Strange ultrafilters.

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Abstract

We deal with some natural properties of ultrafilters which trivially fail for normal ultrafilters.

Throughout the paper all ultrafilters considered are non-principal.
If U is a $\kappa-$complete ultrafilter over κ, then denote by $i_U : V \to M_U \simeq \mathrm{Ult}(V, U)$ the corresponding elementary embedding and the transitive collapse of the ultrapower.
If W is a $\kappa-$complete ultrafilters over κ and $\langle W_\alpha \mid \alpha < \kappa \rangle$ is a sequence of $\kappa-$complete ultrafilters, then $W - \lim \langle W_\alpha \mid \alpha < \kappa \rangle$ is a $\kappa-$complete ultrafilter over κ which consists of all $X \subseteq \kappa$ such that

$$\{\alpha < \kappa \mid X \in W_\alpha\} \in W.$$

Let us address first the following natural question asked by Eyal Kaplan:

Is it possible to have a $\kappa-$complete ultrafilter F over κ such that for some sequence of $\kappa-$complete ultrafilters $\langle W_\alpha \mid \alpha < \kappa \rangle$ over κ different from F we have
$F = F - \lim \langle W_\alpha \mid \alpha < \kappa \rangle$?

Note that this is clearly impossible once F is normal. Also, this is impossible once the family $\langle W_\alpha \mid \alpha < \kappa \rangle$ is *discrete*, i.e. there is a sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ which consists of pairwise disjoint sets such that $A_\alpha \in W_\alpha$, for every $\alpha < \kappa$.
However, it turns out that the situation occurs quit often.

Theorem 0.1 *Let $F = W - \lim \langle W_\alpha \mid \alpha < \kappa \rangle$, for some discrete (or discrete mod W) family of $\kappa-$complete ultrafilters $W_\alpha, \alpha < \kappa$, over κ. Then there is a family $\langle E_\nu \mid \nu < \kappa \rangle$ of $\kappa-$complete ultrafilters over κ different from F such that*
$F = F - \lim \langle E_\nu \mid \nu < \kappa \rangle$.

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Proof. Consider $i_W : V \to M_W$. Let $i_W(\langle W_\alpha \mid \alpha < \kappa\rangle) = \langle W'_\alpha \mid \alpha < i_W(\kappa)\rangle$. Take now the ultrapower of M_W by $W'_{[id]_W}$. Let

$$\sigma := i_{W'_{[id]_W}} : M_W \to N$$

be the corresponding elementary embedding. The family $\langle W_\alpha \mid \alpha < \kappa\rangle$ is discrete, so it is not hard to see that $W'_{[id]_W}$ differs from $i_W(F)$ and

$$\sigma \circ i_W = i_F \text{ and } N = M_F.$$

Consider now $\sigma(W'_{[id]_W})$. It is a $i_F(\kappa)-$complete ultrafilter over $i_F(\kappa)$ in M_F different from $i_F(F)$. In V, we pick a sequence $\langle E_\nu \mid \nu < \kappa\rangle$ of $\kappa-$complete ultrafilters over κ which represents $\sigma(W'_{[id]_W})$ in the ultrapower M_F.
Let $i_F(\langle E_\nu \mid \nu < \kappa\rangle) = \langle E'_\nu \mid \nu < i_F(\kappa)\rangle$. Then $\sigma(W'_{[id]_W}) = E'_{[id]_F}$.
Now,

$$Z \in F - \lim \langle E_\nu \mid \nu < \kappa\rangle \Leftrightarrow \{\nu < \kappa \mid Z \in E_\nu\} \in F \Leftrightarrow$$

$$i_F(Z) \in E'_{[id]_F} = \sigma(W'_{[id]_W}) \Leftrightarrow \sigma(i_W(Z)) \in \sigma(W'_{[id]_W}) \Leftrightarrow$$

$$i_W(Z) \in W'_{[id]_W} \Leftrightarrow \{\alpha < \kappa \mid Z \in W_\alpha\} \in W \Leftrightarrow Z \in W - \lim \langle W_\alpha \mid \alpha < \kappa\rangle = F.$$

So, $\langle E_\nu \mid \nu < \kappa\rangle$ is as desired.
□

Remark 0.2 1. Note that the family $\langle E_\nu \mid \nu < \kappa\rangle$ have same ultrafilters, i.e. the function $\nu \mapsto E_\nu$ is not one-to -one. Moreover, it cannot be one-to -one on a set of ν's in F.
2. We do not know to achieve $F = F - \lim \langle E_\nu \mid \nu < \kappa\rangle$ with a family consisting of different ultrafilters. Clearly this is impossible once the family is discrete.

Let show now the following negative result.

Proposition 0.3 *Suppose that U, W and $\langle E_\alpha \mid \alpha < \kappa\rangle$ are $\kappa-$complete ultrafilters over κ such that $U =_{R-K} E_\alpha$ and $U \neq E_\alpha$, for every $\alpha < \kappa$. Then $U \neq W - \lim \langle E_\alpha \mid \alpha < \kappa\rangle$.*

Proof. Suppose otherwise. Then $U = W - \lim \langle E_\alpha \mid \alpha < \kappa\rangle$.
Observe first that if $U' =_{R-K} U$, then

$$U' = W - \lim \langle E'_\alpha \mid \alpha < \kappa\rangle,$$

for some $\langle E'_\beta \mid \beta < \kappa \rangle$. Thus, let $U' =_{R-K} U$ and let $h : \kappa \to \kappa$ be a one to one function witnessing this, say $h_* U = U'$. Set $E'_\alpha = h_* E_\alpha$, for every $\alpha < \kappa$.
Let $Y \subseteq \kappa$. Then

$$Y \in U' \Leftrightarrow h^{-1\prime\prime} Y = X \in U \Leftrightarrow \{\alpha < \kappa \mid X \in E_\alpha\} \in W \Leftrightarrow$$

$$\{\alpha < \kappa \mid Y = h'' X \in h_* E_\alpha\} \in W \Leftrightarrow \{\alpha < \kappa \mid Y \in E'_\alpha = h_* E_\alpha\} \in W.$$

Next, consider $i_U : V \to M_U \simeq {}^\kappa V/U$. Set $i := i_U$ and $M := M_U$. Let $\eta = [id]_U$.
If there is $\eta' < \eta$ and $f_{\eta'} : \kappa \to \kappa$ such that $i(f_{\eta'})(\eta') = \eta$, then let η^* be the least such η'. Note that then there will be no $\eta' < \eta^*$ such that for some $f : \kappa \to \kappa$, $i(f)(\eta') = \eta^*$, since otherwise $i(f_{\eta^*} \circ f)(\eta') = \eta$, which contradicts the minimality of η^*.

