following:

Lemma 6.1 *(a)* $|\{tp_{\alpha,n,k}(b) \mid b \in H(\chi^{+k})\}| = \kappa_{\alpha n}^{+k+1}$

(b) $|\{tp_{\alpha,n,\kappa}(a,b) \mid a,b \in H(\chi^{+k})\}| = \kappa_{\alpha n}^{+k+1}$

Lemma 6.2 *Let* $A \prec \mathfrak{A}_{\alpha,n,k+1}$ *and* $|A| \geq \kappa_{\alpha n}^{+k+1}$. *Then the following holds:*

(a) for every $a, b \in H(\chi^{+k})$ *there* $c, d \in A \cap H(\chi^{+k})$ *with* $tp_{\alpha,n,k}(a,b) = tp_{\alpha,n,k}(c,d)$

(b) for every $a \in A$ *and* $b \in H(\chi^{+k})$ *there is* $d \in A \cap H(\chi^{+k})$ *so that* $tp_{\alpha,n,k}(a \cap H(\chi^{+k}),\ b) = tp_{\alpha,n,k}(a \cap H(\chi^{+k}),\ d)$.

Lemma 6.3 *Suppose that* $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_{\alpha n}^{+k+1}$. *Let* τ *be an ordinal less than* $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$ *those* $k+1$*-type is realized unboundedly often below* $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. *Then there are* τ' *and* $A' \prec A \cap H(\chi^{+k})$ *such that* $\tau', A' \in A$ *and* $\langle \tau', A' \rangle$ *and* $\langle \tau, A \cap H(\chi^{+k}) \rangle$ *realize the same* $tp_{\alpha,n,k}$. *Moreover, if* $|A| \in A$, *then we can find such* A' *of cardinality* $|A|$.

Lemma 6.4 *Suppose that* $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_{\alpha n}^{+k+1}$, $B \prec \mathfrak{A}_{\alpha,n,k}$, *and* $C \in \mathcal{P}(B) \cap A \cap H(\chi^{+k})$. *Then there is* D *so that*

(a) $D \in A$

(b) $C \subseteq D$

(c) $D \prec H(\chi^{+k})$.

(d) $tp_{\alpha,n,k}(C,B) = tp_{\alpha,n,k}(C,D)$.

7 The Main Forcing.

Our aim here will be to define a forcing that turns the connections defined in Section 3 into pcf-structure. In order to do this, we realize connections as isomorphism functions between suitable structures.
It is rather natural to define (see 7.1.1 below) such isomorphisms for automatically connected blocks.
A slight complication here is that rather than connecting (i.e. making correspond by isomorphisms) blocks of cardinals provided by a connection, we connect the one element Prikry sequences (i.e. indiscernibles) of higher blocks to blocks below. The reason for doing this is

that the cardinals outside of blocks of indiscernibles naturally belong to pcf-generators for some κ_α^+'s, since GCH is assumed in the ground model. This will not allow us to proceed all the way up to $\kappa_{\omega_1}^+$.

Turn now to the manual connection.

Let us explain the reason for using it at all.

The point is that without it there will be plenty blocks that left unconnected to higher ones. Just note that the functions $g_\alpha, \alpha < \omega_1, \alpha = 0$ or α is a successor ordinal, which were used for the connections, satisfy

$$g_\alpha(n) \geq \sum_{m=0}^{n} g_\beta(m) \ ,$$

for $\beta > \alpha$ and actually,

$$g_\alpha(n) > \sum_{m=0}^{n} g_\beta(m) \ ,$$

must hold at many places. This strong inequality generates unconnected (automatically) blocks from the level α to the level β.

Now this unconnected blocks (or more precisely the cardinals inside them) will be then in pcf-generators of some κ_ξ^+, for limit ξ's below ω_1. Then the Localization Property ([12]) will not allow to climb all the way up to $\kappa_{\omega_1}^+$.

A complication with manual connections is that in contrast with the automatic ones, the number of blocks does not fit together nicely.

For example - for some $\alpha < \beta < \omega_1$, α limit non-zero ordinal and $\alpha + 2 < \beta < \alpha + \omega$, the very first block of the first level $[\kappa_{\alpha+2}^+, \kappa_{\alpha+2,0,0,\omega_1}^+]$ of $\alpha + 2$ may correspond (by a manual connection) to say 10-th block of the second level of β, i.e. to $[\kappa_{\beta,1,9,\omega_1}^{++}, \kappa_{\beta,1,10,\omega_1}^+]$.

Now, by No Hole Principle ([12]), the blocks of β starting from 9 and below (or their regular cardinals) should be connected to those below the first of $\alpha + 2$. However, $\alpha + 2$ has no blocks below its first one.

The solution is to drop down to $\alpha + 1$. In order to so, we need a variation of *a drop in cofinality* which was used in Section 4 of [6].

