Intermediate models of Prikry generic extensions

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Abstract

We prove that if $\mathbf{V} \subseteq \mathbf{V}[h]$ is a generic extension by Prikry forcing then every transitive intermediate model M of **ZFC**, where $\mathbf{V} \subsetneqq M \subseteq \mathbf{V}[h]$, is again a Prikry generic extension of $\mathbf{V}$. Moreover the family of intermediate models is parametrised by $\mathscr{P}(\omega)/\text{finite}$. The result is proved by studying $\mathbf{V}$-constructibility degrees of sets in $\mathbf{V}[h]$ using parameters in $\mathbf{V}$.

Introduction

For a generic extension $\mathbf{V} \subseteq \mathbf{V}[G]$ one may study the family of all intermediate models M where $\mathbf{V} \subseteq M \subseteq \mathbf{V}[G]$ and M is a transitive model of **ZFC**. It is well-known that every intermediate model is itself a set generic extension of $\mathbf{V}$ and hence of the form $\mathbf{V}[X]$ for some $X \in \mathbf{V}[G]$. The structure of the family of intermediate models depends on the initial forcing extension. If $\mathbf{V}[G]$ is obtained by adjoining a Sacks real, $\mathbf{V}$ and $\mathbf{V}[G]$ are the only intermediate models. In the case of Cohen and Solovay-random reals the family displays a rather amorphous structure with respect to the inclusion relation $\subseteq$. In this paper we show that for Prikry forcing there is good control over the intermediate models:

Theorem 1. *Let κ be a measurable cardinal in the ground model $\mathbf{V}$ and let $h \subseteq \kappa$ be a Prikry sequence over $\mathbf{V}$. Then every intermediate model M of the Prikry extension $\mathbf{V} \subseteq \mathbf{V}[h]$ is of the form $M = \mathbf{V}[h']$ for some subset of the Prikry sequence. Moreover, it is true in $\mathbf{V}[h]$ that the structure*

$$\langle \{M : M \text{ is an intermediate model, } \mathbf{V} \subseteq M \subseteq \mathbf{V}[h]\}, \subseteq \rangle$$

is order isomorphic to $\langle \mathscr{P}(\omega)/\text{finite}, \subseteq^* \rangle$.

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Note that every *infinite* subset of a Prikry sequence h in $\mathbf{V}[h]$ is a Prikry sequence itself, by the Matias criterion, see for example T. Jech [3],21.14.

Let us state an equivalent formulation of Theorem 1 in forcing terms:

Theorem 2. *Every subforcing of the Prikry forcing is either trivial or it is equivalent to the the Prikry forcing with the same normal measure.*

Note that this may break down without normality, i.e. once tree Prikry forcings are used. Also there are always non-trivial subforcings of the supercompact Prikry forcing. Namely, if κ is a supercompact cardinal, then every $(\kappa,\infty)-$ distributive forcing notion $(Q,\leq)$ is a subforcing of the supercompact Prikry forcing with a normal ultrafilter over $\mathcal{P}_\kappa(2^{|Q|})$, see, for example, [5] 6.21.

If $\mathbf{V}[X]$ and $\mathbf{V}[Y]$ are intermediate models then $\mathbf{V}[X]\subseteq\mathbf{V}[Y]$ iff $X\in\mathbf{V}[Y]$, that is, iff X is $\mathbf{V}$-constructible from Y. So we are lead to study the degrees of $\mathbf{V}$-constructibility in Prikry extensions of the ground model $\mathbf{V}$. Generic extensions by Cohen or Solovay-random reals display a rather amorphous structure of the constructibility degrees over the ground universe. Some other extensions, notably Sacks forcing and its iterations, contain a more definite structure of the constructibility degrees, as shown for example in [1].

Formally the partial order $\leqslant_{\mathbf{V}}$ of $\mathbf{V}$-constructibility in a generic extension of $\mathbf{V}$ is defined by: $X\leqslant_{\mathbf{V}} Y$ iff $X\in\mathbf{V}[Y]=\bigcup_{z\in\mathbf{V},\ z\subseteq\mathtt{Ord}}\mathbf{L}[z,Y]$, and $\mathbf{L}[z,Y]$ is the class of all sets Gödel-constructible relative to $\langle z,Y\rangle$. Thus $X\leqslant_{\mathbf{V}} Y$ means that there is a set $z\in\mathbf{V}$, $z\subseteq\mathtt{Ord}$, such that $X\in\mathbf{L}[z,Y]$. Here $\mathtt{Ord}$ denotes the ordinals in $\mathbf{V}$. The equivalence relation $\equiv_{\mathbf{V}}$ is defined so that $X\equiv_{\mathbf{V}} Y$ iff both $X\leqslant_{\mathbf{V}} Y$ and $Y\leqslant_{\mathbf{V}} X$. The equivalence classes of $\equiv_{\mathbf{V}}$ are called *degrees of* $\mathbf{V}$*-constructibility*, or just $\mathbf{V}$*-degrees*.

Prikry forcing over a ground model $\mathbf{V}$ with a measurable cardinal κ produces a *Prikry sequence*, that is, a cofinal set $h\subseteq\kappa$ of order type ω. For more details, see Section 3. The next theorem is our main result; it obviously implies Theorem 1. Theorem 3 says that every set in the Prikry extension is $\mathbf{V}$-constructibly equivalent to a subsequence of the Prikry sequence, and moreover, the structure of the $\mathbf{V}$-degrees is isomorphic to the structure of subsets of ω under inclusion modulo finite.

Theorem 3. *Suppose that* $h\subseteq\kappa$ *is a Prikry sequence over the ground model* $\mathbf{V}$. *Then in the Prikry extension* $\mathbf{V}[h]$ *of* $\mathbf{V}$:

(i) *for every set* X *there exists a set* $d\subseteq h$ *satisfying* $X\equiv_{\mathbf{V}} d$;

(ii) *if* $c,c'\subseteq h$ *then* $c'\leqslant_{\mathbf{V}} c$ *iff* $c'\smallsetminus c$ *is finite.*

We obtain Theorem 3 as a consequence of the two following theorems:

Theorem 4 (= Theorem 3 for sets $X\subseteq\kappa$)**.** *Suppose that* $h\subseteq\kappa$ *is a Prikry sequence over the ground model* $\mathbf{V}$. *Then in the Prikry extension* $\mathbf{V}[h]$ *of* $\mathbf{V}$:

(i) *for every set* $X \subseteq \kappa$ *there exists a set* $d \subseteq h$ *satisfying* $X \equiv_{\mathbf{V}} d$;

(ii) *if* $c, c' \subseteq h$ *then* $c' \leqslant_{\mathbf{V}} c$ *iff* $c' \smallsetminus c$ *is finite.*

Theorem 5 (reduction to sets $X \subseteq \kappa$)**.** *Suppose that* $h \subseteq \kappa$ *is a Prikry sequence over the ground model* $\mathbf{V}$. *Then in the Prikry extension* $\mathbf{V}[h]$ *of* $\mathbf{V}$: *for every set* X *there exists a set* $Y \subseteq \kappa$ *satisfying* $X \equiv_{\mathbf{V}} Y$.

The proof of Theorem 4 (Sections 1 – 7) is based on indiscernible subsets of κ. We make use of a representation of subsets of κ in the Prikry extension by means of certain functions defined on $[\kappa]^{\text{fin}}$ in the ground universe. This is similar to some extent to the analysis of degrees of constructibility in iterated Sacks extensions, but the technique is completely different.

The proof of Theorem 5 (Sections 8 – 11) will involve some general forcing arguments, including Solovay's technique of representation of a generic extension as a generic extension of any of its subextensions.

We use standard set theoretic notation, as for example in [3]. In particular $[X]^n$ denotes the collection of all n-element subsets of a set or class X, and $[X]^{\text{fin}} = \bigcup_{n\in\omega} [X]^n$ (the collection of all finite subsets). $|x|$ is the number of elements in a finite set x. It is assumed that the reader has some acquaintance with forcing, as well as elementary definitions and facts related to measurable cardinals, normal ultrafilters, and ultrapowers.

1 Good indiscernible sets

Let U be a normal ultrafilter U on a measurable cardinal κ.

Recall that a non-empty set $I \subseteq \kappa$ is an *indiscernible set w.r.t.* a family of sets $\mathscr{F}$ iff for any set $B \in \mathscr{F}$ and any $n \geq 1$ either every s in $[I]^n$ belongs to B or every s in $[I]^n$ does not belong to B. Rowbottom's theorem, known since the early 1960s, implies that if κ and U are as above then for any family $\mathscr{F}$ of cardinality less than κ there exists a set $I \in U$ which is an indiscernible set w.r.t. $\mathscr{F}$. We employ this basic result to find a slightly more convenient type of indiscernible sets.

Proposition 6. *Suppose that* $\mathscr{F}$ *is a family of cardinality less than* κ.

Then there exists a **good set of indiscernibles** $I \in U$ *w.r.t.* $\mathscr{F}$, *that is, for any* $n \geq 1$, *any* $B \in \mathscr{F}$ *and any sets* $a \in [\kappa]^{\text{fin}}$ *and* $x, y \in [I]^n$, *if* $\mathtt{max}\, a < \mathtt{min}\, x$ *and* $\mathtt{max}\, a < \mathtt{min}\, y$ *then* $a \cup x \in B \Longleftrightarrow a \cup y \in B$.

Note that the ordinals in a are not assumed to be members of the set I.