For every $\delta < i(\kappa)$, denote by U_δ the ultrafilter $\{X \subseteq \kappa \mid \delta \in i(X)\}$.
Then $U_{\eta^*} \geq_{R-K} U$, as witnessed by f_{η^*}, but also $U_{\eta^*} \leq_{R-K} U$, since U_{η^*} is defined from i. Hence $U_{\eta^*} =_{R-K} U$.

By the observation above, we can replace then U by U_{η^*}. Assume for simplicity that already $U = U_{\eta^*}$.
Let $\alpha < \kappa$. Consider E_α. Pick $\delta_\alpha < i(\kappa)$ such that $E_\alpha = \{X \subseteq \kappa \mid \delta_\alpha \in i(X)\}$.
We have $E_\alpha =_{R-K} U$, so there is $h_\alpha : \kappa \to \kappa$ one to one such that $\delta_\alpha = i(h_\alpha)(\eta)$. Then $\eta = i(h_\alpha^{-1})(\delta_\alpha)$ which implies by the choice of η that $\delta_\alpha > \eta$.

Let $\pi : \kappa \to \kappa$ be a projection of U to the normal measure U_κ. Consider now the following set:

$$Z = \{\nu < \kappa \mid \forall \nu' < \nu \forall \alpha < \pi(\nu)(h_\alpha(\nu') \neq \nu)\}.$$

Then, by the choice of η, $Z \in U$, since for every $h : \kappa \to \kappa$ and in particular for every $h_\alpha, \alpha < \kappa$, we have $i(h)(\eta') \neq \eta$, whenever $\eta' < \eta$, and so, $\eta \in i(Z)$.
On the other hand,

$$i(\kappa \setminus Z) = \{\nu < i(\kappa) \mid \exists \nu' < \nu \exists \alpha < i(\pi)(\nu)(i(h)_\alpha(\nu') = \nu)\}.$$

So, if $\alpha < \kappa$, then $\kappa \setminus Z \in E_\alpha$, provided $i(\pi)(\delta_\alpha) > \alpha$, since $i(h_\alpha)(\eta) = \delta_\alpha$ and $\eta < \delta_\alpha$. In particular, this holds if π is not a constant function mod E_α.
Unfortunately, we do not see a reason why this should be the case.

In order to overcome the problem, let us use more involved argument. The idea would be to replace Z by another, similar set, but without π.

An ordinal $\alpha < i(\kappa)$ is called *a generator* of the embedding i iff for every $n, 1 \leq n < \omega$, every $g : [\kappa]^n \to \kappa$ and for every $\vec{\nu} \in [\alpha]^n$, $i(g)(\vec{\nu}) \neq \alpha$.

Now, either η is a generator or there are an increasing sequence of generators $\langle \eta_0, ..., \eta_{n-1} \rangle$ below η and a function $g_\eta : [\kappa]^n \to \kappa$ such that $\eta = i(g_\eta)(\eta_0, ..., \eta_{n-1})$. Let us deal with the later case. The former one is similar and a bit simpler.
There may be several possibilities for sequences of generators and functions g_η as above. Pick first $\eta'_0 < \eta$ to be the least generator such that there is a finite sequence of generators $a \in [\eta'_0]^{<\omega}$ such that for some function $g : [\kappa]^{|a|+1} \to \kappa$ we have $\eta = i(g)(a^\frown \eta'_0)$.
Next, let $\eta'_1 < \eta$ to be the least generator $< \eta'_0$ such that there is a finite sequence of generators $a \in [\eta'_1]^{<\omega}$ such that for some function $g : [\kappa]^{|a|+2} \to \kappa$ we have $\eta = i(g)(a^\frown \langle \eta'_1, \eta'_0 \rangle)$. Continue further by recursion. After finitely many steps, we will construct a sequence $\eta'_0 > \eta'_1 > ... > \eta'_{n-1}$ of generators such that each member is the smallest possible (in the above sense) and for some function $g : [\kappa]^n \to \kappa$ we have $\eta = i(g)(\langle \eta'_{n-1}, ..., \eta'_0 \rangle)$.
Set now $\eta_{n-1} := \eta'_0, ..., \eta_0 = \eta'_{n-1}$.

Claim 1 $\eta = \eta_{n-1} + \eta_{n-2} + ... + \eta_0$.

Proof. First note that $\eta \leq \eta_{n-1} + \eta_{n-2} + ... + \eta_0$, since it is easy to find $f : \kappa \to \kappa$ such that $i(f)(\eta_{n-1} + \eta_{n-2} + ... + \eta_0) = \eta$.
Next let $\eta = \xi_{m-1} + ... + \xi_0$ be the Cantor normal form of η. By the minimality of η_{n-1}, we must have $\eta_{n-1} = \xi_{m-1}$. Then again, minimality of η_{n-2} implies that also $\eta_{n-2} = \xi_{m-2}$. Finally, we will have $n = m$ and $\eta_0 = \xi_0$.
$\square$ of the claim.

By the claim then, for almost all $\alpha < \kappa$, $\delta_\alpha = \eta^\alpha_{n-1} + \eta^\alpha_{n-2} + ... + \eta^\alpha_0$, since $U = U - \lim \langle E_\alpha \mid \alpha < \kappa \rangle$, and $\eta^\alpha_{n-1} \geq \eta_{n-1}$, since $\eta < \delta_\alpha$. Assume that this holds for every $\alpha < \kappa$.
Let $\pi_1 : \kappa \to \kappa$ be the projection of an ordinal to its largest component in the Cantor normal form, i.e. $\pi_1(\xi_{m-1} + \xi_{m-2} + ... + \xi_0) = \xi_{m-1}$. Then $i(\pi_1)(\eta) = \eta_{n-1}$ and $i(\pi')(\delta_\alpha) = \eta^\alpha_{n-1}$, for every $\alpha < \kappa$. Also note that $\kappa \leq \eta_{n-1} \leq \eta^\alpha_{n-1}$, for every $\alpha < \kappa$.

