THE FIRST MEASURABLE CAN BE THE FIRST INACCESSIBLE CARDINAL

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Abstract. In [7] the second and third author showed that if the least inaccessible cardinal is the least measurable cardinal, then there is an inner model with $o(\kappa) \geq 2$. In this paper we improve this to $o(\kappa) \geq \kappa + 1$ and show that if κ is a κ^{++}-supercompact cardinal, then there is a symmetric extension in which it is the least inaccessible and the least measurable cardinal.

1. Introduction

Large cardinal axioms form the yardstick with which we measure the consistency strength of various mathematical statements. In other words, given a mathematical statement, we can use large cardinals to give lower and upper bounds as to the question "how strong of a mathematical foundation is required to prove the statement is possibly true?". Perhaps the most famous large cardinal axiom is the one positing the existence of an inaccessible cardinal, or a "Tarski–Grothendieck universe".[1]

Measurable cardinals, in the standard context of set theory, where the axiom of choice is taken as true, can be defined by two equivalent formulations: the existence of ultrafilters;[2] or as the critical points of elementary embeddings. The equivalence, which relies heavily on Łoś' theorem, can fail without the axiom of choice. In the 1960s, Jech proved in [8] that ω_1, the least uncountable cardinal which can never be a critical point of an elementary embedding,[3] could be a measurable cardinal.

A sequence of results under the axiom of determinacy, starting with Solovay's proof for the measurability of ω_1 and ω_2, and reaching its final form in Steel's theorem that assuming $V = L(\mathbb{R})$, every uncountable regular cardinals below Θ is measurable ([14, Theorem 8.27]), shows that measurable cardinals are common in some natural models of ZF.

In [6], the second and third authors isolated the notion of a "critical cardinal" which is a critical point of an embedding, and studied the consequences of critical cardinals without the axiom of choice.

That work led the question of how small can the least measurable cardinal be, if the axiom of choice is allowed to fail. Of course, it can be the least uncountable cardinal, but that is not a satisfying answer to the question. In [7] the two authors

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[1] Without the axiom of choice the many definitions of an inaccessible cardinal which are equivalent in ZFC will no longer need to be equivalent (see [4] for details). In this work "κ is inaccessible" means that $V_\kappa \models \mathsf{ZF}_2$, that is ZF formulated in second-order logic, or equivalently there is no $x \in V_\kappa$ and a function $f\colon x \to \kappa$ whose image is a cofinal subset of κ.

[2] A cardinal κ is measurable if and only if there is a non-principal κ-complete ultrafilter on κ.

[3] At least not if we require the embedding to be definable or at least amenable, that is.

show that the least Mahlo cardinal[4] can be the least measurable cardinal as well, and that the large cardinal strength of this assertion is merely "there exists a measurable cardinal", that is to say that in order to produce a model where the least measurable cardinal is the least Mahlo cardinal, it was enough to start with a model in which a single measurable cardinal exists. However, in trying to reduce the measurability even lower we run into an intriguing situation. If the least measurable cardinal is also the least inaccessible cardinal, then we must have began with a model with many measurable cardinals.

This means that producing a model where the least measurable cardinal is the least inaccessible cardinal would require us to work harder. The difficulty does not lie in the fact that this the first inaccessible, but rather in the fact that the cardinal is not Mahlo. Once the set of inaccessible, or even regular, cardinals is negligible, all manner of difficulties start to arise. This phenomenon, and therefore the question that we are concerned with here, is not unique to situations where the axiom of choice fails. For example, even in ZFC it is not known if the tree property can hold at a non-Mahlo weakly inaccessible cardinal, whether it is consistent that there is no Suslin tree on a non-Mahlo inaccessible cardinal, and it is known that the failure of diamond principles at the least inaccessible cardinal also has a perhaps-surprisingly strong large cardinal lower bounds.

If one replaces a (strongly) inaccessible by a weakly inaccessible, then starting from $\mathsf{AD} + V = L(\mathbb{R})$, Apter constructed a model of ZF in which the least measurable and the least weakly inaccessible coincide (see [1]). Namely, basing on the aforementioned result of Steel [14], he uses the Prikry forcing (in ZF context) to turn any given set $A \subseteq \Theta$ of measurable cardinals into singulars and preserving measurability of the rest.

1.1. **In this paper.** In this work we establish upper bounds for the statements "there is a measurable cardinal that is not a Mahlo cardinal" and "the least inaccessible cardinal is the least measurable cardinal", as well as far improved lower bounds. Indeed, we show that if κ is a κ^{++}-supercompact cardinal, then there is a symmetric extension in which it is a non-Mahlo measurable cardinal, and a further symmetric extension in which it is also the least inaccessible cardinal. In both of these, we show that $\mathsf{DC}_{<\kappa}$ holds, and ZFC holds below κ. We also show that the lower bound required for these results is at least as high as "there is an inner model with $o(\kappa) \geq \kappa$".

We use supercompact Radin forcing to construct the symmetric extension, and the Mitchell covering lemma (for "there is no inner model with $o(\alpha) = \alpha^{++}$") to provide the lower bounds.

Some questions and conjectures are given at the end as well.

1.2. **Technical preliminaries.** We assume that the readers are familiar with the techniques of forcing and symmetric extensions, but we include a brief outline of the latter. Fixing a forcing notion $\mathbb{P}$, an automorphism, π, of $\mathbb{P}$ acts on the $\mathbb{P}$-names in a recursive definition given by

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

Unsurprisingly, as $\Vdash$ is defined from the order, $p \Vdash \varphi(\dot{x})$ if and only if $\pi p \Vdash \varphi(\pi\dot{x})$.

Fixing a group of automorphisms, $\mathcal{G}$, we define $\operatorname{sym}(\dot{x})$, for a $\mathbb{P}$-name $\dot{x}$, as $\{\pi \in \mathcal{G} \mid \pi\dot{x} = \dot{x}\}$. Since interpretation of names by generic filter satisfies the equation

$$(\pi^{-1}\dot{x})^G = \dot{x}^{\pi``G},$$

[4]This is "the next step" after inaccessibility, in the sense that the set of inaccessible cardinals below a Mahlo cardinal is not negligible.

where G Is a generic filter, we get that $\dot{x}^G = \dot{x}^{\pi``G}$ when $\pi\dot{x} = \dot{x}$.