Proof. For any ordinal $\alpha < \kappa$ the family $\mathscr{F}_\alpha$ of all sets of the form

$$\{x \in [\kappa]^n : a \cup x \in B \wedge \mathtt{max}\, a \leqslant \alpha < \mathtt{min}\, x\}, \quad \text{where } a \in [\alpha+1]^{\text{fin}} \text{ and } B \in \mathscr{F},$$

has cardinality less than κ. Therefore there exists a set $I_\alpha \in U$, $I_\alpha \subseteq \kappa \smallsetminus \alpha$ that is an indiscernible set *w.r.t.* $\mathscr{F}_\alpha$. Consider the diagonal intersection $I = \underset{\alpha<\kappa}{\boldsymbol{\Delta}} I_\alpha$. Thus $\xi \in I$ iff $\xi > 0$ and $\xi \in \bigcap_{\alpha<\xi} I_\alpha$. By the normality of U, I still belongs to U. To check the good indiscernibility, let n, B, a, x, y be as in Proposition 6. Then $\mu = \mathtt{max}\, a < \mathtt{min}\, x$, and hence $x \subseteq I_\mu$ by the definition of I. Similarly $y \subseteq I_\mu$. Then $a \cup x \in B \iff a \cup y \in B$ for any $B \in \mathscr{F}$ by the choice of I_μ. (If $a = \varnothing$ then we take $\mu = 0$ in this argument.) $\square$

2 Canonization of functions

As above, let U be a normal ultrafilter U on a measurable cardinal κ.

The next theorem will be our main technical tool. For any sets $x, s \subseteq \mathtt{Ord}$ we define the order preserving restriction $x \mathbin{/\!\!/} s \subseteq x$ ("x restricted to s") as follows. Put elements of x in the increasing order: $x = \{\xi_\gamma : \gamma < \delta\}$. Now define $x \mathbin{/\!\!/} s = \{\xi_\gamma : \gamma \in s\}$. Note that if $y \subseteq x \in [\mathtt{Ord}]^n$ then there is a unique set $s \subseteq n$ such that $y = x \mathbin{/\!\!/} s$.

Theorem 7. *Suppose that F is a function defined on $[\kappa]^{\rm fin}$. Then for any $n \geq 1$ and $a \in [\kappa]^{\rm fin}$ there exist sets $J_n(a) \in U$ and $\mathtt{bas}_n(a) \subseteq n$ such that for all $x, y \in [J_n(a)]^n$ with $\mathtt{max}\, a < \mathtt{min}\, x, \mathtt{min}\, y$ we have $F(a \cup x) = F(a \cup y)$ if and only if $x \mathbin{/\!\!/} \mathtt{bas}_n(a) = y \mathbin{/\!\!/} \mathtt{bas}_n(a)$.*

Proof. Let ϑ be any cardinal bigger than κ such that $\mathbf{V}_\vartheta$ (the ϑ-th level of the von Neumann hierarchy) contains F. Let $\mathscr{F}$ be the collection of all sets $z \in \mathbf{V}_\vartheta$ definable in $\mathbf{V}_\vartheta$ by an $\in$-formula with F as the only parameter; $\mathscr{F}$ is countable. Let $I \in U$ be given by Proposition 6 for such an $\mathscr{F}$.

We prove the theorem by induction on n.

Suppose that $n = 1$. Fix a set $a \in [\kappa]^{\rm fin}$. Put $J_1(a) = I$. Take any distinct ordinals $\xi, \eta \in I$ bigger than $\mathtt{max}\, a$. If $F(a \cup \{\xi\}) = F(a \cup \{\eta\})$ then by the choice of I we have $F(a \cup \{\xi\}) = F(a \cup \{\eta\})$ for *every* pair of ordinals $\xi \neq \eta$ in I. Indeed take the set

$$\{a \cup \{\xi, \eta\} : a \in [\kappa]^{\rm fin} \wedge \mathtt{min}\, a < \xi, \eta < \kappa \wedge F(a \cup \{\xi\}) = F(a \cup \{\eta\})\}$$

as B in Proposition 6. Therefore $\mathtt{bas}_1(a) = \varnothing$ is as required. If $F(a \cup \{\xi\}) \neq F(a \cup \{\eta\})$ then similarly $\mathtt{bas}_1(a) = \{0\}$ works.

Now the induction step $n \to n+1$. The idea is to reduce the level $n+1$ to n for bigger sets a. Fix $a \in [\kappa]^{\rm fin}$. Take any $\xi < \kappa$, $\mathtt{max}\, a < \xi$. By the induction hypothesis there exist sets $J_n(a \cup \{\xi\}) \in U$ and $\mathtt{bas}_n(a \cup \{\xi\}) \subseteq n$ such that

$$F(a \cup \{\xi\} \cup x) = F(a \cup \{\xi\} \cup y) \quad \text{iff} \quad x \mathbin{/\!\!/} \mathtt{bas}_n(a \cup \{\xi\}) = y \mathbin{/\!\!/} \mathtt{bas}_n(a \cup \{\xi\})$$

holds for any pair of sets $x, y \in [J_n(a \cup \{\xi\})]^n$ with $\xi < \mathtt{min}\, x, \mathtt{min}\, y$.

Obviously there exist sets $J \in U$ and $s \subseteq n$ such that $\mathtt{max}\, a < \mathtt{min}\, J$ and $\mathtt{bas}_n(a \cup \{\xi\}) = s$ for all $\xi \in J$. The set $J' = I \cap J \cap \mathop{\boldsymbol{\Delta}}_{\gamma \in J} J_n(a \cup \{\gamma\})$ belongs to U since U is a normal filter. (Note that $\xi \in J'$ iff $\xi \in I \cap J$, $\xi > 0$, and $\xi \in J_n(a \cup \{\gamma\})$ for all $\gamma \in J$, $\gamma < \xi$.) Moreover we have

$$F(a \cup \{\xi\} \cup x) = F(a \cup \{\xi\} \cup y) \quad \text{iff} \quad x \mathbin{/\!/} s = y \mathbin{/\!/} s \tag{1}$$

for any $\xi \in J'$ and any pair of sets $x, y \in [J']^n$ with $\xi < \mathtt{min}\, x, \mathtt{min}\, y$.

We put $J_{n+1}(a) = J'$. To define $\mathtt{bas}_{n+1}(a)$, take any pair of ordinals $\alpha \neq \gamma$ in J' bigger than $\mathtt{max}\, a$. Also take any $z \in [J']^n$ with $\mathtt{min}\, z > \alpha, \gamma$.

Case 1: $F(a \cup \{\alpha\} \cup z) = F(a \cup \{\gamma\} \cup z)$. Let us show that $\mathtt{bas}_{n+1}(a) = \{1 + k : k \in s\}$ works. Take any $x', y' \in [J']^{n+1}$ with $\xi = \mathtt{min}\, x' > \mathtt{max}\, a$ and $\eta = \mathtt{min}\, y' > \mathtt{max}\, a$. Then $x = x' \smallsetminus \{\xi\}$ and $y = y' \smallsetminus \{\eta\}$ belong to $[J']^n$.

Suppose, for instance, that $\eta \leqslant \xi$. Then still $\eta < \mathtt{min}\, x$. Therefore, by the case assumption, the choice of I, and the fact that $J' \subseteq I$, the equality $F(a \cup x') = F(a \cup x'')$ holds, where $x'' = \{\eta\} \cup x$. Further by (1) $F(a \cup y') = F(a \cup x'')$ iff $x \mathbin{/\!/} s = y \mathbin{/\!/} s$. And finally the equality $x' \mathbin{/\!/} \mathtt{bas}_{n+1}(a) = y' \mathbin{/\!/} \mathtt{bas}_{n+1}(a)$ is equivalent to $x \mathbin{/\!/} s = y \mathbin{/\!/} s$.

Case 2: $F(a \cup \{\alpha\} \cup z) \neq F(a \cup \{\gamma\} \cup z)$. A similar argument shows that $\mathtt{bas}_{n+1}(a) = \{0\} \cup \{1 + k : k \in s\}$ works. □

3 Prikry extension

For more details on measurable cardinals and Prikry forcing, see for example [3].

Let $\mathbf{V}$ be the model for which we prove Theorems 3, 4, 5. Until the end of the paper we'll consider $\mathbf{V}$ as the ground universe of all sets. In particular all ordinals are ordinals in $\mathbf{V}$ and generally all sets are sets in $\mathbf{V}$ or depending on the context sets in generic extensions of $\mathbf{V}$. We suppose that κ is a measurable cardinal in $\mathbf{V}$ and U is a normal ultrafilter over κ in $\mathbf{V}$.

Recall that the Prikry forcing $= (U)$ associated to U consists of all pairs $p = \langle a_p, A_p \rangle$ of sets $a_p \in [\kappa]^{\mathrm{fin}}$ and $A_p \in U$ such that $\mathtt{max}\, a_p < \mathtt{min}\, A_p$. The order is as follows: $p \leqslant q$ (meaning that p is stronger) iff $a_q \subseteq_{\mathtt{end}} a_p$ (meaning that a_p is an end-extension of a_q, that is, $a_q \subseteq a_p$ and $\mathtt{max}\, a_q < \mathtt{min}\,(a_p \smallsetminus a_q)$), $A_p \subseteq A_q$, and $a_p \smallsetminus a_q \subseteq A_q$.

-generic extensions are called *Prikry extensions*.