The situation (a bit simplified one) is as follows. Suppose that we would like to have a nice scale of functions $\langle f_\xi \mid \xi < \mu^{+3} \rangle$ in $\prod_{n<\omega} \mu^{+n+2}$, for some regular cardinals μ_n's unbounded in μ such that $\{\mu_n^{+r_n} \mid n < \omega, r_n < n+2\}$ correspond to μ^+. We must have that $\{\mu^{+n+2} \mid n < \omega\}$ corresponds to μ^{+3}. But what about μ^{++}? Usually, the set which correspond to it is obtained from the one for μ^{+3} by reducing each member by one, i.e. $\{\mu^{+n+1} \mid n < \omega\}$. But here we have that $\{\mu^{+n+1} \mid n < \omega\}$ corresponds to μ^+. So, going down is needed in order to realize such configuration. Namely, for ξ's less than μ^{+3} of cofinality μ^{++}, the cofinality of $f_\xi(n)$'s should drop down below μ_n.

We turn now to the definition of the main forcing $\mathcal{P}$. Let us split the definition into ω–many steps. First we define pure conditions $\mathcal{P}_0$, at the next step $\mathcal{P}_1$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_0$, then $\mathcal{P}_2$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_1$, etc. Finally $\mathcal{P}$ will be $\bigcup_{n<\omega} \mathcal{P}_n$.

Definition 7.1 The set $\mathcal{P}_0$ consists of all sequences

$$\langle p_\alpha \mid \alpha < \omega_1 \text{ and } \alpha \text{ is a successor ordinal })\rangle$$

such that $p_\alpha = \langle p_{\alpha\beta} \mid \alpha < \beta < \omega_1 \text{ is a successor ordinal } \rangle$, and for all $n < \omega, \alpha < \beta < \omega_1$ is a successor ordinal ,
$p_{\alpha\beta} = \langle p_{\alpha\beta x} \mid x \in \mathit{connect}(\alpha, \beta)\rangle$, where for every $x \in \mathit{connect}(\alpha, \beta)$,
$p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha\beta x}, f_{\alpha\beta x}\rangle$ is such that:

1. (Non-dropping connection)
   If $x \in \mathit{connect}(\alpha, \beta)$, $x = ((n_1, k_1), (n_2, k_2))$, for some $k_1, k_2, n_1, n_2 < \omega$, [4] then
   (a) $A_{\alpha\beta x} = A_{\alpha n_1}$, i.e. it does not depend on β, x, but rather on on level n_1 of α (and α itself).
   It is a set of measure one for some measure of the extender $E_{\kappa_{\alpha n_1}}$. Denote the corresponding coordinate in $E_{\kappa_{\alpha n_1}}$ by $\mathit{coor}(A_{\alpha n_1})$.
   (b) $a_{\alpha\beta x} = a_{\alpha\beta n_1}$, i.e. it depends on α, β and n_1 only.
   (c) $a_{\alpha\beta n_1}$ is an isomorphism between a $(\prod_{k \leq n_1} A_{\beta k})$–name of a generic suitable structure $\underset{\sim}{\mathfrak{X}}^{\beta}_{\alpha\beta n_1}$ of size $< \kappa_{\alpha n_1}$ over β of the level n_1 and a suitable structure $\mathfrak{X}^{\alpha}_{\alpha\beta n_1}$ of α of the level n_1.
   (d) For each $k \leq n_1$ and $\eta \in A_{\beta k}$ let us denote by $\rho_{\beta k}$ the projection of η to the normal measure of the extender $E_{\beta k}$.
   For each $m < g_\beta(k)$ and $\gamma \leq \omega_1$ let $\rho_{\beta k m \gamma}$ be $\pi_{\mathit{coor}(A_{\beta k})\kappa_{\beta k m \gamma}}(\eta)$, i.e. the indiscernible which corresponds to $\kappa_{\beta k m \gamma}$, where $\mathit{coor}(A_{\beta k})$ is the coordinate of $E_{\kappa_\beta k}$ to which $A_{\beta k}$ belongs.[5]
   We require that for each

[4]Note that then $m(\alpha, \beta) \leq n_1, n_2 \leq n_1$ and the level n_1 of β is also automatically connected to the level n_1 of α.

[5]It is not hard to arrange that $\rho_{\beta k}$ already determines all $\rho_{\beta k m \gamma}$'s, for every $k \leq n_1, m < g_\beta(k)$ and $\beta < \gamma \leq \omega_1$.

$\langle \eta_0, ..., \eta_{n_1} \rangle \in \prod_{k \leq n_1} A_{\beta k}$, for every $k \leq n_1, m < g_\beta(k)$ and $\beta < \gamma \leq \omega_1$, $a_{\alpha\beta n_1}[\langle \eta_0, ..., \eta_{n_2} \rangle]$ (i.e. the interpretation of $a_{\alpha\beta n_2}$ according to $\langle \eta_0, ..., \eta_{n_2} \rangle$) is the isomorphism between $\underset{\sim}{\mathfrak{X}}^{\beta}_{\alpha\beta n_1}[\langle \eta_0, ..., \eta_{n_2} \rangle]$ and $\mathfrak{X}^{\alpha}_{\alpha\beta n_1}$ which maps models of sizes $\rho_{\beta k m \gamma}$ and $(\rho_{\beta k m \gamma})^+$ to models over the level n_1 of α of cardinalities $\kappa_{\alpha n_1 m^* \gamma}$ and $(\kappa_{\alpha n_1 m^* \gamma})^+$ respectively, where $m^* = (g_\alpha(n_1) - \sum_{s=k}^{n_1} g_\beta(s)) + m$ (i.e. we start as far right as possible).

