Short Extenders Forcings II

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Abstract

A model with $\mathrm{otp}(\mathrm{pcf}(\mathfrak{a})) = \omega_1 + 1$ is constructed, for countable set $\mathfrak{a}$ of regular cardinals.

1 Preliminary Settings

Let $\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ be an an increasing continuous sequence of singular cardinals of cofinality ω so that for each $\alpha < \omega_1$, if $\alpha = 0$ or α is a successor ordinal, then κ_α is a limit of an increasing sequence $\langle \kappa_{\alpha,n} \mid n < \omega \rangle$ of cardinals such that

(1) $\kappa_{\alpha,n}$ is strong up to a 2-Mahlo cardinal $< \kappa_{\alpha,n+1}$,

(2) $\kappa_{\alpha,0} > \kappa_{\alpha-1}$.

Fix a sequence $\langle g_\alpha \mid \alpha < \omega_1,\ \alpha = 0$ or it is a successor ordinal$\rangle$ of functions from ω to ω such that for every $\alpha, \beta, \alpha < \beta$ which are zero or successor ordinals below ω_1 the following holds

(a) $\langle g_\alpha(n) \mid n < \omega \rangle$ is increasing

(b) there is $m(\alpha, \beta) < \omega$ such that for every $n \geq m(\alpha, \beta)$

$$g_\alpha(n) \geq \sum_{m=0}^{n} g_\beta(m) \ .$$

(c) $g_\alpha(0) = 1$

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The easiest way is probably to force such a sequence.
Conditions are of the form

$\langle n, \{h_\alpha | \alpha \in I\}\rangle$, where $n < \omega, I$ is a finite subset of ω_1 and $h_\alpha : n \to \omega$.

The order is defined as follows:

$\langle n, \{h_\alpha | \alpha \in I\}\rangle \leq \langle m, \{t_\beta | \beta \in J\}\rangle$ iff $n \leq m, I \subseteq J$, for every $\alpha \leq \beta, \alpha, \beta \in I$, we have $t_\alpha | n = h_\alpha$ and if $n \leq k < m$ then require that $t_\alpha(k) \geq \sum_{0 \leq s \leq k} t_\beta(s)$.

It is possible to construct such a sequence in ZFC. Pick first a sequence $\langle h_\alpha | \alpha < \omega_1\rangle$ of functions from ω to ω such that

(1) $\langle h_\alpha(n) \mid n < \omega\rangle$ is non-decreasing and converges to infinity;

(2) if $\alpha < \beta$ then $h_\alpha > h_\beta$ mod finite.

Replace now each h_α by h'_α such that $h'_\alpha(n) = h_\alpha(n) + n + 1$.

Define $g_\alpha(n)$ to be $2^{(2^{\ldots (2^{h'_\alpha(n)})\ldots})}$ where the number of powers is $h'_\alpha(n)$.

Let us argue that it is as required. Let $\alpha < \beta$. Pick $m(\alpha, \beta)$ to be such that for every $n \geq m(\alpha, \beta)$ we have $h'_\alpha(n) > h'_\beta(n)$.

Let $n \geq m(\alpha, \beta)$. Consider $\sum_{0 \leq s \leq n} g_\beta(s)$.

Then

$$\sum_{0 \leq s \leq n} g_\beta(s) \leq (n+1) \cdot g_\beta(n) \leq (g_\beta(n))^2 \leq 2^{g_\beta(n)} \leq g_\alpha(n).$$

In order to motivate the further construction let us consider first two simple (relatively) situation. The first will deal with only two cardinals κ_0 and κ_1, and the second with ω-many of them- $\langle \kappa_n \mid n < \omega\rangle$.

2 Two cardinals

We would like to blow up the powers (or pp) of both κ_0, κ_1 to κ_1^{++}. Organize this as follows. We have $\langle \kappa_{1,n} \mid n < \omega\rangle$. For each $n < \omega$ fix some regular $\kappa_{1,n,1}, \kappa_{1,n+1} > \kappa_{1,n,1} \geq \kappa_{1,n}^{+n+2}$. It will be the one connected to κ_1^{++} at the level n of κ_1. Denote by $\underset{\sim}{\rho}_{1,n,1}$ the canonical name of the indiscernible for $\kappa_{1,n,1}$, i.e. the cardinal corresponding to $\kappa_{1,n,1}$ in the one element Prikry forcing or more precisely in the short extenders forcing of [2].

So the sequence $\langle \rho_{1,n,1} \mid n < \omega\rangle$ will correspond to κ_1^{++} after the forcing.

Now turn to the 0–level. We have here $\langle \kappa_{0,n} \mid n < \omega\rangle$. Instead of a direct connection to κ_1^{++} let us arrange a connection to elements of the interval $[\kappa_0^+, \kappa_1)$ and then via $\langle \rho_{1,n,1} \mid n < \omega\rangle$ it will continue automatically further to κ_1^{++}.

Specify first an interval $[\kappa_{0,0}, \kappa_{0,0,1}]$ that will correspond to $[\kappa_0^+, \rho_{1,0,1}]$, for some regular large enough $\kappa_{0,0,1} < \kappa_{0,1}$, say a Mahlo or even a measurable (note that there are plenty of such cardinals blow $\kappa_{0,1}$ since it is strong). Connection between them will be arranged and $\rho_{1,0,1}$ will correspond to $\kappa_{0,0,1}$. No more cardinals (i.e. those above $\rho_{0,0,1}$) will be connected to the 0–level of κ_0.

Turn to the next level of κ_0. We like to connect 0 and 1– levels of κ_1 to the 1–level of κ_0. In order to do this let us reserve two blocks of cardinals $[\kappa_{0,1}, \kappa_{0,0,0,1}]$ and $[\kappa_{0,0,1,0}, \kappa_{0,0,1,1}]$ such that $\kappa_{0,0,0,1} < \kappa_{0,0,1,0} < \kappa_{0,2}$ and $\kappa_{0,0,0,1}, \kappa_{0,0,1,0}, \kappa_{0,0,1,1}$ are large enough (again Mahlo or measurables). Now,the interval $[\kappa_0^+, \rho_{1,0,1}]$ will be connected to the first block $[\kappa_{0,1}, \kappa_{0,0,0,1}]$ and the interval $[\rho_{1,0,1}^+, \rho_{1,1,1}]$ to the second block $[\kappa_{0,0,1,0}, \kappa_{0,0,1,1}]$ with $\rho_{1,0,1}$ corresponding to $\kappa_{0,0,0,1}$ and $\rho_{1,1,1}$ to $\kappa_{0,0,1,1}$. No further cardinals from κ_1 will be connected to this level of κ_0.

Continue further in a similar fashion: connect the levels $0, 1, 2$ of κ_1 to the 2–level of κ_0 by specifying three blocks at this level, etc.

3 ω–many cardinals

We would like to blow up the powers (or pp) of all $\kappa_n, n < \omega$ to κ_ω^+. Organize this as follows. For each $k < \omega$, pick the first block for κ_k to be the interval $[\kappa_{k,0}, \kappa_{k,0,0,\omega}]$, where $\kappa_{k,0,0,\omega}$ is large enough cardinal below $\kappa_{k,1}$ which is a limit of an increasing sequence of large enough regular cardinals $\langle \kappa_{k,0,m,l} \mid m < g_k(0), l < \omega\rangle$ between $\kappa_{k,0}$ and $\kappa_{k,0,0,\omega+1}$, where $g_k : \omega \to \omega$ and each value $g_k(i)$ will be defined by induction at stage i. We set $g_k(0) = 1$.
Denote by $\rho_{k,0,0,l}$ the indiscernible (that will be forced further) for $\kappa_{k,0,0,l}$, for every $l \in \omega + 1$. Connect the interval $[\kappa_{m-1}^+, \rho_{m,0,0,\omega}]$ to $[\kappa_{k,0,0,m}, \kappa_{k,0,0,\omega}]$, for every $m, k < m - 1 < \omega$ so that κ_{m-1}^+ corresponds to $\kappa_{k,0,0,m}$, $\rho_{m,0,0,l}$ corresponds to $\kappa_{k,0,0,l}$, for each $l, m < l < \omega$ and $\rho_{m,0,0,\omega}$ corresponds to $\kappa_{k,0,0,\omega}$. The obvious commutativity is required.

Turn to the second levels of κ_k's. For each $k > 0$ we define $g_k(1) = 1$ and make no connections to m's above k. For $k = 0$ set $g_0(1) = 2$. Then at the level second level of κ_0, we reserve two blocks (instead of one) $[\kappa_{k,1}, \kappa_{k,1,0,\omega}]$ and $[\kappa_{k,1,1,0}, \kappa_{k,1,1,\omega}]$, where $\kappa_{k,1,0,\omega} < \kappa_{k,1,1,0} < \kappa_{1,2}$, $\kappa_{k,1,0,\omega}$ is a limit of increasing sequence of large enough regular cardinals $\langle \kappa_{k,1,0,l} \mid l < \omega\rangle$ above $\kappa_{k,1}$ and $\kappa_{k,1,1,\omega}$ is a limit of an increasing sequence of large enough regular cardinals $\langle \kappa_{k,1,1,l} \mid l < \omega\rangle$ above $\kappa_{k,1,0,\omega}$.
Then for each m,such that $k < m - 1 < \omega$ we connect the interval $[\kappa_{m-1}^+, \rho_{m,0,0,\omega}]$ with $[\kappa_{k,1,0,m}, \kappa_{k,1,0,\omega}]$ and $[\rho_{m,0,0,\omega}^{++}, \rho_{m,1,0,\omega}]$ with the second block starting from $\kappa_{k,1,1,m}$. Require for the first block that κ_{m-1}^+ corresponds to $\kappa_{k,1,0,m}$, $\rho_{m,0,0,l}$ corresponds to $\kappa_{k,1,0,l}$, for each $l, m < l < \omega$ and $\rho_{m,0,0,\omega}$ corresponds to $\kappa_{k,1,0,\omega}$. For the second block let us require that $\rho_{m,0,0,\omega}^{++}$ corresponds to $\kappa_{k,1,1,m}$, $\rho_{m,1,0,l}$ corresponds to $\kappa_{k,1,1,l}$, for each $l, m < l < \omega$ and $\rho_{m,1,0,\omega}$

corresponds to $\kappa_{k,1,1,\omega}$.

At third levels of κ_k's let us do the following. For each $k > 1$ $g_k(2) = 1$ and make no connections to m's above k.
If $k = 1$, then set $g_1(2) = 2$ and proceed exactly as at the second levels with $k = 0$ replaced by $k = 1$.
If $k = 0$ then set $g_0(2) = 4$, reserve 4 blocks at the third level of κ_0 and arrange the connections to this blocks in the similar to that used for κ_0 above, covering three levels of κ_m's, for $m > k$.

At the forth levels we do a similar connection only stepping up by one, etc.

It is not hard under the same lines to generalize the above construction from ω–many cardinals to η–many for every countable η.
A structure suggest below in order to deal with to ω_1–many cardinals will require drops with infinite repetitions.

4 ω_1–many cardinals

We would like to blow up the powers (or pp) of all $\kappa_\alpha, \alpha < \omega_1$ to $\kappa^+_{\omega_1}$.
The first tusk will be to arrange a pcf–structure that will be realized. It requires some work since we allow only finitely many blocks at each level. Note that in view of [9] one cannot allow infinitely many blocks at least not under the large cardinals assumptions used here (below a strong or a little bit more).

Organize the things as follows.
Let $n < \omega$ and $1 \leq \alpha < \omega_1$ be a successor ordinal or $\alpha = 0$. We reserve at level n a splitting into $g_\alpha(n)$–blocks one above another:

$$\langle \kappa_{\alpha,n,m,i} \mid m < g_\alpha(n), i \leq \omega_1 \rangle,$$

so that

1. $\kappa_{\alpha,n} < \kappa_{\alpha,n,0,0}$,
2. $\kappa_{\alpha,n,m,i'} < \kappa_{\alpha,n,m,i}$, for every $m < g_\alpha(n), i' < i \leq \omega_1$,
3. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n,m+1,0}$, for every $m < g_\alpha(n)$,
4. for every successor ordinal $i < \omega_1$ or if $i = 0$ let $\kappa_{\alpha,n,m,i}$ be large enough (say a Mahlo or even measurable),

5. for every limit $i, 0 < i \leq \omega_1$ let $\kappa_{\alpha,n,m,i} = \sup(\{\kappa_{\alpha,n,m,i'} \mid i' < i\})$,

6. $\kappa_{\alpha,n,m,\omega_1} < \kappa_{\alpha,n+1}$, for every $m < g_\alpha(n)$.

Further by $\alpha < \omega_1$ we will mean always a successor ordinal or 0.

Let us incorporate indiscernibles that will be generated by extender based forcings into the blocks as follows. Denote as above the indiscernible for $\kappa_{\alpha,n,m,i}$ by $\rho_{\alpha,n,m,i}$.
$[\kappa^+_{\alpha-1}, \rho^+_{\alpha,0,0,\omega_1}]$ will the first block of α of the level 0 (if $\alpha = 0$, then let it be $[\omega_1, \rho^+_{0,0,0,\omega_1}]$). Then for every $m < g_\alpha(0)$ let m–th block of α of the level 0 be $[\rho^{++}_{\alpha 0 m-1 \omega_1}, \rho^+_{\alpha,0,m,\omega_1}]$. The first block of the level 1 of α will be $[\rho^{++}_{\alpha 0 g_\alpha(0)-1\omega_1}, \rho^+_{\alpha,1,0,\omega_1}]$. In general the first block of the level $n > 0$ of α will be $[\rho^{++}_{\alpha n-1 g_\alpha(n-1)-1\omega_1}, \rho^+_{\alpha,n,0,\omega_1}]$. The m-th block ($m > 0$) of the level $n > 0$ of α will be $[\rho^{++}_{\alpha n m-1\omega_1}, \rho^+_{\alpha,n,m,\omega_1}]$.
Special attention will be devoted to the very last blocks of each level,
i.e. to $[\rho^{++}_{\alpha n g_\alpha(n)-2,\omega_1}, \rho^+_{\alpha,n,g_\alpha(n)-1,\omega_1}]$.

