SPFA by finite conditions. *

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In [2] Itay Neeman presented a new way of iterating of proper forcings. We would like to generalize it here to semi-proper.

1 Forcing conditions

Let us start with the following simple observation.

Lemma 1.1 *Suppose that A, B are elementary submodels of $\langle H_\chi, <\rangle$, with χ large enough, and $B \in A$. Let $Q \in A, B$ be a forcing notion. Suppose that B is closed under $|Q|-$sequences of its elements (or even ${}^{\eta>}B \subseteq B$, once Q satisfies $\eta-$c.c.). Let $G \subseteq Q$ be generic. Then $(A \cap B)[G] = A[G] \cap B[G]$.*

Proof. Clearly $(A \cap B)[G] \subseteq A[G] \cap B[G]$. Let us show the opposite inclusion. It is enough to deal with ordinals.
Note that $B[G] \cap V = B$, since $Q \subseteq B$. Let $\delta \in A[G] \cap B[G] \cap On = A[G] \cap B \cap On$. Pick a canonical name $\underset{\sim}{\delta} \in A$ such that $\underset{\sim}{\delta}_G = \delta$. It is of the form $\langle\langle q_i, \check{\tau}_i\rangle \mid i < \rho \leq |Q|\rangle$, where τ_i's are in B and $\langle q_i \mid i < \rho\rangle$ is a maximal antichain.
Now, $\underset{\sim}{\delta} \in B$, since ${}^{|Q|}B \subseteq B$. So, $\underset{\sim}{\delta} \in A \cap B$. Hence, $\delta = \underset{\sim}{\delta}_G \in (A \cap B)[G]$ and we are done.
□

Remark 1.2 1. If Q is a proper forcing and A is countable, then $A[G] \cap V = A$, $(A \cap B)[G] \cap V = A \cap B$ and, since $B[G] \cap V = B$, we have $(A \cap B)[G] = A[G] \cap B[G]$ immediately.

*Itay Neeman suggested a different construction which is more close to his original in [2].

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2. Note that if one does not require that $B \in A$, then it is possible to have $(A \cap B)[G] \subsetneqq A[G] \cap B[G]$. even for a semi-proper forcing notion Q and B of the form V_λ, with an inaccessible λ.

   Thus, suppose that we have an increasing continuous sequence $\langle \lambda_\alpha \mid \alpha < \delta \rangle$ in A such that

   (a) δ is a regular cardinal,

   (b) $\aleph_2 \leq \delta < \lambda_0$,

   (c) $\lambda_{\alpha+1}$ is an inaccessible, for every $\alpha < \delta$,

   (d) $V_{\lambda_\alpha} \prec H_\chi$, for every $\alpha < \delta$,

   (e) there is a semi-proper forcing $Q \in A$ that changes the cofinality of δ to ω (say, δ is a measurable or the Namba forcing for δ is a semi-proper).

   Let $\eta = A \cap \omega_1$. Set $\lambda = \lambda_{\beta+1}$, for some $\beta \in [\eta + 1, \delta)$.

   Suppose now that a generic subset G of Q chooses η to be a member of the generic cofinal in δ sequence. Then η, and hence, also λ_η will be in $A[G]$, but not in $(A \cap B)[G]$, since $\lambda_\eta \geq \sup(A \cap B \cap On)$. Just, in general, if $M \prec H_\chi$ and $Q \in M$, then $\sup(M \cap On) = \sup(M[G] \cap On)$. Because, whenever $\underset{\sim}{\sigma} \in M$ is a name of an ordinal, then the set

   $$X = \{\xi \mid \exists q \in Q(q \Vdash \underset{\sim}{\sigma} = \xi)\} \in M.$$

   Hence $\sup(X) \in M$. But $\Vdash \underset{\sim}{\sigma} < \sup(X)$.
   It is not hard to modify the construction in order to insure $A[G] \cap \omega_1 = A \cap \omega_1$. Thus, in case of a measurable δ and $Q =$Prikry forcing, let us first pick a semi-generic condition $\langle t, T \rangle$ for A, and let $\eta' = \min(T)$. Set now $\lambda = \lambda_{\beta+1}$, for some $\beta \in [\eta' + 1, \delta)$, and let η' be a member of a generic Prikry sequence. Then, as above, η', and hence, also $\lambda_{\eta'}$ will be in $A[G]$, but not in $(A \cap B)[G]$, since $\lambda_{\eta'} \geq \sup(A \cap B \cap On)$.

