Two Stationary Sets with Different Gaps of the Power Function

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Abstract

Starting with a strong cardinal a model with a cardinal κ of cofinality $\aleph_1$ such that both sets $\{\alpha < \kappa \mid 2^\alpha = \alpha^{++}\}$ and $\{\alpha < \kappa \mid 2^\alpha = \alpha^{+++}\}$ are stationary is constructed. Based on this construction and using supercompacts a model in which both sets $\{\alpha < \kappa \mid 2^\alpha = \alpha^{+}\}$ and $\{\alpha < \kappa \mid 2^\alpha = \alpha^{++}\}$ are stationary is obtained.

0 Introduction

The classical theorem of Silver states that if κ is a singular cardinal of uncountable cofinality and $2^\kappa > \kappa^+$, then the set $\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$ contains a club. But what if $2^\kappa = \kappa^+$, can both sets

$$\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$$

and

$$\{\alpha < \kappa \mid 2^\alpha > \alpha^+\}$$

be stationary?

The purpose of the present paper is to provide an affirmative answer. The construction uses supercompact cardinals. By [4], at least Woodin cardinals are necessary.

The main issue will be to construct a model with a cardinal κ of cofinality $\aleph_1$ such that both sets $\{\alpha < \kappa \mid 2^\alpha = \alpha^{++}\}$ and $\{\alpha < \kappa \mid 2^\alpha = \alpha^{+++}\}$ are stationary. We start from a regular cardinal κ having a coherent sequence of $\kappa^{+\omega+3}$ extenders of the length $\aleph_1$. A variation of the extender based Magidor forcing (see [5], [6]) is used to change its cofinality

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to $\aleph_1$ blowing up powers of cardinals over the generic Magidor sequence below. The point will be to arrange a different behaviour on stationary sets. For this a variation of the short extenders forcing for $\kappa^{+\omega+3}$ will be used. Finally, for a model with $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ and $\{\alpha < \kappa \mid 2^\alpha = \alpha^{++}\}$ both stationary, we plugin supercompacts cardinals into the previous construction and use them to collapse α^+ to α by changing its cofinality to ω, with α's in the generic Magidor sequence for κ.

1 Preliminary Settings

1.1 Cofinal sequences and stationary set

Let us attach to every δ, $0 < \delta < \omega_1$, a successor ordinal $\delta^* < \delta$ so that for every successor ordinal $\tau < \omega_1$ the set of δ's with $\tau = \delta^*$ is stationary. Clearly, the set

$$\begin{aligned} C &= \{\mu < \omega_1 | \mu \text{ is limit ordinal and for every successor } \tau < \mu \\ &\quad \text{the set of } \delta\text{'s below } \mu \text{ with } \delta^* = \tau \text{ is unbounded in } \mu\} \end{aligned}$$

is closed unbounded.

Let S be a subset of $\lim(\lim(C))$. It can be nonstationary, but in the interesting cases S will be stationary, costationary. For every $\mu \in C$ fix a cofinal sequence $\langle \mu'_n | n < \omega \rangle$. Let $\mu \in C$. We define another cofinal sequence $\langle \mu_n | n < \omega \rangle$ as follows:

$$\mu_0 = 0, \quad \mu_{n+1} = \min\{\delta < \mu \mid \delta \text{ limit}, \quad \delta \notin S, \quad \delta^* = \mu_n + 1 \quad \text{and} \quad \delta \geq \mu'_{n+1}\} \ .$$

The advantage of using such improved cofinal sequences is that once we have μ_n then $\langle \mu_k | k \leq n \rangle$ is uniquely determined without need in μ. Thus $\mu_{n-1} = \mu^*_n - 1$, $\mu_{n-2} = \mu^*_{n-1} - 1$, etc.

1.2 The extenders sequence

We assume that the ground model satisfies GCH and has a coherent sequence

$$\vec{E} = \langle E(\alpha, \beta) \mid \alpha \leq \kappa, \quad \alpha \in \operatorname{dom} \vec{E}, \quad \beta < \omega_1 \rangle$$

such that for every $\alpha \leq \kappa$ and $\beta < \omega_1$ the following hold:

(a) $E(\alpha, \beta)$ is an $(\alpha, \alpha^{+\omega+3})$-extender over α.

(b) (coherence)

$$j_{E(\alpha,\beta)}(\vec{E}) \restriction (\alpha+1) = \langle E(\alpha',\beta') \mid (\alpha' < \alpha) \quad \text{or} \quad (\alpha' = \alpha \quad \text{and} \quad \beta' < \beta)\rangle \ ,$$

where $j_{E(\alpha,\beta)} : V \to M \simeq \mathrm{Ult}(V, E(\alpha,\beta))$ is the elementary embedding by $E(\alpha,\beta)$

(c) there are disjoint subsets $\langle E_{\alpha,i} \mid i < \omega_1\rangle$ of α such that $E_{\alpha,\beta}$ belongs to the normal measure $E(\alpha,\beta)(\alpha)$ of $E(\alpha,\beta)$ and for every $\gamma \leq \alpha$, $\gamma \in \mathrm{dom}\ \vec{E}$, $i < \omega_1$

$$E_{\alpha i} \cap \gamma = E_{\gamma i} \ ,$$

where for $\tau < \alpha^{+\omega+3}$ the τ-th measure $E(\alpha,\beta)(\tau)$ of $E(\alpha,\beta)$ is the set $\{X \subseteq \alpha \mid \tau \in j_{E(\alpha,\beta)}(X)\}$.

(d) $\vec{E} \restriction (\alpha,\beta) = j_{E(\alpha,\beta)}(f)(\alpha)$ for some $f \in {}^{\alpha}V_{\alpha}$, i.e., essentially it depends on the normal measure $E(\alpha,\beta)(\alpha)$, where

$$\vec{E} \restriction (\alpha,\beta) = \langle \vec{E}(\alpha',\beta') \mid (\alpha' < \alpha) \quad \text{or} \quad (\alpha' = \alpha \quad \text{and} \quad \beta' < \beta)\rangle \ .$$

1.3 Types

Let χ be a regular cardinal large enough (thus $\kappa^{+\omega+4}$ will do it). For $k \leq \omega$ we consider a structure $\mathfrak{a}_k = \langle L_{\chi^{+k}}[\vec{E}], \epsilon, \vec{E},$ the enumeration of $[\kappa^{+\omega+3}]^{\leq\kappa}$, and of $[\kappa^{+\omega+2}]^{\leq\kappa}$,

$$\langle \chi^{+n} \mid n \leq k\rangle, \kappa, 0, 1, \ldots, \alpha \ldots \mid \alpha < \kappa^{+k}\rangle$$

in an appropriate language which we denote $\mathcal{L}_k$.