In the final (after the main forcing) model we will have the following structure. Every element of the set $\{\kappa^+_\beta \mid \alpha < \beta < \omega_1\}$ will be represented at all the levels up to level α. A countable set with uncountable pcf over α will be the set of indiscernibles

$$\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\} \ .$$

For every successor ordinal β, $\alpha < \beta < \omega_1$, each indiscernible $\rho^+_{\beta,n,m,\omega_1}$ $(n < \omega, m < g_\beta(n))$ will be in the pcf of this set. Thus, we will have the following:

$$\mathrm{pcf}(\{\rho^+_{\alpha,n,m,\omega_1} \mid n < \omega, m < g_\alpha(n)\}) =$$

$$= \{\rho^+_{\beta,n,m,\omega_1} \mid \alpha < \beta < \omega_1, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_{\omega_1}\}.$$

Actually for each limit ordinal $\gamma, \alpha < \gamma \leq \omega_1$, the following will hold:

$$\mathrm{pcf}(\{\rho^+_{\alpha,n,m,\gamma} \mid n < \omega, m < g_\alpha(n)\}) =$$

$$= \{\rho^+_{\beta,n,m,\gamma} \mid \alpha < \beta < \gamma, \beta \text{ is a successor ordinal}, n < \omega, m < g_\beta(n)\} \cup \{\kappa^+_\gamma\}.$$

Note that for $\gamma < \omega_1$ the set on the right side of equality is countable.

Let us establish the first connection between the levels and blocks by induction.
Start with a connection of the level 1 to to the level 0.
Consider $m(0,1)$, i.e. the least $m < \omega$ such that for every $n \geq m$ we have

$$g_0(n) \geq \sum_{k=0}^{n} g_1(k) \ .$$

This is a place from which blocks of the second level fit nicely inside those of the first level. Let us arrange the connection as follows. Connect the all the blocks of the levels $n, n \leq m(0,1)$ of κ_1 to the blocks of the level $m(0,1)$ of κ_0 moving to the right as much as possible, i.e. if $r = g_0(m(0,1)) - \sum_{k=0}^{m(0,1)} g_1(k)$, then the first block of κ_1 is connected to the r–th block of the level $m(0,1)$ of κ_0, the second block of κ_1 is connected to $r+1$-th block of the level $m(0,1)$ of κ_0 etc., the last block of the level $m(0,1)$ of κ_1 will be connected to the last block of the level $m(0,1)$ of κ_0.

Let us deal now with a level $\alpha > 1$. Fix an enumeration $\langle \alpha_i \mid i < \omega \rangle$ of α (if $\alpha < \omega$, then the construction is the same). Connect blocks from $[\kappa_{\alpha-1}, \kappa_\alpha]$ (further refired as of α) to blocks from $[\kappa_{\alpha_0 - 1}, \kappa_{\alpha_0}]$ (further refired as of α_0) exactly as above (i.e. κ_1 and κ_0).
Let us deal now with α_1. We would like to have a tree order at least on the very last blocks of each level. Thus we would not allow a block of α to be connected to two unconnected blocks of α_0 and α_1. Split into two cases.
Case 1. $\alpha_1 > \alpha_0$. Let $l(\alpha_0, \alpha_1)$ be the first level where the connection between α_0 and α_1 starts. Then, by induction, $l(\alpha_0, \alpha_1) \geq m(\alpha_0, \alpha_1)$. Let $l(\alpha_0, \alpha)$ be the first level where the connection between α_0 and α starts. By the definition we have $l(\alpha_0, \alpha) = m(\alpha_0, \alpha)$. Consider $m(\alpha_1, \alpha)$. It is tempting to start the connection between α and α_1 with the levels $m(\alpha_1, \alpha)$, but we would like to avoid a situation when the last block of a level n of α is connected to last blocks of levels n of both α_0 and α_1, which are disconnected, i.e. the connection order is not a tree order. So set $l(\alpha_1, \alpha) = \max(l(\alpha_0, \alpha_1), m(\alpha_1, \alpha))$. Note that $m(\alpha_0, \alpha) = l(\alpha_0, \alpha) \leq l(\alpha_1, \alpha)$, since $l(\alpha_0, \alpha_1) \geq m(\alpha_0, \alpha_1)$.
Also note that there is a commutativity here, and for each $n \geq l(\alpha_1, \alpha)$, blocks of α of levels $\leq n$ are connected to the level n of α_1 and the levels $\leq n$ of α_1 are connected to the level n of α_0.
Case 2. $\alpha_1 < \alpha_0$.
The treatment is similar only now α_0 is connected to α_1. Set

$$l(\alpha_1, \alpha) = \max(l(\alpha_0, \alpha_1), l(\alpha_0, \alpha), m(\alpha_1, \alpha)).$$

Continue in the same fashion by induction.

Let us called the established connection *automatic connection.* Last blocks ordered by this connection form a tree order by the construction.
It will be shown below (in Lemma 4.4) that there is no ω_1–branches. Note that we required that $g_\alpha(0) = 1$ for all α's, and hence first levels fit nicely one with an other. However, the automatic connection is defined so that if $\alpha < \alpha' < \omega_1$ and for some $n < \omega$ n-th levels

of α and α' are connected, then for every $m, n \leq m < \omega$, m-th levels of α and α' are automatically connected as well. Hence the ability to connect first levels does not imply that they will be actually connected by the automatic connection.

Let $\alpha < \omega_1, n < \omega$ and $m < g_\alpha(n)$. Set

$$a_\alpha(n,m) = \{(\alpha', n', m') \mid \alpha' < \alpha, \text{ the block } m' \text{ of } n' \text{ of } \alpha'$$

$$\text{is connected automatically to those of } m \text{ of } n \text{ of } \alpha\}.$$

Lemma 4.1 *Let $\alpha < \omega_1, n_1, n_2 < \omega$, $m_1 < g_\alpha(n_1), m_2 < g_\alpha(n_2)$ and ($n_1 \neq n_2$ or $n_1 = n_2$ but $m_1 \neq m_2$). Then $a_\alpha(n_1, m_1) \cap a_\alpha(n_2, m_2) = \emptyset$.*

Proof. Let $\langle \alpha_k \mid k < \omega \rangle$ be the enumeration of α which was used in the definition of the automatic connection. Clearly, the connection to α_0 cannot map different blocks of α to a same block. Consider α_1. If $\alpha_1 < \alpha_0$, then be start the connection from α to α_1 not before the level $m(\alpha_0, \alpha_1)$. For any further $\beta \leq \alpha_1$, and any level $n < \omega$ of β to which both α_0 and α_1 are connected, we must to have $m(\alpha_0, \beta) \leq n$ and $m(\alpha_1, \beta) \leq n$. Then all the blocks of α up to (and including) the level n are connected with the blocks of the level n of α_1 starting with the most right block of the level n of α_1, and then the blocks of α_1 up to and including the level n of α_1 are connected to the level n of β again starting with the most right block of the level n of β. So we have a kind of commutativity there. Hence no collisions occur over a level n of β.
The same argument works if $\alpha_1 > \alpha_0$. Just replace α_0 by α_1 above.

Suppose now that $k > 0$ and $\{\alpha_0, ..., \alpha_k\}$ provide the empty intersection. Let us argue that adding α_{k+1} does not change this. Split the set $\{\alpha_0, ..., \alpha_k\}$ into two sets $\{\alpha_{i_0}, ..., \alpha_{i_s}\}$, $\{\alpha_{j_0}, ..., \alpha_{j_r}\}$ such that the members of the first are below α_{k+1} and the members of the second one are above it. Consider $\beta \leq \alpha_{k+1}$ and a level n of β where a potential intersection can occur once α is connected to α_{k+1}. Let $\{\alpha_{l_0}, ..., \alpha_{l_t}\}$ be the subset of $\{\alpha_{i_0}, ..., \alpha_{i_s}\}$ which consists of all the elements $\geq \beta$.
Then $n \geq m(\alpha_{k+1}, \alpha_{j_q})$ and $n \geq m(\alpha_{i_p}, \alpha_{k+1})$, for every $q \leq r, p \leq s$. Also we can assume that $n \geq m(\beta, \alpha_{k+1})$. Otherwise there is no connection from α_{k+1} to the level n of β. There must be some $\alpha^* \in \{\alpha_{l_0}, ..., \alpha_{l_t}\} \cup \{\alpha_{j_0}, ..., \alpha_{j_r}\}$ with $m(\beta, \alpha^*) \leq n$, since otherwise only α_{k+1} will be connected to the level n of β and then the intersection will not have elements there. We deal now with $\alpha^*, \alpha_{k+1}, \beta$ and n exactly as above.
□

Note that many blocks remain unconnected. If no further connection will be made, then the following will occur. Unconnected blocks of an $\alpha < \omega_1$ will correspond to κ_α^+. By [8] we will have here always $\max(\mathrm{pcf}(\kappa_\alpha^+ \mid \alpha < \beta)) = \kappa_\beta^+$, for every $\beta < \omega_1$, due to the initial large cardinal assumptions. So, eventually there will be $\beta < \omega_1$ such that all blocks of all $\alpha < \beta$ will correspond to κ_β^+. It is clearly bad for our purpose.

We would like to extend the automatic connection such that for every α, if ρ and η are the last members of different blocks for α (it does not matter if levels are the same or not), then $\mathfrak{b}_{\rho^+} \neq \mathfrak{b}_{\eta^+}$. A problematic for us situation is once a connection was established in a way that for some $\alpha < \omega_1$ there are two different blocks for α that are connected to same blocks for unboundedly many levels below α. A problem will be then with a chain condition over α. Note that by Localization Property (see [12] or [1]) once pcf of a countable set is uncountable, there will be countable sets which correspond to cardinals much above their sup. Our construction uses only finitely many blocks at each level. If the connection is not built properly, then some countable set of blocks that should be connected with $\aleph_1$–many may turn to be connected with a single block of some $\alpha < \omega_1$ which will spoil everything.

Let us do the following. We force using a c.c.c. forcing a new connection based on the automatic connection.

Definition 4.2 Let Q be a set consisting of all pairs of finite functions q, ρ such that

1. $\mathrm{dom}(\rho) \subseteq [\omega_1]^2$,
2. $l(\alpha, \beta) \leq \rho(\alpha, \beta) < \omega$, for every $\alpha < \beta$ in the domain of ρ.
   Intuitively, $\rho(\alpha, \beta)$ will specify the place from which the automatic connection between α and β will step into the play.
3. $\mathrm{dom}(q) \subseteq \omega_1 \times (\omega \times \omega)$,
4. $q(\alpha, n, m)$ is a finite subset of $\alpha \times \omega \times \omega$ such that
   (a) if $\langle \beta, r, s \rangle \in q(\alpha, n, m)$, then $s < g_\beta(r)$.
   This will mean that s-th block of the level r of β is connected to m-th block of the level n of α.
   (b) $(\alpha, \beta) \in \mathrm{dom}(\rho)$ iff $\alpha < \beta$ and $\alpha, \beta \in \mathrm{dom}(\mathrm{dom}(q))$.
   (c) If $\langle \beta, r, s \rangle \in q(\alpha, n, m)$, then $\langle \beta, r, g_\beta(r) - 1 \rangle \in q(\alpha, n', m')$, for some $n', m' < \omega$.
   Note that in the automatic connection last blocks of β if connected then are connected to last blocks.

(d) If $\langle\beta, r, s\rangle \in q(\alpha, n, m)$ and $\langle\beta, r, s\rangle$ is automatically connected to some block m' of a level n' of α and $\rho(\beta, \alpha) \leq n$, then $n = n'$ and $m = m'$. I.e. we do not change the automatic connection above $\rho(\beta, \alpha)$. Note that then we must have $n \leq r$.

(e) If $\langle\beta, r, s\rangle \in q(\alpha, n, m)$, $\langle\beta', r', s'\rangle \in q(\alpha', n', m')$ and $r' > r$ then for some s'' (and then also for $s'' = g_\beta(r') - 1$) $\langle\beta, r', s''\rangle \in q(\alpha, n, m)$.

This condition basically requires that the last connected level is same in each component of $\mathrm{dom}(q)$.

(f) If $\langle\beta, r, s\rangle \in q(\alpha, n, m)$, $\langle\alpha, n, m\rangle \in q(\alpha', n', m')$, then $\langle\beta, r, s\rangle \in q(\alpha', n', m')$.

This just the transitivity of the connection.

(g) If $\beta < \alpha < \alpha'$, $\langle\beta, r, s\rangle \in q(\alpha, n, m)$, $\langle\alpha, n, m\rangle$ is automatically connected with $\langle\alpha', n', m'\rangle$, $\langle\alpha', n', m'\rangle \in \mathrm{dom}(q)$ and $\rho(\alpha, \alpha') \leq n$, then $\langle\beta, r, s\rangle \in q(\alpha', n', m')$, or $\langle\beta, r, s\rangle$ is automatically connected with $\langle\alpha', n', m'\rangle$ and $\rho(\beta, \alpha') \leq r$.

(h) If $\beta < \alpha < \alpha'$, $\langle\beta, r, s\rangle \in q(\alpha', n', m')$, $\langle\alpha, n, m\rangle$ is automatically connected with $\langle\alpha', n', m'\rangle$, $\langle\alpha, n, m\rangle \in \mathrm{dom}(q)$ and $\rho(\alpha, \alpha') \leq n$, then $\langle\beta, r, s\rangle \in q(\alpha, n, m)$.