We want, therefore, to isolate a notion which allows us to say when a $\mathbb{P}$-name is interpreted the same way by "most" generic filters. Towards that goal, $\mathcal{F}$ is a *filter of subgroups* if it is a non-empty collection of subgroups of $\mathcal{G}$ which is closed under supergroups and finite intersections. We say that it is a *normal* filter of subgroups if whenever $H \in \mathcal{F}$ and $\pi \in \mathcal{G}$, then $\pi H \pi^{-1} \in \mathcal{F}$ as well.[5]

Let a $\mathbb{P}$-name, $\dot{x}$, be called *symmetric* if $\operatorname{sym}(\dot{x}) \in \mathcal{F}$, and if this property holds hereditarily for all names appearing in $\dot{x}$, we say that it is *hereditarily symmetric*. The class of hereditarily symmetric names is denoted by HS and if G is a generic filter, $\mathsf{HS}^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}\}$ is a transitive model of ZF intermediate between V and $V[G]$, and we refer to it as a *symmetric extension*.

We say that a class M is *κ-closed* (in V, or in a larger class) if whenever $\gamma < \kappa$ and $f\colon \gamma \to M$, then $f \in M$. We say that a forcing notion is κ-closed if every descending sequence of fewer than κ conditions has a lower bound, and a tree is κ-closed if it is κ-closed as a forcing notion (which necessitate reversing its order in our convention).

In these symmetric extensions we often want to preserve some fragments of the axiom of choice, and in this work we will be primarily focused on *Dependent Choice*. Amongst the many interesting equivalences, DC_κ can be formulated as "Every κ-closed tree has a maximal node or a chain of type κ", and $\mathsf{DC}_{<\kappa}$ means that for every $\lambda < \kappa$, DC_λ holds.

For a set of ordinals, x, we write $\operatorname{acc} x$ to denote its accumulation points, i.e. $\{\alpha \mid \alpha = \sup \alpha \cap x\}$. We denote by π_x the Mostowski collapse of x, i.e. the order isomorphism with its order type. And we say that ξ is a successor in x if $\pi_x(\xi)$ is a successor ordinal.

Finally, for $x, y \in \mathcal{P}_\kappa\lambda$ we write $x \underset{\sim}{\subsetneq} y$ to mean $x \subseteq y$ and $|x| < |y \cap \kappa|$. In our context, $y \cap \kappa$ will be a cardinal.

2. Measurable, but not Mahlo

This section will be devoted for the proof of the following theorem.

Theorem 2.1. *Assume* GCH *and let κ be κ^{++}-supercompact. Then, there is a symmetric extension in which κ is non-Mahlo inaccessible and there is a normal measure on κ.*

The idea of the proof of this theorem is to work in an intermediate model between the generic extension by the supercompact Radin club and the standard Radin club. Let us begin by recalling some basic facts about the supercompact Radin forcing and its connection (in this specific case) with the standard Radin forcing.

2.1. Radin Forcing.

We will follow Krueger's presentation of supercompact Radin forcing ([11]). Since κ is κ^{++}-supercompact, by [11, Proposition 2.2], there are $o^{\mathcal{W}}\colon \kappa + 1 \to \mathrm{Ord}$, with $o^{\mathcal{W}}(\kappa) = \kappa^{++}$, and a *coherent sequence* of measures, $\mathcal{W} = \langle W(\alpha, i) \mid \alpha \le \kappa, i < o^{\mathcal{W}}(\alpha)\rangle$. In our context this means that:[6]

(1) For every $\alpha \le \kappa$ and $i < o^{\mathcal{W}}(\alpha)$, $W(\alpha, i)$ is a normal measure on $\mathcal{P}_\alpha\alpha^+$.
(2) For every $\alpha \le \kappa$ and $i < o^{\mathcal{W}}(\alpha)$,
$$j_{W(\alpha,i)}(\mathcal{W})(\alpha) = \mathcal{W}(\alpha) \upharpoonright i = \langle W(\alpha, k) \mid k < i\rangle.$$

Given a coherent sequence $\mathcal{W}$ the *supercompact Radin forcing*, $\mathbb{R}(\mathcal{W})$, as defined by Krueger, is the forcing whose conditions are pairs of the form $\langle \vec{d}, A\rangle$. The set A

[5] The clash in terminology with normal filters in the sense of diagonal intersections is known. We will refer to ultrafilters on ordinals as measures whenever any confusion may arise.

[6] In [11] two additional conditions are required that hold automatically in our context.

lies in $\bigcap_{\zeta<o^{\mathcal{W}}(\kappa)}\mathcal{W}(\kappa,\zeta)$, and $\vec{d}$ is a finite sequence of either members of $\mathcal{P}_\kappa\kappa^+$ or pairs of the form $\langle a_i, A_i\rangle$ which reflect our Radin forcing to some $\alpha<\kappa$. We will work under the implicit assumption that if $a\in\mathcal{P}_\alpha\alpha^+$, then $a\cap\alpha$ is a cardinal and $|a|=|a\cap\alpha|^+$, note that these sets form a large set in all of our measures, and so we can simply restrict our conditions to these sets. For the precise definition of the conditions and the order see [11, Section 3].

For our purposes, we will need the fact that the conditions have a notion of *length* (which is the length of $\vec{d}$) which is getting larger as we strengthen the condition, and a notion of a *direct extension* which is a length preserving extension, denoted by $q\leq^* p$. We will also write $p^\frown\eta$, where η is a sequence, to denote the weakest extension of p whose stem is $\operatorname{stem} p$ followed by η.

For a normal measure over $\mathcal{P}_\kappa\kappa^+$, W, let us denote by $W\restriction\kappa$ the projection of this measure to a normal measure on κ:

$$W\restriction\kappa=\{A\subseteq\kappa\mid\exists B\in W,\ A=\{x\cap\kappa\mid x\in B\}\}.$$

While this projection always induces a normal measure on κ, the coherence of the sequence of projections is more subtle.

Lemma 2.2. *Assume* GCH. *Let $\mathcal{W}$ be a coherent sequence of $\mathcal{P}_\kappa\kappa^+$-supercompact measures with $o^{\mathcal{W}}(\alpha)<\alpha^{++}$ for all $\alpha<\kappa$. Then, $\langle W(\alpha,i)\restriction\alpha\mid\alpha\leq\kappa, i<o^{\mathcal{W}}(\alpha)\rangle$ is coherent.*

Proof. Let ι be the ultrapower embedding by $W(\alpha,i)\restriction\alpha$, j the ultrapower embedding by $W(\alpha,i)$, and k be the quotient map. Namely, $k([f])=j(f)(\alpha)$. We have the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "j"'] & N=\mathrm{Ult}(V,W(\alpha,i)\restriction\alpha) \arrow[d, "k"] \\ & M=\mathrm{Ult}(V,W(\alpha,i)) \end{tikzcd}$$

The map k must have a critical point (as for example the V-cofinality of $\iota(\alpha^+)$ is α^+ while the V-cofinality of $j(\alpha^+)$ is α^{++}). The critical point of k must be an N-cardinal which is not in the image of k. Therefore $\operatorname{crit} k\geq\operatorname{crit} j=\alpha$. Equality, however, is impossible, as for $\mathrm{id}\colon\alpha\to\alpha$, $j(\mathrm{id})(\alpha)=\alpha$, and it cannot be α^+ as for $s\colon\alpha\to\alpha$, $\forall\zeta$, $s(\zeta)=\zeta^+$, $[s]=\alpha^+$ (using the fact that N computes α^+ correctly) and clearly $j(s)(\alpha)=\alpha^+$.