Suppose first that for almost all $\alpha < \kappa$, $\eta_{n-1} < \eta^\alpha_{n-1}$.
Then, also $\eta < \eta^\alpha_{n-1}$. Thus, η_{n-1} is a generator, and hence, it cannot be written as a finite sum of smaller ordinals. Namely,

$$Y = \{\nu < \kappa \mid \forall m < \omega \forall \xi_0 < ... < \xi_{m-1} < \pi_1(\nu)(\xi_{m-1} + ... + \xi_0 < \pi'(\nu))\} \in U,$$

and so, $Y \in E_\alpha$ for almost every $\alpha < \kappa$. This means, in M_1, that

$$\forall m < \omega \forall \xi_0 < ... < \xi_{m-1} < \pi'(\nu)(\xi_{m-1} + ... + \xi_0 < \eta^\alpha_{n-1}),$$

and in particular, $\eta = \eta_{n-1} + ... + \eta_0 < \eta^\alpha_{n-1}$.

Now we are ready to redefine Z. Set

$$Z' = \{\nu < \kappa \mid \forall \nu' < \pi_1(\nu) \forall \alpha < \pi'(\nu)(h_\alpha(\nu') \neq \nu)\}.$$

Then

$$i(\kappa \setminus Z') = \{\nu < i(\kappa) \mid \exists \nu' < i(\pi_1)(\nu) \exists \alpha < i(\pi_1)(\nu)(h'_\alpha(\nu') = \nu)\},$$

where $\langle h'_\alpha \mid \alpha < i(\kappa)\rangle = i(\langle h_\alpha \mid \alpha < \kappa\rangle)$.
Now, if $\alpha < \kappa$, then $\kappa \setminus Z \in E_\alpha$, since $i(\pi_1)(\delta_\alpha) = \eta^\alpha_{n-1} \geq \kappa > \alpha$, $i(h_\alpha)(\eta) = \delta_\alpha$ and $\eta < i(\pi_1)(\delta_\alpha) = \eta^\alpha_{n-1}$.
Let us argue that $Z' \in U$.

Claim 2 $Z' \in U$.

Proof. We show that for every $\alpha < \eta_{n-1}$ and every $\eta' < \eta_{n-1}$, $h'_\alpha(\eta') \neq \eta$. It will be enough to argue that $h'_\alpha(\eta') \neq \eta_{n-1}$, since if $h'_\alpha(\eta') = \eta$, then the projection to the largest component of the Cantor normal form will give η_{n-1}.

Consider the extender G derived from i using ordinals below η_{n-1}, i.e.

$$G = \langle U_a \mid a \in [\eta_{n-1}]^{<\omega}\rangle$$

and its ultrapower $i_G : V \to N_G$.
Another way of stating this is to consider the transitive collapse of

$$\{i(g)(a) \mid a \in [\eta_{n-1}]^{<\omega}\}.$$

Let $k : N_G \to M$ be the natural embedding, i.e. $k(i_G(g)(a)) = i(g)(a)$. Then, $crit(k) = \eta_{n-1}$, since η_{n-1} is a generator, and so, $\eta_{n-1} \neq i(g)(a)$, for $a \in [\eta_{n-1}]^{<\omega}, g : [\kappa]^{|a|} \to \kappa$, but every $\eta' < \eta$ is trivially of such a form, and so does not move by k.

Consider $\langle h_\alpha \mid \alpha < \kappa\rangle$. Let $i_G(\langle h_\alpha \mid \alpha < \kappa\rangle)$ be $\langle h''_\alpha \mid \alpha < i_G(\kappa)\rangle$. Let $\alpha < \eta_{n-1}$ and $\eta' < \eta_{n-1}$. Consider $h''_\alpha(\eta') = \mu$. Apply k to it. Then $k(h''_\alpha(\eta')) = h'_\alpha(\eta') = k(\mu)$, since neither $\alpha < \eta_{n-1}$ nor $\eta' < \eta_{n-1}$ are moved by k. Now, if $k(\mu) = \eta$, then η_{n-1} will in the range of k as the image the projection to the largest component of the Cantor normal form of μ, which is clearly impossible. So, $k(\mu) \neq \eta$, which means that $h'_\alpha(\eta') \neq \eta$ whenever $\alpha < \eta_{n-1}$ and $\eta' < \eta_{n-1}$.
$\square$ of the claim.

Suppose now that that for almost all $\alpha < \kappa$, $\eta_{n-1} = \eta^\alpha_{n-1}$.
Let us assume for simplicity that $n = 2$ and for almost all $\alpha < \kappa$, $\eta_1 < \eta^\alpha_1$ and $\eta_2 = \eta^\alpha_2$. Assume that this holds for every $\alpha < \kappa$.
The crucial is that there is no $f : \kappa \to \kappa$ such that $\eta_1 = i(f)(\eta_2)$, since if this was the case, then we were able to reduce η_1.

Let $\pi_2 : \kappa \to \kappa$ be the projection of an ordinal to its second largest component in the Cantor normal form, i.e. $\pi_2(\xi_{m-1} + \xi_{m-2} + ... + \xi_0) = \xi_{m-2}$. Then $i(\pi_2)(\eta) = \eta_2$ and $i(\pi_2)(\delta_\alpha) = \eta^\alpha_2$, for every $\alpha < \kappa$. Also note that $\kappa \leq \eta_2 \leq \eta^\alpha_2$, for every $\alpha < \kappa$.

Set

$$Z_2 = \{\nu < \kappa \mid \forall \nu' < \pi_2(\nu) \forall \alpha < \pi_2(\nu)(h_\alpha(\pi_1(\nu) + \nu') \neq \nu)\}.$$

Then

$$i(\kappa \setminus Z_2) = \{\nu < i(\kappa) \mid \exists \nu' < i(\pi_2)(\nu) \exists \alpha < i(\pi_2)(\nu)(h'_\alpha(i(\pi_1)(\nu) + \nu') = \nu)\},$$

where $\langle h'_\alpha \mid \alpha < i(\kappa)\rangle = i(\langle h_\alpha \mid \alpha < \kappa\rangle)$.
Now, if $\alpha < \kappa$, then $\kappa \setminus Z_2 \in E_\alpha$, since $i(\pi_2)(\delta_\alpha) = \eta^\alpha_2 \geq \kappa > \alpha$, $i(h_\alpha)(\eta) = \delta_\alpha$ and $\eta = \eta_2 + \eta_1 + \eta_0, \eta_1 + \eta_0 < i(\pi_2)(\delta_\alpha) = \eta^\alpha_1$.
Let us argue that $Z_2 \in U$.