(i) If $\langle\beta, r, s\rangle \in q(\alpha, n, m)$, then $r < n$ (i.e. we connect to higher levels) unless $\langle\beta, r, s\rangle$ is automatically connected to $\langle\alpha, n, m\rangle$ and $\rho(\beta, \alpha) \leq n$(in which case $r \geq n$).

This condition is helpful in the chain condition argument. It allows not to mix automatically connected elements with the rest.

The next two conditions insure a closure of q under connections.

(j) If $\langle\alpha', n', m'\rangle, \langle\alpha, n, m\rangle \in \mathrm{dom}(q)$ and $\alpha > \alpha'$, then $\langle\alpha', n', m'\rangle \in q(\alpha, n^*, m^*)$ for some $n^*, m^* < \omega$.

(k) If $\langle\alpha', n', m'\rangle, \langle\alpha, n, m\rangle \in \mathrm{dom}(q)$,$\langle\beta, r, s\rangle \in q(\alpha', n', m')$ and $\beta < \alpha$, then $\langle\beta, r, s\rangle \in q(\alpha, n^*, m^*)$ for some $n^*, m^* < \omega$.

In particular, if $\beta < \alpha < \alpha'$ and s-th block of a level r of β is connected (by q) to α', then it is connected to α and we have the commutativity here.

5. Let $n = \max(\rho(\alpha, \beta) \mid (\alpha, \beta) \in \mathrm{dom}(\rho))$. Then, for every $\alpha < \beta$ in the domain of ρ, all blocks of α of levels $\leq n$ are connected to blocks of β in q.

6. If $\langle\alpha', n', m'\rangle, \langle\alpha, n, m\rangle \in \mathrm{dom}(q)$,$\alpha' < \alpha$, $\langle\alpha', n', m'\rangle$, $\langle\alpha, n, m\rangle$ are automatically connected and $n' \geq \rho(\alpha', \alpha)$ (i.e. they remain connected), then for every $\langle\beta, r, s\rangle$ with $\beta < \alpha$

   (a) if $\langle\beta, r, s\rangle \in q(\alpha, n, m)$, then $\langle\beta, r, s\rangle \in q(\alpha', n', m')$;

(b) if $\langle\beta, r, s\rangle \in q(\alpha', n', m')$, then $\langle\beta, r, s\rangle \in q(\alpha, n'', m'')$, for some $n'', m'' < \omega$ such that $\langle\alpha', n', m'\rangle$, $\langle\alpha, n'', m''\rangle$ are automatically connected (and since $n' \geq \rho(\alpha', \alpha)$ they remain connected).

Let us define the order on Q.

Definition 4.3 Let $\langle q_1, \rho_1\rangle, \langle q_2, \rho_2\rangle \in Q$. Set $\langle q_1, \rho_1\rangle \geq \langle q_2, \rho_2\rangle$ iff

1. $\rho_1 \supseteq \rho_2$,
2. $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_2)$,
3. for every $\langle\alpha, n, m\rangle \in \mathrm{dom}(q_2)$ we have $q_2(\alpha, n, m) = q_1(\alpha, n, m)$.

Let us give a bit more intuition behind the definition of Q and explain the reason of adding ρ instead of just using the function l of the automatic connection.
The point is to prevent a situation like this: let $\gamma < \beta < \alpha$, $\alpha, \gamma \in \mathrm{dom}(\mathrm{dom}(q))$, $\beta \notin \mathrm{dom}(\mathrm{dom}(q))$ and we like to add it, for some $q \in Q$. Suppose that $l(\gamma, \alpha) = n < l(\beta, \alpha)$ and the level n of γ is connected automatically in q to all the blocks of α up to and including the level n. We need to add β. In order to do this the level n of γ should be connected to β. Then, due to the commutativity, the established connection is continued to α to the level n or below. One may try to use blocks of β of the level n and below for this purpose, but the total number of such blocks may be less than the number of blocks of the level n of γ, i.e. of $g_\gamma(n)$. So some non connected automatically to α blocks of higher levels of β should be used. There may be no such blocks at all or even if there are still this may conflict with automatic connections of bigger than α ordinals in the domain of q.
Once we have ρ, it is possible just to "fix" the automatic connection setting $\rho(\beta, \alpha)$ (i.e. the point from which the automatic connection starts actually to work) higher enough.
Requirement (5) of 4.2 is needed in order deal with a situation once β as above is already in q and so $\rho(\gamma, \beta), \rho(\beta, \alpha)$ are already determined.

Lemma 4.4 *Q satisfies c.c.c.*

Proof. Note that the automatic connection between last blocks of levels is a tree order.
Let us argue that there is no $\aleph_1$–branches. Suppose otherwise. Let $\langle\langle\alpha_i, n_i\rangle \mid i < \omega_1\rangle$ be a sequence such that for every $i < i' < \omega_1$, the last block of the level n_i of α_i is connected automatically to the last block of the level $n_{i'}$ of $\alpha_{i'}$. By the definition of this connection then $n_i = n_{i'}$ for every $i < i' < \omega_1$. Let $n = n_i$. Also, by the same definition, $m(\alpha_i, \alpha_{i'}) \leq$

$l(\alpha_i, \alpha_{i'}) \leq n$. Then for each $k \geq n$ the number of blocks of the level k of $\alpha_{i'}$ (and actually of all the levels $\leq k$) is less or equal than those of the level k of α_i. Then there are some $i(k) < \omega_1$ and $n(k) < \omega$ such that for every $i \geq i(k)$ the number of blocks of the level k of α_i is $n(k)$. Set $i^* = \sup\{i(k) \mid n \leq k < \omega\}$. Then for every i,$i^* < i < \omega_1$,$k \geq n$ we will have that the number of blocks of the level k of α_i is the same as those of the level k of α_{i^*}. But this impossible, since the function $g_{\alpha_{i^*}}$ dominates g_{α_i}.

Suppose that $\langle\langle q_i, \rho_i\rangle \mid i < \omega_1\rangle$ is a sequence of ω_1 elements of Q. Let us concentrate on q_i's. Set $b_i = \mathrm{dom}(\mathrm{dom}(q_i)) \cup \mathrm{dom}\,\mathrm{dom}(\mathrm{rng}(q_i)$ (i.e. the finite sequence of ordinals of $\mathrm{dom}(q_i)$ and of its range). Form a Δ-system. Suppose that $\langle b_i \mid i < \omega_1\rangle$ is already a Δ-system and let b^* be its kernel. Assume also that q_i's are isomorphic over $\omega \cup \sup(b^*)$. Consider now the set consisting of the last blocks (both of the domain and of the range of q_i):

$$c_i = \{\langle \alpha, n, g_\alpha(n) - 1\rangle \mid \langle \alpha, n, g_\alpha(n) - 1\rangle \in \mathrm{dom}(q_i) \cup \mathrm{rng}(q_i) \text{ and } \alpha \notin b^*\}.$$

Clearly $c_i \cap c_{i'} = \emptyset$, for every $i \neq i'$. Then, by the argument of Baumgartner, Malitz, Reinhardt, see [10], Lemma 16.18, there are $i \neq i'$ such that any $x \in c_i$ is incompatible (in our context are not connected automatically) with any $y \in c_{i'}$.
We claim that $q_i, q_{i'}$ are compatible. First let us argue that no two elements $\langle \beta_i, r_i, s_i\rangle$ in the domain or range of $q_i, \beta_i \notin b^*$ and $\langle \beta_j, r_j, s_j\rangle$ in the domain or range of $q_j, \beta_j \notin b^*$ are connected automatically. Suppose otherwise. Let, for example, $\beta_i < \beta_j$. Then $r_j \leq r_i$ and $\langle \beta_j, r_i, g_{\beta_j}(r_i) - 1\rangle$ is automatically connected to $\langle \beta_i, r_i, g_{\beta_i}(r_i) - 1\rangle$, by the definition of the automatic connection. Note that $\langle \beta_j, r_i, g_{\beta_j}(r_i) - 1\rangle$ must appear in q_j (in its domain or range) since the level r_i must appear in q_j as it appears in q_i and they are isomorphic, and once a level appears then there must be the last block of this level as well. Hence $\langle \beta_j, r_i, g_{\beta_j}(r_i) - 1\rangle \in c_j$. But $\langle \beta_i, r_i, g_{\beta_i}(r_i) - 1\rangle \in c_i$ and they are compatible. Contradiction. Now form a condition extending both q_i and q_j by connecting their isomorphic parts.
□

Let G be a generic subset of Q. It naturally defines a connection between blocks. Namely we connect s-th block of a level r of β with m-th block of a level n of α iff for some $(q, \rho) \in G$, $\langle \beta, r, s\rangle \in q(\alpha, n, m)$. Let us call further the part of this connection that is not the automatic connection by *manual connection* .
Denote for $\alpha, n < \omega, m < g_\alpha(m), \alpha_1 < \alpha_2 < \omega_1$,

$$\mathit{connect}(\alpha, n, m) = \{\langle \beta, n_1, m_1\rangle \mid \exists (q, \rho) \in G \quad \langle \beta, n_1, m_1\rangle \in q(\alpha, n, m)\}, \text{ or } \langle \beta, n_1, m_1\rangle$$

$$\text{is automatically connected to } \langle a, n, m\rangle \text{ and } \rho(\beta, \alpha) \leq n_1\},$$

$$connect(\alpha_1, \alpha_2) = \{(n_1, m_1), (n_2, m_2)) \mid \langle \alpha_1, n_1, m_1 \rangle \in connect(\alpha_2, n_2, m_2)\},$$

$$aconnect(\alpha_1, \alpha_2) = \{(n_1, m_1), (n_2, m_2)) \in connect(\alpha_1, \alpha_2) \mid \langle \alpha_1, n_1, m_1 \rangle, \langle \alpha_2, n_2, m_2 \rangle$$

$$\text{are automatically connected and for some } (q, \rho) \in G \text{ we have } \rho(\alpha_1, \alpha_2) \leq n_1\}.$$

$$mconnect(\alpha_1, \alpha_2) = connect(\alpha_1, \alpha_2) \setminus aconnect(\alpha_1, \alpha_2).$$

Let us refer further to elements of $mconnect(\alpha_1, \alpha_2)$ connected by the *manual connection* .

Lemma 4.5 *Suppose that $\langle \beta, r, s \rangle$ is a block of β and $\alpha > \beta$. Then for some $n, m < \omega$ we have $\langle \beta, r, s \rangle \in connect(\alpha, n, m)$.*

Proof. Let $\langle q, \rho \rangle \in Q$. We will construct a stronger condition $\langle q^*, \rho^* \rangle$ with $\langle \alpha, n, m \rangle \in \mathrm{dom}(q^*)$ and $\langle \beta, r, s \rangle \in q^*(\alpha, n, m)$, or $\rho^*(\beta, \alpha) \leq r$ and $\langle \beta, r, s \rangle$ is automatically connected with $\langle \alpha, n, m \rangle$, for some $n, m < \omega$.
If $\langle \beta, r, s \rangle$ is automatically connected with $\langle \alpha, n, m \rangle$, for some $n, m < \omega$ and $(\beta, \alpha) \notin \mathrm{dom}(\rho)$ or $(\beta, \alpha) \in \mathrm{dom}(\rho)$, $\rho(\beta, \alpha) \leq r$, then just set $\rho^*(\beta, \alpha) = r$ or $\rho^*(\beta, \alpha) = \rho(\beta, \alpha)$, if defined and we are done.
Suppose now that the above is not the case.
If both $\alpha, \beta \in \mathrm{dom}(\mathrm{dom}(q))$, then just apply (5) of 4.2 in order to produce q^*. Suppose that $\alpha \in \mathrm{dom}(\mathrm{dom}(q))$, if not then we add first α exactly in the fashion in which β will be added below.
Let us add β. Pick γ to be the largest element of $\mathrm{dom}(\mathrm{dom}(q))$ below β and assume that α is the first element of $\mathrm{dom}(\mathrm{dom}(q))$ above β (if not just replace α by such element extend and use transitivity).
We set $\rho(\gamma, \beta)$ to be the same as $\rho(\beta, \alpha)$ and be at least $\max(\rho(\gamma, \alpha), l(\gamma, \beta), l(\beta, \alpha)$. This will leave enough room in order to insure the commutativity between γ, β, α.
□

Lemma 4.6 *Let $\beta < \alpha < \alpha' < \omega_1, r, s, n', m' < \omega$. Suppose that $\langle \beta, r, s \rangle \in connect(\alpha', n', m')$. Then, for some $n, m < \omega$, with $\langle \alpha, n, m \rangle \in aconnect(\alpha, n', m')$ we have $\langle \beta, r, s \rangle \in connect(\alpha, n, m)$.*

Proof. This follows from 4.2(4h) by the density argument. Thus if for some $(q, \rho) \in G$ we have $\langle \beta, r, s \rangle \in q(\alpha', n', m')$, then once $n \geq \rho(\alpha, \alpha')$ 4.2(4h) implies $\langle \beta, r, s \rangle \in q(\alpha, n, m)$, for some $m < g_\alpha(n)$ such that $\langle \alpha, n, m \rangle$ is automatically connected with $\langle \alpha', n', m' \rangle$.
If $\langle \beta, r, s \rangle$ is automatically connected with $\langle \alpha', n', m' \rangle$, then by the density argument one can find a desired $\langle \alpha, n, m \rangle$.
□

Lemma 4.7 *The connection defined with G has no ω_1–branches.*

Proof. Suppose otherwise. Let $\langle \alpha_i, n_i, m_i \rangle \mid i < \omega_1$ be an ω_1–branch, i.e. $\langle \alpha_i, n_i, m_i \rangle \in$ *connect*(α_j, n_j, m_j), for every $i < j < \omega_1$. Assume without loss of generality that $\alpha_i \geq i$, for every $i < \omega_1$. Let $q_i \in G$ be such that $\langle \alpha_i, n_i, m_i \rangle \in \mathrm{dom}(q_i)$. Then $q_i(\alpha_i, n_i, m_i)$ is finite. Spit it into q_i^0 and q_i^1 such that $q_i^0 = q_i \cap i$ and $q_i^1 = q_i \setminus i$. Shrink to a stationary $S \subseteq \omega_1$ stabilizing q_i^0's. Then for every $i < j < \omega_1$, $\langle \alpha_i, n_i, m_i \rangle$ will be connected automatically with $\langle \alpha_j, n_j, m_j \rangle$. So we have an ω_1 chain under the automatic connection, which is impossible.
□

Lemma 4.8 *For every $\alpha < \omega_1$, $n, n' < \omega$ and $m < g_\alpha(n), m' < g_\alpha(n')$. connect$(\alpha, n, m) \cap$ connect(α, n', m') is bounded in α, unless $n = n'$ and $m = m'$.*

Proof. Note that the automatic connection has this property (even we have disjoint sets by 4.1). The additions made (if at all) are finite.
□

In order to realize the defined above connection there is a need in dropping cofinalities technics. Thus, for example, for some α the very first block of α may be connected (by the manual connection) to the last block of a level $n > 0$ of $\alpha + 1$. So in order to accommodate all the blocks of levels $\leq n$ of $\alpha + 1$ on and below the very first block of α there is a need to drop down below α. Note that on $\alpha - 1$ there is enough places to which such blocks are connected automatically, just starting with a higher enough level of $\alpha - 1$.