Therefore, $\operatorname{crit} k\geq(\alpha^{++})^N$. A simple computation shows that $|(\alpha^{++})^N|^V=\alpha^+$ and thus it has to move under k, so $\operatorname{crit} k=(\alpha^{++})^N$.

Since $o^{\mathcal{W}}(\zeta)<\zeta^{++}$ for all $\zeta<\alpha$, we have that $\iota(o)(\alpha)=i<(\alpha^{++})^N=\operatorname{crit} k$. In particular, for every $\zeta<i$,

$$k(\iota(W)(\alpha,\zeta)\restriction\alpha)=j(W)(\alpha,\zeta)\restriction\alpha=W(\alpha,\zeta)\restriction\alpha,$$

where the last equality follows from the fact that $\mathcal{P}(\alpha)\subseteq N$ and $\operatorname{crit} k>\alpha$. □

We will use $\overline{\mathcal{W}}$ to denote the sequence of projected measures.

Remark 2.3. *Lemma 2.2 does not make sense for longer sequences. Under* GCH, *$o(\kappa)\leq\kappa^{++}$, but a coherent sequence of measures on $\mathcal{P}_\kappa\kappa^+$ can be longer.*

Lemma 2.4. *In the Radin generic extension by $\mathbb{R}(\mathcal{W})$, κ remains inaccessible.*

Proof. This is a standard argument, and the result appeared implicitly already at some papers in the literature.

First, let us derive a *strong Prikry Property* from the standard Prikry Proprety. We will prove it only for conditions of length 0 where the proof of the general case can be obtained using the factorization property (see [11, Section 4]).

Claim 2.5. *Let $p = \langle \varnothing, A\rangle$ be a condition of length 0 and let D be a dense open set in the supercompact Radin forcing. Then, there are:*

(1) *a direct extension $q \leq^* p$,*
(2) *a natural number $n < \omega$ and*
(3) *a rooted tree of height n, $T \subseteq (\mathcal{P}_\kappa \kappa^+)^{\leq n}$, such that for each $\eta \in T$ with $|\eta| < n$, there is $\zeta_\eta < o^{\mathcal{W}}(\kappa)$ such that $\{x \in \mathcal{P}_\kappa \kappa^+ \mid \eta^\frown \langle x\rangle \in T\} \in W(\kappa, \zeta_\eta)$.*

such that for every η in the top level of T, there is a direct extension of $q^\frown \eta$ in D.

Proof. By applying the Prikry Property and the σ-closure of the measures, we can conclude that there is $\langle \varnothing, A^q\rangle = q \leq^* p$ that decides the minimal length of a condition in the generic extension which is in D. Let n be this length.

For every $x \in A^q$, let us check whether there is a direct extension of $q^\frown\langle x\rangle$ that forces that there is a condition r in the generic of length n, in D, such that $\min d^r = x$, and x is minimal in the generic club with such property. Note that this set must be positive (with respect to the filter $\bigcap_{\zeta < o^{\mathcal{W}}(\kappa)} W(\kappa, \zeta)$), as otherwise we could shrink A^q to avoid it and get a contradiction. So, there is some $\zeta < o^{\mathcal{W}}(\kappa)$ such that for a large set T_0 of x with respect to U_ζ, there is $q_x = \langle (x, A^x_0), A^x_1\rangle \leq^* q^\frown\langle x\rangle$ that forces the existence of a condition r in $D \cap \dot{G}$ with length n and $\min d^r = x$.

For each such $x \in T_0$ we repeat the process, and find an ordinal $\zeta_x < o^{\mathcal{W}}(\kappa)$ and measure one many y (relative to $W(\kappa, \zeta_x)$) with the property that there is a direct extension of $q_x^\frown\langle y\rangle$ forcing $\langle x, y\rangle$ to be the first two elements in d^r for $r \in D \cap \dot{G}$ of length n. Continuing this way for n steps we get the existence of T, which proves the claim. □

Let us consider the name $\dot{f}$ for a function from λ to κ for $\lambda < \kappa$. For each $\alpha < \lambda$, let D_α be the dense open set of all conditions deciding a value for $\dot{f}(\check{\alpha})$. Applying our version of the strong Prikry Property for each $\alpha < \lambda$ we obtain a direct extension $q \leq^* p$ and a sequence of trees $\langle T_\alpha \mid \alpha < \lambda\rangle$ of various finite heights. We can attach to each one of the nodes of the trees $\eta \in T_\alpha$, the corresponding direct extension $q_\eta \leq q^\frown\eta$ from D_α.

Let us consider all ζ_η for $\eta \in T$. As there are κ^+ such ordinals and $o^{\mathcal{W}}(\kappa) = \kappa^{++}$, there is an ordinal $\zeta^* < \kappa^{++}$ bounding all of them.

In the ultrapower by $W(\kappa, \zeta^*)$ the we have $j``T_\alpha$ for all $\alpha < \lambda$ as well as the corresponding $j(\eta) \mapsto j(q_\eta)$.

Consider $[\mathrm{id}]_{W(\kappa,\zeta^*)} = j``\kappa^+$. It is easy to verify that one can add this element to each one of the $j(q_\eta)$ for each $\eta \in T_\alpha$.

Moreover, as all measures mentioned by the trees are below ζ^*, for each $\alpha < \lambda$, $\{j(\eta) \in T_\alpha \mid \eta \text{ is a node}\}$ forms a maximal antichains in the Radin forcing below $j``\kappa^+$. Thus for each element in $T_\alpha$, $j(\eta)$, the condition $j(q)^\frown\langle j(\eta), j``\kappa^+\rangle = j(q)^\frown\langle j``\kappa^+, j(\eta)\rangle$ is forcing a value to $j(\dot{f})(\check{\alpha})$, as it extends the condition $j(q_\eta)$. By elementarity this value must be the j-image of the one that q_η forced for $\dot{f}(\check{\alpha})$ and thus below κ.