This means, in particular, that once a non-direct extension was made at the level n_1 of α, then $\rho_{\beta k m \gamma}$ and $(\rho_{\beta k m \gamma})^+$ will correspond to $\rho_{\alpha n_1 m^* \gamma}$ and $(\rho_{\alpha n_1 m^* \gamma})^+$ respectively.

Models of sizes from the interval $((\rho_{\beta k m-1 \omega_1})^+, \rho_{\beta k m \beta+1})$ will be connected with models of sizes in the interval $(\kappa^+_{\alpha n_1 m^* 0}, \kappa_{\alpha n_1 m^* \beta+1})$, if $m > 0$.

If $m = 0$ and $k > 0$, then models of sizes from $((\rho_{\beta k-1 g_\beta(k-1)-1 \omega_1})^+, \kappa_{\beta k-1}) \cup [\kappa_{\beta k-1}, \rho_{\beta k 0 \beta+1})$ will be connected with $(\kappa^+_{\alpha n_1 m^* 0}, \kappa_{\alpha n_1 m^* \beta+1})$.

If $m = 0$ and $k = 0$, then $(\kappa_{\beta-1}, \rho_{\beta 0 0 \beta+1})$ will be connected with $(\kappa_{\alpha n_1 m^* \beta}, \kappa_{\alpha n_1 m^* \beta+1})$.

(e) $f_{\alpha\beta x} = f_{\alpha\beta n_1}$ is a $(\prod_{k \leq n_1} A_{\beta k})-$name of a partial function from κ_{β, n_1} to κ_{α, n_1} of cardinality at most $\kappa_{\beta-1}$.

2. (Dropping connection)
Let $x \in dconnect(\alpha)$, for some odd ordinal $\alpha < \omega_1$. Let $x = ((n, k), (n', k'))$, for some $n < n' < \omega, k, k' < \omega$.
The cardinals corresponding is similar to the case of the non-dropping connection. Pick some limit $\beta > \alpha$ witnessing $x \in dconnect(\alpha)$. Then the first $n-$levels of β fit nicely into $n-$th level of α. Consider the places which correspond to the first $n-$levels of β at the level n' of $\alpha + 1$. They are connected naturally to $n-$th level of α. The blocks below are connected via dropping to $\alpha - 1$. The assignment function a_α is defined accordingly incorporating this drop, i.e. involving intervals correspondings and simple names (in order to ensure the desired degree of completeness.

Let α be a successor even ordinal $< \omega_1$. We describe here the process of dropping down to $\alpha - 1$.

Suppose for simplicity that $n = 0$, i.e. we deal with the first level of α. Then at $\alpha + 1$ we have a level $n' > 0$ and a block k' of this level such that all blocks below k' of the level n' (if there are any) and all the blocks of smaller levels should drop to some level m of $\alpha - 1$.

Describe the assignment function $a = a_{\alpha+1,n',k'}$. As usual it is an isomorphism between a name of a generic suitable structure of size $< \kappa_{\alpha_0}$ over $\alpha+1$ and a suitable structure $\mathfrak{X}^{\alpha}_{\alpha+1,n',k'}$ of α.

Note that the size is $< \kappa_{\alpha_0}$ and not $< \kappa_{\alpha-1,m}$, as in the dropping defined in [6]. We need here to have more closure since drops in cofinality occur here at infinitely many places and all of them to $\alpha - 1 < \alpha$.

Assume that we have square sequences for singular ordinals at relevant places. Given $X \in \operatorname{dom} a$ with $\sup X$ of cofinality that should drop, we consider $C_{\sup X}$ of the square sequence. Use structures of the central piste from X with sup inside $C_{\sup X}$.

Let $a(X)$ be such that $\operatorname{cof}(\sup a(X))$ is the cardinal which corresponds to $\operatorname{cof}(\sup X)$ at $\alpha - 1$. Consider $C_{\sup a(X)}$. Structures from the central piste from X with sup in $C_{\sup X}$ will correspond to those of $a(X)$ and $C_{\sup a(X)}$ in a way that the assignment function does for $\alpha+1$, $\alpha-1$.

In order further to add a model Y below X, we need first to specify the points of $C_{\sup X}$ between which Y sits. The the assignment function between $\alpha-1, \alpha+1$ is used (or is first extended) to provide the corresponding points on $C_{\sup a(X)}$. Then $a(Y)$ is picked accordingly.

3. Let β be a successor ordinal. Assume that for some successor or zero ordinal $\alpha < \beta$ and $x \in \mathit{connect}(\alpha, \beta)$, $a_{\alpha,\beta,x}$ is defined. Then for every $Z \in \operatorname{dom}(a_{\alpha,\beta,x})$, for every $k < \omega$
the set
$\{(\gamma, y) \mid \gamma < \beta, x \in \mathit{connect}(\gamma, \beta)$, $a_{\gamma,\beta,y}$ is defined, $Z \in \operatorname{dom}(a_{\gamma,\beta,y})$ and
$\neg(a_{\gamma,\beta,y}(Z) \cap H(\chi^{+k}_{\gamma,n_1}) \preceq H(\chi^{+k}_{\gamma,n_1}))\}$ is finite,
where $y = ((n_1, m_1), (n_2, m_2))$ and χ_{γ,n_1} is a regular cardinal large enough in the interval $(\kappa^{++}_{\gamma,n_1,g_\gamma(n_1)-1,\omega_1}, \kappa_{\gamma,n_1+1})$

4. Let $\alpha < \beta$ be successor ordinals and
$x \in \mathit{connect}(\alpha, \beta)$. Then
$\operatorname{dom}(a_{\alpha\beta x}) \subseteq \operatorname{dom}(a_{\alpha\beta x})$.