Claim 3 $Z_2 \in U$.

Proof. We show that for every $\alpha < \eta_1$ and every $\eta' < \eta_1$, $h'_\alpha(\eta_2 + \eta') \neq \eta$.

Consider the extender H derived from i using ordinals below η_1 and $\{\eta_2\}$, i.e.

$$H = \langle U_{a^\frown \eta_2} \mid a \in [\eta_1]^{<\omega}\rangle$$

and its ultrapower $i_H : V \to N_H$.
Another way of stating this is to consider the transitive collapse of

$$\{i(g)(a^\frown \eta_2) \mid a \in [\eta_1]^{<\omega}\}.$$

Let $k : N_H \to M$ be the natural embedding, i.e. $k(i_H(g)(a^\frown \eta'_2)) = i(g)(a^\frown \eta_2)$, where η'_2 is the image of η_2 under the transitive collapse.
Then, $crit(k) = \eta_1$, since by the smallnest assumptions we made on η_1, $\eta_1 \neq i(g)(a^\frown \eta_2)$, for $a \in [\eta_1]^{<\omega}, g : [\kappa]^{|a|+1} \to \kappa$, but every $\eta' < \eta_1$ is trivially of such a form, and so does not move by k.

Consider $\langle h_\alpha \mid \alpha < \kappa\rangle$. Let $i_H(\langle h_\alpha \mid \alpha < \kappa\rangle)$ be $\langle h''_\alpha \mid \alpha < i_H(\kappa)\rangle$. Let $\alpha < \eta_1$ and $\eta' < \eta_1$. Consider $h''_\alpha(\eta'_2 + \eta') = \mu$. Apply k to it. Then $k(h''_\alpha(\eta')) = h'_\alpha(\eta_2 + \eta') = k(\mu)$, since neither $\alpha < \eta_1$ nor $\eta' < \eta_1$ are moved by k. Now, if $k(\mu) = \eta$, then η_1 will in the range of k as the image the projection to the second largest component of the Cantor normal form of μ, which is clearly impossible. So, $k(\mu) \neq \eta$, which means that $h'_\alpha(\eta_2 + \eta') \neq \eta$ whenever $\alpha < \eta_1$ and $\eta' < \eta_1$.
$\square$ of the claim.
$\square$

We address now the following issue, raised by Eyal Kaplan:

Let F be a $\kappa-$complete ultrafilter over κ and $n, 0 < n < \omega$. How many ways to project F^n onto F are?

Clearly, we have the projections to each of n many coordinates. But are there any other projections?
It is not hard to see that once F is normal, then - no.
Let us deal with general F's.
Start with $n = 1$.

Proposition 0.4 *Let U be a $\kappa-$complete non-principal ultrafilter over κ, $i_U : V \to M_U \simeq {}^\kappa V/U$ the corresponding elementary embedding. For each $\alpha < i_U(\kappa)$, let $U_\alpha = \{X \subseteq \kappa \mid \alpha \in i_U(X)\}$. Then $U_\alpha = U$ iff $\alpha = [id]_U$.*

Proof. Suppose otherwise. Let $\alpha < i_U(\kappa), \alpha \neq [id]_U$ be such that $U_\alpha = U$. Denote $[id]_U$ by η. Pick $f : \kappa \to \kappa$ which represents α in M_U, i.e. $[f]_U = i_U(f)(\eta) = \alpha$. Then f is one to one on a set in U, since $U_\alpha = U$, and so, the ultrapower by U_α is the same as those U, i.e. M_U. Suppose for simplicity that f is one to one on κ. Then either

$$\{\nu < \kappa \mid f(\nu) > \nu\} \in U$$

or

$$\{\nu < \kappa \mid f(\nu) < \nu\} \in U.$$

Suppose that

$$\{\nu < \kappa \mid f(\nu) > \nu\} \in U,$$

i.e. f is increasing on a set in U. If the second possibility occurs then we can just replace f by f^{-1} and proceed as in the former case.

Let

$$A := \{\nu < \kappa \mid f(\nu) > \nu\} \in U.$$

Note that for every $B \in U$, we have $f''B \in U_\alpha = U$.
For every $n < \omega$, define a set $A^{(n)} \in U$ by induction as follows. Set $A^{(0)} = A$, $A^{(n+1)} = f''A^{(n)}$.
Let

$$A^* = \bigcap_{n<\omega} A^{(n)}.$$

Then $A^* \in U$.
Pick any $\nu \in A^*$. Then $\nu \in A^{(1)}$, hence there is $\nu_1 \in A$ such that $f(\nu_1) = \nu$. This ν_1 is unique, since f is one to one. Also, $\nu_1 < \nu$, since f is increasing on A.
Now, $\nu \in A^{(2)}$, hence there is $\nu_2 \in A$ such that $f(f(\nu_2)) = \nu$. Then $f(\nu_2) = \nu_1$, since f is one to one, and $\nu_2 < \nu_1$, since f is increasing on A.
Continue further by induction. We will obtain an infinite decreasing sequence

$$\nu > \nu_1 > \nu_2 > ...$$

which impossible.
Contradiction.
□

Consider now $n = 2$.
Note that intuitively, if we have say three copies of F inside $F \times F$ at different places, then their envelope (the ultrafilter they generate) should be F^3. But F^3 is not Rudin - Kiesler below F^2.
However, it turns out that it is possible to have three (and much more) copies of an ultrafilter inside its square, as will be shown below.