For an ordinal $\xi < \chi$ (usually ξ will be below $\kappa^{+\omega+3}$) we denote by $tp_k(\xi)$ the $\mathcal{L}_k$-type realized by ξ in $\mathfrak{a}_k$.

Let $\mathcal{L}'_k$ be the language obtained from $\mathcal{L}_k$ by adding a new constant c'. For $\delta < \chi$ let $\mathfrak{a}_{k,\delta}$ be the $\mathfrak{L}'_k$-structure obtained from $\mathfrak{a}_k$ by interpreting c' as δ. The type $tp_k(\delta,\xi)$ is the $\mathcal{L}'_k$-type realized by ξ in $\mathfrak{a}_{k,\delta}$. Further, we shall identify types with ordinals corresponding to them in some fixed well ordering of $\mathcal{P}(\kappa^{+k})$.

1.3.1 Definition

Let $k \leq \omega$ and $\beta < \kappa^{+\omega+3}$ (or $\beta < \kappa^{+\omega+2}$). β is called k-good iff

1. for every $\gamma < \beta$ $tp_k(\gamma,\beta)$ is realized unboundedly often below $\kappa^{+\omega+3}$ (or respectively, $\kappa^{+\omega+2}$);
2. for every bounded $a \subseteq \beta$ of cardinality $\leq \kappa$ there is $\alpha < \beta$ corresponding to a in the enumeration of $[\kappa^{+\omega+3}]^{\leq\kappa}$ (or respectively $[\kappa^{+\omega+2}]^{\leq\kappa}$).

The next two lemmas are proved in [1].

1.3.2 Lemma

The set $\{\beta < \kappa^{+\omega+i} \mid \beta$ *is* ω*-good*$\}$ *contains a club, for every* $i < 2$.

1.3.3 Lemma

Let $0 < k \leq \omega$ *and* β *be* k*-good. Then there are arbitrarily large* $k-1$*-good ordinals below* β.

Let now $\alpha < \omega_1$. For each $k \leq \omega$ and $i < \omega_1$ k-good types are defined in the ultrapower by $E(\kappa, i)$ in the same way. Only $\vec{E}$ should be replaced by $\vec{E} \restriction (\kappa, i)$. Note that a k-good ordinal will be such in the ultrapower by $E(\kappa, i)$, since $L_{\chi^{+k}}[\vec{E} \restriction (\kappa, i)]$ is definable in $L_{\chi^{+k}}[\vec{E}]$.

1.4 Adding Box Sequences

As in [3], we add generic box sequences over κ and below. A variation of Jensen's forcing is used for this purpose. We force a club into $\kappa^{+\omega+3}$ and a box sequence on it simultaneously.

1.4.1 Definition

$p = \langle c, \langle c_\alpha \mid \alpha \in \lim c \rangle\rangle \in \mathrm{Box}'(\kappa^{+\omega+3})$ iff

1. $c \subseteq \kappa^{+\omega+3}$ is closed of cardinality $\kappa^{\omega+2}$.

2. for every $\alpha \in \lim(c)$ the following holds:

   (a) $c_\alpha \subseteq \alpha \cap c$ is a club

   (b) if β is a limit point of c_α then $c_\beta = c_\alpha \cap \beta$

   (c) *otp* $c_\alpha \leq \kappa^{+\omega+2}$ and if $cf\alpha < \kappa^{+\omega+2}$ then *otp* $c_\alpha < \kappa^{+\omega+2}$

   (d) if β is a successor point of c_α then $cf\beta = \kappa^{+\omega+2}$.

$Box'(\kappa^{+\omega+3})$ is ordered by end-extension.

1.4.2 Definition

The set $\mathcal{P}'(1)$ consists of pairs $\langle A^{00}, A^{10}\rangle$ such that

1. $A^{00} \prec \langle H(\kappa^{+\omega+8}), <, \cdots\rangle$, $|A^{00}| = \kappa^{+\omega+2}$, ${}^{\kappa^{+\omega+1}\geq}(A^{00}) \subseteq A^{00}$

2. A^{10} is an elementary chain of elementary submodels of A^{00} which are closed under $\kappa^{+\omega+1}$-sequences and $A^{00} \in A^{10}$.

1.4.3 Definition

Let $\langle A^{00}, A^{10}\rangle$, $\langle B^{00}, B^{10}\rangle \in \mathcal{P}'(1)$. Then $\langle A^{00}, A^{10}\rangle \geq \langle B^{00}, B^{10}\rangle$ iff A^{10} is an end-extension of B^{10}.

$\mathcal{P}'(1)$ is the first stage of the forcing used in [2]. As in [3], we combine Box$'$ with $\mathcal{P}'$.

1.4.4 Definition

$\mathcal{P}''(1)$ consists of $\langle\langle A^{00}, A^{10}\rangle, \langle c_\alpha \mid \alpha \in \lim(\{B \cap \kappa^{+\omega+3} \mid B \in A^{10}\})\rangle$ such that

1. $\langle A^{00}, A^{10}\rangle \in \mathcal{P}'(1)$
2. $\langle c_\alpha \mid \alpha \in \lim(\{B \cap \kappa^{+\omega+3} \mid B \in A^{10}\})\rangle \in \mathrm{Box}'(\kappa^{+\omega+2})$.

The ordering on $\mathcal{P}''(1)$ is defined in the obvious fashion.

As in [3], $\langle \mathcal{P}''(1), \leq\ \rangle$ will be $\kappa^{+\omega+2}+1$-strategically closed. Force in Backward Easton fashion with analogs of $\langle \mathcal{P}''(1), \leq\ \rangle$ below κ and then over κ. Let $\langle \mathcal{P}, \leq\ \rangle$ denote this forcing. Take G_1 to be a generic subset of $\mathcal{P}$ and denote by $G_{1,<\alpha}$, $G_{1,\alpha}$, $G_{1,>\alpha}$ corresponding parts of G_1 below α, over α and above α respectively, where $\alpha \leq \kappa$.

2 The Main Preparation Forcing

The next definition combines the Extender Based Magidor forcing with one of [3, Sec. 5].

2.0.5 Definition

The set $\mathcal{P}^*$ consists of sequences of the form $\langle\langle\gamma, p^{\gamma,i}\rangle \mid \gamma \in s_i, i < \omega_1\rangle, \langle A_i, a_i \mid i < \omega_1\rangle\rangle$ satisfying the following conditions:

1. $\langle\langle\langle\gamma, p^{\gamma,i}\rangle \mid \gamma \in s_i\rangle, A_i \mid i < \omega_1\rangle \in \mathcal{P}_E$ (the extender based Magidor forcing with $\vec{E}$) is a condition over κ. In particular, $A_i \in E(\kappa, i)(\max(s_i))$, for every $i < \omega_1$. Only instead of the common support here at each level i we have its own support s_i. This is used further (clause 2) in order to allow different domains for assignment functions and then eventually identify certain conditions in the final forcing.