$\square$ of remark.

Let κ be a Mahlo cardinal. Fix an increasing continuous chain $\langle \mathfrak{M}_\alpha \mid \alpha < \kappa \rangle$ of $V_{\kappa+1}$ such that

1. $|\mathfrak{M}_\alpha| < \kappa$,

2. $\mathfrak{M}_\alpha \cap V_\kappa = V_{\kappa_\alpha}$, for some $\kappa_\alpha < \kappa$,

3. if κ_α is a regular cardinal, then it is an inaccessible,

4. κ_0 and each $\kappa_{\alpha+1}$ are inaccessible cardinals.

Lemma 1.3 *Suppose $\rho < \eta \leq \kappa$ and κ_ρ,κ_η are regular cardinals. Assume that $A \prec V_{\kappa_\eta}$ is countable. Then, there is $A' \prec V_{\kappa_\rho}$ which realizes the same type over $A \cap V_{\kappa_\rho}$ in V_{κ_ρ}, as A does in V_{κ_η}.* [1]

Proof. Note that $A \cap V_{\kappa_\rho} \in V_{\kappa_\rho}$ since A is countable, κ_ρ is regular and hence inaccessible. Pick some $A' \in \mathfrak{M}_\rho$ that realizes in V_{κ_ρ} the same type as A does in V_{κ_η} over $A \cap V_{\kappa_\rho}$. We have $\mathfrak{M}_\eta \models A \prec V_\kappa$, hence $\mathfrak{M}_\rho \models A' \prec V_\kappa$. But then $A' \subseteq \mathfrak{M}_\rho \cap V_\kappa = V_{\kappa_\rho}$. Now it follows that $A' \prec V_{\kappa_\rho} \prec V_{\kappa_\eta} \prec V_\kappa$.
$\square$

The following notion is relevant in a non-proper context.

Definition 1.4 Suppose that $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is an iteration with initial segments in V_κ. Let $A \preceq V_\kappa$ and $X \in V_\kappa$.
We say α *is reachable from A in 0–steps* iff $\alpha \in A$. Define α is reachable from A in 1–step iff there are $\nu \in A \cap \alpha$ with $P_\nu \in A$ and $p \in P_\nu$, $p \Vdash_{P_\nu} \alpha \in A[\underset{\sim}{G}(P_\nu)]$. Call such p and ν a 1*–step reachability witnesses for α.* Continue by induction. α is reachable from A in $n+1$–steps iff there are n-steps reachability witnesses $\langle \langle p_k, \nu_k \rangle \mid k \leq n \rangle$, $\nu < \alpha$ and $p \in P_\nu$ such that

1. $p_i \in P_{\nu_i}$, for every $i \leq n$,
2. $p_i \leq p_j \restriction \nu_i$, for all $i \leq j \leq n$,
3. $\nu_i < \nu_j < \nu$, for all $i < j \leq n$,
4. p_i, ν_i are a 1–step reachability witnesses for ν_{i+1} with model $A[\underset{\sim}{G}(P_{\nu_{i-1}})]$, if $i > 0$ or with A, if $i = 0$, for every $i < n$,
5. $p_n \Vdash_{P_{\nu_n}} \nu, P_\nu \in A[\underset{\sim}{G}(P_{\nu_n})]$,
6. $p \restriction \nu_n \geq p_n$,
7. $p \Vdash_{P_\nu} \alpha \in A[\underset{\sim}{G}(P_\nu)]$.

[1]By the type over $A \cap V_{\kappa_\rho}$ that A realizes in V_{κ_η}, we mean the set of the Gödel numbers of formulas $\varphi(v)$ in the language of set theory enriched by adding constants for every element of $A \cap V_{\kappa_\rho}$, such that $V_{\kappa_\eta} \models \varphi(A)$.In particular, A and A' are isomorphic over $A \cap A' = A \cap V_{\kappa_\rho}$.

Let us call α *is reachable from* A iff for some $n < \omega$, α is reachable from A in n–steps.

Given some $q \in P_\kappa$, let us define reachability from A relatively to q similar only requiring that witnesses extend q.

If $X \in V_\kappa$, then X *is reachable from* A iff rank(X) and $f_{\mathrm{rank}(X)}(X)$ are reachable from A, where $f_\xi : V_\xi \leftrightarrow |V_\xi|$ is some fixed in advance well ordering.