In this respect $\alpha = 0$ should be treated separately, since $\alpha - 1$ does not exist and so no place to drop. Let us just assume that all blocks of 0 are connected to blocks of 1 automatically. This can be achieved easily by changing g_0, g_1 a bit in order to fit together nicely. In addition do not allow to use blocks of 0 in the forcing Q above.

5 The preparation forcing.

We would like to use a generic set for the forcing $\mathcal{P}'$ of Chapter 3 (Preserving Strong Cardinals) of [6] in order to supply models for the main forcing defined further. Some degree of strongness of $\kappa_{\alpha,n}$ will be needed as well, for every successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$.
Two ways were described in Chapter 3 of [6]. Either can be applied for our purpose.
The first one is as follows.
Assume that for some regular cardinal θ the following set is stationary:

$$S = \{\nu < \theta \mid \nu \text{ is a superstrong with the target } \theta (\text{i.e. there is } i : V \to M, crit(i) = \nu$$

$$\text{and } M \supseteq V_\theta)\}.$$

Return to the definition of κ_γ's and $\kappa_{\gamma,k}$'s. Let us choose them by induction such that all $\kappa_{\gamma,k}$'s are from S. Suppose that $\langle \kappa_{\gamma,k} \mid k < \omega \rangle$ is defined. Then $\kappa_\gamma = \bigcup_{k<\omega} \kappa_{\gamma,k}$. Let $\tilde{\kappa}_\gamma$ be the next element of S. Pick $\kappa_{\gamma+1,0}$ to an element of S above $\tilde{\kappa}_\gamma$.
Force with $\mathcal{P}'(\theta)$ with a smallest size of models say $\aleph_8$. Then, by Lemma 3.0.23 of Chapter 3 (Preserving Strong Cardinals) of [6], each $\kappa_{\alpha,n}$ will remain $\tilde{\kappa}_\alpha$–strong (and even $\kappa^+_{\omega_1}$–strong). Moreover, $\mathcal{P}'(\tilde{\kappa}_\alpha)$ is a nice subforcing of $\mathcal{P}'(\theta)$ by Lemma 3.0.18 of Chapter 3 (Preserving Strong Cardinals) of [6], since $V_{\tilde{\kappa}_\alpha} \preceq V_\theta$ due to the choice of $\tilde{\kappa}_\alpha$ in S.

An other way, which uses initial assumptions below $\mathbf{0}^\P$, is as follows.
Let θ be a 2-Mahlo cardinal and $\kappa < \theta$ be a strong up to θ cardinal. Pick $\delta, \kappa < \delta < \theta$ a Mahlo cardinal such that $V_\delta \prec_{\Sigma_1} V_\theta$. By Lemma 3.0.15 of Chapter 3 (Preserving Strong Cardinals) of [6] or just directly, there will unboundedly many cardinals $\eta < \kappa$ with $\delta_\eta < \kappa$ such that the function $\eta \mapsto \delta_\eta$ represents δ and $V_{\delta_\eta} \prec_{\Sigma_1} V_\theta$. Then, by Lemma 3.0.18 of Chapter 3 of [6], $\mathcal{P}'(\delta_\eta)$ is a nice subforcing of $\mathcal{P}'(\theta)$.
Denote by S the set of all such η's.
Force now with $\mathcal{P}'(\theta)$. Let G' be a generic. By Lemma 3.0.24 of Chapter 3 of [6], embeddings wich witness δ-strongness of κ for large enough δ's below θ extend in $V[G']$. Then, below κ in $V[G']$, we will have unboundedly many η's which are strong up to δ_η for which $V_{\delta_\eta}[G' \cap V_{\delta_\eta}] \prec_{\Sigma_1} V_\theta[G']$, since every $\eta \in S$ is like this.
Now we define by induction $\kappa_{\gamma,k}$'s, κ_γ's and $\tilde{\kappa}_\gamma$'s using this η's and δ_η's.
Thus, suppose that $\langle \kappa_{\gamma,k} \mid k < \omega \rangle$ is defined. Then $\kappa_\gamma = \bigcup_{k<\omega} \kappa_{\gamma,k}$. Let $\tilde{\kappa}_\gamma = \delta_\eta$ for some such $\eta > \kappa_\gamma$. Pick $\kappa_{\gamma+1,0}$ to be the first $\eta \in S$ above $\tilde{\kappa}_\gamma$ and $\kappa_{\gamma+1,1}$ to be the first $\eta \in S$ above $\delta_{\kappa_{\gamma+1,0}}$, etc.

6 Types of Models

Force with $\mathcal{P}'$. Let $G' \subseteq \mathcal{P}'$ be a generic subset. Work in $V[G']$. For each successor or zero ordinal $\alpha < \omega_1$ and $n < \omega$ let us fix a $(\kappa_{\alpha,n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1})$– extender $E_{\alpha n}$, i.e. an extender with the critical point $\kappa_{\alpha,n}$ which ultrapower contains $V_{\kappa_{\alpha,n,g_\alpha(n)-1,\omega_1}+2}$.
An alternative approach will be instead of using a single extender for a level n of α, to use a separate extender $E_{\alpha,n,k}$, for every block $k < g_\alpha(n)$. The choice of $\kappa_{\alpha,n}$'s should be changed then slightly in order to insure strongness of $\kappa_{\alpha,n,k,0}$, for each $k < g_\alpha(n)$.
Also, using GCH (we assume GCH in V and then it will holds in $V[G']$ as well), fix an enumeration $\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle$ of $[\kappa_{\alpha n}]^{<\kappa_{\alpha n}}$ so that for every successor cardinal $\delta < \kappa_{\alpha n}$ the initial segment $\langle x_\gamma \mid \gamma < \delta\rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears stationary many times in each cofinality $< \delta$ in the enumeration. Let $j_{\alpha n}(\langle x_\gamma \mid \gamma < \kappa_{\alpha n}\rangle) = \langle x_\gamma \mid \gamma < j_{\alpha n}(\kappa_{\alpha n})\rangle$, where $j_{\alpha n}$ is a canonical embedding of $E_{\alpha n}$. Then $\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle$ will enumerate $[\kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}]^{\leq\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}}$.

For every $k \leq \omega$, we consider a structure

$$\mathfrak{A}_{\alpha,n,k} = \langle H(\chi^{+k}), \in, \subseteq, \leq, E_{\alpha n}, \kappa_{\alpha n}, \kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}, \langle \kappa_{\alpha,n,m,i} \mid m < g_\alpha(n), i \leq \omega_1\rangle, \chi,$$

$$\langle x_\gamma \mid \gamma < \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}\rangle, G', \theta, \langle \kappa_{\beta m} \mid \beta < \omega_1 \text{ is a successor ordinal or zero }, m < \omega\rangle,$$

$0, 1, \ldots, \xi, \ldots \mid \xi < \kappa^{+k}_{\alpha n}\rangle$
in an appropriate language which we denote $\mathcal{L}_{\alpha,n,k}$, with a large enough regular cardinal χ. Note that we have G' inside, so suitable structures may be chosen inside G' or $G' \cap \mathcal{P}'(\kappa_{\alpha,n})$.

Let $\mathcal{L}'_{\alpha,n,k}$ be the expansion of $\mathcal{L}_{\alpha,n,k}$ by adding a new constant c'. For $a \in H(\chi^{+k})$ of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$ let $\mathfrak{A}_{\alpha,n,k,a}$ be the expansion of $\mathfrak{A}_{\alpha,n,k}$ obtained by interpreting c' as a.

Let $a, b \in H(\chi^{+k})$ be two sets of cardinality less or equal than $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. Denote by $tp_{\alpha,n,k}(b)$ the $\mathcal{L}_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k}$. Further we identify it with the ordinal coding it and refer to it as the k-type of b. Let $tp_{\alpha,n,k}(a,b)$ be a the $\mathcal{L}'_{\alpha,n,k}$-type realized by b in $\mathfrak{A}_{\alpha,n,k,a}$. Note that coding a, b by ordinals we can transform this to the ordinal types of [2].

Now, repeating the usual arguments we obtain the following:

Lemma 6.1 *(a)* $|\{tp_{\alpha,n,k}(b) \mid b \in H(\chi^{+k})\}| = \kappa^{+k+1}_{\alpha n}$

(b) $|\{tp_{\alpha,n,\kappa}(a,b) \mid a, b \in H(\chi^{+k})\}| = \kappa^{+k+1}_{\alpha n}$

Lemma 6.2 *Let* $A \prec \mathfrak{A}_{\alpha,n,k+1}$ *and* $|A| \geq \kappa^{+k+1}_{\alpha n}$*. Then the following holds:*

(a) *for every* $a, b \in H(\chi^{+k})$ *there* $c, d \in A \cap H(\chi^{+k})$ *with* $tp_{\alpha,n,k}(a, b) = tp_{\alpha,n,k}(c, d)$

(b) *for every* $a \in A$ *and* $b \in H(\chi^{+k})$ *there is* $d \in A \cap H(\chi^{+k})$ *so that* $tp_{\alpha,n,k}(a \cap H(\chi^{+k}),$ $b)=tp_{\alpha,n,k}(a \cap H(\chi^{+k}),\ d)$.

Lemma 6.3 *Suppose that* $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_{\alpha n}^{+k+1}$. *Let* τ *be a cardinal in the interval* $[\kappa_{\alpha n}, \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}]$ *those* $k+1$*-type is realized unboundedly often below* $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. *Then there are* $\tau' < \tau$ *and* $A' \prec A \cap H(\chi^{+k})$ *such that* $\tau', A' \in A$ *and* $\langle \tau', A' \rangle$ *and* $\langle \tau, A \cap H(\chi^{+k}) \rangle$ *realize the same* $tp_{\alpha,n,k}$. *Moreover, if* $|A| \in A$, *then we can find such* A' *of cardinality* $|A|$.

Lemma 6.4 *Suppose that* $A \prec \mathfrak{A}_{\alpha,n,k+1}, |A| \geq \kappa_n^{+k+1}$, $B \prec \mathfrak{A}_{\alpha,n,k}$, *and* $C \in \mathcal{P}(B) \cap A \cap H(\chi^{+k})$. *Then there is* D *so that*

(a) $D \in A$

(b) $C \subseteq D$

(c) $D \prec A \cap H(\chi^{+k}) \prec H(\chi^{+k})$.

(d) $tp_{\alpha,n,k}(C, B) = tp_{\alpha,n,k}(C, D)$.

The next definition is analogous to those of [**?**] which in turn is similar to those of [2], but deals with cardinals rather than ordinals. The first two cases are added here for notational simplicity.

Definition 6.5 Let $k \leq n$ and $\nu = \kappa_{\alpha n}^{+\beta+1}$ for some $\beta \leq \kappa^{++}_{\alpha,n,g_\alpha(n)-1,\omega_1}$. The cardinal ν is called *k-good* iff $\nu = \kappa_{\alpha n}^{+n+1}$ (i.e. $\beta = n+1$) or $\nu = \delta_n$ (i.e.$\beta = \kappa_n^{+n+2}$) or the following holds

(1) β is a limit ordinal of cofinality at least $\kappa^{++}_{\alpha n}$

(2) for every $\gamma < \beta$ $tp_{\alpha,n,k}(\gamma, \beta)$ is realized unboundedly many times in $\kappa^{+n+2}_{\alpha,n}$ or equivalently $tp_{\alpha,n,k}(\kappa_n^{+\gamma+1}, \nu)$ is realized unboundedly many times in $\kappa^{+}_{\alpha,n,g_\alpha(n)-1,\omega_1}$.

ν is called *good* iff for some $k \leq n$ ν is k-good.

The following lemma was proved in [2] in context of ordinals, but is true easily for cardinals as well.

Lemma 6.6 *Suppose that a cardinal* $\nu = \kappa_n^{+\beta+1}$ *is* k*-good for some* $k, 0 < k \leq n$ *and* $\beta, n+1 < \beta < \kappa_n^{+n+2}$. *Then there are arbitrary large* $k-1$*- good cardinals below* $\kappa_n^{+\beta}$.