   Conditions with lower part, i.e. below κ are defined completely the same with only the addition of an assignment function. Thus a condition over κ_i ($i < \omega_1$) is of the form

   $$\langle\langle\gamma, p^{\gamma,j}\rangle \mid j \leq i, \gamma \in s_j\rangle, \langle A_j, a_j \mid j < i\rangle, a_i\rangle\ .$$

It is possible to make the definition uniform by adding the assignment function also at ω_1, i.e. a_{ω_1}, but it should be just the identity function. We have here finite sequences $p^{\gamma,i}$ indexed with two indexes. The meaning is that $p^{\gamma,i}$ is the sequence for γ at the level i.

The reason for this double indexing is that we prefer not to keep the full trace of γ from the level ω_1 down. In the usual extender based Magidor (or Radin) forcings (see [6], [5]) to each such γ corresponds to an ω_1-sequence in a generic extension. The basic reason for trying to avoid this here is that for every $k < \omega$ we would like to have at most finitely many ordinals on the trace of γ not being k-good, see (2.5) below. Without requiring this it will be problematic to add new ordinals to the support of a condition.

2. Let $i < \omega_1$ be a limit ordinal. Then $a_i = \langle a_{ij} \mid j < i \rangle$ will be a sequence of so-called assignment functions.

It satisfies the following conditions with sets S and C from 1.1:

(2.1) If $i \in S$ then $doma_{ij} = s_i$ for every $j < i$

(2.2) If $i \notin S$ then $doma_{ij} = s_i \cap \kappa^{+\omega+2}$ for every $j < i$

(2.3) $rnga_{ij} \subseteq s_j$ and if $j \notin S$ then $rnga_{ij} \subseteq s_j \cap \kappa^{+\omega+2}$

(2.4) Each a_{ij} is identity on $\kappa^{+\omega+1}$ and, if $i \notin C$ then a_{ij} is identity.

Thus, for $i \notin C$, the forcing between levels i and $\sup(C \cap i)$ is just the extender based Magidor forcing with addition of a_ℓ's for $\ell \leq \sup(C \cap i)$.

Denote by $doma_i$ the set s_i if $i \in S$ and the set $s_i \cap \kappa^{+\omega+2}$ otherwise.

(2.5) Let $\alpha \in \bigcup_{i<\omega_1} s_i$. Then for every $i < \omega_1$ and $k < \omega$ for all but finitely many $j < i$ $a_{ij}(\alpha)$ is k-good, if defined.

This requirement is crucial for identifying conditions further, and for showing the $\kappa_i^{+\omega+2}$ chain condition of the final forcing. It is parallel to the existence of nondecreasing converging to infinity sequence $\langle k_n \mid n < \omega \rangle$ of natural numbers, that was used in previous constructions with short extenders forcing, see [1],[2],[3].

(2.6) (cofinality correspondence) First $\kappa^{+\omega+3}$ corresponds under a_i to $\kappa_{i_n}^{+\omega+2}$ for each $n < \omega$, where $\langle i_n \mid n < \omega \rangle$ is the cofinal in i sequence picked in 1.1. Let $\alpha \in doma_i$ and $j < i$. Find $n < \omega$ with $j \in (i_n, i_{n+1}]$. We describe below the possibilities for $a_{ij}(\alpha)$ and its cofinality.

(a) If $\alpha \leq \kappa^{+\omega+1}$, then $a_{ij}(\alpha) = \alpha$.

(b) If $cf\alpha = \kappa^{+\omega+2}$ (for example, in case of $i \in S$, $doma_i \subseteq \kappa^{+\omega+3}$, and so there may be a lot of such α's; if $i \notin S$ then this will describe the images of $\kappa^{+\omega+2}$ itself); then $cfa_{ij}(\alpha) = \underset{\sim}{\kappa}{}^{+\omega+2}_{i^*_{n+1}}$ where $\langle \underset{\sim}{\kappa}_\ell \mid \ell < \omega_1 \rangle$ denotes the canonical name of the generic Magidor sequence for the normal measures of the extenders.

(c) $\alpha \in (\kappa^{+\omega+1}, \kappa^{+\omega+2})$ may be a name as well. Thus, for some k, $i < k < \omega_1$ and $\ell < \omega$ such that $k^*_\ell < j$ and $k_\ell > i$, $\underset{\sim}{\alpha}$ may correspond to some ordinal of level k. We cannot, in general recognize this situation over level i alone. So, it is allowed just to have names $\underset{\sim}{\alpha}$ of ordinals of cofinality $\underset{\sim}{\kappa}{}^{+\omega+2}_{k^*_\ell}$. For every j, $k^*_\ell < j < i$, let $cfa_{ij}(\underset{\sim}{\alpha}) = \underset{\sim}{\kappa}{}^{+\omega+2}_{k^*_\ell}$ for each such $\underset{\sim}{\alpha}$ and k, ℓ corresponding to it. Below we shall return again to this case and explain in particular the situation when $j \in (i^*_n, i_n)$ with $i^*_n < k^*_\ell$.

(d) We need to allow a similar to (c) behavior also for α's which are not names. This is crucial for the proof of the Prikry condition. Thus there may be cardinals $\delta < \kappa$ of countable cofinality such that some α's of cofinality $\delta^{+\omega+2}$ in the interval $(\kappa^{+\omega+1}, \kappa^{+\omega+2})$ correspond under a_{ij} to ordinals of cofinality $\delta^{+\omega+2}$.

(e) If $cf\alpha < \kappa$, for $\alpha \in doma_i$, then $\max p^\alpha \geq cf\alpha$.

This means that the correspondence for such α starts working above the level of $cf\alpha$.

(2.7) For every $i < \omega_1$

$$\langle \langle \gamma, p^{\gamma, j} \rangle \mid \gamma \in s_j \rangle, \langle A_j, a_j \mid j < i \rangle, a_i \rangle$$

belongs to the ultrapower by $E(\kappa, i)(mc(s_i))$. In particular, require that $mc(s_i)$ projects on each $\gamma \in s_j$ for every $j < i$. Fix a function F_i representing it. We shall use F_i in the definition of nondirect extensions. The idea is that once a $\kappa_i \in A_i$ is decided, then we can move the condition down to κ_i using $F_i(\kappa_i)$. This will include the assignment functions $\langle a_j \mid j \leq i \rangle$.