Remark 1.5 The only reachable ordinals from A will be members of A, once forcing notions under the consideration are proper instead of semi-proper.

Let us turn to the definition of forcing. We would like to keep it in form of an iteration. There will be finite sets of models "side conditions". However, we prefer to spread them among the coordinates of the iteration. A complication in the semi-proper context is that uncountable ordinals of a given model may increase once passing to a generic extension. We deal with this using *reachability* and *jumps* (defined below).

Definition 1.6 Define by induction on $\tau \leq \kappa$ an iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \tau, \beta < \tau \rangle$.

1. If $\tau = \tau' + 1$ and $\kappa_{\tau'}$ is a singular, then let $P_\tau = P_{\tau'} * Cohen(\omega)$ (or just let $Q_{\tau'}$ to be trivial).

2. If τ is a limit ordinal or if $\tau = \tau' + 1$ and $\kappa_{\tau'}$ is a regular, then set $p = \langle \underset{\sim}{p}_\beta \mid \beta < \tau \rangle \in P_\tau$ iff

   (a) for each $\gamma < \tau$, with κ_γ regular, we have

      i. $p \restriction \gamma = \langle \underset{\sim}{p}_\beta \mid \beta < \gamma \rangle \in P_\gamma$,

      ii. $p \restriction \gamma \Vdash \underset{\sim}{Q}_\gamma$ is a semi-proper forcing notion and $|\underset{\sim}{Q}_\gamma| \leq \kappa_{\gamma+1}$.

   (b) There are three finite sets $s(p), j(p)$ and $m(p)$ (this sets can be read from p except when τ is a singular) such that

      i. $s(p) \subseteq \tau$ called the *support* of p is such that for each $\gamma \in s(p)$
      $p \restriction \gamma \Vdash \underset{\sim}{p}_\gamma \in \underset{\sim}{Q}_\gamma$,

      ii. $j(p) \subseteq \tau$ called the *jumps* of p.
      This will be places in p where change of sequences of models will be allowed.

      iii. $m(p) = \{A_0, ..., A_{k(p)-1}\}$, $k(p) < \omega$, is a finite set called the *models* of p.
      The intuition behind $s(p)$ and $m(p)$ is as follows. $s(p)$ provides finitely many places where essential information, i.e. elements of the forcing notions $\underset{\sim}{Q}_\beta$ are given. The iteration does not have finite support however, so certain

restrictions are made at the rest of coordinates. Basically we would like to have for each $\beta \in \tau \setminus s(p)$ a finite sequence of models over the coordinate β and to require that in all further extensions r of p with $\beta \in s(r)$, $\underset{\sim}{r}_\beta$ is semi-generic over all this models. But having many different models spread over coordinates β results in non-semi-properness of the iteration (already at length ω). Actually, $\aleph_1$ is collapsed.

A solution is to keep a finite information and to it spread over coordinates in $\tau \setminus s(p)$.

An additional complication here relatively to the proper forcing case is that we cannot keep a single sequence of models that is $\in$–increasing, as it is done in [2]. We address this issue again below.

Let us state the requirements on $m(p)$. For every $i < k(p)$ the following hold:

A. $A_i \in V$,

B. $|A_i| = \aleph_0$ or $A_i = V_\delta$ for some inaccessible $\delta < \kappa_\tau$,

C. for every $i < k(p)$,$s, 1 \leq s \leq n_i$,
$A_i \prec V_{\kappa_\tau}$, if κ_τ is an inaccessible,
and $A_i \prec V_{\kappa_{\tau+1}}$, otherwise, i.e. whenever τ is a limit ordinal and κ_τ is a singular cardinal,