5. Let $\alpha < \beta$ be successor ordinals and
$x \in \mathit{connect}(\alpha, \beta)$. Then $\operatorname{dom}(f_{\alpha\beta x}) \cap \operatorname{dom}(a_{\alpha\beta x}) = \emptyset$.

6. (Commutativity of connections) Let $\alpha < \beta < \gamma < \omega_1$ be successor ordinals and $n < \omega$. Assume that $k_\alpha-$th block of $n_\alpha-$th level of α is connected to $k_\beta-$th block of a level

n_β of β and to $k_\gamma-$th block of a level n_γ of γ. Suppose that in addition that $k_\beta-$th block of a level n_β of β and $k_\gamma-$th block of a level n_γ of γ are connected.
Then for each $Z \in \mathrm{dom}(a_{\alpha\gamma((n_\alpha,k_\alpha),(n_\gamma,k_\gamma))})$ we have $Z \in \mathrm{dom}(a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,kn_\gamma))})$ and

$$a_{\alpha\gamma((n_\alpha,k_\alpha),(n_\gamma,k_\gamma))}(Z) = a_{\alpha\beta((n_\alpha,k_\alpha),(n_\beta,k_\beta))}(\underset{\sim}{a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}(Z)}),$$

where $\underset{\sim}{a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}(Z)}$ is a name of the indiscernible which corresponds to $a_{\beta\gamma((n_\beta,k_\beta),(n_\gamma,k_\gamma))}(Z)$.

Definition 7.2 (One element extension.)
Suppose $p = \langle p_\alpha \mid \alpha < \omega_1 \text{ is a successor ordinal } \rangle\rangle \in \mathcal{P}_0$. Let $\alpha < \beta < \omega_1$ be successor ordinals. Let $x = ((n_\alpha, m_\alpha), (n_\beta, m_\beta)) \in \mathit{connect}(\alpha, \beta)$. Let $p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha,x}, f_{\alpha\beta x}\rangle$ and $\eta \in A_{\alpha,x}$.
Assume that $n_\alpha = 0$. In general, if $n_\alpha \neq 0$, then taking a non-direct extension over the level n_α we would like simultaneously to make a non-direct extension at each level $n < n_\alpha$ over α and, in addition, $\alpha - 1$, if α is even, or $\alpha + 1$, if α is odd.
Define $p^\frown\eta$, the one element non direct extension of p by η, to be $q = \langle q_\xi \mid \xi < \omega_1$ and ($\xi = 0$ or ξ is a successor ordinal)$\rangle$ so that

1. for every $\xi, \zeta, \alpha < \xi < \zeta < \omega_1, y \in \mathit{connect}(\xi, \zeta), \quad p_{\xi\zeta y} = q_{\xi\zeta y}$,
2. for every $y \in \mathit{connect}(\alpha, \gamma)$ with the level on α bigger than n_α we have $p_{\alpha\beta y} = q_{\alpha\beta y}$.
3. for every successor ordinal $\gamma, \alpha < \gamma < \omega_1$,

   $$q_{\alpha\gamma y} = f_{\alpha\gamma y} \cup \{\langle \tau, \pi^{E_{\kappa_\alpha,n_\alpha}}_{mc(\alpha,n),a_{\alpha\gamma y}(\tau)}(\eta)\rangle \mid \tau \in \mathrm{dom}(a_{\alpha\gamma y})\},$$

   where $y \in \mathit{connect}(\alpha, \gamma)$ and the level of y over α is n_α as those of x.
4. Let $\alpha', \tau, \alpha' > \alpha > \tau$, be successor ordinals. Then connections $a_{\tau\alpha' y}$ of p will split now in q into connections from α' to α followed by a connection from α to τ. Namely, let $\langle \tau, r, s\rangle$ be connected with $\langle \alpha', n', m'\rangle$. For each (n, m) such that $((n, m), (n', m')) \in \mathit{connect}(\alpha', \alpha)$ and $\langle \tau, r, s\rangle \in \mathit{connect}(\alpha, n, m)$ split $a_{(\alpha',n',m'),(\gamma,r,s)}$ into $a_{(\alpha',n',m'),(\alpha,n,m)}$ followed by $a_{(\gamma,r,s),(\alpha,n,m)}$.
5. For each level $n' < n_\alpha$ of α, the same things occur, i.e. 2-4 above hold with (n_α, m_α) replaced by (n', k'), where k' is any block of the level n'.

Definition 7.3 Set $\mathcal{P}_1$ to be the set all $p^\frown\eta$ as in Definition 7.2. Proceed by induction. For each $n < \omega$, once $\mathcal{P}_n$ is defined, define $\mathcal{P}_{n+1}$ to be the set of all $p^\frown\eta$, where $p \in \mathcal{P}_n$. Finally set $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}_n$.