We conclude that $j(q)^\frown\langle j``\kappa^+\rangle$ forces $j(\dot{f})$ to have a range boudned by κ. Reflecting this, we obtain a $W(\kappa, \zeta^*)$-large set such that adding each x in this set to the stem forces the range of f to be bounded by $x \cap \kappa$. □

Let G be generic for $\mathbb{R}(\mathcal{W})$. Let us denote the generic continuous and increasing sequence in $\mathcal{P}_\kappa\kappa^+$ by C_G, so

$$C_G = \{x \mid \exists p \in G, x \in \operatorname{stem} p\}.$$

Let $\overline{C} = \{x \cap \kappa \mid x \in C_G\}$. Since C_G is continuous and cofinal, $\overline{C} \subseteq \kappa$ is a club. To prove that $\overline{C}$ is a Radin club for $\mathbb{R}(\overline{\mathcal{W}})$ we will need to use the Mathias Criterion for genericity.

Recall that a condition $p \in \mathbb{R}(\overline{\mathcal{W}})$ is compatible with $\overline{C} \subseteq \kappa$ if $\operatorname{stem} p \subseteq \overline{C}$ and whenever $d_i < d_{i+1}$ are two successive points in $\operatorname{stem} p$, then $\overline{C} \cap (\alpha_i, \alpha_{i+1}) \subseteq A_{i+1}$ if $d_{i+1} = \langle \alpha_{i+1}, A_{i+1} \rangle$ or else $\overline{C} \cap (\alpha_i, \alpha_{i+1}) = \varnothing$.

Fact 2.6 (Mathias Criterion)**.** *Let $\overline{C} \subseteq \kappa$ be a club. Let $\overline{G}$ be the collection of all conditions in $\mathbb{R}(\overline{\mathcal{W}})$ compatible with $\overline{C}$. Then $\overline{G}$ is a generic filter iff*

(1) *For every $\alpha \in \operatorname{acc} \overline{C}$, $\overline{C} \cap \alpha$ is generic for $\mathbb{R}(\overline{\mathcal{W}} \restriction \alpha + 1)$.*
(2) *For every $A \in \bigcap_{\zeta < o^{\mathcal{W}}(\kappa)} \mathcal{W}(\kappa, \zeta)$, there is $\eta < \kappa$ such that $\overline{C} \setminus \eta \subseteq A$.*

Lemma 2.7. *$\overline{C}$ is a generic Radin club for $\mathbb{R}(\overline{\mathcal{W}})$.*

Proof. This follows from the Mathias Criterion for genericity of the Radin club. Indeed, let us prove by induction on $\alpha \in \operatorname{acc} \overline{C}$ that the criteria holds. Let $A \in \bigcap \overline{\mathcal{W}}$, when by the definition of $\overline{W}$, the set $\tilde{A} = \{x \in \mathcal{P}_\alpha \alpha^+ \mid x \cap \alpha \in A\} \in \bigcap \mathcal{W}$. For every condition $p \in \mathbb{R}(\mathcal{W})$ with $\alpha \in \operatorname{stem} p$ there is a direct extension q such that the large set associated with α is contained in $\tilde{A}$. In particular, q forces that a tail of elements in $\overline{C} \cap \alpha$, is contained in A. $\square$

Lemma 2.8. *Every $\alpha \in \operatorname{acc} C_G$ is singular in $V[G]$.*

Proof. First, by the factorization argument, this statement is equivalent to the statement that forcing with supercomapct Radin forcing for the coherent sequence $\mathcal{W} \restriction \alpha + 1$ with top cardinal α and $o^{\mathcal{W}}(\alpha) = \zeta < \alpha^{++}$ singularizes α.

Claim 2.9. *Fix $\alpha \leq \kappa$. Assume that $\zeta < \alpha^{++}$. The measures $\{W(\alpha, \xi) \mid \xi < \zeta\}$ are discrete in the sense that there is a partition of $\mathcal{P}_\alpha \alpha^+$, $\langle B_i \mid i < \zeta \rangle$ such that $B_i \in W(\alpha, j)$ iff $i = j$.*

Proof. Let $\langle \mathcal{U}_i \mid i < i_* \rangle$ be an enumeration $\langle W(\alpha, \xi) \mid \xi < \zeta \rangle$ with $i^* \leq \alpha^+$. For each $i < j < i_*$ let $B_{i,j} \in \mathcal{U}_i \setminus \mathcal{U}_j$ and $B_{j,i} = \mathcal{P}_\alpha \alpha^+ \setminus B_{i,j}$. Let $B_{i,i} = \mathcal{P}_\alpha \alpha^+$. Let $B_i = \{x \in \mathcal{P}_\alpha \alpha^+ \mid i \in x\} \cap \triangle_{j < i_*} B_{i,j}$. So, $B_i \in \mathcal{U}_i$ by the normality of $\mathcal{U}_i$.

Moreover, for $i < j$, $B_i \cap B_j = \varnothing$. Indeed, if $x \in B_i \cap B_j$ then $i, j \in x$ and thus $x \in B_{i,j}$ and $x \in B_{j,i}$, a contradiction. $\square$

Let $h \colon i_* \to \zeta$ be the bijection used in the proof above. As the set $B_* = \bigcup_{i < \alpha^+} B_i$ belongs to $\bigcap_{\xi < \zeta} W(\alpha, \xi)$, for any large enough $y \in C_G$, with $y \cap \kappa < \alpha$, $y \in B_*$. So, for such y, we can find the unique $\xi < \zeta$ such that $y \in B_i$ for $\xi = h(i)$. Without loss of generality, all the elements of the Radin club below α belong to B_*. Moreover, we may assume (by shrinking B_i if necessary), that for every $x \in B_i$, $\pi_x(\bigcup_{h(\xi) < h(i)} B_\xi)$ belongs to the intersection of measures of $x \cap \kappa$, that is, $\bigcap_{\alpha < o^{\mathcal{W}}(x \cap \kappa)} W(x \cap \kappa, \alpha)$. Let us now split into cases.

Case 0: If $\operatorname{cf} \zeta < \alpha$, let $\langle \delta_i \mid i < \operatorname{cf} \zeta \rangle$. Let y_i be the least element in the Radin club below α such that $y_i \in B_{h^{-1}(\delta_i)}$ (generically, there must be such an element). If $\sup_{i < \operatorname{cf} \zeta} y_i \cap \kappa < \alpha$, then by the closure of C_G, $y_* = \bigcup y_i \in C_G$ and it lies below α. So, $y_* \in B_\rho$ for some ρ, but this is impossible as for all but boundedly many $i < \operatorname{cf} \rho$, $\delta_i \geq h(\rho)$. This is a contradiction, as by genericity, for any large enough element in C_G below y_* belongs to $\pi_{y_*}(\bigcup_{h(\xi) < h(\rho)} B_\xi)$ and in particular do not belong to $B_{h^{-1}(\delta_i)}$ for $\delta_i \geq h(\xi)$.