(c) For each $\gamma < \tau$ let us specify a sequence of models based on a set of models from $m(p)$ which will stand over the coordinate γ.
Let $\gamma < \tau$. Take all models $A \in m(p)$ such that
$p \restriction \gamma \Vdash_{P_\gamma} \kappa_\gamma, P_\gamma, \underset{\sim}{Q}_\gamma$ are reachable from A. Denote the set of all such models by $\tilde{A}''_\gamma$.
Let $G(P_\gamma) \subseteq P_\gamma$ be a generic with $p \restriction \gamma \in G(P_\gamma)$. Set $\tilde{A}'_\gamma = \{A[G(P_\gamma)] \cap V_{\kappa_{\gamma+1}}[G(P_\gamma)] \mid A \in \tilde{A}''_\gamma, \kappa_{\gamma+1} \in A[G(P_\gamma)]\} \cup \{A[G(P_\gamma)] \mid A \in \tilde{A}''_\gamma, A[G(P_\gamma)] \subseteq V_{\kappa_{\gamma+1}}[G(P_\gamma)]$. Let $\tilde{A}_\gamma$ be the set obtained from $\tilde{A}'_\gamma$ by adding to it all intersections of countable members of $\tilde{A}'_\gamma$ with uncountable ones.
Note that 1.1 does not always apply here, since some V_δ's may be reachable from A's, but not elements.
Back in V, let $\underset{\sim}{\tilde{A}}_\gamma$ be a name of such $\tilde{A}_\gamma$.
We require the following:

i. $p \restriction \gamma \Vdash_{P_\gamma}$ the set $\underset{\sim}{\tilde{A}}_\gamma$ is well ordered according to $\in$ $-$relation.
Let us denote by $\langle \underset{\sim}{A}_{\gamma 1}, \underset{\sim}{A}_{\gamma 2}, ..., \underset{\sim}{A}_{\gamma \underset{\sim}{n}_\gamma} \rangle$ the sequence obtained from $\underset{\sim}{\tilde{A}}_\gamma$ by this well order.

ii. $p \restriction \gamma \Vdash_{P_\gamma} (\underset{\sim}{A}_{\gamma 1}[\underset{\sim}{G}(P_\gamma)] \in \underset{\sim}{A}_{\gamma 2}[\underset{\sim}{G}(P_\gamma)] \in ... \in \underset{\sim}{A}_{\gamma \underset{\sim}{n}_\gamma}[\underset{\sim}{G}(P_\gamma)]$ and the sequence is closed under intersections of its countable models with uncountable ones.)

iii. If $\gamma \in s(p)$, then $p \restriction \gamma \Vdash_{P_\gamma} \forall k(1 \leq k \leq \underset{\sim}{n}_\gamma \to (\underset{\sim}{p}_\gamma \in \underset{\sim}{A}_{\gamma k}[\underset{\sim}{G}(P_\gamma)] \vee \underset{\sim}{p}_\gamma$ is a semi-generic over $\underset{\sim}{A}_{\gamma k}[\underset{\sim}{G}(P_\gamma)]))$.

(d) (Jumps) Suppose that for some $\gamma < \tau, k, l < \omega$ we have $p \restriction \gamma \Vdash_{P_\gamma} \underset{\sim}{A}_{\gamma k}[\underset{\sim}{G}(P_\gamma)] \in \underset{\sim}{A}_{\gamma l}[\underset{\sim}{G}(P_\gamma)]$ and $\underset{\sim}{A}_{\gamma l}[\underset{\sim}{G}(P_\gamma)]$ countable.
Let $\gamma^* \leq \gamma$ be the maximal element of $j(p)$ below γ, if exists or 0 otherwise. Let $A \in m(p)$ be any model such that $\underset{\sim}{A}_{\gamma l}$ is obtained from A, as in (2c) above(i.e. as a member of $\tilde{A}'_\gamma$ or $\tilde{A}_\gamma$, forced by $p \restriction \gamma$). Pick the least element $\gamma^{**} \geq \gamma^*$ which is reachable from A. We require that there is $B \in m(p)$ such that

i. $p \restriction \gamma \Vdash_{P_\gamma} \underset{\sim}{A}_{\gamma k}$ can be obtained from B, as in (2c) above.

ii. $p \restriction \gamma \Vdash_{P_{\gamma^{**}}} B \in A[\underset{\sim}{G}(P_{\gamma^{**}})]$.
In particular, if $j(p) \cap \gamma + 1 = \emptyset$, then just $B \in A$.