We'll make use of the following basic fact about the Prikry forcing.

Proposition 8. *If $p \in$ and φ is a closed formula of the -forcing language then there is a condition $q \in$, $q \leqslant p$ which decides φ and satisfies $a_p = a_q$.* □

The first part of the following proposition is an immediate corollary, using the κ-completeness.

Proposition 9. (1) *In a Prikry extension* $\mathbf{V}[G]$ *of the ground universe* $\mathbf{V}$*, if* $X \subseteq \alpha < \kappa$ *then* $X \in \mathbf{V}$*. Therefore every* $\mathbf{V}$*-cardinal* $\vartheta < \kappa$ *remains a cardinal in the extension, and with the same cofinality.*

(2) *Moreover,* κ *itself remains a cardinal, too, but its cofinality changes to* ω*.*

(3) *Finally all cardinals* $\vartheta > \kappa$ *remain cardinals, and the cofinality does not change provided it was bigger than* κ *in* $\mathbf{V}$*.* $\square$

Proof. Given a set $G \subseteq$, we let $h_G = \bigcup_{p \in G} a_p$, a subset of κ. If G is a generic filter then by an easy forcing argument h_G is a set of order type ω cofinal in κ, and hence $\texttt{cof}\,\kappa = \omega$ in the extension. Finally to prove (3) note that satisfies the κ^+-c.c. simply because any two conditions $p, q \in$ such that $a_p = a_q$ are compatible in . $\square$

Sets of the form h_G, $G \subseteq$ being a Prikry generic filter over a ground model $\mathbf{V}$ containing κ and U, are called *Prikry sequences* (over $\mathbf{V}$). It is known that every Prikry sequence $h = h_G$ is a cofinal subset of κ of order type ω, so that it can equivalently be considered as the sequence

$$h = \langle h(0), h(1), h(2), \dots, h(n), h(n+1), \dots \rangle\,, \quad \text{in the increasing order}\,.$$

In particular, $h(0)$ is the least element of h, and so on.

By $\underline{G}$ we denote a name for the canonical generic subset of . Let $\underline{h}$ be a name for h_G. Then forces that $\underline{h} \subseteq \check{\kappa}$ is a set of order type $\check{\omega}$ cofinal in $\check{\kappa}$, and $\underline{G} = \{p \in \check{} : a_p \subset_{\texttt{end}} \underline{h} \wedge \underline{h} \setminus a_p \subseteq A_p\}$. And $\check{x}$ (x being a set in the ground model, for example, ω or κ) is a canonical name of x in the forcing language.

4 Coding subsets of κ in the Prikry extension

Beginning the proof of Theorem 4, we fix, for Sections 4 to 7, a -name $\underline{X}$ of a subset of $\check{\kappa}$. And κ, U as in Section 3 remain to be fixed. Our goal is to code $\underline{X}$ by a subset of the canonical Prikry sequence $\underline{h}$.

Crucial assumption 10. Suppose to the contrary that a condition $p^* = \langle a^*, A^* \rangle \in$ forces otherwise, that is, forces that there is no set $d \subseteq \underline{h}$ satisfying $\underline{X} \equiv_{\mathbf{V}} d$. $\square$

Unfortunately, working in this assumption towards contradiction, a^* and, occasionally, A^* would inconveniently contaminate each and every essential formula. Therefore we would like to work in a somewhat narrower case:

Crucial assumption 11. Suppose that $p^* = \langle \varnothing, \kappa \rangle$ in Crucial assumption 10.

We are going to derive a contradiction by getting a condition $p \in$, actually of the form $p = \langle \varnothing, I \rangle$ (where $I \in U$ will be a certain indiscernible set), which forces the opposite, that is, the existence of a set $d \subseteq \underline{h}$ satisfying $\underline{X} \equiv_{\mathbf{V}} d$. $\square$

To explain why this assumption does not lead to any loss of generality, note first of all that p^* still can be (inconveniently) imported in the arguments below in a certain uniform way — and we'll show how to do this in the definition of F just below. However there is a more formal argument. The point is that there exists another -name $\underline{X}'$ such that whatever p^* forces for $\underline{X}$ is forced for $\underline{X}'$ by $\langle\varnothing,\kappa\rangle$. To define $\underline{X}'$, note that the canonical Prikry sequence h_G in a Prikry extension $\mathbf{V}[G]$ is almost included in A^* since $A^* \in U$. Therefore, cutting an appropiate finite initial segment b of h_G and adjoining a^* we obtain another cofinal set h'. By a standard forcing argument, h' will be a Prikry sequence (for a suitably defined generic filter G'), compatible with p^*. Define X' to be $\underline{X}$ interpreted in the sense of h' (or G'). And then define $\underline{X}'$ to be a name of X' in the Prikry forcing language. Then under Crucial assumption 10 $\langle\varnothing,\kappa\rangle$ forces that there is no set $d \subseteq \underline{h}$ satisfying $\underline{X}' \equiv_{\mathbf{V}} d$.

After these preliminaries, we begin to work towards contradiction under Crucial assumptions 10 and 11. Define, for each $x \in [\kappa]^{\text{fin}}$,

$$F(x) = \{\xi < \kappa : \exists p \in \ (a_p = x \wedge p \mathrel{||\!\!-} \check{\xi} \in \underline{X})\}. \tag{2}$$

It follows from Theorem 7 that there exist a set $J = \bigcap_n J_n(\varnothing) \in U$ and a sequence $\{\mathtt{bas}_n\}_{n\in\omega}$ of sets $\mathtt{bas}_n = \mathtt{bas}_n(\varnothing) \subseteq n$ such that the equivalence

$$F(x) = F(y) \iff x \mathbin{/\!\!/} \mathtt{bas}_n = y \mathbin{/\!\!/} \mathtt{bas}_n \tag{3}$$

holds for all n and $x, y \in [J]^n$.

Remark 12. Working with an arbitrary condition $p^* = \langle a^*, A^*\rangle \in$, we would have to define $J = \bigcap_n J_n(a^*)$ and $\mathtt{bas}_n = \mathtt{bas}_n(a^*)$, of course. □

It is interesting to figure out whether $\mathtt{bas}_n = \mathtt{bas}_k \cap n$ holds for $n < k$. But fortunately a somewhat weaker result of the next lemma will suffice for our goals.

Lemma 13. *If* $n < k$ *then* $\mathtt{bas}_n \subseteq \mathtt{bas}_k$.

Proof. By (3), it suffices to show that $F(x) = F(y)$ holds for any sets $x, y \subseteq I$ satisfying $|x| = |y| = n$ and $x \mathbin{/\!\!/} s = y \mathbin{/\!\!/} s$, where $s = \mathtt{bas}_{n+1} \cap n$. Suppose otherwise: $F(x) \neq F(y)$, that is, for example, there is an ordinal $\xi \in F(x) \smallsetminus F(y)$. Then there exists a condition $p \in$ with $a_p = x$ that forces $\check{\xi} \in \underline{X}$, and by Proposition 8 there is a condition $q \in$ with $a_q = y$ that forces $\check{\xi} \notin \underline{X}$. We may assume that $A_p = A_q \subseteq I$ and $\xi < \mu = \mathtt{min}\, A_p$. Then the sets $x' = x \cup \{\mu\}$ and $y' = y \cup \{\mu\}$ satisfy $|x'| = |y'| = n+1$ and $x' \mathbin{/\!\!/} \mathtt{bas}_{n+1} = y' \mathbin{/\!\!/} \mathtt{bas}_{n+1}$. It follows that $F(x') = F(y')$ by (3).

Consider conditions $p' = \langle x', A_p \smallsetminus \{\mu\}\rangle$ and $q' = \langle y', A_q \smallsetminus \{\mu\}\rangle$. Obviously $p' \leqslant p$ in , therefore p' forces $\check{\xi} \in \underline{X}$, and then $\xi \in F(x') = F(y')$. It follows that there is a condition $r \in$ with $a_r = y'$ that still forces $\check{\xi} \in \underline{X}$. This is a contradiction because $q' \mathrel{||\!\!-} \check{\xi} \notin \underline{X}$ (indeed $q' \leqslant q$) and $a_{q'} = a_r = y'$. □

5 Further definitions and technical lemmas

Now fix a cardinal $\vartheta > \kappa$. Then $\mathbf{V}_\vartheta$ is a transitive set containing $\kappa, F, , J$, the ultrafilter U, the sequence $\{\mathtt{bas}_n\}_{n\in\omega}$, and the relation $p \mathrel{||\!\!-} \check\xi \in \underline{X}$ (of two arguments p and ξ). Let $\mathscr{F}$ be the family of all subsets of $\mathbf{V}_\vartheta$ definable in $\mathbf{V}_\vartheta$ by an $\in$-formula with those seven sets involved as parameters. Let $I \in U$ satisfy Proposition 6 with these initial conditions. We can assume that $I \subseteq J$.

The following is the key technical instrument.

Lemma 14. *If $\xi < \gamma < \kappa$, $p, q \in$, $a_p \cap \gamma = a_q \cap \gamma$, $(a_p \cup a_q) \smallsetminus \gamma \subseteq I$, and $A_p \cup A_q \subseteq I \smallsetminus \gamma$. Then $p \mathrel{||\!\!-} \check\xi \in \underline{X}$ iff $q \mathrel{||\!\!-} \check\xi \in \underline{X}$.*

In particular if a condition $p \in$ satisfying $a_p \smallsetminus \gamma \subseteq I$ and $A_p \subseteq I \smallsetminus \gamma$ forces $\check\xi \in \underline{X}$ then so does $q = \langle a_p \cap \gamma, I \smallsetminus \gamma\rangle$.