(2.8) Let $i < \omega_1$ be a limit ordinal, $n < \omega$ and $j \in (i^*_n, i_n)$ be equal to k^*_m for some $k \in (i, \omega_1)$, with $k_{m+1} > i$ and $m < \omega$. Suppose that $\underset{\sim}{\alpha} \in doma_i$ is a name of an ordinal of cofinality $\underset{\sim}{\kappa}{}^{+\omega+2}_j$. We like to keep the cofinality of $a_{ij'}(\alpha)$ to be $\kappa_j^{+\omega+2}$ for every j', $j < j' < i$.

It is automatically holds in the extender based Magidor forcing for j''s above i_n. Since then $j' \in [i^*_{n+1}, i)$ and $j = k^*_m < i_n$. But once $j' \leq i_{n+1}$, then usually (in extender based Magidor forcing) such cofinality disappears in finite sequences (p^γ's) of the forcing condition. We require the following:

(a) the set $A(j) = \{\underset{\sim}{\alpha} \in dom a_i \mid cf \underset{\sim}{\alpha} = \underset{\sim}{\kappa}_j^{+\omega+2}\}$ has a maximal element $\underset{\sim}{\alpha}(j)$ in the extender ordering $\le_{E(\kappa,i)}$. Each element of this set is reachable from $\underset{\sim}{\alpha}(j)$ via walks over the fixed in 1.4 generic box sequence.

(b) for each j and $\underset{\sim}{\alpha}(j)$ as above, there are κ-many possibilities for the actual value of $\underset{\sim}{\alpha}(j)$. Just according to the value of $\underset{\sim}{\kappa}_j$, which may be one of κ-many ordinals. Let $\widehat{\alpha}(j)$ be the least ordinal above the possible values of $\underset{\sim}{\alpha}(j)$. We require that $cf \widehat{\alpha}(j) = \kappa$, $\widehat{\alpha}(j) \in dom a_i$ and $cf(a_{ii_n}(\widehat{\alpha}(j))) = \underset{\sim}{\kappa}_{i_n^*}$.

(c) $\widehat{\alpha}(j)+1 \in dom a_i$ and $a_{ii_n}(\widehat{\alpha}(j)+1)$ is a supremum of a model $M \prec H(\chi)$, $|M| = \underset{\sim}{\kappa}_{i_n}^{+\omega+1}$ intersected with $\kappa_{i_n}^{+\omega+2}$.

(d) The walks (via the fixed generic box sequence) from $\widehat{\alpha}(j)$ to $\underset{\sim}{\alpha}(j)$ and below are copied by a_{ii_n} to the walks from $a_{ii_n}(\widehat{\alpha}+1)$ to $a_{ii_n}(\underset{\sim}{\alpha}(j))$. Require also that all the images of members of $A(j)$ with distances $\le \underset{\sim}{\kappa}_{i_n}$ from $\underset{\sim}{\alpha}(j)$ (i.e. such that all the ordinals of the walk are below $\underset{\sim}{\kappa}_{i_n}$)) remain below $a_{ii_n}(\widehat{\alpha}(j))$. Note that there are at most $\aleph_0$ possible $k^* = j$ as above.

The next condition is a weakening of order preservation. The full order preservation is usually used in context of extender based forcing inside supports or assignment functions.

(2.9) (order preservation) Let $i \le \omega_1$, $\beta_1, \beta_2 \in dom a_i$ and $\beta_1 < \beta_2$.

(a) for every $j < i$, $a_{ij}(\beta_1) < a_{ij}(\beta_2)$ provided either

(α) no value $\underset{\sim}{\kappa}_k$ is decided for $k \ge j^*$.

or

(β) both $a_{ij}(\beta_1)$, $a_{ij}(\beta_2)$ are names of ordinals and not real ordinals.

Notice, that we do not require "$a_{ij}(\beta_1) < a_{ij}(\beta_2)$ for every $j < i$". So it well may be a case that for unboundedly many j's in i $a_{ij}(\beta_1) \not< a_{ij}(\beta_2)$. For example if for some $k^* < i$ there are unboundedly many j's in i with $j^* = k^*$.

(b) there exists $n_0 < \omega$ such that for every n, $n_0 \le n < \omega$

$$a_{ii_n}(\beta_1) < a_{ii_n}(\beta_2) .$$

Recall that $\langle i^*_n \mid n < \omega \rangle$ is increasing cofinal in i sequence, so only finitely many of i_n's will have a chance to escape clause (a) (α). Also, it is possible to arrange that for relevant β's $a_{ii_n}(\beta)$ will be a name of ordinal and not an ordinal for all but finitely many n's. It is allowed for $a_{ij}(\beta)$ to be an actual ordinal and not a name in general. The use of this will be made in the proof of the Prikry condition. More specifically, in carrying out the recursion construction going over all nondirect extensions in the proof.

Monotonicity generally used in short extender forcings, and will be used here also, in order to prove an appropriate chain conditions.

(2.10) (commutativity) Let $i < \omega_1$, be a limit ordinal, $j' < j < i$ and $\underset{\sim}{\kappa}_k$ was not yet decided for any $k \geq \max(j^*, j'^*)$. Then for every $\beta \in doma_i$, $a_{ij}(\beta) \in doma_j$ and $a_{jj'}(a_{ij}(\beta)) = a_{ij'}(\beta)$. Note that such β may be a name, as well as, $a_{ij}(\beta)$.

The commutativity will not hold, in general, once $\underset{\sim}{\kappa}_k$ is decided for some $k \geq \max(j^*, j'^*)$, since then a_{ij} or $a_{ij'}$ need not be one to one.

We require also for every $\nu \in A_i$ with $\nu > \max(p_\beta)$ (i.e. which is permitted for β) the corresponding equality holds at the level ν^0 with $a_{ij}, a_{ij'}$ and $a_{jj'}$ replaced by the corresponding values of $F_i(\nu)$. In addition, for every $\nu \in A_j$ with $\nu > \max(p_{a_{ij}(\beta)}, p_{a_{ij'}(\beta)})$, if ρ is the projection of ν to the coordinate $a_{ij}(\beta)$, η is the projection of ν to $a_{ij'}(\beta)$ and $b_{jj'}$ is the corresponding to $a_{jj'}$ part of $F_j(\nu)$, then

$$b_{jj'}(\rho) = \eta \ .$$

For $i \in S$, $\kappa_i^{+\omega+3}$ will correspond to $\langle \kappa_{i_n}^{+\omega+2} \mid n < \omega \rangle$, and $\kappa_i^{+\omega+2}$ to $\langle \kappa_{i^*_n}^{+\omega+2} \mid n < \omega \rangle$, i.e.