Definition 1.7 Let $\tau \leq \kappa$ and let $p = \langle \underset{\sim}{p}_\beta \mid \beta < \tau \rangle, p' = \langle \underset{\sim}{p}'_\beta \mid \beta < \tau \rangle \in P_\kappa$. Set $p \geq p'$ (p is stronger than p') iff

1. $m(p) \supseteq m(p')$,

2. $s(p) \supseteq s(p')$,

3. $j(p) \supseteq j(p')$,

4. for every $\beta \in s(p')$ we have $p \restriction \beta \Vdash_{P_\beta} \underset{\sim}{p}_\beta \geq_{\underset{\sim}{Q}_\beta} \underset{\sim}{p}'_\beta$,

Lemma 1.8 *Let $\rho \leq \tau \leq \kappa$. Then the forcing P_ρ is a complete subforcing of P_τ.*

Proof. Let $\rho < \tau \leq \kappa$.
We define a dense subset $D_{\tau\rho}$ of P_τ and a projection function $\pi_{\tau\rho}$ from $D_{\tau\rho}$ to a dense subset of P_ρ.
Let $D'_{\tau\rho}$ be the set of all elements $p \in P_\tau$ such that

1. $V_{\kappa_{\rho+1}} \in m(p)$ and, if κ_ρ is a regular (and so inaccessible) cardinal then $V_{\kappa_\rho} \in m(p)$,

2. there exists $A \in m(p)$ which satisfies the following:

   (a) $|A| = \aleph_0$,

   (b) for all $B \in m(p) \setminus \{A\}$ we have $B \in A$,

Clearly each condition in P_τ can be extended by adding $V_{\kappa_{\rho+1}}$,V_{κ_ρ}, if κ_ρ is a regular, and such model A. So, $D'_{\tau\rho}$ is dense.
Define now $D_{\tau\rho}$. First, we use Lemma 1.3 to reflect A to $\mathfrak{M}_\rho$ over $A \cap V_\rho$, if κ_ρ is a regular, and reflect A to $\mathfrak{M}_{\rho+1}$ over $A \cap V_{\rho+1}$, if κ_ρ is a singular. Let A' be the result.[2]
Let $m'(p)$ denotes the set of models that correspond to $m(p)$ under this reflection.
Extend p to p^* by adding to p all the models from the set $m(p) \cup m(p')$.
Set $D_{\tau\rho} = \{p^* \mid p \in D'_{\tau\rho}\}$.
Let $q \in D_{\tau\rho}$. Define $\pi_{\tau\rho}(q)$ to be the restriction of q to V_{κ_ρ}, if κ_ρ is a regular, and to $V_{\kappa_{\rho+1}}$, if κ_ρ is a singular.
It is easy to see that $\pi_{\tau\rho}$ is as desired.
□

Lemma 1.9 *Let $\alpha < \kappa$, $G_\alpha \subseteq P_\alpha$ generic and $A_0 \in A_1 \in ... \in A_n$ be an $\in$- increasing sequence of elementary submodels of $V_\kappa[G_\alpha]$ consisting of countable models and models of the form $V_\delta[G_\alpha]$ for some inaccessible $\delta < \kappa$ such that $Q_\alpha \in A_i$, for every $i \leq n$. Suppose that the sequence is closed under intersections of each of its countable members with the first uncountable one, i.e. if $i^* \leq n$ is the least with A_{i^*} uncountable, then for every $i \leq n$ with A_i countable we have $A_{i^*} \cap A_i = A_k$, for some $k \leq n$(actually then necessary $k < i^*$).*
Then there is $q \in Q_\alpha$ which is Q_α-semi-generic for every model on sequence.

Proof. Clearly, any condition is generic over uncountable A_i. So our worry is only about countable ones.
Consider $\langle A_j \mid j < i^*\rangle$. This an $\in$-increasing sequence of countable models. It is easy to find q which Q_α-semi-generic for all of them. Just pick $q_0 \in A_1$ to be Q_α-semi-generic over A_0. Then extend it to $q_1 \in A_2$ which is Q_α-semi-generic over A_1 and so on.
We claim q is Q_α-semi-generic over every model A_i, $i \leq n$. Just note that $\mathcal{P}(Q_\alpha) \subseteq A_{i^*}$. So for every countable A_i, if $D \in A_i$ is a dense subset of Q_α, then $D \in A_i \cap A_{i^*}$. Hence a condition is semi-generic over A_i iff it is a semi-generic over $A_i \cap A_{i^*}$. By the assumption, $A_i \cap A_{i^*} = A_k$, for some $k < i^*$. So, q is Q_α-semi-generic over A_i.
□

Lemma 1.10 *The forcing P_α is semi-proper for every $\alpha \leq \kappa$.*

[2]Note that in view of Remark 1.2(2), it is not enough just to intersect models with V_{κ_ρ}, even if κ_ρ is a regular, and then to use Lemma 1.1. However, it will be fine to do this once V_{κ_ρ} is a member of each model with supremum above κ_ρ.