Definition 7.4 Let $p, q \in \mathcal{P}$.

1. We say that p is a direct extension of q and denote this by $p \geq^* q$ iff p is obtained from q by extending $a_{\alpha\beta x}, f_{\alpha\beta x}$'s and by shrinking the sets of measures one probably by passing to bigger measure first.

2. The forcing order $\geq$ is defined as follows:
   $p \geq q$ iff there are $q_1, ..., q_n \in \mathcal{P}, \eta_1, ..., \eta_n$ such that

   (a) $q \leq^* q_1$,

   (b) for every $k, 1 \leq k \leq n, \quad q_k{}^\frown\eta_k \in \mathcal{P}$,

   (c) for every $k, 1 \leq k < n, \quad q_k{}^\frown\eta_k \leq^* q_{k+1}$,

   (d) $q_n{}^\frown\eta_n \leq^* p$.

For each even $\alpha < \omega_1$ we split $\mathcal{P}$ into $(\mathcal{P}\backslash\kappa_\alpha) * \mathcal{P} \restriction \kappa_{\alpha+1}$, where $\mathcal{P}\backslash\kappa_\alpha$ is the part of $\mathcal{P}$ is defined as $\mathcal{P}$ but with $\kappa_{\alpha+1}$ replacing κ_0, i.e. everything is above κ_α and the first cardinal we deal with is $\kappa_{\alpha+1,0}$. $\mathcal{P} \restriction \kappa_{\alpha+1}$ is defined in $V[G']^{\mathcal{P}\backslash\kappa_\alpha}$, where $G' = G(\mathcal{P}')$ is a generic subset of the preparation forcing $\mathcal{P}'$. We just replace for every $\beta > \alpha + 1$ the connection of β to levels of γ's $\gamma \leq \alpha + 1$ by its connection to $\alpha + 1$ followed by the corresponding connection from $\alpha + 1$ to γ.

Lemma 7.5 $\langle \mathcal{P}, \leq, \leq^* \rangle$ *is a Prikry type forcing notion.*

Proof. Work in $V[G(\mathcal{P}')]$. Suppose otherwise.
Let $p \in P$ and σ be a statement of the forcing language such that there is no $q \geq^* p$ which decides σ.
Suppose for simplicity that $p \in \mathcal{P}_0$.
We peak an elementary chain of elementary submodels of H_χ (for χ big enough)

$$\langle M(\kappa_{\alpha n}, \xi) \mid \alpha < \omega_1 \text{ is a successor ordinal }, n < \omega, \xi \leq \kappa_{\alpha n} \rangle$$

such that

1. $p, \sigma \in M(\kappa_{00}, 0)$,

2. $|M(\kappa_{\alpha n}, \xi)| = \kappa_{\alpha n}$,

3. if ξ is a limit ordinal then $M(\kappa_{\alpha n}, \xi) = \bigcup_{\xi' < \xi} M(\kappa_{\alpha n}, \xi')$,

4. $\langle M(\kappa_{\alpha n}, \xi') \mid \xi' < \xi \rangle \in M(\kappa_{\alpha n}, \xi)$, for every successor ξ,

5. $\langle M(\kappa_{\alpha n}, \xi) \mid \xi \leq \kappa_{\alpha n} \rangle \in M(\kappa_{\alpha n+1}, 0)$,

6. $\langle M(\kappa_{\alpha n}, \xi) \in G(\mathcal{P}')$.

7. Let $M(\kappa_{\omega_1}) = \bigcup_{\alpha < \omega_1, n < \omega} M(\kappa_{\alpha n}, \kappa_{\alpha n})$.
   Then

   (a) $M(\kappa_{\omega_1}) \in G(\mathcal{P}')$,

   (b) each model $M(\kappa_{\alpha n}, \xi)$ is on the main piste of $M(\kappa_{\omega_1})$.

Let $p = \langle p_\alpha \mid \alpha < \omega_1, \alpha \text{ is a successor ordinal} \rangle$. Start with p_1.

Let A_1 be its set of measure one. Proceed by induction on $\vec{\nu} \in [A_1]^{<\omega}$. Let $\vec{\nu}_0$ be the least. Ask whether there is a direct extension of

$$(p_1^\frown \vec{\nu}_0)^\frown p \setminus \{1\}$$

which decides σ.
If there is no such extension, then proceed to the next element of $[A_1]^{<\omega}$.
If there is such an extension $q_1^{0\frown} q^0 \setminus \{1\}$, then we continue to the next element $\vec{\nu}_1 \in [A_1]^{<\omega}$ but replacing $p \setminus \{1\}$ by $q_0 \setminus \{1\}$ and p_2 by a relevant part of it, as in the usual Extender Based Prikry forcing.

Continue in a similar fashion. Denote the final condition by $q(1)$. Shrink the measure one set A_1 in order to have the same decision or no decision of σ. By the assumption made, no decision.