Proof. Suppose this is not the case. Then since $A_q \subseteq I$ we can w.l.o.g. assume that $p \mathrel{||\!\!-} \check\xi \in \underline{X}$ and $q \mathrel{||\!\!-} \check\xi \notin \underline{X}$. Put $a = a_p \cap \gamma = a_q \cap \gamma$. The remaining parts $y = a_q \smallsetminus a$ and $x = a_p \smallsetminus a$ are finite subsets of $I \smallsetminus \gamma$. We can assume that $|x| = |y|$ as otherwise the condition with the shorter part can be appropriately strengthened.

Put $m = |a|$ and $n = |x| = |y|$. Then $a = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$. The ordinal ξ can be equal to one of α_i or belong to one of intervals $[0, \alpha_0)$, (α_i, α_{i+1}), $\xi > \alpha_m$ — totally $2m+1$ options. Let *the order structure of the triple a, x, ξ* be the following information: $\mathtt{max}\, a < \mathtt{min}\, x$, $\xi < \mathtt{min}\, a_x$, and the choice between the $2m+1$ options mentioned just above.

Consider the set B of all unions of the form $\{\eta\} \cup u \cup v$ such that $\eta < \kappa$, $u \in [\kappa]^m$, $v \in [\kappa]^n$, the order structure of the triple u, v, η is the same as the order structure of the triple a, x, ξ, and there is a condition $r \in$ such that $a_r = u \cup v$ and $r \mathrel{||\!\!-} \check\eta \in \underline{X}$. Then $B \in \mathscr{F}$. Moreover $\{\xi\} \cup a \cup x \in B$ is witnessed by the condition $r = p$. It follows that $\{\xi\} \cup a \cup y \in B$ as well by the choice of I. (Note that $x \cup y \subseteq I$. But $a \subseteq I$ and $\xi \in I$ are not assumed.) Therefore there is a condition $r \in$ such that $a_r = a \cup y = a_q$ and $r \mathrel{||\!\!-} \check\xi \in \underline{X}$. Thus conditions q, r with $a_q = a_r$ are incompatible. But this is a contradiction. $\square$

Lemma 15. *If $\gamma < \kappa$ and $a_0 \subseteq \gamma$ is finite then the condition $p_0 = \langle a_0, I \smallsetminus \gamma\rangle$ forces $\underline{X} \cap \check\gamma = \check F(\underline{h} \cap \check\gamma) \cap \check\gamma$.*

Proof. Fix any ordinal $\xi < \gamma$. Suppose that a condition $p \in$, $p \leqslant p_0$ forces $\check\xi \in \underline{X}$. Then $A_p \subseteq I \smallsetminus \gamma$, and hence $\gamma \leqslant \mathtt{min}\, A_p$. Note that $\underline{h} \cap \check\gamma$ is forced by p to be equal to $\check a$, where $a = a_p \cap \gamma$. Therefore we have to show that $\xi \in F(a)$, that is, there exists a condition $q \in$ with $a_q = a$ which forces $\check\xi \in \underline{X}$. Yet $q = \langle a, I \smallsetminus \gamma\rangle$ is such a condition by Lemma 14.

Conversely suppose that a condition $p \in$, $p \leqslant p_0$ forces $\check\xi \notin \underline{X}$. As above, we have to prove that $\xi \notin F(a)$, where $a = a_p \cap \gamma$. Otherwise there is a condition

$q \in$ with $a_q = a$ such that $q \mid\!\!\vdash \check\xi \in \underline{X}$. It can be assumed that $A_q \subseteq I \smallsetminus \gamma$. Then Lemma 14 leads to contradiction. $\square$

For any $\alpha \in I$ let $\alpha^\dagger$ be the next element of I.

Corollary 16. *If $p = \langle a, A\rangle \in$, $A \subseteq I$, and $\gamma = (\mathtt{max}\, a)^\dagger$ then p forces $\underline{X} \cap \check\gamma = \check F(\check a) \cap \check\gamma$.* $\square$

6 Getting the set from a subsequence

Put $S = \bigcup_n \mathtt{bas}_n$. It follows from Lemma 13 that for any n there exists a number $k = k_n$ such that $S \cap n = \mathtt{bas}_k \cap n$ for all $k \geq k_n$; in particular $S \cap n = \mathtt{bas}_{k_n} \cap n$.

Let $\underline{d}$ be a name for $\underline{h} \mathbin{/\!/} \check S$. This is a subsequence of $\underline{h}$ in the extension.

Proposition 17. *The condition $\langle \varnothing, I\rangle$ forces $\underline{X} \leqslant_{\mathbf{V}} \underline{d}$.*

Proof. Argue in the Prikry extension $\mathbf{V}[G]$ of the ground universe $\mathbf{V}$, where $G \subseteq$ is a generic filter containing $\langle \varnothing, I\rangle$. Define $h = h_G$, $X = \underline{X}[G]$, and $d = h \mathbin{/\!/} S$. Then $h \subseteq I$ since G contains $\langle \varnothing, I\rangle$.

For any m let $X_m = \bigcup_x (F(x) \cap \mathtt{max}\, x)$, where the union is taken over all finite sets $x \subseteq I$ satisfying $|x| \geq m$ and $x \mathbin{/\!/} S \subseteq_{\mathtt{end}} d$. The sequence of sets X_m belongs to $\mathbf{V}[d]$, of course, and hence so does the set

$$X' = \bigcup_n \bigcap_{m \geq n} X_m = \{\xi < \kappa : \exists n\, \forall m \geq n\, (\xi \in X_m)\}.$$

It remains to prove that $X = X'$.

Suppose that $\xi \in X$. Then $\check\xi \in \underline{X}$ is forced by a condition $p = \langle a, A\rangle \in G$. Note that a is a finite initial segment of h, hence $a \subseteq I$, and in addition $h \smallsetminus a \subseteq A$. Assuming w.l.o.g. that $\mathtt{max}\, a > \xi$, we assert that $\xi \in X_m$ for any $m \geq n = |a|$. Indeed let x be the set of the first m elements of h. Then $|x| = m$, $a \subseteq_{\mathtt{end}} x \subseteq I$, $\xi < \mathtt{max}\, x$, $x \smallsetminus a \subseteq A$, and obviously $x \mathbin{/\!/} S \subseteq_{\mathtt{end}} d$. (This is true for any $x \subseteq_{\mathtt{end}} h$.) Thus to show $\xi \in X_m$ we have only to prove that $\xi \in F(x)$.

Consider any condition $q = \langle x, A_q\rangle \in G$ with $a_q = x$. Let $A' = A_q \cap A$; then $q' = \langle x, A'\rangle$ also belongs to G and easily $q' \leqslant p$. (Recall that $x \smallsetminus a \subseteq A$.) Therefore q' forces $\check\xi \in \underline{X}$, and this implies $\xi \in F(x)$, as required.

Conversely, suppose that $\xi \in X'$, and this is witnessed by some n, so that $\xi \in X_m$ for all $m \geq n$. Take $m_0 \geq n$ big enough for the set y_0 of the first m_0 elements of h to satisfy $\xi < \mathtt{max}\, y_0$. By definition there is a finite set $x \subseteq I$ such that $\xi < \mathtt{max}\, x$, $m = |x| \geq m_0$, $x \mathbin{/\!/} S \subseteq_{\mathtt{end}} d$, and $\xi \in F(x)$. Let y be the set of the first m elements of h. Then $y \subseteq I$ and $\xi < \mathtt{max}\, y$ by the choice of m_0. In addition $x \mathbin{/\!/} S = y \mathbin{/\!/} S = h$, so that $x \mathbin{/\!/} \mathtt{bas}_n = y \mathbin{/\!/} \mathtt{bas}_n$, and hence $F(x) = F(y)$ by (3). Thus $\xi \in F(y)$. This means the existence of a condition $q \in$ with $a_q = y$ such that $q \mid\!\!\vdash \check\xi \in \underline{X}$. We can w.l.o.g. assume that $A_q \subseteq I$.

This does not immediately imply $\xi \in X$ because we cannot claim that $q \in G$. However let $\gamma = 1 + \mathtt{max}\, y$. The condition $p = \langle y, I \smallsetminus \gamma \rangle$ obviously belongs to G. Moreover p forces $\check{\xi} \in \underline{X}$ by Lemma 14 because so does q. Thus $\xi \in X$, as required. $\square$

The next lemma on subsets of h_G proves claim (ii) of Theorem 4.

Lemma 18. *In the Prikry extension* $\mathbf{V}[G]$*,*

(i) *for every* $c \subseteq h = h_G$ *there is a unique* $E \subseteq \omega$ *in* $\mathbf{V}$ *such that* $c = h /\!\!/ E$;

(ii) *if* $c, c' \subseteq h$ *then* $c' \leqslant_{\mathbf{V}} c$ *iff* $c' \smallsetminus c$ *is finite.*

Proof. (i) Obviously in $\mathbf{V}[G]$ there is a unique set $E \subseteq \omega$ with $c = h /\!\!/ E$. It belongs to $\mathbf{V}$ because the Prikry forcing does not add new bounded subsets of κ.