Theorem 0.5 *Let $\langle W_\alpha \mid \alpha < \kappa \rangle$ be a discrete family of $\kappa-$complete ultrafilters over κ and W be a $\kappa-$complete ultrafilters over κ. Assume that $W >_{R-K} W_\alpha$, for every $\alpha < \kappa$.*
Let $F = W - \lim \langle W_\alpha \mid \alpha < \kappa \rangle$.
Then there is a function $g : [\kappa]^2 \to \kappa$ such that

1. *$g_* F \times F = F$, i.e. g projects $F \times F$ to F,*

2. *g is different (mod F) from the projections of $F \times F$ to the first and to the second coordinate.*

Proof. We preserve the notation of Theorem 0.1. The discreteness of the family $\langle W_\alpha \mid \alpha < \kappa \rangle$ implies that $F \geq_{R-K} W$. Hence $F >_{R-K} W_\alpha$, for every $\alpha < \kappa$. Then, in M_W, $i_W(F) >_{R-K}$

$W'_{[id]_W}$. Applying σ, we get that $i_F(F) >_{R-K} \sigma(W'_{[id]_W})$.
Pick some $h : i_F(\kappa) \to i_F(\kappa)$ witnessing this.
Now, we form the second ultrapower by taking the ultrapower of M_F by $i_F(F)$. Clearly, $M_{F\times F}$ is this ultrapower and $i_{F\times F} = i_{i_F(F)} \circ i_F$.
Set $\eta = [h]_{i_F(F)}$. Then $i_F(F) >_{R-K} \sigma(W'_{[id]_W})$ implies that $i_F(\kappa) \leq \eta \neq [id]_{i_F(F)}$.
Now

$$Z \in F \Leftrightarrow \{\alpha < \kappa \mid Z \in W_\alpha\} \in W \Leftrightarrow i_W(Z) \in W'_{[id]_W}$$

$$\Leftrightarrow \sigma(i_W(Z)) \in \sigma(W'_{[id]_W}) \Leftrightarrow i_F(Z) \in \sigma(W'_{[id]_W}) \Leftrightarrow \eta \in i_{i_F(F)}(i_F(Z)) \Leftrightarrow \eta \in i_{F\times F}(Z).$$

Pick a function $g : [\kappa]^2 \to \kappa$ which represents η in $M_{F\times F}$. Then $g_* F \times F = F$. Namely, let $A \in F \times F$ and $Z = g''A$. We have $[id]_{F\times F} \in i_{F\times F}(A)$. But, $i_{F\times F}(g)([id]_{F\times F}) = \eta$, so $\eta \in i_{F\times F}(Z)$, and then, by above $Z \in F$.
Clearly, $[id]_F < \eta$ and we argued that due to $<_{R-K}$, also $\eta \neq [id]_{i_F(F)}$.
So we are done.
□

The theorem has the following somewhat curious corollary:

Corollary 0.6 *Let F be as in the previous theorem. Let P_F be the Prikry forcing with F and $\vec{\xi}$ a Prikry sequence. Then, in $V[\vec{\xi}]$ there is another Prikry sequence $\vec{\eta}$ for F(over V) which is disjoint from $\vec{\xi}$.*

Proof. Let us use g of the theorem to construct $\vec{\eta}$ from $\vec{\xi}$. Set $\eta_n = g(\xi_{2n}, \xi_{2n+1})$, for every $n < \omega$. The properties of g imply that the sequence $\vec{\eta}$ is as desired.
□

Note that the sequence $\vec{\eta}$ is not maximal, i.e. $V[\vec{\eta}] \neq V[\vec{\xi}]$.
Clearly the above situation is impossible once F is normal.

Theorem 0.7 *Let $\langle W_\alpha \mid \alpha < \kappa \rangle$ be a discrete family of $\kappa-$complete ultrafilters over κ and W be a $\kappa-$complete ultrafilters over κ. Let $s, 1 \leq s < \omega$. Assume that $W >_{R-K} W^s_\alpha$, for every $\alpha < \kappa$.*
Let $F = W - \lim \langle W_\alpha \mid \alpha < \kappa \rangle$.
Then there is a function $g : [\kappa]^2 \to [\kappa]^s$ such that

1. *g is different (mod F) from the projections of $F \times F$ to the first and to the second coordinate.*

2. *$g_*F \times F$ is a $\kappa-$complete ultrafilter over $[\kappa]^s$ such that for every $\ell, 1 \leq \ell \leq s$, the $\ell-$th component of $g_*F \times F$, i.e. the projection of $g_*F \times F$ to its $\ell-$th coordinate*

$$\{Z \subseteq \kappa \mid \exists Y \in g_*F \times F(Z = \{\nu_\ell \mid \langle \nu_1, ..., \nu_\ell, ..., \nu_s \rangle \in Y\})\},$$

is equal to F.

Proof. We proceed as in Theorem 0.5. The discreteness of the family $\langle W_\alpha \mid \alpha < \kappa \rangle$ implies that $F \geq_{R-K} W$. Hence $F >_{R-K} W_\alpha^s$, for every $\alpha < \kappa$. Then, in M_W, $i_W(F) >_{R-K} W'^s_{[id]_W}$. Applying σ, we get that $i_F(F) >_{R-K} \sigma(W'^s_{[id]_W}) = (\sigma(W'_{[id]_W}))^s$.
Pick some $h : i_F(\kappa) \to [i_F(\kappa)]^s$ witnessing this.
Now, we form the second ultrapower by taking the ultrapower of M_F by $i_F(F)$. Clearly, $M_{F\times F}$ is this ultrapower and $i_{F\times F} = i_{i_F(F)} \circ i_F$.
Set $\langle \eta_1, ..., \eta_s \rangle = [h]_{i_F(F)}$. Then $i_F(F) >_{R-K} \sigma(W'^s_{[id]_W}) = (\sigma(W'_{[id]_W}))^s$ implies that $i_F(\kappa) \leq \eta_1 < ... < \eta_\ell < ...\eta_s$ and $\eta_\ell \neq [id]_{i_F(F)}$, for every $\ell, 1 \leq \ell \leq s$.
Now, for every $\ell, 1 \leq \ell \leq s$,

$$Z \in F \Leftrightarrow \{\alpha < \kappa \mid Z \in W_\alpha\} \in W \Leftrightarrow i_W(Z) \in W'_{[id]_W}$$

$$\Leftrightarrow \sigma(i_W(Z)) \in \sigma(W'_{[id]_W}) \Leftrightarrow i_F(Z) \in \sigma(W'_{[id]_W}) \Leftrightarrow \eta_\ell \in i_{i_F(F)}(i_F(Z)) \Leftrightarrow \eta_\ell \in i_{F\times F}(Z).$$