Case 1: if $\operatorname{cf} \zeta \in \{\alpha, \alpha^+\}$. Let z_* be the element in C_G with $z_* \cap \kappa = \alpha$. Let $\langle \delta_i \mid i < \operatorname{cf} \zeta \rangle$ be a cofinal sequence at ζ. Let us shrink B_i so that for all $x \in B_{h^{-1}(\delta_i)}$, $\sup(\pi_{z_*}(x) \cap \operatorname{cf} \zeta) > i$.

Pick $y_0 \in C_G$ arbitrary and let us recursively define $y_{n+1} \in C_G$ to be an element of $B_{h^{-1}(\delta_\xi)}$ for $\xi = \pi_{z*}(y_n \cap \operatorname{cf} \zeta)$.

Let us show that $\bigcup_{n < \omega} y_n = z_*$. Indeed, let $\bigcup_{n < \omega} y_n = y_*$ and let us assume that $y_* \cap \kappa < \alpha$. Then, there is ξ such that $y_* \in B_\xi$.

Let $\alpha_* = y_* \cap \kappa$. Then, as before, $h(\xi) = \delta_{\alpha_*}$ (as otherwise, if $\alpha_* > h(\xi)$ there is $n < \omega$ such that $y_n \cap \kappa$ is strictly larger than $h(\xi)$ and if $\alpha_* < h(\xi)$, than for all large n, the ξ' such that $y_n \in B_{h(\xi')}$ is bounded). But this is impossible as for all $x \in B_\xi$, $\sup(\pi_{z*}(x) \cap \operatorname{cf}\zeta) > \delta_{\alpha_*}$.

□

The following lemma is due to Radin, [13, Claim 8]

Lemma 2.10. *In the generic extension by $\overline{\mathcal{W}}$, κ remains measurable.*

2.2. **Symmetric Model.** So, to summarize, we obtained two models: in the full Radin generic extension by $\mathcal{W}$, $V[G]$, κ is an inaccessible cardinal. Moreover, for every $\alpha < \kappa$ in the normal Radin club, $\overline{C}$, α is singular.

In the submodel $V[\overline{G}]$—the generic extension by the Radin club obtained from the projected measures, κ is measurable. We would like now to consider an intermediate symmetric model, W_1, in which κ remains measurable, $\overline{G}$ exists but every α in the Radin club is singular.

To make the definition of the automorphisms easier to understand we will adopt the conventions that a condition is a finite sequence of pairs, $\vec{d}$, whose last element is $\langle \kappa^+, A\rangle$ where A was previously the full measure set, and that for $d_i \in \mathcal{P}_\kappa \kappa^+$ we will freely confuse between d_i and $\langle d_i, \varnothing\rangle$. The stem of p under this convention, therefore, is p without its last coordinate.

Let us work in the symmetric model with respect to symmetries as the ones from [6, Section 5]. Note, that unlike the case in [6], here the filter of groups is actually κ^+-complete. For the completeness of this paper, let us spell out the group of automorphisms and the normal filter of groups.

Definition 2.11. *Let $g\colon \kappa^+ \to \kappa^+$ be a bijection, then g lifts to $g_1\colon \mathcal{P}_\kappa\kappa^+ \to \mathcal{P}_\kappa\kappa^+$ by $g_1(x) = g``x$. Going further, we can lift $g_1$ to $g_2\colon \mathcal{P}(\mathcal{P}_\kappa\kappa^+) \to \mathcal{P}(\mathcal{P}_\kappa\kappa^+)$ defined by $g_2(A) = g_1``A$. We define σ_g as the pointwise application of g_1 and g_2. Namely,*

$$\sigma_g(\langle\langle a_0, B_0\rangle, \ldots, \langle a_n, B_n\rangle\rangle) = \langle\langle g_1(a_0), g_2(B_0)\rangle, \ldots, \langle g_1(a_n), g_2(B_n)\rangle\rangle.$$

It is easy to verify that σ_g is an automorphism of a dense subset of $\mathbb{R}(\mathcal{W})$, and so extends to an automorphism of the Boolean completion. So we can let $\mathcal{G}$ be the group of all the automorphisms of the form σ_g for some bijection $g\colon \kappa^+ \to \kappa^+$.

Definition 2.12. *For every $\alpha < \kappa^+$ let H_α be the group of automorphisms σ_g for g such that $g \restriction \alpha = \mathrm{id}$. Let $\mathcal{F} = \langle \{H_\alpha \mid \alpha < \kappa^+\}\rangle$.*

Proposition 2.13. *$\mathcal{F}$ is a normal filter of groups over $\mathcal{G}$.*

Proof. Note that any $\sigma_h \in \sigma_g H_\alpha \sigma_g^{-1}$ must have the property that $h \restriction (g``\alpha) = \mathrm{id}$. Since $\kappa^+$ is regular, let $\delta = \sup g``\alpha$, then $H_\delta \subseteq \sigma_g H_\alpha \sigma_g^{-1}$. □

Claim 2.14. *If $a \in C_G$, then $\mathcal{P}_{a\cap\kappa} a^{V[C_G]} \in W_1$.*

Proof. Let $a \in C_G$. Then, $\delta = \sup a \cap \kappa^+$ is bounded. Thus, H_δ must fix the canonical name for a and any subset of a will have a name fixed by H_δ as well. □

Claim 2.15. *$\overline{C} \in W_1$.*

Proof. $\{\langle p, \check{\alpha}\rangle \mid \alpha \in x \in \operatorname{stem} p\}$ is a name for $\overline{C}$, and it is preserved by H_κ. □

In order to show that κ remains measurable in the symmetric extension, we need to show that every symmetric subset of κ is introduced by a small forcing over the model $V[\overline{C}]$.

Lemma 2.16. *Let a be a symmetric set of ordinals. Then, there is a forcing notion of cardinality $< \kappa$, $\mathbb{Q}$ such that a is introduced by $\mathbb{Q}$ over $V[\overline{C}]$.*

Proof. Let $\dot{X} \in \mathsf{HS}$ be a name for a set of ordinals. So, there is $\zeta < \kappa^+$ such that H_ζ witnesses that $\dot{X} \in \mathsf{HS}$.

Fix a well order of $H(\kappa^{++})$ and let D be the club of all $x \in \mathcal{P}_\kappa \kappa^+$ which satisfy $x = \mathrm{Hull}^{H(\kappa^{++})}(x) \cap \kappa^+$. Shrinking D to a measure one set, we may assume that $\operatorname{otp} x = (x \cap \kappa)^+$ for all $x \in D$. Let $y \in C_G$ such that $\zeta \leq \sup y$ and every $y' \in C_G$ above y is in D.