(ii) Let, by (i), $c = h /\!\!/ E$ and $c' = h /\!\!/ K$, where $E, K \subseteq \omega$ are sets in $\mathbf{V}$. Suppose on the contrary that $K \smallsetminus E$ is infinite but $h /\!\!/ K \leqslant_{\mathbf{V}} h /\!\!/ E$. Then there exist a set $x \in \mathbf{V}$ and an ordinal α such that $h /\!\!/ K$ is the α-th element in the canonical Gödel wellordering of $\mathbf{L}[x, h /\!\!/ E]$. This is forced by a condition $p \in G$, so that p forces that $\underline{h} /\!\!/ \check{K}$ is the $\check{\alpha}$-th element in the canonical Gödel wellordering of $\mathbf{L}[\check{x}, \underline{h} /\!\!/ \check{E}]$.

As $K \smallsetminus E$ is infinite, there is an element $n \in K \smallsetminus E$ such that $n \geq |a_p|$. We can w.l.o.g. assume that $n = |a_p|$ (otherwise consider a suitable stronger condition). Choose ordinals ξ, η in the set A_p such that $\mathtt{max}\, a_p < \xi < \eta$. Define a pair of conditions $q, r \in$ so that $a_q = a_p \cup \{\xi\}$, $a_r = a_p \cup \{\eta\}$, and $A_q = A_r = A_p \smallsetminus (\eta + 1)$, that is, $\eta < \mathtt{min}\, A_q = \mathtt{min}\, A_r$. Both q and r are stronger than p.

Let $h \subseteq \kappa$ be a Prikry sequence compatible with q (in the sense that $a_q \subseteq_{\mathtt{end}} h$ and $h \smallsetminus a_q \subseteq A_q$). Clearly $\xi \in h$ but $\eta \notin h$. Then $h' = h \cup \{\eta\} \smallsetminus \{\xi\}$ is a Prikry sequence compatible with r by an ordinary forcing argument. Note that $h /\!\!/ E = h' /\!\!/ E$ since $n \notin E$. It follows that $h /\!\!/ K = h' /\!\!/ K$ because p forces $\underline{h} /\!\!/ \check{K}$ be an absolute function of $\underline{h} /\!\!/ \check{E}$. But on the other hand $h /\!\!/ K \neq h' /\!\!/ K$ because $n \in K$ and the n-th element ξ of h is not equal to the n-th element η of h'. This contradiction completes the proof. $\square$

7 Getting the subsequence from the set

Here we prove the opposite reduction:

Proposition 19. *The condition* $\langle \varnothing, I \rangle$ *forces* $\underline{d} \leqslant_{\mathbf{V}} \underline{X}$.

Proof. Argue in the Prikry extension $\mathbf{V}[G]$ of the ground universe $\mathbf{V}$, where $G \subseteq$ is a generic filter containing $\langle \varnothing, I \rangle$. Define $h = h_G$, $X = \underline{X}[G]$, and $d = h /\!\!/ S$. Then $h \subseteq I$ since G contains $\langle \varnothing, I \rangle$. Let W be the family of all finite non-empty sets $x \subseteq I$ such that $X \cap \gamma = F(x) \cap \gamma$, where $\gamma = (\mathtt{max}\, x)^{\dagger}$.

Lemma 20. *If x is an initial segment of h then $x \in W$.*

Proof. Easily by Lemma 15. $\square$ *(Lemma 20)*

As obviously $W \leqslant_{\mathbf{V}} X$, we have to prove that $d \leqslant_{\mathbf{V}} W$ in $\mathbf{V}[G]$.

Lemma 21. *Suppose that $0 \in \mathtt{bas}_1$. Then the only ordinal $\vartheta \in I$ such that $\{\vartheta\} \in W$ is equal to $h(0)$, the least element of h.*

Proof. $\{h(0)\} \in W$ by Lemma 20. Suppose for a contradiction that there are two different ordinals ϑ as in the lemma. In other words there exist ordinals $\xi < \eta$ in I such that $F(\{\xi\}) \cap \xi^\dagger = X \cap \xi^\dagger$ and $F(\{\eta\}) \cap \eta^\dagger = X \cap \eta^\dagger$. Since $\xi^\dagger < \eta^\dagger$, clearly $F(\{\xi\}) \cap \xi^\dagger = F(\{\eta\}) \cap \xi^\dagger$. It follows from the assumption $0 \in \mathtt{bas}_1$ that $F(\{\xi\}) = F(\{\eta\})$ iff $\xi = \eta$. Therefore to get a contradiction it suffices to prove that $F(\{\xi\}) = F(\{\eta\})$.

Case 1: $\eta = \xi^\dagger$, so that $F(\{\xi\}) \cap \eta = F(\{\eta\}) \cap \eta$. By the indiscernibility of I this holds for *every* pair of ordinals $\xi < \eta$ in I. It follows that $F(\{\xi\}) = F(\{\eta\})$. Indeed take any $\zeta \in I$ bigger than η. Then $F(\{\xi\}) \cap \zeta = F(\{\zeta\}) \cap \zeta = F(\{\eta\}) \cap \zeta$.

Case 2: $\gamma = \xi^\dagger < \eta$. Then $F(\{\xi\}) \cap \gamma = F(\{\eta\}) \cap \gamma$ for all ordinals $\xi < \gamma < \eta$ in I by the indiscernibility. And once again $F(\{\xi\}) = F(\{\eta\})$. Indeed take any pair of ordinals $\gamma < \zeta$ in I with $\gamma > \mathtt{max}\{\xi, \eta\}$. Then $F(\{\xi\}) \cap \gamma = F(\{\zeta\}) \cap \gamma = F(\{\eta\}) \cap \gamma$. $\square$ *(Lemma 21)*

Lemma 22. *Suppose that $x, y \subseteq I$ are finite sets, $|x| = |y| = n \geq 2$, $\mathtt{max}\, x = \mathtt{max}\, y$, and $F(x) \neq F(y)$. Then $F(x) \cap \gamma \neq F(y) \cap \gamma$, where $\gamma = (\mathtt{max}\, x)^\dagger$.*

Proof. Otherwise by the indiscernibility of I we would have $F(x) \cap \gamma = F(y) \cap \gamma$ for all $\gamma \in I$, $\gamma > \mathtt{max}\, x$. $\square$ *(Lemma 22)*

By the way Lemma 22 is trivially true for $n = 1$ since in this case $\mathtt{max}\, x = \mathtt{max}\, y$ implies $x = y$. On the other hand the next lemma in the case $n = 1$ easily follows from Lemma 21.

Lemma 23. *Suppose that sets $x, y \in W$ satisfy $|x| = |y| = n \geq 2$. Then $F(x) = F(y)$, therefore $x \,/\!\!/\, \mathtt{bas}_n = y \,/\!\!/\, \mathtt{bas}_n$.*

Proof. First of all note that the second claim is equivalent to the first claim by (3), and hence it suffices to prove only one of them. For instance if $\mathtt{bas}_n = 0$ then $x \,/\!\!/\, \mathtt{bas}_n = y \,/\!\!/\, \mathtt{bas}_n = \varnothing$ is obvious, so in the rest of the proof of the lemma we'll suppose that $\mathtt{bas}_n \neq \varnothing$ and prove the equality $F(x) = F(y)$.

Let $\mu = \mathtt{max}\, x$ and $\nu = \mathtt{max}\, y$. We assume w.l.o.g. that $\mu \leqslant \nu$. Then

$$F(x) \cap \mu^\dagger = X \cap \mu^\dagger \quad \text{and} \quad F(y) \cap \nu^\dagger = X \cap \nu^\dagger,$$

because $x, y \in W$. Therefore $F(x) \cap \mu^\dagger = F(y) \cap \mu^\dagger$.

If $\max x = \max y = \mu$ then $F(x) \cap \mu^\dagger = F(y) \cap \mu^\dagger$ because x, y belong to W, hence $F(x) = F(y)$ by Lemma 22. Thus it will be assumed that $\mu = \max x < \max y = \nu$, in addition to the earlier assumption $\mathtt{bas}_n \neq \varnothing$.

By the indiscernibility we can suppose that all elements of the sets x, y have limit indices in the sense of the natural increasing order of I — this allows us to move them, if necessary, without any change in their common configuration within I. We have several cases.

Case 1: the number $n-1$ does not belong to $\mathtt{bas}_n$. Then the sets $u = x /\!\!/ \mathtt{bas}_n$ and $v = y /\!\!/ \mathtt{bas}_n$ do not contain ordinals resp. μ and ν. Thus $z = x \cup \{\nu\} \smallsetminus \{\mu\}$ satisfies $z /\!\!/ \mathtt{bas}_n = x /\!\!/ \mathtt{bas}_n = u$, therefore $F(x) = F(z)$. In particular, $F(z) \cap \gamma = F(x) \cap \gamma = F(y) \cap \gamma$, where $\gamma = \mu^\dagger$. On the other hand $\max z = \max y = \nu$, therefore $F(y) = F(z)$ by Lemma 22 (because $F(z) \cap \gamma = F(y) \cap \gamma$), and finally $F(x) = F(y)$, as required.

Case 2: the number $n-1$ belongs to $\mathtt{bas}_n$. Then the sets $u = x /\!\!/ \mathtt{bas}_n$ and $v = y /\!\!/ \mathtt{bas}_n$ are different (since $\mu < \nu$), therefore $F(x) \neq F(y)$. We are going to prove $F(x) = F(y)$ even in this case, which is thereby self-contradictory.