$$\kappa_i^{+\omega+3} = cf\left(\prod_{n<\omega} \kappa_{i_n}^{+\omega+2} \Big/ \text{finite}\right)$$

$$\kappa_i^{+\omega+2} = cf\left(\prod_{n<\omega} \kappa_{i^*_n}^{+\omega+2} \Big/ \text{finite}\right) \ .$$

We would like to preserve both $\kappa_i^{+\omega+2}$ and $\kappa_i^{+\omega+3}$. This will be insured by showing $\kappa_i^{+\omega+2}$ – c.c. of the relevant forcing. But in order to obtain such chain conditions, we deal with submodels of sizes $\kappa_i^{+\omega+2}$ and $\kappa_i^{+\omega+1}$ in a fashion of [3, Sec. 5]. Thus the assignment functions a_i's ($i \in S \cup \{\omega_1\}$ will be actually applied to such models or just to ordinals coding them in some canonical fashion. We require the following:

(3) there is some

$$\langle\langle A^{00}, A^{10}\rangle, \langle c_\nu \mid \nu \in \lim A^{10}\rangle\rangle$$

in $G_{1\kappa}$ such that:

(3.1) there is a maximal under inclusion model A in s. It is a limit element of A^{10} and $cf(A \cap \kappa^{+\omega+3}) = \kappa^{+\omega+2}$. This A is what is called the maximal coordinate of the condition (or, if one likes to work with codes, it is coded by the maximal coordinate of the condition).

(3.2) for every i, j with $j < i < \omega_1$ $a_{ij}(A)$ is a model depending on the value of $\underset{\sim}{\kappa}_j$ only and such that

$$cf(a_{ij}(A) \cap \kappa^{+\omega+3}) = cf(a_{ij}(\kappa^{+\omega+2}))\ .$$

(3.3) conditions (2.1)–(2.10) above are valid for models in s just via ordinal coding them with "$<$" replaced by "$\subseteq$" and "$\in$".

(3.4) For each limit point B of A^{10} which is in s, for every $i \leq \omega_1$ and $n < \omega$, fix the element $C^{i_n}_{a_{ii_n}(B)}$ of a box sequence $\vec{C}^{i_n}$ over $\kappa^{+\omega+2} \cap \{\alpha < \kappa^{+\omega+2} \mid cf\alpha \leq \kappa_{i^*_n}\}$. We require that a_{ii_n} preserve the walks via box sequences in the sense of [3, Sec. 5]. Thus the conditions (e)–(i) of 5.10 of this paper are required to hold here.

Note that in the present setting the set F used in [3, 5.10] disappears. But it can be reconstructed easily. Thus for any $\langle\langle A^{00}, A^{10}\rangle, \langle c_\nu \mid \nu \in \lim A^{10}\rangle\rangle \in G_{1\kappa}$, F will be just the set of all p's in $\mathcal{P}^*$ satisfying 2.1 with this particular $\langle\langle A^{00}, A^{10}\rangle$, $\langle\langle c_\nu \mid \nu \in \lim A^{10}\rangle\rangle$.

Definition 2.2. Let $p = \langle\langle\langle\gamma, p^{\gamma,i}\rangle \mid i < \omega_1, \gamma \in s_i\rangle$, $\langle A_i, a_i \mid i < \omega_1\rangle\rangle$ and

$$q = \langle\langle\langle\gamma, q^{\gamma,i}\rangle \mid i < \omega_1, \gamma \in t_i\rangle,\ \langle B_i, b_i \mid i < \omega_1\rangle\rangle$$

be two elements of $\mathcal{P}^*$ over κ. Then $p \geq^* q$ iff

(1) $s_i \supseteq t_i$

(2) for every $\gamma \in t_i$, $i < \omega_1$ we have $p^{\gamma,i} = q^{\gamma,i}$

(3) $a_i \supseteq b_i$

(4) $B_i \supseteq \pi_{mc(s_i),mc(t_i)}$"$(A_i)$

where $\pi_{mc(s_i),mc(t_i)}$ is the canonical projection on $E(\kappa, i)(mc(s_i))$ onto $E(\kappa, i)(mc(t_i))$.

The order $\geq^*$ is defined similar for elements $\mathcal{P}^*$ which are not over κ.

Let us define now a one-step extension of a condition over κ. The general definition of the forcing order $\leq$ on $\mathcal{P}^*$ is defined then by a simple induction as in [5]. Again, the absent assignment function over the level ω_1 can be viewed as the identity function making the definition of extensions below κ the same as those over κ itself.

Definition 2.3. Let $p = \langle\langle\langle\gamma, p^{\gamma,i}\rangle \mid \gamma \in s_i\rangle, \langle A_i, a_i \mid i < \omega_1\rangle\rangle$ be an element of $\mathcal{P}^*$ over κ, $i < \omega_1$ and $\lambda \in A_i$. Define $p_{\langle\lambda\rangle}$ the one element extension of p by λ. $p_{\langle\lambda\rangle} = p_1{}^\frown p_0$, where p_1 the lower part of $p_{\langle\lambda\rangle}$ and p_0 the part over κ are defined as follows

$$(1) \qquad p_0 = \langle\langle\langle\gamma, p_0^{\gamma,j}\rangle \mid \gamma \in s_j \ \ i < j < \omega_1\rangle, \langle A_j \backslash \lambda + 1, a_j^0 \mid i < j < \omega_1\rangle$$

is such that

(a) $a_j^0 = \langle a_{jk} \mid i < k < j\rangle$ for every j, $i < j < \omega_1$.

The meaning is that we shall use the same assignment functions but only up to the level i.

(b) for every j, $i < j < \omega_1$ and $\gamma \in s$

(α) if $\max(p^{\gamma,j}) \geq \kappa_i$, then $p_0^{\gamma,j} = p^{\gamma,j}$, where $\kappa_i = (\lambda)^0$ is the projection of λ to the normal measure of $E(\kappa, i)$.

(β) if $\max(p^{\gamma,j}) < \kappa_i$, then

$$p^{\gamma,j} = p^{\gamma,j}{}^\frown\langle\pi_{mc(s_i),a_{ji}(\gamma)}(\lambda)\rangle \ .$$

So, in this case we extend the γ-th sequence by adding to it the value at the level i. In contrast to the usual extender based Magidor forcing, we do not add $\pi_{mc(s_i),\gamma}(\lambda)$ but rather first evaluate the assignment of γ by a_{ji} and then add the corresponding value to it from the level i. Note that $(\pi_{mc(s_i),a_{ji}(\gamma)}(\lambda))^0 = \kappa_i = (\pi_{mc(s_i),\gamma}(\lambda))^0$. So, adding $\pi_{mc(s_i),a_{ji}(\gamma)}(\lambda)$ to $p_{\gamma,j}$ and not $\pi_{mc(s_i),\gamma}(\lambda)$ (as in the usual extender based forcings) is not important as far as one is concerned with the "being permitted" issue.