We claim that for every $y' \in C_G$ such that $y \subsetneq y'$, $y' \cap \zeta$ is fully determined by $y' \cap \kappa$. Indeed, for every $\overline{\zeta} < \zeta$ in y let $h_{\overline{\zeta}} \colon \kappa \to \overline{\zeta}$ be the least bijection in $H(\kappa^{++})$, then y' is closed under this bijection and its inverse.

Let p be a condition that forces the above holds, so y belongs to the stem of p, so for every $\sigma_f \in H_\zeta$, $p \Vdash \sigma_f(\dot{X}) = \dot{X}$ and that every $y' \in C_G$ above y is in D. By enlarging ζ, if needed, we may assume that $\sup y = \zeta$.

Let $q \leq p$ such that $q \Vdash \check{\beta} \in \dot{X}$. Then whenever $q' \leq p$ such that q' agrees with q on $C_G \restriction y$ and $\overline{C}$, it is impossible that $q' \Vdash \check{\beta} \notin \dot{X}$.

Indeed, by extending q and q' if needed, we may assume that

$$q = \langle d_0, \ldots, d_n \rangle, q' = \langle d'_0, \ldots, d'_n \rangle$$

with $d_i = \langle y_i, B_i \rangle$, $d'_i = \langle y'_i, B'_i \rangle$. For some $i < n$, $y_i = y$, and so for all $j < i$, $y_j = y'_j$ and for every $j \geq i$, $y_j \cap \kappa = y'_j \cap \kappa$. Given such $j \geq i$, we have that for unboundedly many $\overline{\zeta} < \zeta$, $\overline{\zeta} \in y_j \cap y'_j$, so $\overline{\zeta} \cap y_j = \overline{\zeta} \cap y'_j$, and so this holds for all $\overline{\zeta} \leq \zeta$ as well. Therefore, $\operatorname{otp} y_j = \operatorname{otp} y'_j$, so we can find $f \colon \kappa^+ \to \kappa^+$ such that $f \restriction \zeta + 1 = \mathrm{id}$ and $f``y_j = y'_j$ and $\sigma_f(\operatorname{stem} p) = \operatorname{stem} p$. Repeating the argument recursively for $i \leq j < n$ we find some $\sigma_f \in H_\zeta$ such that $\sigma_f(q)$ is compatible with q', and therefore they must agree on the truth of $\check{\beta} \in \dot{X}$, as both names are preserved by σ_f.[7] □

As every set of ordinals is added by a small forcing, any normal measure in $V[\overline{C}_G]$ on κ extends to a measure in W_1, as shown by Jech in [8] (see also [7, Theorem 2.7]).

Lemma 2.17. *W_1 is κ-closed in $V[C_G]$.*

Proof. Suppose that for some $\delta < \kappa$ and $f \in V[C_G]$, $f \colon \delta \to W_1$. We will show that $f \in W_1$ as well. In $V[C_G]$ we can find a function $s \colon \delta \to \kappa^+$ such that for all $\alpha < \delta$, $s(\alpha) = \beta$ if and only of if β is the least for which there is some $\dot{x} \in \mathsf{HS}$ such that $H_\beta \subseteq \operatorname{sym}(\dot{x})$ and $\dot{x}^G = f(\alpha)$, we fix some $\zeta < \kappa^+$ such that $s``\delta \subseteq \zeta$.

Let $p \in G$ be a condition such that

$$p \Vdash \dot{f} \colon \check{\delta} \to \mathsf{HS}, \check{H}_{\dot{s}(\check{\alpha})} \subseteq \operatorname{sym}(\dot{f}(\check{\alpha})), \text{ and } \dot{s}``\check{\delta} \subseteq \check{\zeta},$$

where $\dot{f}^G = f$ and $\dot{s}^G = s$. We may assume, by increasing ζ is necessary, that for every $\langle a, A \rangle \in \operatorname{stem} p$, $\sup a < \zeta$.

Using [11, Lemma 3.1], if $\delta < a \cap \kappa$ for every $\langle a, A \rangle \in \operatorname{stem} p$, we can find a descending sequence of direct extensions p_α for $\alpha \leq \delta$, where for some $\dot{f}_\alpha \in \mathsf{HS}$, $p_\alpha \Vdash \dot{f} \restriction \check{\alpha} = \dot{f}_\alpha$. In particular, $p_\delta \Vdash \dot{f} = \dot{f}_\delta$. As this can be done densely below p, it means that $f \in W_1$ as wanted. If, however, it is not the case that $\delta < a \cap \kappa$ for all a in the stem of p, then by applying Lemma 4.1 and Lemma 5.8 from [11] we can split p into two parts $p^{\leq m}$ and $p^{>m}$, factorize $\mathbb{R}(\mathcal{W}) \restriction p = \mathbb{R}(\mathcal{W}) \restriction p^{\leq m} \times \mathbb{R}(\mathcal{W}) \restriction p^{>m}$ and make the above argument in $\mathbb{R}(\mathcal{W}) \restriction p^{>m}$. In either case, however, we obtain that W_1 is κ-closed in $V[C_G]$, as wanted. □

This lemma provides us with two important corollaries.

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[7] Here the main advantage of using coherent sequence instead of measure sequence manifests itself: this permutation moves the large set B_k to a large set with respect to the intersection of measures on y'_k.

Corollary 2.18. $W_1 \models \mathsf{DC}_{<\kappa}$.

Proof. By [10, Lemma 3.2], since $V[C_G]$ is a model of ZFC and W_1 is κ-closed, $W_1 \models \mathsf{DC}_{<\kappa}$. □

Corollary 2.19. $V_\kappa^{V[C_G]} = V_\kappa^{W_1}$. *In particular,* $V_\kappa^{W_1} \models \mathsf{ZFC}$. □

3. The least inaccessible cardinal

Let us strengthen Theorem 2.1 by collapsing cardinals below κ to make it the least inaccessible.

Theorem 3.1. *There is a symmetric extension of W_1 where κ remains measurable and is the least inaccessible cardinal. In particular, if* GCH *holds and κ is a κ^{++}-supercompact, then there is a symmetric extension in which κ is a measurable cardinal which is the least inaccessible cardinal.*

Proof. Recall that every successor point in $\overline{C}_G \subseteq \kappa$, the Radin club, is regular and every limit point is singular. Since $V[\overline{C}_G] \subseteq W_1$, we can use that to define the symmetric extension. Let $C = \overline{C}_G \cup \{\omega\}$ be enumerated as $\{\rho_\alpha \mid \alpha < \kappa\}$, and define $\mathbb{P}$ to be the Easton-support product $\prod_{\alpha<\kappa} \mathrm{Col}(\rho_\alpha^+, {<}\rho_{\alpha+1})$. Note that $\mathbb{P} \subseteq V_\kappa^{W_1} = V_\kappa^{V[C_G]}$, so all of its initial segments are well-orderable, and behave as expected. We will also write $\mathbb{P}_{\leq\delta}$ ($\mathbb{P}_{<\delta}$) and $\mathbb{P}^{>\delta}$ ($\mathbb{P}^{\geq\delta}$) to indicate the factorization of $\mathbb{P}$ into the initial segment of the product up to δ and its remainder.