Pick a function $g_\ell : [\kappa]^2 \to \kappa$ which represents η_ℓ in $M_{F\times F}$. Then $(g_\ell)_*F \times F = F$. Namely, let $A \in F \times F$ and $Z = g''A$. We have $[id]_{F\times F} \in i_{F\times F}(A)$. But, $i_{F\times F}(g_\ell)([id]_{F\times F}) = \eta_\ell$, so $\eta_\ell \in i_{F\times F}(Z)$, and then, by above $Z \in F$.
Clearly, $[id]_F < \eta_\ell$ and we argued that due to $<_{R-K}$, also $\eta_\ell \neq [id]_{i_F(F)}$.
Set $g = (g_1, ..., g_s)$. Then it is as desired.
□

The theorem has somewhat curious corollaries:

Corollary 0.8 *Let $s, 1 \leq s < \omega$. Then there are $\kappa-$complete ultrafilters F over κ and $\tilde{F}$ over $[\kappa]^s$ such that*

1. *all projections of $\tilde{F}$ to its coordinates are F,*

2. *$F \times F >_{R-K} \tilde{F}$.*

Clearly, if $s > 1$ then $\tilde{F}$ cannot be the product of its coordinates.

Corollary 0.9 *Let F be as in the previous theorem. Let P_F be the Prikry forcing with F and $\vec{\xi}$ a Prikry sequence. Then, in $V[\vec{\xi}]$ there are s pairwise disjoint Prikry sequences $\langle \vec{\eta_\ell} \mid 1 \leq \ell \leq s\rangle$ for F (over V) which are also disjoint from $\vec{\xi}$.*

Proof. Let us use g_ℓ's of the theorem to construct $\vec{\eta_\ell}$ from $\vec{\xi}$. Set $\eta_{\ell n} = g_\ell(\xi_{2n}, \xi_{2n+1})$, for every $n < \omega$. The properties of g_ℓ imply that the sequence $\vec{\eta_\ell}$ is as desired.
□

Let us replace a finite s by an infinite. In order to do so we will need to go beyond just measurability of κ. Consider the case $s = \kappa$, i.e. we aim will be to construct F such that $F \times F$ has $\kappa-$many different projections to F.
A similar argument (with canonical functions) can be used to obtain κ^+-many.
The analog of Corollary 0.9 with $\kappa-$many disjoint Prikry sequences will follow.
It is possible to produce such a model by forcing over a model with $o(\kappa) = \kappa$. Instead, let us make a stronger assumption and proceed without forcing.
Assume, for simplicity GCH. Suppose that there is a (κ, κ^{+3})-extender E with ultrapower closed under $\kappa-$sequences of its elements, i.e.
there is $j : V \to M \simeq \mathrm{Ult}(V, E)$ such that

1. κ is the critical point of j,

2. $M \supseteq V_{\kappa+3}$,

3. ${}^\kappa M \subseteq M$.

For every $\alpha < j(\kappa)$, set

$$E_\alpha = \{Z \subseteq \kappa \mid \alpha \in j(Z)\}.$$

The number of ultrafilter over κ is κ^{++}. So, there is $\mu^* < \kappa^{+3}$ such that for every $\mu, \mu^* \leq \mu < \kappa^{+3}$, the ultrafilter E_μ appears κ^{+3} many times below κ^{+3}.
Pick now an increasing sequence $\langle \mu_\xi \mid \xi < \kappa\rangle$ such that

1. $\mu^* \leq \mu_\xi < \kappa^{+3}$, for every $\xi < \kappa^{+3}$,

2. $E_{\mu_\xi} \neq E_{\mu_\zeta}$, whenever $\xi \neq \zeta$.

Note that the family $\langle E_{\mu_\xi} \mid \xi < \kappa\rangle$ is discrete, since each of E_{μ_ξ}'s is a $P-$point.

There is a set $A = \{\tau_\nu \mid \nu < \kappa \cdot \kappa\} \subseteq [\mu^*, \kappa^{+3})$ of order type $\kappa \cdot \kappa$ such that $E_{\tau_\nu} = E_{\mu_\xi}$, for all $\nu \in [\kappa \cdot \xi, \kappa \cdot \xi + \kappa)$. Using the $\kappa-$closure of M, find $\delta, \sup(A) \leq \delta < \kappa^{+3}$ which codes A, and so, $E_\delta >_{R-K} E_\gamma$, for every $\gamma \in A$.

Now let W be E_δ and $W_\alpha = E_{\mu_\alpha}$, for every $\alpha < \kappa$.
Repeat the argument of Theorem 0.7. We will obtain F over κ and $\tilde{F}$ over $[\kappa]^\kappa$ such that

1. all projections of $\tilde{F}$ to its coordinates are F,
2. $F \times F >_{R-K} \tilde{F}$.

This implies:

Corollary 0.10 *Let P_F be the Prikry forcing with F and $\vec{\xi}$ a Prikry sequence. Then, in $V[\vec{\xi}]$ there are κ pairwise disjoint Prikry sequences $\langle \vec{\eta_\gamma} \mid 1 \leq \gamma \leq \kappa \rangle$ for F (over V) which are also disjoint from $\vec{\xi}$.*

Let us show it is possible to have two disjoint maximal Prikry sequences once a normal measure is replaced by a non-normal.

Theorem 0.11 *Let U be a normal measure over κ and let $P_{U\times U}$ be the Prikry forcing with $U \times U$. Then in $V^{P_{U\times U}}$ there disjoint maximal Prikry sequences for $P_{U\times U}$, i.e. there are sequences $\vec{\eta} = \langle \eta_n \mid n < \omega \rangle$, $\vec{\eta'} = \langle \eta'_n \mid n < \omega \rangle$ such that*

1. *$\{\eta_n \mid n < \omega\} \cap \{\eta_n \mid n < \omega\} = \emptyset$,*
2. *$\vec{\eta}$ is $P_{U\times U}$ generic over V,*
3. *$\vec{\eta'}$ is $P_{U\times U}$ generic over V,*
4. *$V[\vec{\eta}] = V[\vec{\eta'}]$.*

Proof.
Recall that
$$X \in U \times U \Leftrightarrow \{\alpha < \kappa \mid \{\beta < \kappa \mid (\alpha, \beta) \in X\} \in U\} \in U.$$
So,
$$[\kappa]^2 = \{(\alpha, \beta) \mid \alpha < \beta\} \in U \times U.$$
Force with $P_{U\times U}$. Let
$$\vec{\eta} = \langle \eta_n \mid n < \omega \rangle$$
be a generic Prikry sequence.
Assume for simplicity that all its members come from $[\kappa]^2$.