Case 2a: $\mathtt{bas}_n = \{n-1\}$. Then $u = \{\mu\}$ and $v = \{\nu\}$. In this case the value of $F(y)$ does not depend on the values of ordinals in $y \smallsetminus \{\nu\}$, and hence we can assume that $\mu^\dagger < \min y$. (Otherwise shift them suitably, using the "limit indices" assumption above.) Now the same argument as in the proof of Lemma 21 (Case 2) shows that still $F(x) = F(y)$.

Case 2b: the set $\mathtt{bas}_n$ contains both the number $n-1$ and at least one more element. Accordingly the sets u and v contain both resp. μ and ν and elements other than resp. μ and ν.

Case 2b1: $u \smallsetminus \{\mu\} = v \smallsetminus \{\nu\}$. Let $z = v \cup (x \smallsetminus u)$. In other words z consists of those elements of y which belong to $v = y /\!\!/ \mathtt{bas}_n$ and those elements of x which do *not* belong to $u = x /\!\!/ \mathtt{bas}_n$. In our assumptions (including the Case 2b1 assumption), we have $z /\!\!/ \mathtt{bas}_n = y /\!\!/ \mathtt{bas}_n = v$, and hence $F(z) = F(y)$. It follows that $F(x) \cap \mu^\dagger = F(z) \cap \mu^\dagger$ — because $F(x) \cap \mu^\dagger = F(y) \cap \mu^\dagger$, see above. On the other hand, the only difference between x and z is that $\mu = \max x < \nu = \max z = \max y$, and in the rest $x \smallsetminus \{\mu\} = z \smallsetminus \{\nu\}$. Put $w = x \smallsetminus \{\mu\} = z \smallsetminus \{\nu\}$, so that $x = w \cup \{\mu\}$ and $z = w \cup \{\nu\}$.

According to the "limit indices" assumption, the ordinal $\gamma = \mu^\dagger$ satisfies $\gamma < \nu$ strictly, and, recall, $F(w \cup \{\mu\}) \cap \gamma = F(w \cup \{\nu\}) \cap \gamma$. By the indiscernibility, this equality holds for any triple of ordinals $\mu < \gamma < \nu$ in I such that $\sup w < \mu$. Choose any pair of $\gamma' < \nu'$ in I such that $\nu < \gamma'$. Then immediately

$$F(w \cup \{\mu\}) \cap \gamma' = F(w \cup \{\nu'\}) \cap \gamma' \quad \text{and} \quad F(w \cup \{\nu\}) \cap \gamma' = F(w \cup \{\nu'\}) \cap \gamma' .$$

(Consider the triples $\mu < \gamma' < \nu'$ and $\nu < \gamma' < \nu'$.) It follows that

$$F(x) \cap \gamma' = F(w \cup \{\mu\}) \cap \gamma' = F(w \cup \{\nu\}) \cap \gamma' = F(z) \cap \gamma' ,$$

and hence $F(x) = F(z)$ because γ' can be arbitrarily large in κ. And finally $F(x) = F(y)$, as required.

Case 2b2: $u \smallsetminus \{\mu\} \neq v \smallsetminus \{\nu\}$, and then (since $|u| = |v|$) there exists an ordinal $\alpha \in u \smallsetminus v$, $\alpha < \mu$. According to the "limit indices" assumption, the ordinal $\beta = \alpha^\dagger$ does not occur in x and does not occur in y. Put $x' = x \cup \{\beta\} \smallsetminus \{\alpha\}$, and if $\alpha \in y$ (but $\notin v$) then $y' = y \cup \{\beta\} \smallsetminus \{\alpha\}$ as well. (And if $\alpha \notin y$ then we keep $y' = y$.) Then $x \mathbin{/\!\!/} \mathtt{bas}_n \neq x' \mathbin{/\!\!/} \mathtt{bas}_n$, hence $F(x) \neq F(x')$, and further $F(x) \cap \gamma \neq F(x') \cap \gamma$ by Lemma 22, where $\gamma = \mu^\dagger$. On the other hand, $y \mathbin{/\!\!/} \mathtt{bas}_n = y' \mathbin{/\!\!/} \mathtt{bas}_n$, because the substitution of β for α does not alter the set $v = y \mathbin{/\!\!/} \mathtt{bas}_n$. Therefore $F(y) = F(y')$. And finally the order configuration of the complex x, y, γ is clearly similar to the configuration of x', y', γ, and hence the equalities $F(x) \cap \gamma = F(y) \cap \gamma$ and $F(x') \cap \gamma = F(y') \cap \gamma$ hold or fail simultaneously, contradiction to the above. $\square$ (*Lemma 23*)

We are ready to accomplish **the proof of Proposition 19**.

Fix a number $m \geq 1$ and, arguing in the Prikry extension $\mathbf{V}[G]$, show how the set $D_m = d \mathbin{/\!\!/} m$ of m first elements of the subsequence $d = h \mathbin{/\!\!/} S$ can be recovered starting from X. We assume that d is infinite as otherwise there is nothing to prove. Then there is a least number $n = n_m \geq m$ such that $|S \cap n| \geq m$, and further there is a least number $k = k_m \geq n_m$ such that $\mathtt{bas}_k \cap n = S \cap n$.

Consider the set W_k of all k-element sets $x \in W$. In particular the set $x_k = \{h(i) : i < k\}$ of first k elements of the whole Prikry sequence $h = h_G$ belongs to W_k by Lemma 20. Suppose that $x, y \in W_k$. Then $x \mathbin{/\!\!/} \mathtt{bas}_k = y \mathbin{/\!\!/} \mathtt{bas}_k$ by Lemma 23, and hence the first m elements of the sets $x \mathbin{/\!\!/} S$ and $y \mathbin{/\!\!/} S$ are the same by the choice of k and n. In other words, for any $x \in W_k$ the first m elements of the sets $x \mathbin{/\!\!/} S$ and $x_k \mathbin{/\!\!/} S$ are the same. But the first m elements of the set $x_k \mathbin{/\!\!/} S$ are equal to the set D_m of the first m elements of d.

Thus the following plan of computing D_m in the Prikry extension works: compute $n = n_m$ and $k = k_m$ as above, take any $x \in W_k$ and take the first m elements of the set $x \mathbin{/\!\!/} S$.

$\square$ (*Proposition 19*)

Propositions 17 and 19 end **the proof of claim** (i) **of Theorem 4**. Indeed, we have proved that the condition $\langle \varnothing, I \rangle$ forces $\underline{d} \equiv_{\mathbf{V}} \underline{X}$, and this contradicts Crucial assumptions 10 and 11.

$\square$ (*Theorem 4*)

8 Reduction theorem: the scheme of the proof

Here we start **the proof of Theorem 5**. It obviously suffices to prove Theorem 5 for sets $X \subseteq \mathtt{Ord}$. And this proof will go on by induction on the least cardinal $\lambda \geq \kappa$ such that $X \subseteq \lambda$, and the case $\lambda = \kappa$ is obvious.

We have to carry out the step: prove the result for λ assuming it holds for all cardinals λ', $\kappa \leqslant \lambda' < \lambda$. Thus we suppose that

(I) $G \subseteq$ is a generic set over the ground model $\mathbf{V}$,

(II) $h = h_G \subseteq \kappa$ is a corresponding Prikry sequence,

(III) $\lambda > \kappa$ is a cardinal, and

(IV) if λ' is a cardinal, $\kappa \leqslant \lambda' < \lambda$, and $X \subseteq \lambda'$, $X \in \mathbf{V}[G]$, then there exists a set $Y \subseteq \kappa$ such that $\mathbf{V}[X'] = \mathbf{V}[Y]$ (the inductive hypothesis).

Lemma 24 (the inductive step lemma)**.** *Under these assumptions, if* $X \subseteq \lambda$, $X \in \mathbf{V}[G]$*, then there is a set* $Y \subseteq \kappa$ *such that* $\mathbf{V}[X] = \mathbf{V}[Y]$.

The proof of the lemma will be different in two cases, the first of which (Sections 9 and 10) is the case when $\mathtt{cof}\,\lambda > \kappa$ strictly while the second (Section 11) will be the case when $\mathtt{cof}\,\lambda \leqslant \kappa$.[1]

9 Large cofinality inductive step

Here we prove Lemma 24 in the case when $\mathtt{cof}\,\lambda > \kappa$. Fix a set $X \subseteq \lambda$ in $\mathbf{V}[h]$.

The following is a warmup lemma; it presents a key argument in a particular, simplified case.

Lemma 25. *Under the conditions of Lemma 24, if* $\mathtt{cof}\,\lambda > \kappa$ *and* $x_\xi = X \cap \xi \in \mathbf{V}$ *for all* $\xi < \lambda$ *then* $X \in \mathbf{V}$.

Proof. Let $\underline{X}$ be a name for X, so that $X = \underline{X}[G]$. For each $\xi < \lambda$, let P_ξ be the set of all conditions $p \in$ which force $\underline{X} \cap \check{\xi} = \check{x}_\xi$. Thus every set P_ξ belongs to $\mathbf{V}$ and is non-empty (contains a condition in G). Moreover $\xi < \eta \Longrightarrow P_\eta \subseteq P_\xi$.

Case 1: there is a condition $p \in \bigcap_{\xi<\lambda} P_\xi$. Then clearly p decides every formula of the form $\check{\alpha} \in \underline{X}$, $\alpha < \lambda$, and we are done.