7 The Main Forcing.

Suitable structures and suitable generic structures are defined similar to those in Sections 1.2 or 2.4 of [6].

We would like to define the main forcing $\mathcal{P}$. Let us split the definition into ω–many steps. First we define pure conditions $\mathcal{P}_0$, at the next step $\mathcal{P}_1$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_0$, then $\mathcal{P}_2$ will be the set of all one step non direct extensions of elements of $\mathcal{P}_1$, etc. Finally $\mathcal{P}$ will be $\bigcup_{n<\omega}\mathcal{P}_n$.

Definition 7.1 The set $\mathcal{P}_0$ consists of all sequences

$$\langle p_\alpha \mid \alpha < \omega_1 \text{ and } (\alpha = 0 \text{ or } \alpha \text{ is a successor ordinal })\rangle$$

such that

1. $p_\alpha = \langle p_{\alpha\beta} \mid \alpha < \beta < \omega_1 \text{ is a successor ordinal } \rangle$ and for all $n < \omega, \alpha < \beta < \omega_1$ is a successor ordinal the following hold:

   (a) $p_{\alpha\beta} = \langle p_{\alpha\beta x} \mid x \in \mathit{connect}(\alpha, \beta)\rangle$, where for every $x \in \mathit{connect}(\alpha, \beta)$ we have $p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha x}, f_{\alpha\beta x}\rangle$ is such that:

      i. if $x \in \mathit{aconnect}(\alpha, \beta)$, $x = ((k_1, n_1), (k_2, n_2))$, for some $k_1, k_2, n_1, n_2 < \omega$, then

         A. $\mathrm{dom}(a_{\alpha\beta x})$ is a suitable generic structure for $G(\mathcal{P}') \cap V_{\tilde{\kappa}_\beta}$, where $\tilde{\kappa}_\beta \geq \kappa_\beta$ was defined in Section 5.

         B. $a_{\alpha\beta x}$ is an homomorphism between a generic suitable structure $\mathrm{dom}(a_{\alpha\beta x})$ from the the interval $[\kappa^+_{\beta-1}, \kappa_{\beta,n_1}]$ and those at the level of n_1 of α.
         We require that for every $X, Y \in \mathrm{dom}(a_{\alpha\beta x})$,
         $a_{\alpha\beta x}(X) = a_{\alpha\beta x}(Y)$ iff $X \cap \kappa_{\beta,n_1} = Y \cap \kappa_{\beta,n_1}$. Moreover the function a' on $\{X \cap V_{\kappa_{\beta,n_1}} \mid X \in \mathrm{dom}(a_{\alpha\beta x})\}$ defined by $a'(X \cap V_{\kappa_{\beta,n_1}}) = a_{\alpha\beta x}(X)$ is an isomorphism. [1]
         Recall that once automatic connection acts between α and β, then $n_2 \leq n_1$.
         Actually the domain of $a_{\alpha\beta x}$ may be rather a name in the part of the forcing above κ_α. Namely it may depend on Prikry sequences for β up to the level n_2 of β. So together $\langle a_{\alpha\beta x} \mid x \in \mathit{connect}(\alpha, \beta)\rangle$ may depend

[1] The reason for using taller models than just those in $V_{\kappa_{\beta,n_1}}$ (and $a_{\alpha\beta x}$ instead of a') is that below, in the proof of the Prikry condition, such taller models are used essentially.

on full Prikry sequences of β, i.e. on the forcing $\mathcal{P} \setminus \kappa_\beta$. This is needed in order to prove the Prikry condition of our forcing. However conditions are in V and this is used in the chain condition argument below.

C. $a_{\alpha\beta x}(\kappa^+_{\beta-1})$ is β-th measurable (or β-th Mahlo) of the block k_1 of the level n_1 of α.

I.e. connections for β always start leaving the first β-places over α untouched. This is needed for commutativity of connections. Thus, say that we have also $\gamma > \beta$. Then $\kappa^+_{\gamma-1}$ will correspond to γ-th measurable of blocks over β and over α. $a_{\alpha\beta x}$ will move then the indiscernible corresponding to γ-th measurable of blocks over β to γ-th measurable of blocks over α.

D. $|a_{\alpha\beta x}| < \kappa_{\alpha,x}$;

E. $A_{\alpha,x}$ is the set of measure one for $E_{\alpha,n_1,\eta}$, for some η which is above (in the order of the extender E_{α,n_1}) [2] of $\max(\mathrm{rng}(a_{\alpha\beta x}))$.

Note that $A_{\alpha,x}$ does not depend on β, i.e. we have the same set of measure one for each β. Further let us denote this η by $mc(\alpha,x)$ (the maximal coordinate of α, x).

We require that $A_{\alpha,x}$ does not depend on (k_1, β, n_2, k_2) and depends only on α, n_1.

F. $f_{\alpha\beta x}$ is a partial function from $\kappa_{\beta,x}$ to $\kappa_{\alpha,x}$ of cardinality at most $\kappa_{\beta-1}$.

ii. If $x \in mconnect(\alpha,\beta)$, $x = ((k_1,n_1),(k_2,n_2))$, for some $k_1,k_2,n_1,n_2 < \omega$, then the following holds

A. $\mathrm{dom}(a_{\alpha\beta x})$ is a suitable generic structure for $G(\mathcal{P}') \cap V_{\tilde{\kappa}_\beta}$, where $\tilde{\kappa}_\beta \geq \kappa_\beta$ was defined in Section 5.

B. $a_{\alpha\beta x}$ is an homomorphism between a generic suitable structure $\mathrm{dom}(a_{\alpha\beta x})$ from the the interval $[\kappa^+_{\beta-1}, \kappa_{\beta,n_1}]$ and those at the block k_1 of the level n_1 of α. A drop will occur here to some $\alpha' < \alpha$.

We require that for every $X, Y \in \mathrm{dom}(a_{\alpha\beta x})$,
$a_{\alpha\beta x}(X) = a_{\alpha\beta x}(Y)$ iff $X \cap \kappa_{\beta,n_1} = Y \cap \kappa_{\beta,n_1}$. Moreover the function a' on $\{X \cap V_{\kappa_{\beta,n_1}} \mid X \in \mathrm{dom}(a_{\alpha\beta x})\}$ defined by $a'(X \cap V_{\kappa_{\beta,n_1}}) = a_{\alpha\beta x}(X)$ is an isomorphism.

Again the domain of $a_{\alpha\beta x}$ may be rather a name in the part of the forcing

[2]Once using an alternative approach there will be $E_{\alpha,n_1,k_1,\eta}$ and E_{α,n_1,k_1} instead of $E_{\alpha,n_1,\eta}$ and E_{α,n_1}.

above κ_α. Namely it may depend on Prikry sequences for β up to the level n_2 of β.

C. $a_{\alpha\beta x}(\kappa^+_{\beta-1})$ is β-th measurable (or β-th Mahlo) of the block k_1 of the level n_1 of α.

D. $|a_{\alpha\beta x}| < \kappa_{\alpha'}$;

E. $A_{\alpha,x}$ is the set of measure one for $E_{\alpha,n_1,\eta}$, for some η which is above (in the order of the extender E_{α,n_1}) of $\max(\mathrm{rng}(a_{\alpha\beta x}))$ (see the footnote).
We require that $A_{\alpha,x}$ depends only on α, n_1.
Further let us denote this η by $mc(\alpha, x)$ (the maximal coordinate of α, x).

F. $f_{\alpha\beta x}$ is a partial function from $\kappa_{\beta,x}$ to $\kappa_{\alpha,x}$ of cardinality at most $\kappa_{\beta-1}$.

(b) (Commutativity of connections) Let β, γ be successor ordinals, $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$. Assume that k_α-th block of n_α-th level of α is connected to k_β-th block of a level n_β of β and to k_γ-th block of a level n_γ of γ. Suppose that in addition that k_β-th block of a level n_β of β and k_γ-th block of a level n_γ of γ are connected.
Then for each $Z \in \mathrm{dom}(a_{\alpha\gamma((k_\alpha,n_\alpha),(k_\gamma,n_\gamma))})$ we have $Z \in \mathrm{dom}(a_{\beta\gamma((k_\beta,n_\beta),(k_\gamma,n_\gamma))})$ and

$$a_{\alpha\gamma((k_\alpha,n_\alpha),(k_\gamma,n_\gamma))}(Z) = a_{\alpha\beta((k_\alpha,n_\alpha),(k_\beta,n_\beta))}(a_{\beta\gamma((k_\beta,n_\alpha),(k_\beta,n_\gamma))}(\underset{\sim}{Z})),$$

where $\underset{\sim}{Z}$ is a name of the indiscernible corresponding to Z.

2. We need to deal here simultaneously with ω–many levels for every interval $(\kappa_\alpha, \kappa_{\alpha+1})$, $\alpha < \omega_1$.
Fix $\alpha < \omega_1$. We will allow to use names for models of the interval $[\kappa_\alpha, \kappa_{\alpha+1}]$. Still all the models involved will be in $G' = G'(\mathcal{P}')$ the generic subset of the preparation forcing $\mathcal{P}'$. Names will be of a very particular form as described below.

Start with a description of such names for the first level.
There will be a model $M \in G(\mathcal{P}')$ of cardinality $\kappa_{\alpha+1,0}$ in the domain of the assignment function and an $\in$–increasing continuous sequence $\langle M_\eta \mid \eta < \kappa_{\alpha+1,0}, \eta \text{ is a cardinal} \rangle$ such that $|M_\eta| = \eta$, $M_\eta \in M \cap G(\mathcal{P}')$ and M_η is on the central piste (line) according to M.
For every η in the set of measure one, i.e. in $A_{\alpha,0}$, M_η will appear in the domain once η was added.
In addition there is a sequence $\langle \tilde{B}_\eta \mid \eta < \kappa_{\alpha+1,0}, \eta \text{ is a cardinal} \rangle$ such that

(a) $\tilde{B}_\eta$ is a suitable generic structure consisting of models of cardinalities $< \eta$.
Let B_η denotes the top model of $\tilde{B}_\eta$.

(b) $|B_\eta| = \kappa_\alpha^+$,

(c) $B_\eta \cap M_\eta = B_0 \cap M_0$, moreover $\tilde{B}_\eta, \tilde{B}_{\eta'}$ realize the same type over their intersection and they consist of models of same sizes, for every $\eta' < \kappa_{\alpha+1,0}, \eta'$ is a cardinal .

The intuition behind this is that once η is added, then also $\tilde{B}_\eta$ is.

Require that $\tilde{B}_0$ is in the domain of the assignment function. Also require that the largest model of the domain, call it B, is so that

(a) $|B| = \kappa_\alpha^+$,

(b) $\langle M_\eta \mid \eta < \kappa_{\alpha+1,0}, \eta \text{ is a cardinal } \rangle \in B$.
Note that $M \in B$ just due to the maximality of B.

(c) $\langle \tilde{B}_\eta \mid \eta < \kappa_{\alpha+1,0}, \eta \text{ is a cardinal } \rangle \in B$,

(d) $\min(A_{\alpha,0}) > \sup(B \cap \kappa_{\alpha,0})$.

Describe now names which depend on two levels, i.e. $\kappa_{\alpha+1,0}, \kappa_{\alpha+1,1}$. A general case deals with finitely many levels. It repeats the setting below only with a bit more complicated notation.
As before, we will have a model $M \in G(\mathcal{P}')$ of cardinality $\kappa_{\alpha+1,0}$ in the domain of the assignment function and an $\in$–increasing continuous sequence
$\langle M_\eta \mid \eta < \kappa_{\alpha+1,0}, \eta \text{ is a cardinal } \rangle$ such that $|M_\eta| = \eta$, $M_\eta \in M \cap G(\mathcal{P}')$ and M_η is on the central piste (line) according to M. In addition require that there are sequences $\langle M_{1\eta} \mid \eta < \kappa_{\alpha+1,0}, \eta \text{ is a cardinal } \rangle$, $\langle M_{\eta,\xi} \mid \xi < \kappa_{\alpha+1,1}, \xi \text{ is a cardinal } \rangle$ such that

(a) $\langle M_{1\eta} \mid \eta < \kappa_{\alpha+1,0}, \eta \text{ is a cardinal } \rangle$ is an $\in$–increasing continuous,

(b) $|M_{1,\eta}| = \kappa_{\alpha+1,1}$, for every η,

(c) $M_{1,\eta+1} \in M_{\eta+1}$, and $M_{1,\eta+1}$ is on the central piste according to $M_{\eta+1}$, for every η.

(d) $\langle M_{\eta,\xi} \mid \xi < \kappa_{\alpha+1,1}, \xi \text{ is a cardinal } \rangle$ is an $\in$–increasing continuous, for each η, and such that $|M_{\eta,\xi}| = \xi$, $M_{\eta,\xi} \in M_{1,\eta} \cap G(\mathcal{P}')$ and $M_{\eta,\xi}$ is on the central piste (line) according to $M_{1,\eta}$.

For every $(\eta, \xi) \in A_{\alpha,0} \times A_{\alpha,1}$, $M_\eta, M_{1,\eta}$ and $M_{\eta,\xi}$ will appear in the domain once (η, ξ) was added.

In addition there is a sequence $\langle \tilde{B}_{\eta,\xi} \mid \eta < \kappa_{\alpha+1,0}, \xi < \kappa_{\alpha+1,1}, \eta, \xi$ are cardinals $\rangle$ such that for every cardinal $\eta < \kappa_{\alpha+1,0}$ the following hold:

(a) $\tilde{B}_{\eta,\xi}$ is a suitable generic structure consisting of models of cardinalities $< \xi$. Let $B_{\eta,\xi}$ denotes the top model of $\tilde{B}_{\eta,\xi}$.