Proof. Let $M \prec H(\chi)$ be a countable elementary submodel, $r \in P_\kappa$ and $r, P_\kappa \in M$, for some $\chi > \kappa$ large enough. Extend r to a condition r^* by adding M to every sequence in $m(r)$ (as the largest model under $\in$) and intersections of it with uncountable models. We claim that $r^* \restriction \alpha$ is P_α–semi-generic over M, for every $\alpha \leq \kappa, \alpha \in M$. Let us prove this by induction on α. Let $p \geq r^* \restriction \alpha, p \in P_\alpha$ and $\underset{\sim}{\mu} \in M$ a P_α–name of a countable ordinal.

It is enough to find some $\tilde{p}$ which is compatible with p and forces $\underset{\sim}{\mu} \in M \cap \omega_1$.

Case 1. *α is a successor ordinal or α is a limit ordinal and there are ordinals in $M \cap \alpha$ above members of $s(p) \cup j(p)$.*

Let $\eta = \alpha'$, if $\alpha = \alpha' + 1$ and if α is a limit ordinal, then let η be the first element $M \cap \alpha$ of above members of $s(p) \cup j(p)$.
Force with P_η. Let G_η be generic with $p \restriction \eta \in G_\eta$. By induction, $M[G_\eta] \cap \omega_1 = M \cap \omega_1$.
Work in $(M[G_\eta])^V$(i.e. the ground model of $M[G_\eta]$; note that it may be bigger than M, need not be in V, but it is equal to $M[G_\eta] \cap V$) pick an extension r' of $r \restriction \alpha$ such that $m(r')$ includes the restrictions to $(M[G_\eta])^V$ of sequences of $m(p)$. Recall that there are only finitely many models that are involved in this sequences and all relevant ones are in $(M[G_\eta])^V$ by the choice of η.
Consider the set $D_\eta = \{t \in P_\eta \mid \exists t' \in P_\alpha$ such that $t' \restriction \eta = t \in P_\alpha, t' \geq r'$ and it decides $\underset{\sim}{\mu}\}$. D_η is in $(M[G_\eta])^V$ and is dense in P_η. Pick some $t \in G_\eta \cap D_\eta$ in $M[G_\eta]$. Let $t' \in (M[G_\eta])^V$ be a witness forcing $\underset{\sim}{\mu}$ to be some $\nu < \omega_1$. Then $\nu \in M[G_\eta] \cap \omega_1 = M \cap \omega_1$.
Let us argue that t' is compatible with p. The only problem that may lead to incompatibility is that for some $\beta \in \alpha \setminus s(p)$ we have $\beta \in s(t')$ and t'_β is not semi-generic for some countable model of p_β. Consider such β. Then β must be one of the coordinates of t', since t was in G_η which is generic for P_η. Remember that $t' \in (M[G_\eta])^V$ and hence $\underset{\sim}{t}'_\beta \in (M[G_\eta])^V$ as well. By the definition of order the model M (and actually, $M[G_\eta]$) appears among models of p_β. The part of p_β which consists of elements of M^{P_β} is in fact included into r'_β. But t'_β is Q_β–semi-generic for every countable model of r'_β, and hence of the part of p_β below M^{P_β}. Also $t'_\beta \in (M[G_\eta])^V$, since there are only finitely many places where $s(r')$ is increased and so all of them are inside $(M[G_\eta])^V$. Now everything follows, since p_β is $\in$–increasing, closed under intersections sequence with M inside and Q_β is a semi-proper forcing. Note that there may be a need to add η to the set of jumps, once some of models of t' are in $(M[G_\eta])^V \setminus M$.

Case 2. *α is a limit ordinal and the ordinals of $M \cap \alpha$ are bounded below* $\max(s(p) \cup j(p))$.

By extending p if necessary we can assume that $\max(s(p))$ is above $j(p)$.
A new point here relatively to the iteration of proper forcing notions is that a generic extension $M[G_\alpha]$ can have new ordinals (i.e. ordinals not in M) even if $M[G_\alpha] \cap \omega_1 = M \cap \omega_1$.