Let for every $n < \omega, \eta_n = (\eta_{n0}, \eta_{n1})$.
Define now a new sequence

$$\vec{\eta}' = \langle \eta'_n \mid n < \omega \rangle$$

as follows:
set $\eta'_n = (\eta_{n1}, \eta_{n+1,0})$, for all $n < \omega$.
Clearly, $V[\vec{\eta}] = V[\vec{\eta}']$ and $\vec{\eta}, \vec{\eta}'$ are disjoint as the sets.

Claim 4 $\vec{\eta}'$ is a Prikry sequence for $P_{U \times U}$ over V.

Proof. Let $A \in U \times U$. We need to show that a final segment of $\vec{\eta}'$ is contained in A. Let $\langle t, T \rangle$ be any condition. Assume for simplicity that t is just empty and $T \subseteq A$.

Consider $U^4 = (U \times U) \times ((U \times U))$. It can be written as $U \times (U \times U) \times U$. Let $\pi_{23} : [\kappa]^4 \to [\kappa]^2$ be the projection to 2,3 coordinates, i.e.

$$\pi_{23}(\alpha, \beta, \gamma, \delta) = (\beta, \gamma).$$

Then π_{23} will project U^4 to $U^2 = U \times U$.
In particular, $B := \pi_{23}{}''A \times A \in U \times U$. So, $C := B \cap A \in U \times U$. Let $D = \pi_{23}^{-1}{}''C$. Then

$$\{(\alpha, \beta) \in [\kappa]^2 \mid \{(\gamma, \delta) \in [\kappa]^2 \mid (\alpha, \beta, \gamma, \delta) \in D\} \in U \times U\} \in U \times U.$$

Set

$$X = \{(\alpha, \beta) \in [\kappa]^2 \mid \{(\gamma, \delta) \in [\kappa]^2 \mid (\alpha, \beta, \gamma, \delta) \in D\} \in U \times U\}$$

and

$$Y_{(\alpha,\beta)} = \{(\gamma, \delta) \in [\kappa]^2 \mid (\alpha, \beta, \gamma, \delta) \in D\},$$

for every $(\alpha, \beta) \in X$. Consider

$$Y = \Delta^*_{(\alpha,\beta)\in X} Y_{(\alpha,\beta)} = \{(\gamma, \delta) \in [\kappa]^2 \mid \forall (\alpha, \beta) \in X(\beta < \gamma \to (\gamma, \delta) \in Y_{(\alpha,\beta)})\}.$$

Then $Y \in U \times U$, since in the ultrapower by $U \times U$ we have

$$(\kappa, \kappa_1) \in i_{U \times U}(Y)_{(\alpha,\beta)},$$

for every $(\alpha, \beta) \in i_{U \times U}(X)$ with $\beta < \kappa$, where $\kappa_1 = i_U(\kappa)$. Hence,

$$(\kappa, \kappa_1) \in i_{U \times U}(Y).$$

Take finally $Z := X \cap Y \cap C$.
Then the condition $\langle \langle \rangle, Z \rangle$ will force that $\vec{\eta}'$ will be contained in A.

$\square$ of the claim.

$\square$

Note that once $F = \mathcal{V} \times \mathcal{U}$ and $\mathcal{V} \leq_{R-K} \mathcal{U}$, then it is easy to produce g that satisfies the conclusion of 0.5.

Namely, let s be a projection of $\mathcal{U}$ on $\mathcal{V}$.

Define $g : [\kappa \times \kappa]^2 \to \kappa \times \kappa$ as follows:

$$g((\alpha, \beta), (\gamma, \delta)) = (s(\beta), \delta).$$

We would like to argue that this is basically the only possibility provided the set $\{o(\alpha) \mid \alpha < \kappa\}$ is bounded in κ in the core model.

Start with the following observation:

Theorem 0.12 *Assume that κ is a measurable cardinal and the set $\{o(\alpha) \mid \alpha < \kappa\}$ is bounded in κ in the core model. Let U be a $\kappa-$complete ultrafilter over κ. Then the number of Rudin-Keisler non-equivalent ultrafilters which are $\leq_{R-K} U$ is strictly less than κ.*

Proof. Denote the core model by $\mathcal{K}$. Consider $j := i_U \restriction \mathcal{K}$. Then, by Mitchell [5], j is an iterated ultrapower of $\mathcal{K}$ by its measures. The number of generators[1] of j is less than κ, since the set $\{o(\alpha) \mid \alpha < \kappa\}$ is bounded in κ in the core model, every generator is a critical point of one of the embeddings forming j and ${}^{\kappa}M_U \subseteq M_U$.

Denote the set of generators of j by $Gen(j)$.

Now suppose that $\langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of pairwise different $\kappa-$complete ultrafilters over κ which are $\leq_{R-K} U$.

Then, for every $\alpha < \kappa$ there is $\rho_\alpha, \kappa \leq \rho_\alpha < j(\kappa)$, such that

$$U_\alpha = \{X \subseteq \kappa \mid \rho_\alpha \in i_U(X)\}.$$

Now, the number of generators is less than κ, so all but less than $\kappa-$many ρ_α's are not generators. Suppose for simplicity that non of them is a generator.

Then, for every $\alpha < \kappa$ there is $\vec{\eta}_\alpha \in [Gen(j) \cap \rho_\alpha]^{<\omega}$ and a function $f_\alpha \in \mathcal{K}$ such that

$$\rho_\alpha = j(f_\alpha)(\vec{\eta}_\alpha).$$

Assume that $\vec{\eta}_\alpha$ is such smallest possible set of generators.