(b) $|B_{\eta,\xi}| = \kappa_\alpha^+$,

(c) $B_{\eta,\xi} \cap M_{\eta,\xi} = B_{\eta,0} \cap M_{\eta,0}$, moreover $\tilde{B}_{\eta,\xi}, \tilde{B}_{\eta,\xi'}$ realize the same type over their intersection and they consist of models of same sizes, for every $\eta' < \kappa_{\alpha+1,1}, \xi'$ is a cardinal .

Require that the sequence $\langle \tilde{B}_{\eta,0} \mid \eta < \kappa_{\alpha+1,0}, \eta$ is a cardinal $\rangle$ have the properties of the sequence $\langle \tilde{B}_\eta \mid \eta < \kappa_{\alpha+1,0}, \eta$ is a cardinal $\rangle$ above.

Require that $\tilde{B}_{0,0}$ is in the domain of the assignment function. Also require that the largest model of the domain, call it B, is so that

(a) $|B| = \kappa_\alpha^+$,

(b) $\langle M_\eta \mid \eta < \kappa_{\alpha+1,0}, \eta$ is a cardinal $\rangle \in B$. Note that $M \in B$ just due to the maximality of B.

(c) $\langle M_{1\eta} \mid \eta < \kappa_{\alpha+1,0}, \eta$ is a cardinal $\rangle$, $\langle M_{\eta,\xi} \mid \xi < \kappa_{\alpha+1,1}, \eta, \xi$ are cardinals $\rangle \in B$.

(d) $\langle \tilde{B}_{\eta,\xi} \mid \eta < \kappa_{\alpha+1,0}, \xi < \kappa_{\alpha+1,0}, \eta, \xi$ are cardinals $\rangle \in B$,

(e) $\min(A_{\alpha,0}) > \sup(B \cap \kappa_{\alpha,0})$,

(f) $\min(A_{\alpha,1}) > \sup(B \cap \kappa_{\alpha,1})$.

3. Let $\beta < \omega_1$. Lot of indiscernibles are added between κ_β and $\kappa_{\beta+1}$. Consider for example a level $n < \omega$ level of $\kappa_{\beta+1}$. The number of possibilities (the cardinality of the corresponding set of measure one) is above κ_β, however sizes of suitable structures of further levels of $\kappa_{\beta+1}$ is $< \kappa_\beta$ (even if there is no drops in cofinalities).
So in order to show the Prikry condition of the forcing names of models which depend on choices of members of relevant measure one sets should be used.
Here a model A of suitable structure of a level $m > n$ is allowed to be a name $\underset{\sim}{A}$ which depends on a choice of a sequence $\langle \eta_k \mid k \leq n \rangle$ of elements of measure one sets over levels $k \leq n$. So the actual model A (in V) will be the interpretation $\underset{\sim}{A}[\langle \eta_k \mid k \leq n \rangle]$.
Let us organize this as follows:
if $\underset{\sim}{A}$ is a name as above which is in a suitable structure, then we require that a least model D (all of them are from the generic structure) of cardinality $\kappa_{\beta n}$ with $\underset{\sim}{A} \in D$

appears there. In addition require that also a models B with $\underset{\sim}{A}, D \in B$, $|B| = \kappa_\beta^+$ all along the piste leading to D are there.

4. We describe here situations in which dropping in cofinalities occurs and state the related requirements. All connections from ordinals $\gamma, \alpha+1 < \gamma < \omega_1$ to ordinals below $\alpha+1$ are replaced by connections to $\alpha+1$ (which are very closed) and from $\alpha+1$ down. We split into two cases according to α being a successor or limit.
Case 1. α is a limit ordinal.
Fix then in advance a cofinal in α sequence $\langle \alpha_i \mid i < \omega \rangle$. Denote $\rho(\alpha_i, \alpha)$ by n_i. An isomorphism between suitable structures will move one over first n_0 levels over $\alpha+1$ to those over the level n_0 of α_0 (*aconection*). There may be in addition blocks over levels $\leq n_0$ of α_0 connected manually to blocks of the first n_0 levels of $\alpha+1$. We keep such connections. Note that this may result in moving models of different cardinalities into models of a same cardinality (still " $\in$ "–relation is preserved).
In general if $k < \omega$, then an isomorphism between suitable structures will move one over first n_k levels over $\alpha+1$ to those over the level n_k of α_k (*aconection*). There may be in addition blocks over levels $\leq n_k$ of α_k connected manually to blocks of the first n_k levels of $\alpha+1$. We keep such connections.
The rest of connections,i.e. connections of α with β's below α_0, or $\beta \in (\alpha_k, \alpha_{k+1})$ will obtained using the commutativity. Thus, in order to get to $\beta \in (\alpha_k, \alpha_{k+1})$, let us go down to α_{k+1} first using *aconnect* and then continue from α_{k+1} to β.
Note that by 4.2(4i) *aconnected* elements cannot mix with the rest. Namely, if a block m of a level n of $\alpha+1$ is connected manually to a block s of a level r of β_k, then $n > r$ and so $\langle \alpha+1, n, m \rangle$ cannot be connected automatically to the level r of β_k.
Case 2. $\alpha = \tau + 1$.
In this case send all connections which go from $\alpha+1$ down, first to $\tau+1$ and then from $\tau+1$ further down.

   Suppose now that $\langle \alpha+1, n_1, m_1 \rangle, \langle \alpha+1, n_2, m_2 \rangle$ are both connected to $\langle \beta, r, s \rangle$ at same stage $n < \omega$ (i.e. once a level n and levels below of $\alpha+1$ are considered, in particular $n_1, n_2 \leq n$), where $\beta = \beta_k$, for some $k < \omega$ in **Case 1**, or $\beta = \tau+1$ in **Case 2**. Then this connections should be both manual by 4.2(4i), since the automatic connection cannot connect two different blocks for $\alpha+1$ to same block down, so at least one of the blocks is connected manually and then r is below the level of this block which is at most n. Recall that automatic connection in this case connects between first n-levels of $\alpha+1$ and the level n of β. We have $r < n$, hence the connection to the level r here can be only the manual one.

mi po:

Let us describe the dropping in cofinality that occurs here. Note that there may be infinitely many blocks of $\alpha+1$ connected to the block $\langle \beta, r, s\rangle$. At most one of them is a-connected to $\langle \beta, r, s\rangle$. Organize the dropping in cofinalities as follows. Denote by X the set of blocks of $\alpha+1$ connected to the block $\langle \beta, r, s\rangle$. Let $\langle x_i \mid i < j \leq \omega\rangle$ be the listing of X increasing in the lexicographical order. Let $x_i = \langle \alpha+1, n_i, m_i\rangle$. Assume that x_0 is a-connected to $\langle \beta, r, s\rangle$ (if a-connected block exists). Consider x_1, provided that there is a-connected block, otherwise replace x_1 by x_0.
We let the block x_1 to correspond to $\langle \beta, r, s\rangle$ and the blocks below it (i.e. $\langle \alpha+1, n, m\rangle$ with $n = n_i$ and $m < m_i$, or $n < n_i$) to drop below β to places which are a-connected to $\langle \alpha+1, n_1, m_1-1\rangle$, if $m_1 > 0$, or with $\langle \alpha+1, n_1-1, g_{\alpha+1}(n_1-1)-1\rangle$, if $m_1 = 0$. Note that ω–many possibilities for a drop in cofinality are allowed here. Use splitting into intervals in order to incorporate drops to different levels below β. It is similar to what was done Section 4 of [6]. Such setting allows us to gain a closure once a non-direct extension is made at some γ below β. Just the only drops that remain really active are those to points above γ.
Continue in the same fashion with every $i > 1$.
Denote this last block before x_i by x_i^*. On x_i^* and the blocks below the isomorphism between suitable structures will respect cardinalities, but not on x_i itself. Thus there may be models from x_j $(j > i)$ with images having same cardinalities as those from x_i. Still $\in$–relation is preserved. The thing that prevents cardinals collapses (and actually insures the right chain condition) is the requirement that any model Z (from the domain of an isomorphism) is connected starting from a certain level only by an a-connection. Note that once we have models $Y \in Z$, $|Y| > |Z|$ and $Z' \in Y$ need to be added, where Z, Z', Y are of Δ–system type, then we can go to a level from which both Z, Y are a-connected. At this level and above the difference in cardinalities of Y, Z is respected. So it is possible to find the image for Z' inside the image of Y over the image of $Z \cap Y$, since the image of $Z \cap Y$ has cardinality of the image of Z and so is bounded in the image of Y.

Let us explain splitting into intervals a bit more. Suppose that some cardinality η over $\alpha+1$ drops down below β to cardinalities $\langle \eta_{\beta k} \mid k < \omega\rangle, \eta_{\beta k} < \eta_{\beta k+1}$ and $\bigcup_{k<\omega} \eta_{\beta k} = \kappa_\beta$. Now, given a maximal model $A^{0\eta}$ in the domain of the assignment function (denote it by a). We consider its piste (central line) $C^\eta(A^{0\eta})$. There is an increasing sequence

$\langle B_i \mid i \leq i^* \rangle$ of elements of $C^\eta(A^{0\eta}) \cap \mathrm{dom}(a)$, for some $i^* \leq \omega$, such that $A^{0\eta} = B_{i^*}$ and for every $i < i^*$ there is $k_i < \omega$, $|a(B_i)| = \eta_{\beta k_i}$. We have here a splitting into intervals $(B_i, B_{i+1}]$ along $C^\eta(B_{i+1}) = C^\eta(A^{0\eta}) \cap (B_{i+1} \cup \{B_{i+1}\})$.
Suppose now that a non-direct extension was made at $\eta_{\beta 0}$. Let ν be a member of the corresponding set of measure one which was added. Let A be the largest model of an interval for $\eta_{\beta 0}$ and its image $a(A)$ is of order type ν. We require that there is an increasing sequence $\langle B_i^A \mid i < \nu \rangle$ in the domain of a consisting of models of cardinalities $> \eta_{\beta 0}$ but still of dropping ones, with cardinalities of drops there unbounded in all drops, and such that $\langle a(B_i^A) \mid i < \nu \rangle$ is unbounded in $a(A)$. [3]

Require that if A' is a model in $\mathrm{dom}(a)$ and, as A, it in an interval for $\eta_{\beta 0}$, then $A' \in A$ implies that $\langle B_i^{A'} \mid i < \nu \rangle$ is bounded in $\langle B_i^A \mid i < \nu \rangle$.
We allow different A, A' to have the same sequence of B's provided A' appears on f–side of the condition and $a(A) = a(A')$. This is needed in order to show the Prikry condition.
$a(A) = a(A')$ is allowed (for dropping cardinalities) only
if $a(\mathrm{otp}(C^{|A|}(A))) = a(\mathrm{otp}(C^{|A'|}(A')))$, i.e. if a non-direct extension was made over the level to which the cardinality $|A| = |A'|$ drops, then the values of a on $\mathrm{otp}(C^{|A|}(A))$ and $\mathrm{otp}(C^{|A|}(A'))$ are the same. This essential in order to show the chain condition working with names.

We require the following (needed for the chain condition argument):
if $\nu' < \nu$ are in the set of measure one for $\eta_{\beta 0}$ and A', A correspond to them over β, then $\langle B_i^{A'} \mid i < \nu \rangle$ is bounded in $\langle B_i^A \mid i < \nu \rangle$.
This allows to put together conditions (in the chain condition argument) exactly as in a single drop situation of [6] (Chapter 4).

Definition 7.2 Suppose $p = \langle p_\alpha \mid \alpha < \omega_1 \text{ and } (\alpha = 0 \text{ or } \alpha \text{ is a successor ordinal }) \rangle \in \mathcal{P}_0$, $\alpha < \omega_1$ be zero or a successor ordinal, $\beta, \alpha < \beta < \omega_1$ a successor ordinal and $x = ((n_\alpha, m_\alpha), (n_\beta, m_\beta)) \in \mathit{connect}(\alpha, \beta)$. Let $p_{\alpha\beta x} = \langle a_{\alpha\beta x}, A_{\alpha,x}, f_{\alpha\beta x} \rangle$ and $\eta \in A_{\alpha,x}$. Define $p^\frown \eta$, the one element non direct extension of p by η, to be $q = \langle q_\xi \mid \xi < \omega_1 \text{ and } (\xi = 0 \text{ or } \xi \text{ is a successor ordinal }) \rangle$ so that

1. for every $\xi, \zeta, \alpha < \xi < \zeta < \omega_1, \quad p_{\xi\zeta} = q_{\xi\zeta}$,

[3] The requirement that cardinalities of drops there are unbounded in all drops is needed in order to deal with further non-direct extensions. Thus, if this cardinalities were allowed to be bounded, then picking a non-direct extension in a dropping cardinality above all of them may change completely the sequence with unbounded in $a(A)$ images.

2. for every $y \in connect(\alpha, \beta)$ with the level on α bigger than n_α we have $p_{\alpha\beta y} = q_{\alpha\beta y}$.

3. for every successor ordinal $\beta, \alpha < \beta < \omega_1$, $\quad q_{\alpha\beta y} = f_{\alpha\beta y} \cup \{\langle \tau, \pi_{mc(\alpha,n),a_{\alpha\beta y}(\tau)}(\eta)\rangle \mid \tau \in \mathrm{dom}(a_{\alpha\beta y})\}$, where $y \in connect(\alpha, \beta)$ and the level of y over α is n_α as those of x.

