On spectrum of strongly uniform ultrafilters over a singular cardinal.

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Abstract

We construct a model with $\mathrm{sp}^{str}(\aleph_\omega) = \{\aleph_{\omega+2}\}$ and $\aleph_\omega$ strong limit.

1 Some basic definitions.

Let κ be a strong limit singular cardinal and D be an ultrafilter over κ.
D is called *uniform* iff for every $A \in D$, $|A| = \kappa$.
D is called *strongly uniform* iff there exists an increasing sequence $\vec{\tau} = \langle \tau_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ such that
for every $A \in D$, the set $\{\alpha < \mathrm{cof}(\kappa) \mid |A \cap \tau_\alpha| = \tau_\alpha\}$ is unbounded in $\mathrm{cof}(\kappa)$.
Note that sets $\{\alpha < \mathrm{cof}(\kappa) \mid |A \cap \tau_\alpha| = \tau_\alpha\}$, with $A \in D$ generate a uniform ultrafilter over $\mathrm{cof}(\kappa)$ which is Rudin-Keisler below D.

A subset W of D is called *a bases* of D iff for every $A \in D$ there is $B \in W$ such that $B \subseteq^* A$, i.e. $|B \setminus A| < \kappa$.
$\mathrm{ch}(D) = \min(\{|W| \mid W \text{ is a basis of } D\})$.

$\mathrm{sp}(\kappa) = \{\mathrm{ch}(D) \mid D \text{ is a uniform ultrafilter over } \kappa\}$.
$\mathrm{sp}^{str}(\kappa) = \{\mathrm{ch}(D) \mid D \text{ is a strongly uniform ultrafilter over } \kappa\}$.

$\mathfrak{u}(\kappa) = \min(\{\mathrm{ch}(D) \mid D \text{ is a uniform ultrafilter over } \kappa\})$.
$\mathfrak{u}^{str}(\kappa) = \min(\{\mathrm{ch}(D) \mid D \text{ is a strongly uniform ultrafilter over } \kappa\})$.

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2 Main construction.

Our aim will be to construct a model in which $\aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $\mathfrak{u}^{str}(\aleph_\omega) = \aleph_{\omega+2}$. Hence, $\mathrm{sp}^{str}(\aleph_\omega) = \{\aleph_{\omega+2}\}$ in this model.
Actually more information on uniform ultrafilters over $\aleph_\omega$ will be given.

Assume GCH.

Suppose that E is a $(\kappa, \kappa^{++})-$extender over κ.

In [6], using this type of assumption, models of $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $2^{\aleph_n} = \aleph_{a(n)}, n < \omega$, where $a : \omega \to \omega$, $a(n) > n, m \leq n \to a(n) \leq a(m)$, were constructed.

We will use a particular model of this type here.

Let us give the description of cardinals and the power function of the model used.

Let $\langle \kappa_n \mid n < \omega \rangle$ denote the Prikry sequence of the normal measure of the extender E with $\kappa_0 = \aleph_0$.

The cardinal structure:

κ_0	κ_0^+	κ_0^{++}	κ_0^{+3}	κ_1	κ_1^+	κ_1^{++}	κ_1^{+3}	κ_2	κ_2^+	κ_2^{++}	κ_2^{+3}	κ_3	κ_3^+	κ_3^{++}	$\kappa_3^{+3}\ldots$
$\aleph_0$		$\aleph_1$	$\aleph_2$	$\aleph_3$		$\aleph_4$	$\aleph_5$	$\aleph_6$		$\aleph_7$	$\aleph_8$	$\aleph_9$		$\aleph_{10}$	$\aleph_{11}\ldots.$

I.e. cardinals of the form κ_n^+ are collapsed to κ_n's and cardinals of intervals $(\kappa_n^{+3}, \kappa_{n+1})$ are collapsed to κ_n^{+3}. In particular κ turnes into $\aleph_\omega$.

Now the power function that will be used:

$2^\kappa = \kappa^{++}$, GCH above κ^+, for every $n < \omega$ the following holds:

1. $2^{\kappa_n} = \kappa_n^{++}$,[1] i.e. GCH at κ_n, since κ_n^+ is collapsed,

2. $2^{\kappa_n^{++}} = \kappa_n^{+3}$, again GCH at the successor of κ_n,

3. $2^{\kappa_n^{+3}} = \kappa_{n+1}^{++}$, i.e. GCH fails with a gap 2 here.

Blowing up powers are achieved by adding the corresponding Cohen functions.

We have the following cardinal arithmetic structure in this model:

$$\mathrm{tcf}(\prod_{n<\omega}(\kappa_n^{++}, <_{J^{bd}}) = \kappa^{++} = \aleph_{\omega+2}.$$

The rest relevant products correspond to $\kappa^+ = \aleph_{\omega+1}$.

Let us turn to the analysis of strongly uniform ultrafilters over $\kappa = \aleph_\omega$ in this model.
We proceed in a slightly more general setting. Let D be a uniform ultrafilter on κ.