Note that due to the smallness of $\vec{\eta}_\alpha$, the function f_α can be picked to be one to one, since

[1] an ordinal $\eta, \kappa \leq \eta < j(\kappa)$, is called a generator of j iff for every $n < \omega, f : [\kappa]^n \to \kappa$ in $\mathcal{K}$ and $a \in [\eta]^n$, $j(f)(a) \neq \eta$.

in $\mathcal{K}$, the ultrafilters

$$\{Y \subseteq \kappa \mid Y \in \mathcal{K} \text{ and } \vec{\eta}_\alpha \in j(Y)\}$$

and

$$\{Z \subseteq \kappa \mid Z \in \mathcal{K} \text{ and } \rho_\alpha \in j(Z)\}$$

have the same ultrapower. Then U_α will be Rudin-Keisler equivalent to

$$W_{\vec{\eta}_\alpha} := \{X \mid \vec{\eta}_\alpha \in i_U(X)\},$$

as witnessed by f_α.
Again, all but less that $\kappa-$many $\vec{\eta}_\alpha$'s, and so $W_{\vec{\eta}_\alpha}$, are the same.
Hence, all but less that $\kappa-$many U_α's will be Rudin-Keisler equivalent.
□

Theorem 0.13 *Assume that κ is a measurable cardinal and the set $\{o(\alpha) \mid \alpha < \kappa\}$ is bounded in κ in the core model. Let F, W be $\kappa-$ complete ultrafilters over κ such that $g_* F \times W >_{R-K} F$ for some function $g : [\kappa]^2 \to \kappa$ which is different (mod $F \times W$) from the projections of $F \times W$ to the first coordinate. Assume in addition that if $W \geq_{R-K} F$ then g is different (mod $F \times W$) from any projection which witnesses this.*
Then there are $\kappa-$complete ultrafilters W', $\mathcal{V}$ and $\{\mathcal{U}_\alpha \mid \alpha < \kappa\}$ such that

1. *$W' \leq_{R-K} W$,*
2. *$\mathcal{U}_\alpha =_{R-K} W'$, for every $\alpha < \kappa$,*
3. *$\mathcal{V} \leq_{R-K} F$,*
4. *$F =_{R-K} \mathcal{V} \times W'$,*
5. *$F = \mathcal{V} - \lim \langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle$.*

Proof. Let $g : [\kappa]^2 \to \kappa$ be such projection. Let $\rho = [g]_{F \times W}$. Set, in M_F,

$$\mathcal{U} = \{X \subseteq i_F(\kappa) \mid \rho \in i_{i_F(W)}(X)\}.$$

Then $\mathcal{U} \supseteq i''_F F$. Let the sequence $\langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle$ be a sequence of $\kappa-$complete ultrafilters over κ that represents $\mathcal{U}$ in M_F, i.e.

$$i_F(\langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle)([id]_F) = \mathcal{U}.$$

We have then that

$$F = F - \lim \langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle,$$

since

$$X \in F - \lim \langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle \Leftrightarrow \{\alpha < \kappa \mid X \in \mathcal{U}_\alpha\} \in F \Leftrightarrow i_F(X) \in \mathcal{U} \Leftrightarrow$$

$$\rho \in i_{i_F(W)}(X) \Leftrightarrow [g]_{F \times W} \in i_{F \times W}(X) \Leftrightarrow X \in F.$$

Note that in M_F, $\mathcal{U} \leq_{R-K} i_F(W)$, hence, by elementarity, $\mathcal{U}_\alpha \leq_{R-K} W$ for almost all α's mod F. Assume for simplicity that this is true for every $\alpha < \kappa$.
By 0.12, then the number of Rudin-Keisler non-equivalent ultrafilters among $\mathcal{U}_\alpha$'s is strictly less than κ. So, there is $A \in F$ such that for every $\alpha, \beta \in A$, $\mathcal{U}_\alpha =_{R-K} \mathcal{U}_\beta$.
Let W' be such that $\mathcal{U}_\alpha =_{R-K} W'$, for every $\alpha \in A$.
Let us get rid now from same ultrafilters.
For $\alpha, \beta \in A$, set $\alpha \sim \beta$ iff $\mathcal{U}_\alpha = \mathcal{U}_\beta$. Let t be a function that picks one member from each equivalence class.
If $|\text{rng}(t)| < \kappa$, then there is $\alpha^* \in A$ such that for almost all α mod F, $\mathcal{U}_\alpha = \mathcal{U}_{\alpha^*}$. Then $F = F - lim \langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle$ will imply $F = \mathcal{U}_{\alpha^*}$. Also, in M_F, $i_F(F)$ will be $\mathcal{U}$. Recall that $\mathcal{U}_{\alpha^*} \leq_{R-K} W$. So, $F \leq_{R-K} W$. Then, as in M_F, $i_F(F)$ will be $\mathcal{U}$, g will be a projection of W to F. Which contradicts to the assumption of the theorem.

So, $|\text{rng}(t)| = \kappa$.

Set $\mathcal{V} = t_* F$. Then $\mathcal{V}$ be $\kappa-$complete non-trivial ultrafilter over κ, $\mathcal{V} \leq_{R-K} F$ and

$$F = \mathcal{V} - \lim \langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle.$$

Now, in M_F,

$$i_F(W') =_{R-K} \mathcal{U} \leq_{R-K} i_F(W).$$

Hence, $W' \leq_{R-K} W$.
Finally, applying separation, which holds under (anti) large cardinals assumptions made by [4], to $\mathcal{V}$ and $\langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle$ and using $F = \mathcal{V} - \lim \langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle$ it is not hard to see that

$$\text{Ult}(V, F) = M_F = \text{Ult}(M_{\mathcal{V}}, i_{\mathcal{V}}(\langle \mathcal{U}_\alpha \mid \alpha < \kappa \rangle)([id]_{\mathcal{V}})) = \text{Ult}(M_{\mathcal{V}}, i_{\mathcal{V}}(W')).$$

Hence, $F =_{R-K} \mathcal{V} \times W'$.
□

Remark 0.14 Note that, as in [3], starting with a measurable κ such that the set $\{o(\alpha) \mid \alpha < \kappa\}$ is unbounded in it, it is possible to construct a model with $\kappa-$complete ultrafilters

W, $\langle W_\alpha \mid \alpha < \kappa \rangle$ as in 0.5 and in addition a sequence $\langle W_\alpha \mid \alpha < \kappa \rangle$ is Rudin-Keisler increasing, or alternatively, it can be made of normal ultrafilters. In this type of situations the conclusion of 0.13 will be wrong.

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