Also we prefer here to keep $p^{\gamma,j}$ to be attached to γ and not to drop below κ_i. The reason is that different γ's may be assigned by a_{ji} to the same ordinal. This cases generation of possibly more than one sequence below κ_i for the same ordinal.

$$(2) \qquad p_1 = \langle\langle\langle\gamma, p_1^{\gamma,j}\rangle \mid \gamma \in s_{1j}, j \leq i\rangle, \langle A_{1j}, a_j^1 \mid j < i\rangle, a_i^1\rangle$$

is such that

(a) $s_{1i} = \mathrm{dom} a_i$

(b) $s_{1j} = \{\pi_{mc(s_i),\gamma}(\lambda) \mid \gamma \in s_j \text{ and } \max(p^{\gamma,j}) < \kappa_i\}$

(c) $p_1^{\pi_{mc(s_i),\gamma}(\lambda),j} = p^{\gamma,j}$ for $\gamma \in s_j, j \leq i$ and $\max(p^{\gamma,i}) < \kappa_i$.

(d) $A_{1j} = A_j \cap \kappa_i$ for each $j < i$

(e) $\langle a_j^1 \mid j \leq i \rangle = F_i(\lambda)$, where F_i represents $\langle a_j \mid j \leq i \rangle$, as in Definition 2.1 (2.7).

Once we have a generic set, $p_1^{\gamma,i}$'s, for $\gamma \in s_1$, of this set are put together to form an i-sequence unbounded in κ_i and corresponding to γ.

The linkage with the part below κ_i of $p^{\gamma,j}$ and eventually to the γ-th sequence over κ_j will be via the sequence over κ_i corresponding the $\pi_{mc,a_{ji}(\gamma)}(\lambda)$.

Let $j < \omega_1$ be a limit ordinal. If κ_j is decided, then we can split naturally $\mathcal{P}^*$ into $\mathcal{P}^*_{\leq j}$ – the part at κ_j and below, and $\mathcal{P}^*_{>j}$ the part above κ_j. For $p \in \mathcal{P}^*$, $p \restriction j$ (or $p \restriction \kappa_j$) and $p \restriction \omega_1 \backslash j$ are defined in the obvious fashion. The splitting at successor j's is avoided in order to keep the connection between both parts $\mathcal{P}_{\leq j}$ and $\mathcal{P}_{>j}$. Thus, if j is of the form k^*_{m+1}, for some $k < \omega_1$, and $m < \omega$, then some ordinals from the level k may be connected via the assignment function a_k to $k_m = k^*_{m+1} - 1$ avoiding k^*_{m+1}.

Repeating the proofs of 5.16–5.18 of [3], we obtain the following:

Lemma 2.4. *Let $j < \omega_1$ be limit ordinal and assume that κ_j is decided. Then $\mathcal{P}_{>\kappa_j} * \langle \mathcal{P}^*_{>j}, \leq^* \rangle$ is $\kappa_j^{+\omega+5}$–strategically closed in $V^{\mathcal{P}_{\leq \kappa_j}}$.*

The proof of the next lemma basically repeats the arguments of [5], Sec.6.

Lemma 2.5. *Let $\langle t, p \rangle \in \mathcal{P} * \mathcal{P}^*$ and σ is a statement of the forcing language. Then there is $\langle t^*, p^* \rangle \geq \langle t, p \rangle$ such that $p^* \geq^* p$ and $\langle t^*, p^* \rangle \| \sigma$.*

Lemma 2.6. *For every $j < \omega_1$, as in 2.4, $\mathcal{P}_{>\kappa_j} * \mathcal{P}^*_{>j}$ does not add new subsets to κ_j over $V^{\mathcal{P}_{\leq \kappa_j}}$.*

3 The Main Forcing

Working in $V^{\mathcal{P}}$, we define a partial order $\longrightarrow$ on $\mathcal{P}^*$ such that $\langle \mathcal{P}^*, \rightarrow \rangle$ will be nice subforcing of $\langle \mathcal{P}^*, \leq \rangle$ and will satisfy the desired chain conditions. Thus for $i \in \omega_1 \backslash S$ the forcing $\langle \mathcal{P}_{\leq i} \rightarrow \rangle$ will satisfy $\kappa_i^{+\omega+2}$ – c.c., by Δ-system argument, and for $i \in S$ the forcing $\langle \mathcal{P}_{\leq i}, \rightarrow \rangle$ will satisfy $\kappa_i^{+\omega+2}$ – c.c. by arguments similar to those of [3, Sec. 5]. We start with a definition of equivalences $\longleftrightarrow$.

Definition 3.1 Suppose that $i < \omega_1$ is a limit ordinal and

$$p = \langle\langle \gamma, p^{\gamma,j}\rangle \mid j \leq i \text{ and } \gamma \in s_j\rangle, \langle A_j, a_j \mid j < i\rangle, a_i\rangle \ ,$$
$$q = \langle\langle \gamma, q^{\gamma,j}\rangle \mid j \leq i \text{ and } \gamma \in t_j\rangle, \langle B_j, b_j \mid j < i\rangle, b_i\rangle$$

are two conditions in $\mathcal{P} \restriction i$ over κ_i (i.e. no κ_j is determined for $j < i$ or, in other words, the lower parts of these conditions are empty). We set $p \longleftrightarrow q$ (p, q are equivalent) iff the following holds:

(a) $s_i = t_i = \mathrm{dom} a_i = \mathrm{dom} b_i$,

(b) s_j and t_j are order isomorphic as sets of ordinals for each $j < i$

(c) $A_j = B_j$ for each $j < i$

(d) $p^{\gamma,j} = q^{\sigma(\gamma),j}$ for every $\gamma \in s_j$, $j \leq i$ where σ is the order isomorphism between s_j and t_j.

Clearly, if $i \in S$ then $\sigma = id$

(e) for every $k < \omega$, for all but finitely many $j' < j \leq i$

$$\langle s_j, a_{jj'}, rnga_{j,j'}\rangle \quad \text{and} \quad \langle t_j, b_{jj'}, rngb_{j,j'}\rangle$$

realize the same k-type. Moreover they always (for each $j' < j \leq i$) realize the same 4-type.

(f) for each $j' < j < i$, p and q have the same representing function $F_{jj'}$ (see 2.1 (2.7)).

Note, that in case $i \notin S$, $s \backslash \kappa^{+\omega+2}$ and $t \backslash \kappa^{+\omega+2}$ may be different. They require to realize the same k_n–type, which will allow us to identify such p and q. This in turn will be used to show that $2^{\kappa_i} \leq \kappa_i^{+\omega+2}$.