4. Let $\alpha', \tau, \alpha' > \alpha > \tau$, be successor ordinals or zero. Then connections $a_{\tau\alpha' y}$ of p will split now in q into connections from α' to α followed by a connection from α to τ. Namely, let $\langle \tau, r, s\rangle$ be connected with $\langle \alpha', n', m'\rangle$. For each (n, m) such that $((n, m), (n', m')) \in aconnect(\alpha', \alpha)$ and $\langle \tau, r, s\rangle \in connect(\alpha, n, m)$ (the are such n, m by Lemma 4.6) split $a_{(\alpha',n',m'),(\gamma,r,s)}$ into $a_{(\alpha',n',m'),(\alpha,n,m)}$ followed by $a_{(\gamma,r,s),(\alpha,n,m)}$.

5. For each level $n' < n_\alpha$ of α, the same things occur, i.e. 2-4 above hold with (n_α, m_α) replaced by (n', k'), where k' is any block of the level n'.

6. Each connection which drops in cofinality below the block of η, i.e. below the level n_α of α, we freeze such drops and deal only with drops to cofinalities above ηin a fashion used in Section 6 of [6] for same purpose.

Definition 7.3 Set $\mathcal{P}_1$ to be the set all $p^\frown\eta$ as in Definition 7.2. Proceed by induction. For each $n < \omega$, once $\mathcal{P}_n$ is defined, define $\mathcal{P}_{n+1}$ to be the set of all $p^\frown\eta$, where $p \in \mathcal{P}_n$. Finally set $\mathcal{P} = \bigcup_{n<\omega} \mathcal{P}_n$.

Definition 7.4 Let $p, q \in \mathcal{P}$.

1. We say that p is a direct extension of q and denote this by $p \geq^* q$ iff p is obtained from q by extending $a_{\alpha\beta x}, f_{\alpha\beta x}$'s and by shrinking the sets of measures one probably by passing to bigger measure first.

2. The forcing order $\geq$ is defined as follows:
   $p \geq q$ iff there are $\eta_1, ..., \eta_n$ such that $q^\frown\eta_1^\frown...^\frown\eta_n$ is defined and $p \geq^* q^\frown\eta_1^\frown...^\frown\eta_n$.

For each $\alpha < \omega_1$. $\mathcal{P}$ splits into $(\mathcal{P}\backslash\kappa_\alpha) * \mathcal{P} \restriction \kappa_{\alpha+1}$, where $\mathcal{P}\backslash\kappa_\alpha$ is the part of $\mathcal{P}$ is defined as $\mathcal{P}$ but with $\kappa_{\alpha+1}$ replacing κ_0, i.e. everything is above κ_α and the first cardinal we deal with is $\kappa_{\alpha+1,0}$. $\mathcal{P} \restriction \kappa_{\alpha+1}$ is defined in $V[G']^{\mathcal{P}\backslash\kappa_\alpha}$ as $\mathcal{P}$ was defined in $V[G']$, but cutting everything at $\kappa_{\alpha+1}$, where $G' = G(\mathcal{P}')$ is a generic subset of the preparation forcing $\mathcal{P}'$.

Lemma 7.5 *(a) For every $\alpha < \omega_1$, $(\mathcal{P}'_{\geq\kappa_{\alpha+1}} * \langle \mathcal{P} \setminus \kappa_{\alpha+1}, \leq^* \rangle)/\mathcal{P}'_{<\kappa_{\alpha+1}}$ is $\kappa_\alpha-$ strategically closed above any non direct extension made over $\kappa_{\alpha+1}$, where $\mathcal{P}'_{\geq\kappa_{\alpha+1}}$ is the restriction of $\mathcal{P}'$ to models of sizes $\geq \kappa_{\alpha+1}$ and $\mathcal{P}'_{<\kappa_{\alpha+1}} = \mathcal{P}'/\mathcal{P}'_{\geq\kappa_{\alpha+1}}$, i.e. models of sizes $< \kappa_{\alpha+1}$.*

Moreover, if a non-direct extension was made at a level n of $\alpha+1$ by adding some η from the set of measure one, then it will be η^+– strategically closed.

(b) *For every limit $\alpha<\omega_1$ and $\delta<\alpha$, $(\mathcal{P}'_{\geq\kappa_{\alpha+1}} * \langle \mathcal{P}\setminus\kappa_\alpha, \leq^* \rangle)/\mathcal{P}'_{<\kappa_{\alpha+1}}$ is κ_δ– strategically closed above a non–direct extension made over κ_δ.*

Remark 7.6 It is possible to use the argument of the proof of the next lemma 7.7(The Prikry condition), which appeals to elementary submodels of $H(\chi)[G(\mathcal{P}')]$, in order to show that the forcing $\langle \mathcal{P}\setminus\kappa_{\alpha+1}, \leq, \leq^* \rangle$ (over $V[G(\mathcal{P}')]$) does not add new bounded subsets of $\kappa_{\alpha+1}$.

Proof. Let us deal with (a), the part (b) is similar.

Models of domains of assignment functions in the forcing $\mathcal{P}\setminus\kappa_{\alpha+1}$ have sizes $\geq\kappa_{\alpha+1}$. Such models where added by $\mathcal{P}'_{\geq\kappa_{\alpha+1}}$, i.e. the upper part of $\mathcal{P}'$ or more precisely by the restriction of $\mathcal{P}'$ to models of sizes $\geq\kappa_{\alpha+1}$. The forcing $\mathcal{P}'_{\geq\kappa_{\alpha+1}}$ is $\kappa_{\alpha+1}$–strategically closed. However, the measures in the interval $(\kappa_\alpha,\kappa_{\alpha+1})$ of the forcing $\mathcal{P}\setminus\kappa_{\alpha+1}$ depend on $\mathcal{P}'_{<\kappa_{\alpha+1}}$ and the last forcing is not enough strategically closed. In order to overcome this difficulty, we can just work with $\mathcal{P}'_{<\kappa_{\alpha+1}}$-names of relevant parts. So, there is no need to increase conditions from $\mathcal{P}'_{<\kappa_{\alpha+1}}$ in the process.

Suppose that a non-direct extension was made at a level n of $\alpha+1$ by adding some η from the set of measure one. Then for any successor ordinals $\beta,\gamma,\alpha+1<\beta<\omega_1,\gamma\leq\alpha+1$ connections from β to γ will split to those from β to $\alpha+1$ and then further down from $\alpha+1$ to γ. There may be some blocks of $\alpha+1$ of levels above n which are manually connected to those of β. Then droppings in cofinality occur at such places. Note that droppings in cofinality whenever occur come in intervals and there are intervals of cardinality above η. This provides a degree of closure above η.

□

Let us prove now the Prikry condition.

Lemma 7.7 *$\langle \mathcal{P}, \leq, \leq^* \rangle$ is a Prikry type forcing notion.*

Proof. Let $p\in\mathcal{P}$ and σ be a statement of the forcing language. We need to show that there is a direct extension of p which decides σ. Suppose otherwise. Let us build by induction sequences of extensions of p. Proceed as follows. Climb up from below over α's less than ω_1. Suppose that $\alpha<\omega_1$ and we already took care of every $\beta<\alpha$. Let $p^\alpha\geq^* p$ be the resulting condition. We deal with levels of α by induction again. So let n be a level of α. Pick η from

the set of measure one of p^α at this level. If there is an extension $q = q^{\alpha n\eta}$ of $p^{\alpha\frown}\eta$ which is non-direct only below α or below the level n of α and decides σ, then we keep the "direct addition" of $q^{\alpha n\eta}$ to p^α from the level n of α and up and set back all the rest (i.e. over n and below). Denote it by $p^{\alpha n\eta}$. If there is no such q, then we keep p^α unchanged and set $p^{\alpha n\eta} = p^\alpha$.
Note that once η was added the connections from $\alpha' > \alpha$ to ordinals below α split leaving the part from α' to α closed enough, since droppings in cofinality whenever occur come in intervals and there are intervals of cardinality above η. This provides degree of closure above η.
Names (depending on η) are allowed at levels above n of α, which eliminates a need in completeness there. Recall that sizes of suitable structures at such levels are $< \kappa_\alpha$ even when no dropping in cofinality occur. Here is a new point and we will address it specially below.

So, as usual, we can to accumulate things above α together and to continue the argument. At the next step we pick the least ξ in the set of measure one above η and repeat the above with $p^{\alpha n\eta\frown}\xi$. Define $q^{\alpha n\xi}$ and $p^{\alpha n\xi}$ as above.
Continue further in the same fashion through all elements of the measure one set. There is no problem at limit steps due to the degree of closure above the level n and above α Let $r^{\alpha n}$ be the direct extension of p^α obtained using the process above. Now we shrink the set of measure one in order to have same conclusions about σ (i.e. if it is forced by $r^{\alpha n\frown}\tau^\frown q^{\alpha n\tau}$ or its negation is forced by $r^{\alpha n\frown}\tau^\frown q^{\alpha n\tau}$ or no such extension of $r^{\alpha n\frown}\tau$ can decide σ). Let $p^{\alpha n}$ be the condition obtained from $r^{\alpha n}$ by replacing the set of measure one by this shrunken set. We claim that no decision about σ was made. Just otherwise we shrink more to stabilize the non-direct addition below n. Then actually the decision can be made without extending at the level n. Note only that parts of connecting functions will turn to be names which depend on a choice of an element from the measure one set. By the assumption there is no way to decide σ using non-direct additions from below the level n.
Hence, no decision about σ was made at n as well. Continue then to the levels $n+1, n+2$, etc. and further to bigger α's. Finally this process will produce a direct extension of p which has no extension (direct or undirect) which decides σ. Which is clearly impossible. Contradiction.

There is one new point here that should be addressed. In previous constructions (like in [6]) forcing notions $\langle \mathcal{P}, \leq^* \rangle$ where not closed, but rather the iteration of a preparation forcings $\mathcal{P}'$ with $\langle \mathcal{P}, \leq^* \rangle$ was strategically closed and this suffice for proving the Prikry condition. The reason was that the degree of strategic closure of $\mathcal{P}'$ was above κ and extensions used in the process where below κ. So the number of steps needed for the argument was at most κ which is below the degree of strategic closure.

In present setting this is somewhat different.

Say, for example, that we have $\alpha < \omega_1$ and run the Prikry argument between κ_α and $\kappa_{\alpha+1}$. In the preparation forcing $\mathcal{P}'$ we will have models of sizes $\geq \kappa_{\alpha+1}$, of sizes in the interval $[\kappa_\alpha, \kappa_{\alpha+1})$ and of sizes below κ_α.

The part above $\kappa_{\alpha+1}$ shares enough strategic closure in order to run the usual argument.

Models of sizes below κ_α (and which are below κ_α) can be stabilized since the degree of completeness of extenders used in the interval $[\kappa_\alpha, \kappa_{\alpha+1})$ is above κ_α.

This leaves us with models of sizes inside the interval $[\kappa_\alpha, \kappa_{\alpha+1}]$.

So let $n < \omega$ and suppose that we are at the level n of $\kappa_{\alpha+1}$, i.e. at $\kappa_{\alpha+1,n}$. Again we can use a strategic closure of models of sizes above $(\kappa_{\alpha,n,g_\alpha(n)-1,\omega_1})^+$, but models of smaller sizes (still above κ_α) will be treated as names corresponding to choices of elements of measure one set of a condition.

More precisely, to each ν in the set of measure one of a condition under the consideration will correspond models of a generic subset $G(\mathcal{P}')$ of $\mathcal{P}'$ of sizes $< \kappa_{\alpha+1,n}$. We cannot with $G(\mathcal{P}')$ itself in order to use the strategic closure of $\mathcal{P}'$, but rather with $\mathcal{P}'_{\geq \kappa_{\alpha+1}}$ (i.e. the part of $\mathcal{P}'$ above $\kappa_{\alpha+1}$). Hence models of smaller cardinalities will be actually names in such setting also in $\mathcal{P}'_{\geq \kappa_{\alpha+1}}$.

Let us simplify this as follows. Work already in $V[G(\mathcal{P}')]$, and in order to compensate the luck of strategic closure let us use appropriate elementary submodels instead.

Namely, we peak a sequence

$$\langle M(\kappa_{\alpha+1,n})^{\xi,m} \mid \alpha \leq \xi < \omega_1, m, n < \omega \text{ if } \alpha = \xi \text{ then } m \geq n \rangle$$

such that

1. $|M(\kappa_{\alpha+1,n})^{\xi,m}| = \kappa_{\xi+1,m}$,
2. $M(\kappa_{\alpha+1,n})^{\xi,m} \subseteq M(\kappa_{\alpha+1,n})^{\xi',m'}$ once $\xi < \xi'$ or $\xi = \xi'$ and $m < m'$.
3. $\sup(M(\kappa_{\alpha+1,n})^{\xi,m} \cap On) = \sup(M(\kappa_{\alpha+1,n})^{\alpha,n} \cap On)$,
4. for every $\alpha < \omega_1$, $\langle M(\kappa_{\alpha+1,n})^{\xi,m} \mid \alpha \leq \xi < \omega_1, m, n < \omega \text{ if } \alpha = \xi \text{ then } m \geq n \rangle \in M(\kappa_{\alpha'+1,n'})^{\alpha',n'}$, whenever $\alpha' > \alpha$ or $\alpha' = \alpha$ and $n' > n$.
5. $M(\kappa_{\alpha+1,n})^{\xi,m} \in G(\mathcal{P}')$,

6. $M(\kappa_{\alpha+1,n})^{\xi,m} \prec H(\chi)[G(\mathcal{P}')]$, with χ big enough.