Case 2: not case 1. Then the set $\Xi = \{\xi < \lambda : P_{\xi+1} \subsetneqq P_\xi\}$ is unbounded in λ, and hence $\mathtt{card}\,\Xi > \kappa$ under the assumptions of Lemma 24. For any $\xi \in \Xi$ choose an arbitrary $p_\xi \in P_\xi \smallsetminus P_{\xi+1}$. Then p_ξ does not force $\underline{X} \cap \check{\xi} = \check{x}_\xi$, so pick a condition $q_\xi = \langle b_\xi, B_\xi \rangle \leqslant p_\xi$ which forces $\underline{X} \cap \check{\xi} \neq \check{x}_\xi$. Then the conditions q_ξ, $\xi \in \Xi$, are pairwise incompatible in . Then by a simple cardinality argument in $\mathbf{V}[G]$ there is an unbounded set $\Xi' \subseteq \Xi$ (hence still $\mathtt{card}\,\Xi' > \kappa$), and a finite set $t \subseteq \kappa$ such that $b_\xi = t$ for all $\xi \in \Xi'$. But this is impossible because any two conditions with the same first (finite) component are obviously compatible. $\square$

[1] Formally by $\mathtt{cof}\,\lambda$ we mean the cofinality in $\mathbf{V}$, the ground universe. However by Proposition 9 if $\mathtt{cof}\,\lambda > \kappa$ in $\mathbf{V}$ then $\mathtt{cof}\,\lambda > \kappa$ in any Prikry extension of $\mathbf{V}$. And the converse is true by elementary reasons.

But of course a set $X \subseteq \lambda$ does not necessarily satisfy $X \cap \xi \in \mathbf{V}$ for all $\xi < \lambda$. Still the following is true:

Lemma 26. *Under the conditions of Lemma 24, if* $\mathtt{cof}\,\lambda > \kappa$ *then there is a set* $d \in \mathbf{V}[h]$*,* $d \subseteq h$*, such that* $X \cap \xi \equiv_{\mathbf{V}} d$ *for all sufficiently large* $\xi < \lambda$*.*

Proof. It follows from the inductive hypothesis that, in $\mathbf{V}[G]$, for every $\xi < \lambda$ there is a set $d_\xi \subseteq h$ satisfying $X \cap \xi \equiv_{\mathbf{V}} d_\xi$. By a simple cardinality argument based on $\mathtt{cof}\,\lambda > \kappa$, there is a single set $d \subseteq h$ in $\mathbf{V}[h]$ such that $d_\xi = d$, and hence $X \cap \xi \equiv_{\mathbf{V}} d$, for unboundedly many $\xi < \lambda$, and hence for all sufficiently large $\xi < \lambda$. □

Since a set $X \subseteq \lambda$ in $\mathbf{V}[h]$ has been fixed, let us also fix a set $d \subseteq h$ in $\mathbf{V}[h]$ such that $X \cap \xi \equiv_{\mathbf{V}} d$ for all sufficiently large $\xi < \lambda$ (and still $X \cap \xi \in \mathbf{V}[d]$ for all $\xi < \lambda$ in general).

Lemma 27 (the key lemma)**.** *Then* $X \in \mathbf{V}[d]$*, therefore* $X \equiv_{\mathbf{V}} d$*.*

We precede the proof of the lemma with a few remarks and constructions.

As d is a subset of the given Prikry sequence $h = \{h(0) < h(1) < h(2) < \ldots\}$, there is a unique infinite and coinfinite set $\mathrm{ESS} \subseteq \omega$, $\mathrm{ESS} \in \mathbf{V}[h]$ (the set of "essential" indices) such that $d = h /\!\!/ \mathrm{ESS} = \{h(k) : k \in \mathrm{ESS}\}$. Then in fact $\mathrm{ESS} \in \mathbf{V}$ by Proposition 9. We w.l.o.g. assume that $0 \in \mathrm{ESS}$.

For any n, we let $n^{\oplus}$ be the least number $j \in \mathrm{ESS}$, $j > n$.

Our idea is to consider the universe $\mathbf{V}[G] = \mathbf{V}[h]$ as a generic extension of the subuniverse $\mathbf{V}[d]$, and get Lemma 27 following the proof of Lemma 25.

The technical device we employ is the "quotient forcing" of Solovay [4]. Let $\underline{d} \in \mathbf{V}$ be a canonical -name for $d = h /\!\!/ \mathrm{ESS}$. In $\mathbf{V}[G]$, we define $/d$ to be the set of all conditions $p \in$ **compatible with** d in the sense that p -forces, over $\mathbf{V}$, no any statement regarding $\underline{d}$ contradicting to the factual properties of $d = h /\!\!/ \mathrm{ESS}$ as the interpretation of $\underline{d}$ in $\mathbf{V}[G]$. This looks like unsound definition because of the intended quantifier over statements. Yet can be eliminated by a certain transfinite procedure of discarding "wrong" conditions.

Definition 28. A sequence of sets $A_\xi \subseteq$ is defined in $\mathbf{V}[d]$ by transfinite induction on $\xi \in \mathtt{Ord}$ as follows.

A_0 is the set of all conditions $p \in$ such that, for some $\alpha < \kappa$, either p -forces $\check{\alpha} \in \underline{d}$ but $\alpha \notin d$, or conversely, p -forces $\check{\alpha} \notin \underline{d}$ but $\alpha \in d$.

$A_{\xi+1}$ is the set of all conditions $p \in$ such that A_ξ is dense in below p.

Finally, if $\lambda \in \mathtt{Ord}$ is limit then $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$.

Clearly $A_\xi \subseteq A_\eta$ whenever $\xi < \eta$. Therefore there is an ordinal δ such that $A_\delta = A_{\delta+1}$, and hence $A_\gamma = A_\delta$ for all $\gamma > \delta$. Put $A = A_\delta$ and $/d = \smallsetminus A$ (the set Σ of Solovay [4, Section 4]). Obviously $/d \in \mathbf{V}[d]$. □

The following principal facts were discovered in [4].

1°: $G \subseteq /d$,

2°: if $D \in \mathbf{V}$ is a dense subset of then $D \cap (/d)$ is dense in $/d$,

3°: if $p \in /d$, $q \in$, and $p \leqslant q$ then $q \in /d$,

4°: if $p \in$ then $p \in /d$ iff there is a set $G' \subseteq$, -generic over $\mathbf{V}$, containing p, and such that $h_{G'} /\!\!/ \text{ESS} = d$,

5°: G is a set $(/d)$-generic over $\mathbf{V}[d]$,

6°: any set $G' \subseteq /d$, $(/d)$-generic over $\mathbf{V}[d]$, is also -generic over $\mathbf{V}$.

Claims 1°, 2°, 3°, 5°, and implication $\Longrightarrow$ in 4° are just $(\Sigma 1) - (\Sigma 5)$ in [4, Section 4], while 6° is an easy consequence of 2°. Let us prove the inverse implication in 4°. Suppose that $G' \subseteq$ is a set -generic over $\mathbf{V}$, and $h_{G'} /\!\!/ \text{ESS} = d$. Prove that $G' \subseteq /d$, that is, $G' \cap A_\xi = \varnothing$ for all ξ. That $G' \cap A_0 = \varnothing$ follows from the assumption $h_{G'} /\!\!/ \text{ESS} = d$. The limit step is obvious. Finally assume that $G' \cap A_\xi = \varnothing$ and prove $G' \cap A_{\xi+1} = \varnothing$. Suppose towards the contrary that $p \in G' \cap A_{\xi+1}$. By definition, the set A_ξ is dense below p, thus easily $G' \cap A_\xi \neq \varnothing$, which is a contradiction.

Thus $\mathbf{V}[G]$ is a generic extension of $\mathbf{V}[d]$. This allows us to carry out the proof of Lemma 27 on the base of the following lemma. The proof of Lemma 29 will follow in Section 10.

Lemma 29. *The forcing $/d$ satisfies κ^+-CC in $\mathbf{V}[h]$.*

Proof (Lemma 27)**.** Thus X (the given set) belongs to $\mathbf{V}[G] = \mathbf{V}[d][G]$, and this is a $(/d)$-generic extension of $\mathbf{V}[d]$ by 5° above. Emulating the proof of Lemma 25, we let $\underline{X} \in \mathbf{V}[d]$ be a $(/d)$-name for X. For each $\xi < \lambda$, let P_ξ be the set of all conditions $p \in /d$ which $(/d)$-force $\underline{X} \cap \check{\xi} = \check{x}_\xi$ over $\mathbf{V}[d]$ (where $x_\xi = X \cap \xi \in \mathbf{V}[d]$). Thus $\varnothing \neq P_\xi \in \mathbf{V}[d]$, and $\xi < \eta \Longrightarrow P_\eta \subseteq P_\xi$.

Case 1: there is a condition $p \in \bigcap_{\xi<\lambda} P_\xi$. Then p decides, over $\mathbf{V}[d]$, every formula of the form $\check{\alpha} \in \underline{X}$, $\alpha < \lambda$, so easily $X \in \mathbf{V}[d]$, and we are done.

Case 2: not case 1. Then, in $\mathbf{V}[h]$, there is a pairwise incompatible set in $/d$ of cardinality at least κ^+. (Just as in the proof of Lemma 25.) But this contradicts to Lemma 29.