If $\alpha < \omega_2$, then the treatment of semi-proper and proper cases is identical, but if $\alpha \geq \omega_2$, then in a semi-proper case $M[G_\alpha] \setminus M$ may have ordinals in $s(p)$.
If no ordinal $\zeta, \min(s(p) \setminus M) \leq \zeta < \alpha$ is reachable from M (relatively to p), then take η to be an element of $M \cap \alpha$ above $M \cap s(p)$ and repeat the argument of the previous case.
Otherwise find some $n < \omega$, $\xi < \alpha$ and $q \in P_\xi$ compatible with $p \restriction P_\xi$ which witness reachability from M of some $\zeta < \alpha$ in n–steps and so that $s(p) \subseteq \zeta$ or no element of $\geq \min(s(p) \setminus \zeta + 1)$ is reachable relatively to q.[3] Suppose for simplicity that $n = 1$. The treatment of the general case is similar.
Then $\xi \in M$, by the definition of reachability witnesses. Force with P_ξ. Let G_ξ be a generic with $p \restriction \xi, q \in G_\xi$. By induction, $M[G_\xi] \cap \omega_1 = M \cap \omega_1$. We have $\zeta \in M[G_\xi]$.
Work in $V[G_\xi]$. Take $\eta = \zeta$ and repeat the argument of Case 1. The requirement (2d) of 1.6 applies in order to deal with elements of $j(p)$ which are below $\sup(M \cap \alpha)$, but not in $M[G_\eta]$ (if any).
□

Lemma 1.11 *The forcing P_κ preserves κ.*

Proof. Let $M \preceq H(\chi)$ be an elementary submodel such that $M \cap V_\kappa = V_\delta$, for some inaccessible $\delta = \kappa_\gamma$ below κ, $r \in P_\kappa$ and $r, P_\kappa \in M$, where χ is a big enough cardinal.
Extend r to a condition r^* by adding M to $m(r)$ (as the largest model under $\in$). We claim that r^* is P_κ–generic over M. Let $p \geq r^*$ and $D \in M$ a dense open subset of P_κ. It is enough to find some $\tilde{p}$ which is compatible with p and belongs to $M \cap D$.
Without loss of generality we assume that there is a countable model $A \in m(p)$ such that $A \supseteq s(p)$ and every model of $m(p)$ except A is in A.
Our next tusk will be to reflect A nicely into M. Note that $A \cap M \in M$ so there are many A''s in M that realize same types as A over $A \cap M$. But we will need to specify a particular type. Our worry is about ordinals in $M \cap \kappa = \delta$ beyond those of $A \cap M \cap \kappa = A \cap \delta$ that are reachable from A.

For every $n < \omega$ let η_n be the supremum of all ordinals less than δ which are reachable in n–steps from A. Note that $|\eta_n| \leq |P_{\eta_n}|$. By induction, using inaccessibility of δ it follows that η_n and then also $|P_{\eta_n}| < \delta$.
Let $\eta_\omega = \bigcup_{n<\omega} \eta_n$.

[3]Here is the point that prevents us from just dealing with an $\in$–increasing sequence of models in V, as was done in [2] for proper forcing.

Pick now $A' \in M$ which realizes the same type as A over $M \cap A$ with parameters from $V_{\kappa_{\eta_\omega}+1}$. Also include D as a parameter. Let M' be the image of M under the isomorphism between A and A'. Set $\delta' = M' \cap \kappa$. Clearly, $M' \in M$ and $\delta' < \delta$.

Now we extend p by adding A' and M' to $m(p)$. Consider $p \restriction M$ which defined naturally by leaving only models from sequences of $m(p)$ which are in M. Find some $\tilde{p} \geq p \restriction M$ inside $M \cap D$. By the construction it is compatible with p and so we are done.
The crucial point here is that A and A' agree on all reachable from A coordinates $\gamma < \delta$. Note that no ordinals $\geq \delta$ can be reached from A'. Hence we have a compatibility.
$\square$

Now, if κ is a supercompact and a Laver function $F : \kappa \to V_\kappa$ supplies semi-proper forcings, then SPFA will hold in $V[G(P_\kappa)]$.

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References

[1] M. Foreman, M. Magidor and S. Shelah, Martin Maximum, Ann. of Math. Norte Dame Journal of Formal Logic, 46(3), 2005, pp.357-371.

[2] I. Neeman, Forcing with sequences of models of two types.

[3] S. Shelah, Proper and Improper forcing, Springer 1998.