Proceed by induction. Suppose that $\alpha < \omega_1$ is a successor ordinal and let $q(< \alpha)$ be the condition constructed up to stage α. We build $q(\alpha)$.
Let A_α denote the set of measure one of the α-th coordinate of $q(< \alpha)$. Proceed by induction on $\vec{\nu} \in [A_\alpha]^{<\omega}$. Let $\vec{\nu}_0$ be the least element. Ask whether there exists a direct extension

$q_{<\alpha,0} \geq q(< \alpha)_{<\alpha}$ and a direct extension r of $q(< \alpha)_{\alpha 0}{}^\frown \vec{\nu}_0{}^\frown q_{<\alpha,>\alpha}(< \alpha)$ such that $q_{\alpha,0}{}^\frown r \Vdash \sigma$.
If there are no such $q_{\alpha,\vec{\nu}_0}$ and r, then we continue to the next member of A_α. If there are such $q_{\alpha,\vec{\nu}_0}$ and r, then we continue to the next member of A_α working above r as in the Extender Based Prikry.

Continue in a similar fashion. Denote the final condition by $q_{(\alpha)}$. Shrink A_α to some A' in order to have the same decisions or no decision all the time. If for every $\vec{\nu}$ in A' there is such $q_{\alpha,\vec{\nu}}$, then we combine such $q_{\alpha,\vec{\nu}}$'s into a single name. Sets of measures one can be made to be independent on choice of $\vec{\nu}$'s, since their number is small.

Suppose now that we went through all the coordinates, let $p^* \geq^* p$ be the resulting condition. Find an extension $s \geq p^*$ which decides σ and such that the maximal coordinate where a non-direct extension was taken is as small as possible. Let α denote this maximal coordinate of s. Then the process at stage α ensures that on the set of measure one the same holds. So, we are able to replace a non-direct extension at α by a direct one. Contradiction.

Proceed by induction. Suppose we got to level n of some α. Denote by X the corresponding set of measure one of the condition q built (i.e. $A_{\alpha n}$ of it). Continue by induction on members of X. We use here models $\langle M(\kappa_{\alpha n}, \xi) \mid \xi \leq \kappa_{\alpha n}\rangle$. Thus, if $\nu \in X$, then work inside $M(\kappa_{\alpha n}, \nu+1)$. We ask if there is an extension of $q(\nu)^\frown\nu$ (where $q(\nu)$ was formed on the previous stage) which decides σ and is a direct extension above α, n. If so, then pick such extension and add $M(\kappa_{\alpha n}, \nu+1)$ to be the largest model. Otherwise, we make no change.
Non-direct parts below α, n will be stabilized once all ν's in X are considered. More precisely, we stabilize each $\langle \tau_1^\nu, ..., \tau_s^\nu\rangle$ that is below ν and a direct extension of $q(\nu)^\frown\langle \tau_1^\nu, ..., \tau_s^\nu\rangle^\frown\nu$ decides σ. Isomorphisms between structures (a's) and Cohen functions (f's) below are dealt as names depending on ν's.
Being of models $M(\kappa_{\beta m})$'s on the central line of $M(\kappa_{\omega_1})$ allows freely to take unions.
□

Lemma 7.6 $\langle \mathcal{P}, \leq \rangle$ *does not add new bounded subsets to* κ_0.

Proof. Let $p \in \mathcal{P}$, $\underset{\sim}{z}$ be a $\mathcal{P}-$name and $p \Vdash \underset{\sim}{z}$ is a bounded subset of κ_0. Extending p if necessary we can assume that $p \Vdash \underset{\sim}{z} \subseteq \kappa_{0m}$, for some $m < \omega$. Extend p further, if necessary, and assume that non-direct extensions were made in it at every level $n \leq m$ of 0.
Pick an elementary submodel $M \preceq H\chi$ of cardinality κ_0^+ such that

1. $p, \underset{\sim}{z} \in M$,
2. $M \in G(\mathcal{P}')$,
3. there is an increasing continuous sequence $\langle M_\xi \mid \xi < \kappa_{0m}\rangle$ of elementary submodels of M such that
    (a) $p, \underset{\sim}{z} \in M_0$,

(b) $\langle M_\xi \mid \xi < \kappa_{0m}\rangle$ on the piste of M of models of size κ_0^+,

(c) $M = \bigcup_{\xi<\kappa_{0m}} M_\xi$.

Now, we use the previous lemma 7.5 and build by induction a $\leq^*$ −increasing sequence $\langle p(\xi) \mid \xi < \kappa_{0m}\rangle$ of extensions of p such that $p(\xi) \in M_{\xi+1}$ and $p(\xi)\|\xi \in \underset{\sim}{z}$.
We have enough closure to run the process and eventually the upper bound of $\langle p(\xi) \mid \xi < \kappa_{0m}\rangle$ will decide $\underset{\sim}{z}$ completely.
□

Similar argument gives the following:

Lemma 7.7 *For every $\alpha < \omega_1$, $\langle \mathcal{P} \setminus \kappa_\alpha, \leq \rangle$ does not add new bounded subsets to $\kappa_{\alpha+1}$.*

Define now $\longleftrightarrow$ and $\longrightarrow$.