□ (*Lemma 27, modulo Lemma 29*)

□ (*The case* $\texttt{cof}\,\lambda > \kappa$ *in Lemma 24, modulo Lemma 29*)

10 Proof of the chain condition lemma

This section presents the proof of Lemma 29. We argue in the notation of Section 9. The idea is to define a dense set $\subseteq /d$ which is somewhat simpler that $/d$ itself so that the chain condition for will be rather obvious.

Definition 30. is the set of all conditions $p = \langle a, A\rangle \in$ such that

(A) $\mu = \mathtt{min}\, A \in d$, and hence $\mu = h(n)$, where $n = n(p) \in \text{ESS}$;

(B) $h \smallsetminus \mu \subseteq A$;

(C) $\mathtt{dom}\, a = n$, thus a is a sequence of length $n = n(p)$ defined by (A);

(D) $a \mathbin{/\!\!/} \text{ESS} = d \cap \mu$. □

To explain (D), suppose, for the sake of clarity, that $\text{ESS} = \{0, 2, 4, 6, \dots\}$ (all even numbers), and $a = \{\alpha_0 < \alpha_1 < \dots < \alpha_{n-2} < \alpha_{n-1}\}$. Then $a \mathbin{/\!\!/} \text{ESS} = \{\alpha_0 < \alpha_2 < \alpha_4 < \dots < \alpha_{2k-2} < \alpha_{2k}\}$, where $n = 2k+2$ or $n = 2k+1$. Now, (D) requires that $a \mathbin{/\!\!/} \text{ESS}$ coincides with the set of all members $\xi \in d$ smaller than μ.

Obviously $\in \mathbf{V}[h]$, but we can hardly expect that $\in \mathbf{V}[d]$, and hence cannot replace $/d$ as a forcing over $\mathbf{V}[d]$.

Lemma 31. *$Q \subseteq /d$, and is dense in $/d$, so that if $p = \langle a, A\rangle \in /d$ then there is a condition $q = \langle b, B\rangle \in$ such that $q \leqslant b$.*

Proof. Assume that $p = \langle a, A\rangle \in$. Then $a = \{\alpha_0 < \alpha_1 < \dots < \alpha_{n-2} < \alpha_{n-1}\}$, where $n = n(p) \in \text{ESS}$, and $h(n) = \mu$, where $\mu = \mathtt{min}\, A$, and finally addition $a \mathbin{/\!\!/} \text{ESS} = d \cap \mu$. Prove that $p \in /d$. According to 4° of Section 9, it suffices to find a Prikry sequence $h' : \omega \to \kappa$ over $\mathbf{V}$, compatible with p and such that still $h' \mathbin{/\!\!/} \text{ESS} = d$. We put $h'(j) = h(j)$ for all $j \geq n$ and for all $j < n$, $j \in \text{ESS}$. It needs more work to suitably define $h'(j)$ for numbers $j \in n \smallsetminus \text{ESS}$.

Thus let $j < n$, $j \notin \text{ESS}$. There is $k < n$, $k \in \text{ESS}$, such that $k < j < k^{\oplus} \leqslant n$, and $h'(k) = h(k) = a(k)$, $h'(k^{\oplus}) = h(k^{\oplus})$ $(= a(k^{\oplus})$, if $k^{\oplus} < n$ strictly) by (D). It follows that $h'(k) < a(j) < h'(k^{\oplus})$, and we simply put $h'(j) = a(j)$.

By construction, $h' : \omega \to \kappa$ is increasing, compatible with p, still $h' \mathbin{/\!\!/} \text{ESS} = d$, and $h'(j) = h(j)$ for all but finite j, so that h' is a Prikry sequence over $\mathbf{V}$, as required. This ends the proof of $Q \subseteq /d$.

To prove the density, suppose that $q = \langle b, B\rangle \in /d$. According to 4° of Section 9, there exists a Prikry sequence $h' : \omega \to \kappa$ over $\mathbf{V}$, compatible with q and such that still $h' \mathbin{/\!\!/} \text{ESS} = d$. In particular $h'(k) = h(k)$ for all $k \in \text{ESS}$. The sets $h \smallsetminus B$ and $h' \smallsetminus B$ are finite, and hence there is a number $n \in \text{ESS}$ such that $\mu = h'(n) = h(n) \in d$ and $h \smallsetminus \mu \subseteq B$, $h' \smallsetminus \mu \subseteq B$. Put $A = B \smallsetminus \mu$, thus $\mu = \mathtt{min}\, A$ and still we have both $h \smallsetminus \mu \subseteq A$ and $h' \smallsetminus \mu \subseteq A$.

Let finally $a = h' \restriction n = \langle h'(0), \dots, h'(n-1)\rangle$. Then $a \mathbin{/\!\!/} \text{ESS} = d \cap \mu$ since $d = h' \mathbin{/\!\!/} \text{ESS}$. It follows that $p = \langle a, A\rangle \in$. To show that $p \leqslant q$, note that $A \subseteq B$ by construction, a extends b simply because h' is compatible with q, and $a \smallsetminus b \subseteq B$ simply because $a \subseteq h \subseteq b \cup B$. □

Proof (Lemma 29)**.** Suppose towards the contrary that $W \in \mathbf{V}[h]$, $W \subseteq /d$, is an antichain in $/d$, and $\mathtt{card}\, W = \kappa^+$. By Lemma 31, we can w.l.o.g. assume that

$W \subseteq$. By a cardinality argument, there exist a finite set $a \subseteq \kappa$, an ordinal $\mu < \kappa$, and a set $W' \subseteq W$, such that still $\mathtt{card}\, W' = \kappa^+$, and we have $\min A_p = \mu$ and $a_p = a$ for all $p \in W'$. Consider any pair of conditions $p = \langle a, A_p \rangle \neq q = \langle a, A_q \rangle$ in W'. Then $r = \langle c, A_p \cap A_q \rangle$, a stronger condition, also belongs to , and hence p, q are compatible in , a contradiction. □

11 Small cofinality inductive step

Here we prove Lemma 24 in the case when $\mathtt{cof}\, \lambda \leqslant \kappa$. Let us fix a set $X \subseteq \lambda$ in $\mathbf{V}[G] = \mathbf{V}[h_G]$, and an increasing sequence $\{\vartheta_\xi\}_{\xi<\delta} \in \mathbf{V}$ of cardinals ϑ_ξ, cofinal in λ. The goal is to prove that there is a set $Z \subseteq \kappa$ such that $X \equiv_{\mathbf{V}} Z$.

It follows from the inductive hypothesis (IV) of Section 8 that, in $\mathbf{V}[X]$, for every ordinal $\xi < \delta$ there is a set $Y_\xi \subseteq \kappa$ satisfying $(X \cap \vartheta_\xi) \equiv_{\mathbf{V}} Y_\xi$. Thus there also exist: a parameter $p_\xi \in \mathbf{V}$, and ordinals α_ξ and β_ξ, such that Y_ξ is the α_ξ-th element in the canonical Gödel wellordering of $\mathbf{L}[p_\xi, X \cap \vartheta_\xi]$, and conversely, $X \cap \vartheta_\xi$ is equal to the β_ξ-th element in the canonical Gödel wellordering of $\mathbf{L}[p_\xi, Y_\xi]$, so that the following statement is true in $\mathbf{V}[X]$:

(1) for every Z, if Z is equal to the α_ξ-th element in the canonical Gödel wellordering of $\mathbf{L}[p_\xi, X \cap \vartheta_\xi]$, then $Z \subseteq \kappa$ and $X \cap \vartheta_\xi$ is equal to the β_ξ-th element in the canonical Gödel wellordering of $\mathbf{L}[p_\xi, Z]$.

Moreover, as is a κ^+-CC forcing, and $\delta \leqslant \kappa$, there exist sets P, A, $B \in \mathbf{V}$ of cardinality $\mathtt{card}\, P = \mathtt{card}\, A = \mathtt{card}\, B = \kappa$ in $\mathbf{V}$, such that, in $\mathbf{V}[X]$,

(2) for any $\xi < \delta$, sets p_ξ, α_ξ, β_ξ satisfying (1) do exist in sets resp. P, A, B.

We let $p_\xi(X)$, $\alpha_\xi(X)$, $\beta_\xi(X)$ denote the least such sets, in the sense of fixed wellorderings (in $\mathbf{V}$) of the sets P, A, B. The maps $\xi \longmapsto p_\xi(X)$, $\alpha_\xi(X)$, $\beta_\xi(X)$ are well-defined in $\mathbf{V}[X]$.

Let us now fix, in $\mathbf{V}$, three bijections

$$b_1 : \kappa \xrightarrow{\mathtt{onto}} P\,, \quad b_2 : \kappa \xrightarrow{\mathtt{onto}} A\,, \quad b_3 : \kappa \xrightarrow{\mathtt{onto}} B\,,$$

This converts the maps $\xi \longmapsto p_\xi(X)$, $\alpha_\xi(X)$, $\beta_\xi(X)$ in $\mathbf{V}[X]$ into three functions, say ν_1, ν_2, $\nu_3 : \omega \to \kappa$, which effectively code the maps, and subsequently the sequence of sets $X \cap \vartheta_\xi$, $\xi < \delta$, and the set X itself, so that $X \in \mathbf{V}[\nu_1, \nu_2, \nu_3]$. It remains to code ν_1, ν_2, ν_3 in $\mathbf{V}[X]$ by a single set $Y \subseteq \kappa$.

□ (*The case* $\mathtt{cof}\, \lambda \leqslant \kappa$ *in Lemma 24*)

□ (*Theorems 5 and 3*)

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