[1] κ_n^{+k} here and throughout denotes the $k-$successor as computed in V and not in the extension used.

Suppose that here is $f:\omega\to\omega$ such that for every $A'\in D$ there is $A\in D, A\subseteq A'$ such that the set

$$\{n<\omega \mid |A\cap[\kappa_n,\kappa_{n+1}]|=\kappa_{f(n)}\}$$

is infinite.

Assume first that:

for every $A\in D$, the set

$$\{n<\omega \mid |A\cap[\kappa_n,\kappa_{n+1}]|\geq\kappa_n\}$$

is infinite, in particular, for infinitely many $n<\omega, f(n)\geq n$. Recall that the only cardinals in the interval $[\kappa_n,\kappa_{n+1}]$ are (in the extension!) $\kappa_n,\kappa_n^{++},\kappa_n^{+3},\kappa_{n+1}$. D is an ultrafilter, hence there is $k\in\{0,2,3\}$ such that

for every $A'\in D$ there is $A\in D, A\subseteq A'$ such that the set

$$\{n<\omega \mid |A\cap[\kappa_n,\kappa_{n+1}]|=\kappa_n^{+k}\}$$

is infinite.[2]

Namely,
let $A\in D$ and $X_A=\{n<\omega \mid |A\cap(\kappa_n,\kappa_{n+1})|\geq\kappa_n\}$. For every $n\in X_A$ let $k_n\in\{0,2,3\}$ be such that $|A\cap(\kappa_n,\kappa_{n+1})|=\kappa_n^{+k_n}$.
Set

$$A_k=\bigcup\{A\cap(\kappa_n,\kappa_{n+1}) \mid k_n=k\},$$

for every $k\in\{0,2,3\}$.
Clearly, $A=\bigcup_{k\in\{0,2,3\}}A_k$. Hence there is $k^A\in 0,2,3$ such that $A_{k^A}\in D$.

Pick now $A\in D$ with k^A as small as possible. Then for every $B\in D$, $k^{A\cap B}=k^A$.
Denote such k^A by k^*.
Let $h_n:\sup(A_{k^*}\cap\kappa_{n+1})\leftrightarrow\kappa_n^{+k^*}$, for every $n<\omega$ such that $k_n=k^*$.
Now move D to an isomorphic ultrafilter D' generated by

$$\{\bigcup\{h_n''(B\cap A_{k^*}\cap(\kappa_n,\kappa_{n+1})) \mid k_n=k^*\} \mid B\in D\}.$$

Clearly, $\mathrm{ch}(D')=\mathrm{ch}(D)$. So we can just replace D by D'.

Split now the argument into three cases according to the value of k.

Case 1 *For every $A\in D$ the set $\{n<\omega \mid A$ is unbounded in $\kappa_n^{++}\}$ is infinite.*

The following general proposition that applies to the present case was proved in [4].

[2]The case $|A\cap[\kappa_n,\kappa_{n+1}]|=\kappa_{n+1}$ can be dropped, since its treatment the same as of the case $|A\cap[\kappa_n,\kappa_{n+1}]|=\kappa_n$.

Proposition 2.1 *Suppose that κ is a singular cardinal of cofinality η. Let D be an uniform ultrafilter over κ. Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to κ. Suppose that δ is a regular cardinal such that*

1. $\kappa < \delta \leq 2^\kappa$
2. *there is an increasing sequence of regular cardinals $\langle \delta_\alpha \mid \alpha < \eta \rangle$ such that*
   (a) $\kappa_\alpha < \delta_\alpha \leq \kappa_{\alpha+1} < \delta_{\alpha+1}$, *for every $\alpha < \eta$,*
   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$, *for some filter F on η which extends the filter of co-bounded subsets of η,*
   (c) *for every $A \in D$, the set $\{\alpha < \eta \mid |A \cap \delta_\alpha| = \delta_\alpha\} \in F$*

Then $\mathrm{ch}(D) \geq \delta$.

Now, in the present situation, we have $\mathrm{tcf}(\prod_{n<\omega} \kappa_n^{++}, <_{J^{bd}}) = \kappa^{++}$ and $2^\kappa = \kappa^{++}$. So, by Proposition 2.1, $\mathrm{ch}(D) = \kappa^{++}$.
$\square$ of Case 1.

Case 2. *For every $A \in D$ the set $\{n < \omega \mid A$ is unbounded in $\kappa_n\}$ is infinite.*
Let $\mathcal{W}$ be a generating family for D of cardinality $\aleph_{\omega+1}$.
We will rule this out using the following general proposition from [4]:

Proposition 2.2 *Suppose that κ is a singular cardinal of cofinality η and D is an uniform ultrafilter over κ.*
Let $\langle \kappa_\alpha \mid \alpha < \eta \rangle$ be an increasing sequence of cardinals converging to κ.
Suppose that δ is a regular cardinal such that