Definition 7.8 Let $p, q \in \mathcal{P}$. Set $p \longleftrightarrow q$ iff there is $\alpha < \omega_1$ such that

1. $p \setminus \kappa_\alpha = q \setminus \kappa_\alpha$,
2. for every $k < \omega$, for all but finitely many $\beta \leq \alpha$, for all but finitely many $n < \omega$ the following hold:

   (a) if no non-direct extension was made at the level n of β in p_β and q_β, then $0_{\mathcal{P}\setminus\kappa_\beta} \Vdash_{\mathcal{P}\setminus\kappa_\beta}$ over the level n of β the following hold in p_β and q_β:

      i. f's, A's and $\mathrm{dom}(a)$'s are the same,

      ii. $\mathrm{rng}(a)$'s realize the same k-type;

   (b) if a non-direct extension was made at the level n of β in one of p_β or q_β, then it was made in another as well, and they are equal.

   This means basically that $p \restriction \kappa_{\alpha+1} \longleftrightarrow_{\mathcal{P}\restriction\kappa_{\alpha+1}} q \restriction \kappa_{\alpha+1}$,
   where $\longleftrightarrow_{\mathcal{P}\restriction\kappa_{\alpha+1}}$ states that for each $k < \omega$ all but finitely many coordinates realize the same k-type.

Now we define $\longrightarrow$ in the usual fashion.

Definition 7.9 Let $p, q \in \mathcal{P}$. Set $p \longrightarrow q$ iff there is a sequence of conditions $\langle r_k \mid k < m < \omega\rangle$ so that

(1) $r_0 = p$

(2) $r_{m-1} = q$

(3) for every $k < m-1$,
$r_k \leq r_{k+1}$ or $r_k \longleftrightarrow r_{k+1}$.

Our next task will be to split $\mathcal{P}$ into upper and low parts, and then argue that the former is closed with respect to direct extensions and the later satisfies chain condition.
Let $\alpha < \omega_1$ be a non-zero even ordinal.
The upper part $\mathcal{P} \setminus \kappa_{\alpha+1}$ is defined exactly as $\mathcal{P}$ but starting with $\kappa_{\alpha+1}$ instead of κ_1. Let us turn to the lower part, $\mathcal{P} \restriction \kappa_{\alpha+1}$. We use first the commutativity in order to replace connections from β's above $\alpha+1$ to α and below, by their connections to $\alpha+1$. Then $\alpha' \leq \alpha$ are connected to $\alpha+1$, and so via $\alpha+1$ to rest of $\beta > \alpha+1$.

Recall that α was connected to $\alpha-1$ in a special way which included dropping in cofinality. We split $\mathcal{P} \restriction \kappa_{\alpha+1}$ into $\mathcal{P} \restriction \kappa_{\alpha-1} * \mathcal{P} \restriction \kappa_\alpha$. The part $\langle \mathcal{P} \restriction \kappa_{\alpha-1}, \leq^* \rangle$
is $\kappa^+_{\alpha-1}$-closed and $\langle \mathcal{P} \restriction \kappa_\alpha \rangle$ satisfies $\kappa^{++}_{\alpha-1}$-c.c. These allow us to preserve all the cardinals.

If $\alpha < \omega_1$ is an even ordinal, then it is connected fully to $\alpha+1$. So, $\mathcal{P} \restriction \kappa_{\alpha+1}$ is defined using connections from above to $\alpha+1$. Recall that we have a drop in cofinality from $\alpha+1$ to α. These drop was arranged such that the direct extension order over $\alpha+1$ is $\kappa^{++}_\alpha-$closed in order to ensure preservation of cardinals in the interval $[\kappa_\alpha, \kappa_{\alpha+1}]$.

Lemma 7.10 *Let $\alpha < \omega_1$. Then, in $V^{\mathcal{P}' * \mathcal{P} \setminus \kappa_\alpha}$, the forcing $\langle \mathcal{P} \restriction \kappa_{\alpha+1}, \longrightarrow \rangle$ satisfies κ^{++}_α-c.c.*

Proof. Suppose otherwise. Assume that

$$0_{\mathcal{P} \setminus \kappa_\alpha} \Vdash_{\mathcal{P} \setminus \kappa_\alpha} \{ \underset{\sim}{p}_\xi \mid \xi < \kappa^{++}_\alpha \} \subseteq \mathcal{P} \restriction \kappa_{\alpha+1} \text{ is an antichain .}$$

Force over $V[G(\mathcal{P}')]$ (not over $V[G(\mathcal{P}')][G(\mathcal{P} \setminus \kappa_\alpha)]$!) with the obvious forcing (i.e. initial segments) which produces a κ^{++}_α-chain of members of $G(\mathcal{P}')$ of size κ^+_α. This forcing does not add new sequences of length $\leq \kappa^+_\alpha$.
Pick an elementary submodel $M \preceq H_\chi$ of such generic extension which is a union of an elementary chain $\langle M_\xi \mid \xi < \kappa^{++}_\alpha \rangle$ of its elementary submodels of size κ^+_α which are in $G(\mathcal{P}')$, and such that for every $\xi < \kappa^{++}_\alpha$,
$\langle M_{\xi'} \mid \xi' < \xi \rangle \in M_\xi$ and is on the central piste of M_ξ.
Now we proceed by induction. On stage ξ decide $\underset{\sim}{p}_\xi$ inside M_ξ and add M_ξ as a largest model. The rest of the proof follows completely the lines of the analogues arguments for short extenders forcings (see, for example, Sec 1 of [6]). Eventually, we will have $\xi < \rho < \kappa^{++}_\alpha$ and a condition in $\mathcal{P} \setminus \kappa_\alpha$ which forces compatibility of $\underset{\sim}{p}_\xi$ and $\underset{\sim}{p}_\rho$.
□