Our group, $\mathcal{G}$, is the Easton-support product of $\mathrm{Aut}(\mathrm{Col}(\rho_\alpha^+, {<}\rho_{\alpha+1}))$, acting pointwise on $\mathbb{P}$. The filter is generated by $\mathrm{fix}(\alpha) = \{\pi \in \mathcal{G} \mid \pi \restriction \mathbb{P}_{<\alpha} = \mathrm{id}\}$ for $\alpha < \kappa$. Let W_2 be the symmetric extension of W_1, given by some generic filter.

It is a standard argument that $\mathbb{P}$ is homogeneous, that every proper initial segment of the generic is hereditarily symmetric, and that every set of ordinals in the symmetric extension is added by a proper initial segment of the generic. In particular, Jech's theorem applies and κ remains measurable in W_2.

Let $\alpha < \kappa$ is an uncountable regular cardinal in W_2, letting $\rho = \sup C \cap \alpha$ it must be that $\rho < \alpha$, otherwise α is singular in W_1, so either $\alpha = \rho^+$ or $\alpha = \min C \setminus (\rho+1)$ in which case $\alpha = (\rho^{++})^{W_2}$. In both cases no uncountable regular cardinal below κ is a limit cardinal, so κ is the least inaccessible cardinal. □

Theorem 3.2. $W_2 \models \mathsf{DC}_{<\kappa}$.

Proof. Working in W_1, given any successor ordinal $\alpha < \kappa$ let $\delta = \rho_\alpha$. Decompose $\mathbb{P}$ into $\mathbb{P}_{\leq\alpha} \times \mathbb{P}^{>\alpha}$. Then $W_1 \models |\mathbb{P}_{\leq\alpha}| = \delta$ and $\mathbb{P}^{>\alpha}$ is δ^+-closed. Moreover, we can naturally decompose the symmetric system itself into a product of symmetric systems given by these two component. Since $\mathbb{P}_{\leq\alpha}$ is fixed pointwise, that part is in fact the full generic extension. Therefore W_2 is the generic extension of $W_{2,\alpha}$, the symmetric extension given by $\mathbb{P}^{\geq\alpha}$ component.

Since $W_1 \models \mathsf{DC}_\delta$ and $\mathbb{P}^{\geq\alpha}$ is a δ^+-closed with the filter being δ^+-complete, we get by [10, Lemma 3.1] that $W_{2,\alpha} \models \mathsf{DC}_\delta$. Finally, by [5, Theorem 2.1], we get that $W_2 \models \mathsf{DC}_\delta$ as well. As this holds for unboundedly many $\delta < \kappa$, $W_2 \models \mathsf{DC}_{<\kappa}$. □

4. Lower bounds on the consistency strength

As with many similar results, one is left to wonder if the use of supercompactness is truly necessary, at least in terms of consistency strength needed. The trivial lower bound of a single measurable cardinal was improved by the second and third authors in [7, Theorem 3.6] to show that in the core model there a cardinal with Mitchell order 2 must exist. In this section we improve this result.

Throughout this section, $o(\alpha)$ denotes the Mitchell order of α and K is Mitchell's core model for the anti-large cardinal hypothesis "there is no α such that $o(\alpha) =$

α^{++}". [8] The proof of the theorem relies heavily on Mitchell's covering lemma [12, Theorem 4.19]. We assume that the reader is familiar with the basic definitions and theorems of [12].

Theorem 4.1 (ZF). *Let κ be a strongly inaccessible non-Mahlo measurable cardinal, then $K \models o(\kappa) \geq \kappa + 1$.*

Proof. Let κ be a measurable cardinal, and let U be a κ-complete ultrafilter on κ.[9] Let $C \subseteq \kappa$ be a club of singular cardinals.

Lemma 4.2. $K^{\mathrm{HOD}} = K^{\mathrm{HOD}[C]} = K^{\mathrm{HOD}[C][U \cap \mathrm{HOD}[C]]}$.

Proof. By Vopěnka's theorem [15] (see also [9, Theorem 15.46]), every set or ordinals is generic over HOD. Since K is generically absolute, $K^{\mathrm{HOD}[C]} = K$. Applying this to the set $U \cap \mathrm{HOD}[C]$ (or rather to a set of ordinals encoding it) we obtain the second equality. □

Let $K = K^{\mathrm{HOD}}$. Let $M = \mathrm{HOD}[C]$. The ultrafilter U measures every set in M and thus we can define in V and elementary embedding $j\colon M \to N$, with critical point κ (using the fact that M is a model of ZFC). Using this j we can derive an M-normal measure on κ, $D = \{A \subseteq \kappa \mid A \in M,\ \kappa \in j(A)\}$, containing every club.

Next, since κ is regular, $N \models$ "κ is regular". Therefore, the set of all M-regular cardinals below κ must be in D and in particular, C must contain cardinals which are regular in M and thus in K, but singular in V.

Finally, since $D \in M[U \cap M]$, by the maximality of K, the K-normal measure $D \cap K$ belongs to K.

Let us denote by $A = \mathrm{Reg}^M \cap \kappa = \{\zeta < \kappa \mid M \models \zeta \text{ is regular}\}$. In [7] the argument was that if $\zeta \in A \cap C$, it must be measurable in K. The reason is that ζ is singular in V, so we can find some $t \subseteq \xi$ witnessing that and add it generically to M. Since, as in the lemma above, $K^{M[t]} = K$, by the covering lemma we get that ζ must be measurable in K. In particular $K \models o(\kappa) \geq 2$. By conducting a much more careful analysis of the covering models of K we will see that $o(\kappa)$ must be must higher.

We define a sequence of clubs:

(1) $C_0 = C$,
(2) $C_{\alpha+1} = \operatorname{acc} C_\alpha$,
(3) for limit α, $C_\alpha = \bigcap_{\zeta<\alpha} C_\zeta$.

So $\langle C_\alpha \mid \alpha < \kappa\rangle$ is the sequence of derivatives of C up to κ.

Lemma 4.3. *For every $\zeta \in C_\alpha \cap A$, $K \models o(\zeta) \geq \alpha$.*

Proof. We prove the lemma by induction on α. For $\alpha = 0$ the claim is trivial, and for limit α it follows easily from the definition of C_α.