The conditions (b), (e) and (f) will be crucial for proving the chain condition. Thus, for every $n < \omega$, it will be easy to extend p, q to p', q' such that $p' \restriction i_n = q' \restriction i_n$. Just pick some $\lambda \in A_{i_n} = B_{i_n}$ and consider $p' = p_{\langle\lambda\rangle}$, $q' = q_{\langle\lambda\rangle}$. By (f) above and Definition 2.3, p' and q' will be as desired.

Definition 3.2 Let $p, q \in \mathcal{P}^*$. We set $p \to q$ iff either

(1) $p \leq q$

or

(2) there is a limit $i < \omega_1$ such that κ_i is determined the same way in both p and q, and the following conditions hold:

(2.1) $p \restriction \omega_1 \backslash i \leq q \restriction \omega_1 \backslash i$.

It means that nothing new, not taken into account by "$\leq$", happen above level i.

(2.2) if $\kappa_{\bar{j}} < \kappa_i$ is the largest element decided then it is the same for p and q, and $p \restriction \kappa_{\bar{j}} \rightarrow q \restriction \kappa_{\bar{j}}$.

The last clause turns the definition into inductive one.

(2.3) the maximal model $A(p \restriction \kappa_i)$ of $p \restriction \kappa_i$ appears in $q \restriction \kappa_i$ and for some $q' \leq q \restriction \kappa_i \backslash \kappa_{\bar{j}}$ with $A(p \restriction \kappa_i)$ as a maximal model $p \restriction \kappa_i \backslash \kappa_{\bar{j}} \longleftrightarrow q'$.

The next lemma insures that $\langle \mathcal{P}^*, \rightarrow \rangle$ is a nice subforcing of $\langle \mathcal{P}^*, \leq \rangle$, i.e. every dense open set in $\langle \mathcal{P}^*, \rightarrow \rangle$ generates such a set in $\langle \mathcal{P}^*, \leq \rangle$. The proof is similar to the corresponding lemma of [3, Sec. 5].

Lemma 3.1. *Suppose that $p \rightarrow q \leq q'$ then there is $p' \geq p$ such that $q' \rightarrow p'$, where $p, q, q', p' \in \mathcal{P}^*$.*

Lemma 3.2. *For every limit $i < \omega_1$, inside $V^{\mathcal{P} * \langle \mathcal{P}^*, \rightarrow \rangle}$, $2^{\kappa_i} \geq (\kappa_i^{+\omega+2})^V$ and, if $i \in S$, then $2^{\kappa_i} \geq (\kappa_i^{+\omega+3})^V$.*

Note that actually all the cardinals in the interval $[\kappa_i^{++}, \kappa_i^{+\omega+1}]$ are collapsed to κ_i^+ which itself is preserved as the successor of a singular.

Lemma 3.3. *For every limit $i < \omega_1$, if $i \notin S$ then in $V^{\langle \mathcal{P}^*, \rightarrow \rangle}$ $2^{\kappa_i} < (\kappa_i^{+\omega+3})^V$.*

Proof. In $V_1 = V^{\mathcal{P}}$ we split $\mathcal{P}^*$ into $\mathcal{P}^*_{\leq i}$ and $\mathcal{P}^*_{> i}$. By 2.4, $\langle \mathcal{P}^*_{> i}, \leq \rangle$ does not add new subsets to κ_i. So it will be enough to show that in $V_1^{\mathcal{P}^*_{\leq i}}$ $2^{\kappa_i} < (\kappa_i^{+\omega+3})^V$. Denote $(\kappa_i^{+\omega+3})^V$ by λ. Clearly, $\lambda = (\kappa_i^{+\omega+3})^{V_1}$. Work in V_1. Let $M \prec H(\chi)$, for χ big enough, containing all the relevant information such that $|M| = \kappa_i^{+\omega+2}$, $M \cap \kappa_i^{+\omega+3}$ is an ordinal and M is closed under $\kappa_i^{+\omega+1}$ sequences of its elements. Let $p \in \mathcal{P}^*_{\leq i}$. Then, using elementarity there will be a condition $p^M \in M \cap \mathcal{P}^*_{\leq i}$ such that $p \longleftrightarrow p^M$ and even $p \restriction \kappa_i^{+\omega+2} = p^M \restriction \kappa_i^{+\omega+2}$. This means that $\langle \mathcal{P}^*_{\leq i}, \rightarrow \rangle$ and $\langle \mathcal{P}^*_{\leq i} \cap M, \rightarrow \rangle$ are just the same from forcing point of view. But $|\mathcal{P}^*_{\leq i} \cap M| = \kappa_i^{+\omega+2}$. Hence, $2^{\kappa_i} < \lambda$ in $V_1^{\langle \mathcal{P}^*_{\leq i} \cap M, \rightarrow \rangle}$. So the same holds in $V_1^{\langle \mathcal{P}^*_{\leq i}, \rightarrow \rangle}$. $\square$

Our next task will be to show $\kappa_i^{+\omega+2}$-c.c. of the forcing $\langle \mathcal{P}^*_{\leq i}, \rightarrow \rangle$ in $V^{\mathcal{P}}$ for each limit $i < \omega_1$.

First let us deal with $i \notin S$.

Lemma 3.4. *Suppose that $i \in \omega_1 \backslash S$ is a limit ordinal and the value of κ_i is decided. Then, in $V^{\mathcal{P}}$, $\langle \mathcal{P}^*_{\leq i}, \rightarrow \rangle$ satisfies $\kappa_i^{+\omega+2}$ – c.c.*

Proof. Work in $V^{\mathcal{P}}$. Let $\langle p_\alpha \mid \alpha < \kappa_i^{+\omega+2}\rangle$ be a sequence of elements of $\mathcal{P}^*_{\leq i}$ and

$$p_\alpha = \langle \langle \gamma, p_\alpha^{\gamma,j}\rangle \mid \gamma \in s_{j\alpha}, j \leq i\rangle, A_j^\alpha, a_j^\alpha \mid j < i\rangle\ , a_i^\alpha\rangle\ .$$

We may assume without loss of generality that p_α's are of this form, since the number of possibilities for low parts is small and so they may be assumed to be the same and then just ignored since then the incompatibility if occurs will be due to the upper parts of the conditions.

Shrinking if necessary, we can assume that the following hold:

(1) $A_j^\alpha = A_j^\beta$ for each $\alpha, \beta < \kappa_i^{+\omega+2}$ and $j < i$.

(2) $\langle s_{i\alpha} \mid \alpha < \kappa_i^{+\omega+2}\rangle$ form a Δ-system with a kernel s.