7. $M(\kappa_{\omega_1}) := \bigcup_{\alpha<\omega_1} M(\kappa_{\alpha+1,n})^{\alpha,n}$ is such that

   (a) $M(\kappa_{\omega_1}) \in G(\mathcal{P}')$,

   (b) $M(\kappa_{\omega_1}) \prec H(\chi)[G(\mathcal{P}')]$,

   (c) each model $M(\kappa_{\alpha+1,n})^{\xi,m}$ is on the central piste relatively to $M(\kappa_{\omega_1})$.

This models are used during the induction in the Prikry argument, as explained below. Proceeding along the pistes provides the desired closure, say dealing with $\kappa_{\alpha+1,n}$, for models of sizes at least $\kappa_{\alpha+1,n}$. So we left with models of sizes in the interval $[\kappa_\alpha, \kappa_{\alpha+1,n})$ (at the level n). Now this models are members of $G(\mathcal{P}')$. Still we cannot eliminate completely use of names. Each ν in the set of measure one may have its own models of such sizes.
However, the advantage is that such names are more simple and easier to deal with.

Let us describe now precisely names of this type. It will be essential also for the chain condition argument.
Again let $n < \omega$ be a fixed level. Let X be a set of measure one of this level.
Let M denotes $M(\kappa_{\alpha+1,n})^{\alpha,n}$.
We proceed by induction on members of X. So let at a certain stage we need to consider $\nu \in X$. As usual, if no direct extension (after ν was added) decides a statement σ under the consideration, then nothing is changed.
Suppose otherwise. Then, the condition that we consider, with ν added, extends directly to one that decides σ. Pick such extension inside M. Add on the top of central pistes (lines) a model of size ν^0 in M from the fixed sequence leading to M which consists of models of cardinalities $< |M| = \kappa_{\alpha+1,n}$ from central pistes.
Continue the process. At the next stage the added models of sizes $< \nu$ are removed (and models of sizes $\geq \nu$ are keeped) first, then the next element ν' of X is added and we look for a direct extension that decides σ.
We do the following at limit stages.
First, the conditions generated so far are put together. Note that new models of sizes $< \nu$ are removed after dealing with ν. This guaranties that the result is still a condition.
Then we take the next element of X and continue in the fashion above.
Without loss of generality assume that at each stage in the process the largest (under $\in$) had cardinality κ_α^+ (i.e. the least possible size) and once drops in cofinality occur even of smaller sizes.

Finally, at the end, X is shrunken to a set of measure one X' on which all the decisions about σ are the same.
A new point here will be to use a pressing down function in order to stabilize small models used at each stage in X', provided σ was decided on a measure one set.
Let $\nu_0 = \min(X')$.
Denote the top model at a stage ν by A_ν, and the one of size ν generated as the union of models of sizes ν' from the previous stages $\nu' < \nu$ by N_ν. Then $\langle N_\nu \mid \nu \in X\rangle$ will be an increasing continuous sequence.
We freeze $A_\nu \cap N_\nu$, as well as $\mathrm{otp}(A_\nu)$ etc. So, $A_\nu \cap N_\nu = A_{\nu_0} \cap N_{\nu_0}$, for every $\nu \in X'$ and A_ν, A_{ν_0} are isomorphic over their intersection together with the rest of models added at stages ν_0 and ν respectively.
The condition used will be with measure one set $X' \setminus \{\nu_0\}$, to each $\nu \in X'$ will correspond the sequence of models (of small cardinalities) added in the process at stage ν (this is a name part). Also add A_{ν_0} the models for ν_0. Finally add a top model A of cardinality κ_α^+ on the central piste with all above being its member (i.e. $\in$ but not $\subseteq$) and shrink X' to X'' such that $\min(X'') \geq \sup(A \cap M \cap \kappa_{\alpha+1,n})$.
This will insure that once $\nu \in X''$ was picked, then $A_\nu \cap A = A_{\nu_0} \cap N_{\nu_0}$.

Suppose now that we deal not with a single level n but rather with finitely many of them. Let us illustrate the modification dealing with two levels 0 and 1.
We will use two increasing and continuous sequences of models $\langle M_{0\xi} \mid \xi < \kappa_{\alpha+1,0}\rangle$ and $\langle M_{1\xi} \mid \xi < \kappa_{\alpha+1,0}\rangle$ instead of a single model M above. Require that for every $\xi < \kappa_{\alpha+1,0}$

1. $|M_{0\xi}| = \kappa_{\alpha+1,0}$,
2. $|M_{1\xi}| = \kappa_{\alpha+1,1}$,
3. $M_{1\xi+1} \in M_{0\xi+1}$.

Let X_0 and X_1 denote sets of measures one of levels 0 and 1 respectively. We use $M_{0\xi}$, for every $\xi \in X_0$, and ran the previous argument for members of X_1 (with fixed ξ) inside $M_{1\xi}$.
□

The following Mathias-type property can be shown similar to the Prikry condition.

Lemma 7.8 *Let $p \in \mathcal{P}$ and $D \subseteq \mathcal{P}$ be a dense open. Then there are $p^* \geq^* p$ and a finite sequence $\langle\langle\beta_i, n_i\rangle \in \omega_1 \times \omega \mid 1 \leq i \leq k\rangle$ such that every extension $p^{*\frown}\langle\eta_1, ..., \eta_k\rangle$ of p^* is in*

D, where $p^* \frown \langle \eta_1, ..., \eta_k \rangle$ *is the condition obtained by extending* p^* *non-directly by elements* $\eta_1, ..., \eta_k$ *from sets of measures one at places* $\kappa_{\beta_1, n_1}, ..., \kappa_{\beta_k, n_k}$.

The next lemma will be used in order to show a desired chain condition.

Lemma 7.9 *Let* $\alpha < \omega_1$ *and let* $\omega_1 < \lambda < \kappa_{\alpha+1}$ *be a regular cardinal. Then every set* $A \subseteq V$ *in* $V^{\mathcal{P}' * \mathcal{P} \setminus \kappa_\alpha}$ *of cardinality* λ *contains a subset of cardinality* λ *in* V.

Proof. Note that by Lemma 7.5 all cardinals below $\kappa_{\alpha+1}$ and in particular λ are preserved. Let $A = \{a_\xi \mid \xi < \lambda\}$. For every $\xi < \lambda$ let $D_\xi = \{p \in \mathcal{P} \setminus \kappa_\alpha \mid p \parallel \underset{\sim}{a_\xi}\}$. Define now by induction an $\leq^*$–increasing sequence $\langle p_\xi \mid \xi \leq \lambda \rangle$ and a sequence $\langle \langle \langle \beta_i^\xi, n_i^\xi \rangle \in \omega_1 \times \omega \mid 1 \leq i \leq k_\xi \rangle \mid \xi < \omega_1$ such that $p_\xi, \langle \langle \beta_i^\xi, n_i^\xi \rangle \in \omega_1 \times \omega \mid 1 \leq i \leq k_\xi \rangle$ satisfy the conclusion of Lemma 7.9 with D_ξ.
Now λ is a regular cardinal above ω_1. Hence there are are $S \subseteq \lambda, |S| = \lambda$ and $\langle \langle \beta_i^*, n_i^* \rangle \in \omega_1 \times \omega \mid 1 \leq i \leq k_* \rangle$ such that for every $\xi \in S$ we have $\langle \langle \beta_i^\xi, n_i^\xi \rangle \in \omega_1 \times \omega \mid 1 \leq i \leq k_\xi \rangle = \langle \langle \beta_i^*, n_i^* \rangle \in \omega_1 \times \omega \mid 1 \leq i \leq k_* \rangle$.

Then $p_\lambda \Vdash_{\mathcal{P} \setminus \kappa_\alpha} \langle \underset{\sim}{a_\xi} \mid \xi \in S \rangle \in V$ and we are done.
□

Define $\longleftrightarrow$ and $\longrightarrow$.

Definition 7.10 Let $p, q \in \mathcal{P}$. Set $p \longleftrightarrow q$ iff there is $\alpha < \omega_1$ such that

1. $p \setminus \kappa_\alpha = q \setminus \kappa_\alpha$,
2. $p \restriction \kappa_{\alpha+1} \longleftrightarrow_{\mathcal{P} \restriction \kappa_{\alpha+1}} q \restriction \kappa_{\alpha+1}$,
   where $\longleftrightarrow_{\mathcal{P} \restriction \kappa_{\alpha+1}}$ in the usual fashion requiring that for each $k < \omega$ all but finitely many coordinates realize the same k-type. Moreover always the same 4-type is realized.

Now we define $\longrightarrow$ in the usual fashion.

Definition 7.11 Let $p, q \in \mathcal{P}$. Set $p \longrightarrow q$ iff there is a sequence of conditions $\langle r_k \mid k < m < \omega \rangle$ so that

(1) $r_0 = p$

(2) $r_{m-1} = q$

(3) for every $k < m - 1$,
$r_k \leq r_{k+1}$ or $r_k \longleftrightarrow r_{k+1}$.

Lemma 7.12 *Let $\alpha < \omega_1$. Then, in $V^{\mathcal{P}' * \mathcal{P} \backslash \kappa_\alpha}$, the forcing $\langle \mathcal{P} \restriction \kappa_{\alpha+1}, \longrightarrow \rangle$ satisfies κ_α^{++}-c.c.*

Proof. Suppose otherwise. Let A be an antichain in $\langle \mathcal{P} \restriction \kappa_{\alpha+1}, \longrightarrow \rangle$ of size κ_α^{++}. Apply Lemma 7.9. So there is $B \subseteq A$ of the same cardinality but in $V^{\mathcal{P}'}$. This is impossible since $\langle \mathcal{P} \restriction \kappa_{\alpha+1}, \longrightarrow \rangle$ satisfies κ_α^{++}-c.c. in $V^{\mathcal{P}'}$.

The main issue here is a possibility of adding models to a given condition. The rest of the argument repeats the one of previous settings, see [6]. A new point is that the connection functions are not one to one anymore. Thus the following may happen. Suppose we have $B \in A$, $|B| > |A|$ and we need to add $A' \in B$ such that A, A' are of a Δ-system type over $A \cap A'$. If the connection function a of a condition (i.e. an isomorphism between suitable structures) moves both A and B to models of a same cardinality, then adding of such A' will be just impossible, since then $a(A) \supseteq a(B)$ which leaves no room for A'. Now, in order to overcome this obstacle, note that $|a(A)| = |a(B)|$ (or even $|a(A)| \geq |a(B)|$) may occur only finitely many times (levels). So starting with a certain level n we should have $|a(A)| < |a(B)|$ which allows to add A'.

□

Lemma 7.13 *The forcing $\langle \mathcal{P}, \longrightarrow \rangle$ over $V[G']$ preserves all the cardinals (and every cofinality).*

Proof. Let η be a cardinal in $V[G']$. We show by induction on $\alpha < \omega_1$ that if $\eta \leq \kappa_\alpha$ then it is preserved in the generic extension. Clearly, it is enough to deal only with regular η's. Hence, we need to consider only the following situation:

$$\kappa_\alpha < \eta < \kappa_{\alpha+1},$$

for some $\alpha < \omega_1$. Split the forcing $\mathcal{P}$ into $\mathcal{P} \backslash \kappa_\alpha$ followed by $\mathcal{P} \restriction \kappa_{\alpha+1}$. By Lemma 7.5, $\mathcal{P} \backslash \kappa_\alpha$ does not add new bounded subsets to $\kappa_{\alpha+1}$ (namely, this lemma together with the Prikry condition imply that no new subsets are added to $\kappa_{\alpha+1,0}$, but taking non-direct extensions over $\kappa_{\alpha+1,n}$'s it is easy to push this up to $\kappa_{\alpha+1}$). By Lemma 7.12 the forcing $\mathcal{P} \restriction \kappa_{\alpha+1}$ preserves all the cardinals above κ_α^+. So, the only case that remains is $\eta = \kappa_\alpha^+$. But it is not problematic, since we have here the successor of the singular cardinal and the usual arguments apply.

□

8 Concluding remarks.

The construction of the previous section gives a countable set of regular cardinals $\mathfrak{a}$ with $\text{otp}(\text{pcf}(\mathfrak{a})) = \omega_1 + 1$. It is natural to try to get a bigger order type. The present methods allow to obtain $\omega_1 \cdot \alpha + 1$, for every $\alpha < \omega_1$. Just repeat the construction α– many times (one above another). However it is unclear how to get to $\omega_1 \cdot \omega_1 + 1$ and beyond.

Question 1. Is it possible to increase $\text{otp}(\text{pcf}(\mathfrak{a}))$ beyond $\omega_1 \cdot \omega_1$, for a countable set of regular cardinals $\mathfrak{a}$?

We think that it may be possible under same lines, but using more elaborated techniques, to get any successor order type $< \omega_2$.

Shelah Weak Hypothesis states that the set

$$\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \text{pp}(\eta) > \kappa\}$$

is at most countable. The construction of the previous section provides a counterexample, but very restricted one. The cardinality and even the order type there is ω_1. So the following question is natural:

Question 2. Is it possible to increase $\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \text{pp}(\eta) > \kappa\}$ beyond ω_1, for a cardinal κ?

Note that no upper bound on cardinality of $\{\eta \mid \eta < \kappa, \eta \text{ is a singular cardinal and } \text{pp}(\eta) > \kappa\}$ is known.

Going further beyond ω_1, in view of results of [7] and [9] will require some completely new ideas. The same once one likes to have a set $\{\eta \mid \eta < \kappa, \text{cof}(\eta) > \omega, \text{pp}(\eta) > \kappa\}$ infinite, for some κ.

Question 3. How to move everything down, in particular is it possible to get down to $\aleph_\omega$?

It is possible to add collapses to the present construction, but only very inessential ones. By [8], the supercompact Prikry forcing looks be needed in order to collapse successors of singular cardinals, but this complicates the matters largely. It is unclear how to combine this forcing with sort extenders forcings in a productive way.

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