1. $\kappa < \delta \leq 2^\kappa$
2. *there is an increasing sequences of regular cardinals $\langle \tau_\alpha \mid \alpha < \eta \rangle$ such that*
   (a) $\kappa_\alpha \leq \tau_\alpha < 2^{\tau_\alpha} < \kappa_{\alpha+1}$, *for every $\alpha < \eta$,*
   (b) $\mathrm{tcf}(\prod_{\alpha<\eta} \delta_\alpha, <_F) = \delta$, *where $\delta_\alpha = 2^{\tau_\alpha}$ and F is an ultrafilter on η which extends the filter of co-bounded subsets of η,*
   (c) $\mathfrak{r}(\tau_\alpha) = \delta_\alpha$ *(non-splitting number), i.e. whenever $S \subseteq [\tau_\alpha]^{\tau_\alpha}$ of cardinality $< \delta_\alpha$, then there is $a \in [\tau_\alpha]^{\tau_\alpha}$ such that for every $s \in S$, $|a \cap s| = |(\tau_\alpha \setminus a) \cap s| = \tau_\alpha$. In particular, if $2^{\tau_\alpha} = \tau_\alpha^+$, then $\mathfrak{r}(\tau_\alpha) = \tau_\alpha^+ = \delta_\alpha$.*

(d) *For every $A \in D$, the set $\{\alpha < \eta \mid |A \cap \tau_\alpha| = \tau_\alpha\} \in F$*

Then $\mathrm{ch}(D) \geq \delta$.

Let us take $\eta = \omega$, $\tau_i = \kappa_i$, for every $i < \eta$. We have $\mathrm{tcf}(\prod_{n<\omega}(\kappa_n^{++}, <_{J^{bd}}) = \kappa^{++} = 2^\kappa = \aleph_{\omega+2}$ and $2^{\kappa_i} = \kappa_i^{++}$, for every $i < \omega$.
Recall that we have GCH at κ_i, and so, $\mathfrak{r}(\kappa_i) = 2^{\kappa_i} = \kappa_i^{+2}$, for every $i < \omega$. Hence, the proposition applies and we obtain that $\mathrm{ch}(D) = 2^\kappa$.
$\square$ of Case 2.

Case 3. *For every $A \in D$ the set $\{n < \omega \mid A$ is unbounded in $\kappa_n^{+3}\}$ is infinite.*
Let $\mathcal{W}$ be a generating family for D of cardinality $\aleph_{\omega+1}$.
We will rule this out as in the previous case, using the general proposition 2.2 from [4].
Let us take $\eta = \omega$, $\tau_i = \kappa_i^{+3}$, for every $i < \eta$. We have $\mathrm{tcf}(\prod_{n<\omega}(\kappa_n^{++}, <_{J^{bd}}) = \kappa^{++} = 2^\kappa = \aleph_{\omega+2}$ and $2^{\kappa_i^{+3}} = \kappa_{i+1}^{++}$, for every $i < \omega$.
Recall also, that the power of κ_i^{+3} was blown up using Cohen subsets. So, $\mathfrak{r}(\kappa_i^{+3}) = 2^{\kappa_i^{+3}} = \kappa_{i+1}^{++}$, for every $i < \omega$. Hence, the proposition applies and we obtain that $\mathrm{ch}(D) = 2^\kappa$.
$\square$ of Case 3.

Suppose finally that

there is $A \in D$ such that the set $\{n < \omega \mid |A \cap (\kappa_n, \kappa_{n+1})| < \kappa_n\}$ is co-finite,

i.e. $f(n) < n$, for all but finitely many n's.
Then, for every $m \in \mathrm{rng}(f)$, split (κ_{m-1}, κ_m) into ω many sets $\langle I_{mn} \mid n < \omega\rangle$ each of cardinality κ_m.
Pick $h_n : A \cap (\kappa_n, \kappa_{n+1}) \leftrightarrow I_{f(n)n}$. Use h_n's in the obvious fashion in order to move D to an isomorphic ultrafilter D'. Then, D' falls under one of the cases considered above.

3 Some open problems.

Let κ be singular strong limit cardinal and $2^\kappa > \kappa^+$.

It was shown above that it is possible to have $\mathfrak{u}^{str}(\kappa) = 2^\kappa$. The following remains open:

Question 1. Is it possible to have $\mathfrak{u}(\kappa) = 2^\kappa$ or even $\mathfrak{u}(\kappa) > \kappa^+$?

Question 2. What is $\mathfrak{u}(\kappa)$ in the model of Section 2?

We think that $\mathfrak{u}(\kappa) = \kappa^+$ there.

GCH breaks down below $\aleph_\omega$ in our model. So it is natural to ask the following:

Question 3. Is it possible to have GCH below $\aleph_\omega$ and $\mathfrak{u}^{str}(\aleph_\omega) > \aleph_{\omega+1}$?

The relation of almost inclusion $\subseteq^*$ was used in the definition of a basis and $\mathrm{ch}(D)$. In case of a regular cardinal (with $\kappa^{<\kappa} = \kappa$) it is possible to replace $\subseteq^*$ by $\subseteq$.

Question 4. Can $\subseteq^*$ be replaced by $\subseteq$ for singular?

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