Lemma 7.11 *The forcing $\langle \mathcal{P}, \longrightarrow \rangle$ over $V[G']$ preserves all the cardinals (and every cofinality).*

Proof. Let η be a cardinal in $V[G']$. We show by induction on $\alpha < \omega_1$ that if $\eta \leq \kappa_\alpha$ then it is preserved in the generic extension. Clearly, it is enough to deal only with regular η's. Hence, we need to consider only the following situation:

$$\kappa_\alpha < \eta < \kappa_{\alpha+1},$$

for some $\alpha < \omega_1$. Split the forcing $\mathcal{P}$ into $\mathcal{P}\backslash\kappa_\alpha$ followed by $\mathcal{P} \restriction \kappa_{\alpha+1}$. By Lemma 7.7, $\mathcal{P}\backslash\kappa_\alpha$ does not add new bounded subsets to $\kappa_{\alpha+1}$ (namely, this lemma together with the Prikry condition imply that no new subsets are added to $\kappa_{\alpha+1,0}$, but taking non-direct extensions over $\kappa_{\alpha+1,n}$'s it is easy to push this up to $\kappa_{\alpha+1}$). By Lemma 7.10 the forcing $\mathcal{P} \restriction \kappa_{\alpha+1}$ preserves all the cardinals above κ_α^+. So, the only case that remains is $\eta = \kappa_\alpha^+$. But it is not problematic, since we have here the successor of the singular cardinal and the usual arguments apply.
□

Lemma 7.12 *For every $\alpha < \omega_1$, α non-accumulation point (i.e. $\alpha = 0$ or α non-limit ordinal) the following hold in $V^{\mathcal{P}' * \langle \mathcal{P}, \longrightarrow \rangle}$:*

$$\mathrm{pcf}(\{(\rho_{\alpha n m \omega_1})^+ \mid n < \omega, m < g_\alpha(n)\}) \setminus \kappa_\alpha =$$

$$\{(\rho_{\beta r s \omega_1})^+ \mid \alpha < \beta < \omega_1 \textit{ is a successor ordinal}, r < \omega, s < g_\beta(r)\} \cup \{\kappa_{\omega_1}^+\},$$

moreover, for every limit $\gamma, \alpha < \gamma < \omega_1$,

$$\mathrm{pcf}(\{(\rho_{\alpha n m \gamma})^+ \mid n < \omega, m < g_\alpha(n)\}) \setminus \kappa_\alpha =$$

$$\{(\rho_{\beta r s \gamma})^+ \mid \alpha < \beta < \gamma \textit{ is a successor ordinal}, r < \omega, s < g_\beta(r)\} \cup \{\kappa_\gamma^+\},$$

where $\rho_{\delta t u \xi}$ denotes the indiscernible for $\kappa_{\delta t u \xi}$.

Proof. The proof is by induction on β using the assignment functions (a's) of the conditions and that $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.
□

8 Concluding remarks.

The construction of the previous section gives a countable set of regular cardinals $\mathfrak{a}$ with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) = \omega_1 + 1$. It is natural to try to get a bigger order type. The present methods allow to obtain $\omega_1 \cdot \alpha + 1$, for every $\alpha < \omega_1$. Just repeat the construction α– many times (one above another). However it is unclear how to get to $\omega_1 \cdot \omega_1 + 1$ and beyond. In addition the resulting countable set $\mathfrak{a}$ will have the order type $\omega \cdot \alpha$, and it is unclear whether it is possible to have a set of regular cardinals $\mathfrak{a}$ of order type ω with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) > \omega_1 + 1$.

Question 1. *Is it possible to increase* $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a}))$ *beyond* $\omega_1 \cdot \omega_1$*, for a countable set of regular cardinals* $\mathfrak{a}$*?*

We think that it may be possible under same lines, but using more elaborated techniques, to get any successor order type $< \omega_2$.

Shelah Weak Hypothesis (SWH) states that the set

$$\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$$

is at most countable.

The construction of the previous section provides a counterexample, but very restricted one. The cardinality and even the order type there is ω_1. So the following question is natural:

Question 2. *Is it possible to increase the cardinality of the set*

$$\{\eta \mid \eta < \kappa, \eta \textit{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$$

beyond ω_1*, for a cardinal* κ*?*

Note that no upper bound on cardinality of

$$\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \mathrm{pp}(\eta) > \kappa\}$$

is known.

Going further beyond ω_1, in view of results of [7] and [9] will require some completely new ideas. The same once one likes to have a set $\{\eta \mid \eta < \kappa, \mathrm{cof}(\eta) > \omega, \mathrm{pp}(\eta) > \kappa\}$ infinite, for some κ.

Question 3. *How to move everything down, in particular is it possible to get down to* $\aleph_\omega$*?*

It is possible to add collapses to the present construction, but only very inessential ones. By [8], the supercompact Prikry forcing looks be needed in order to collapse successors of singular cardinals, but this complicates the matters largely. It is unclear how to combine this forcing with short extenders forcings in a productive way.

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