Let us argue for successor ordinal. Let $\zeta \in C_{\alpha+1} \cap A$ (there is such ζ, since as a club $C_{\alpha+1} \in D$). In particular, $\zeta \in C$ and thus it is singular in V. Let $t \subseteq \zeta$ be a cofinal sequence witnessing it with $\operatorname{otp} t = \operatorname{cf}^V \zeta$, as before $K^{M[t]} = K$.

In $M[t]$, let $W \prec H(\lambda)$, for some $\lambda > \kappa$, with $|W| < \zeta$ containing $\alpha \cup \{t, C\}$ and let $x = W \cap \zeta$. Mitchell's covering lemma gives us a system of indiscernibles $I \subseteq \zeta$

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[8] The same argument can be phrased under the more permissive hypothesis of "there is no inner model with a strong cardinal", but as our current result is much weaker than that, there is no need to weaken the hypothesis in this direction.

[9] By [3], it might be that κ does not carry any normal measures, which is why we cannot assume U is normal.

such that $W \subseteq \mathrm{Hull}^{K_\theta}(\rho \cup I)$ for some $\rho < \zeta$ and some θ. Let $g\colon \zeta \to \zeta$ be the function $g(\xi) = \sup(\mathrm{Hull}^{K_\theta}(\xi) \cap \zeta)$.[10]

As the function g belongs to K, the club, $D_g = \{\eta < \zeta \mid g``\eta \subseteq \eta\}$ is in M, Since ζ is regular in M, $C_\alpha \cap D_g$ is still a club of order type ζ.

If $o(\zeta) \leq \alpha < \zeta$, then for almost every $c \in I$, $o(c) < \alpha$. We will show that this is not the case.

Claim 4.4. *If c is a successor in I, then either $o(c) > \alpha$ or c is regular in M.*

Proof. As I is a collection of indiscernibles, every element of I is regular in K. If it is singular in M, then W must contain a sequence s witnessing that and by its regularity in K, there is a covering model of s in M. In this case, it is sufficient to take a Prikry–Magidor sequence. If $o(c) \leq \alpha$, then this sequence for c has order type $\leq \omega^\alpha$ and thus fully contained in the model W and in particular, $I \cap c$ is cofinal in c. □

So, without loss of generality, any $c \in I \setminus \operatorname{acc} I$ is regular in M. Let $\gamma < \eta$ be an arbitrary ordinal. Since $|C_\alpha \cap D_g|^M = \zeta > |I|^M$, we may pick $\delta \in (C_\alpha \cap D_g \setminus \gamma) \setminus I$ and $c = \min(I \setminus \delta)$. So, c must be a successor element of I and thus (by our assumption) regular in M. Let us show that c must be in C_α.

Otherwise, let $\eta = \max(C_\alpha \cap c) \in W$. On one hand, $\eta \geq \delta$, as $\delta \in C_\alpha$. But this implies that there is a sequence of indiscernibles $\vec{u}$ in $I \cap c$ and ordinals $\vec{\tau}$ in ρ such that $\eta = f(\vec{\tau}, \vec{u})$ for some Skolem function f. As both sequences are contained in δ, as $\delta \in D_g$, $g(x) \leq x$, so $\eta < \delta$, a contradiction. So, $c \in C_\alpha \cap A$ and by the inductive hypothesis, $K \models o(c) \geq \alpha$. □

This implies that $K \models o(\kappa) \geq \kappa$. In order to show that $K \models o(\kappa) \geq \kappa + 1$, it is enough to show that $\mathrm{Ult}(M, U) \models o^K(\kappa) \geq \kappa$. Indeed, in this model κ is regular and belongs to $j(C_\alpha)$ for every $\alpha < \kappa$, so in $\mathrm{Ult}(M, U)$, $o^K(\kappa) \geq \kappa$. □

Remark 4.5. *Why are we not continuing the proof to obtain $o(\kappa) \geq \kappa + 2$, or even higher? Unfortunately, the above proof will fail. The proof of Claim 4.4 relies on the fact that α is covered by the small model, W. Once $o(\kappa) \geq \kappa$ is reached, the claim can no longer work, since the covering model is not small. We are then allowed one more step, to obtain $o(\kappa) \geq \kappa + 1$ by using the ultrapower by U.*

Question 4.6.
1. *Can the lower bound be improved?*
2. *What is the exact consistency strength of a strongly inaccessible non-Mahlo measurable cardinal?*

The proof, as written here, depends on the fact that κ is Mahlo in M such that there is a club $C \in M$ such that every element of C can be singularized in a generic extension of M. By a slight modification of the argument, one can provide the following better formulation.

Theorem 4.7. *Assume that there is no inner model of $\exists \alpha,\, o(\alpha) = \alpha^{++}$. Let κ be an inaccessible cardinal such that there is a club $C \subseteq \kappa$ through the singular cardinals below κ and there is a forcing notion singularizing κ while preserving strong limitness. Then $K \models o(\kappa) \geq \kappa$.*

Proof. Let t be a short cofinal sequence at κ and let $W \prec H(\lambda)$ contain t, C and α for some $\alpha < \kappa$. Let us assume, towards a contradiction, that $o(\kappa) < \alpha$.

[10] In Mitchell's theorem, more information is given on the system of indiscernibles, I, and they are produced using certain predicate $\mathcal{C}$. For our proof, we only care about the ordinals obtained from it.

Let I be a set of indiscernibles for W as before, so $W \subseteq \mathrm{Hull}^{K_\theta}(\rho \cup I)$, $\rho < \kappa < \theta$. As before, by Claim 4.4, a successor element of I must either be of Mitchell order above α, or regular in M.

Let us argue, as before, that there are unboundedly many such elements in C, so they are singular in M. Indeed, exactly as before, for arbitrary γ we pick $\delta \in (C \cap D_g \setminus \gamma) \setminus I$ and set $c = \min(I \setminus \delta)$. We claim that $c \in C$.

Otherwise, let $\eta = \max(C \cap c)$ and even though $\eta \in W$, η is not generated from elements of $I \cap c$ and ρ, as $\sup(I \cap c) \leq \delta \leq \eta$, and the supremum is not obtained. □

In [2], the problem of the consistency strength of embedding the forcing for shooting a club through the singular cardinals into a tree Prikry forcing was studied. The consistency strength of this situation was bounded from below by $o(\kappa) \geq \kappa^+ + 1$ and from above by a slight strengthening of a superstrong cardinal. Even though it is unclear whether the construction of [2] can be used in our situation, it seems reasonable that a similar method might be used in order to obtain a model with a non-Mahlo measurable cardinal. Thus we conjecture that the consistency strength of Theorem 2.1 can be further reduced. Also, we expect that the lower bounds for Theorem 2.1 and Theorem 4.1 are not optimal either.

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