(3) for every $\alpha, \beta < \kappa_i^{+\omega+2}$ and $j' < j < i$

$$\langle s_{j\alpha}, a_{jj'}^\alpha\rangle \text{ and } \langle s_{j\beta}, a_{jj'}^\beta\rangle$$

realize the same ω-type. Moreover, if $\sigma_{\alpha\beta}$ is the order isomorphism between $s_{j\alpha}$ and $s_{j\beta}$ then $p_\alpha^{\gamma j} = p_\beta^{\sigma(\gamma)j}$, for every $\gamma \in s_{j\alpha}$

(4) for every $\alpha, \beta < \kappa_i^{+\omega+2}$ and $j < i$ $rnga_{ij}^\alpha$ realizes the same ω-type over $rnga_{ij}^\alpha \restriction s$ as $rnga_{ij}^\beta$ over $rnga_{ij}^\beta \restriction s$. Moreover, if $\sigma_{\alpha\beta}$ is the order isomorphism between $rnga_{ij}^\alpha$ and $rnga_{ij}^\beta$ then $p_\alpha^{\gamma i} = p_\alpha^{\sigma_{\alpha\beta}(\gamma)i}$, for every $\gamma \in s_{i\alpha}$.

(5) for every $j' < j < i, \alpha, \beta < \kappa_i^{+\omega+2}$ p^α and p^β have the same representing function $F_{jj'}$ Note that $F_{jj'}$ is a function from a subset of V_{κ_i} into V_{κ_i} and $2^{\kappa_i} = \kappa_i^+$ in $V^{\mathcal{P}}$.

(6) $s \supseteq s_{i\alpha} \cap \kappa_i^{+\omega+1}$ for every $\alpha < \kappa_i^{+\omega+2}$.

(7) $\min(s_{i\alpha} \backslash s) \geq \alpha$.

Extending, if necessary, we may assume that for every α if $\tau \in s_\alpha$ and $cf\tau \leq \kappa_i$ then a closed cofinal sequence witnessing $cf\tau$ is contained in s_α. This implies $cf(\min(s_\alpha \backslash s)) \geq \kappa_i^+$. Let $\tau_\alpha = \min(s_\alpha \backslash s)$ for $\alpha < \kappa_i^{+\omega+2}$. Consider $\langle a_{ij}^\alpha(\tau_\alpha) \mid j < i\rangle$. By 2.1 (2.5), for every $k < \omega$ for all but finitely many $j < i$ $a_{ij}^\alpha(\tau_\alpha)$ is k-good. Let $\langle j_\ell \mid \ell < \omega\rangle$ be a one-to-one enumeration of i. Then for some nondecreasing converging to infinity sequence of natural numbers $\langle k_n \mid n < \omega\rangle$ we will have that $a_{ij_\ell}^\alpha(\tau_\alpha)$ is k_ℓ–good, for every $\ell < \omega$. Let ℓ_0 be the least $\ell < \omega$ with $k_\ell > 4$. Pick $n_0 < \omega$ to be the least with $i_{n_0} > \max\{j_\ell \mid \ell \leq \ell_0\}$. Then for every j, $i_{n_0} \leq j < i$ $a_{ij}^\alpha(\tau_\alpha)$ is 4-good. Shrinking, if necessary, we can assume that each

$\alpha < \kappa_i^{+\omega+2}$ has the same sequence $\langle k_\ell \mid \ell < \omega \rangle$ and for the same $n_0 < \omega$ we have $a^\alpha_{ij}(\tau_\alpha)$ is 4-good for every j, $i_{n_0} \leq j < i$.

Now we take a nondirect extension of each p_α deciding $\kappa_{i_{n_0}}$ the same way, as well as everything at the level i_{n_0} and below. Denote the resulting conditions by p_α's again.

Let now $\alpha < \beta < \kappa_i^{+\omega+2}$. We would like to show compatibility of p_α and p_β in $\langle \mathcal{P}^*_{\leq i}, \to \rangle$. We proceed as in [1, 2.20]. Thus for every $\ell < \omega$ with $j_\ell > i_{n_0}$, we can find $t_{\alpha\ell}$ realizing the same $k_\ell - 1$ – type over $rng(a^\alpha_{ij_\ell} \restriction s)$ as $rng(a^\alpha_{ij_\ell} \backslash s)$ does so that $\min t_{\alpha\ell} > \max(rnga^\alpha_{ij_\ell})$. Add this t_{α_ℓ} to $rnga^\alpha_{ij_\ell}$. Denote by a'_ℓ the resulting union. Find $t_{\beta\ell}$ so that $\min(rnga^\beta_{ij_\ell} \backslash s) = a^\beta_{ij_\ell}(\tau_\beta) > \max t_{\beta\ell}$, and if $b'_\ell = rnga^\beta_{ij_\ell} \cup t_{\beta\ell}$, then a'_ℓ and b'_ℓ realize the same $k_\ell - 1$ – type.

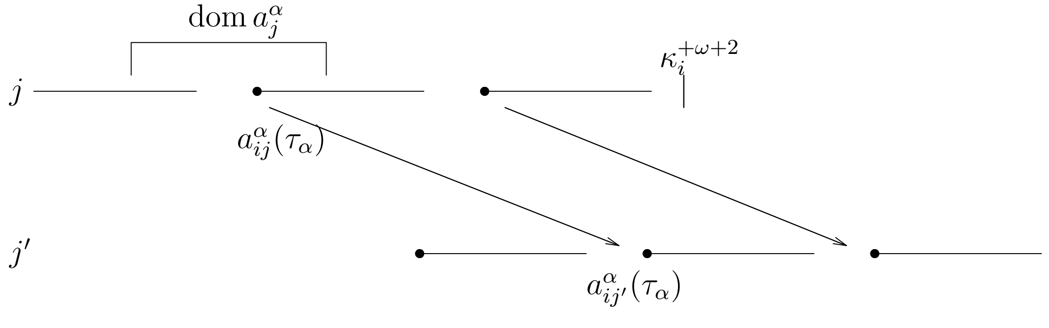
Now we extend in the obvious fashion $a^\alpha_{ij_\ell}$ to a_{ij_ℓ} with range a'_ℓ and $a^\beta_{ij_\ell}$ to b_{ij_ℓ} with range b'_ℓ. If there are no elements of S between i_{n_0} and i, or each $j \in (i_{n_\ell}, i)$ is in S, then this way we obtain equivalent extensions of p_α and p_β. But once there are j's, $i_{\kappa_0} < j < i$ in S and its compliment, then a bit more is needed. In this case ordinals between $\kappa_i^{+\omega+2}$ and $\kappa_i^{+\omega+3}$ should be dealt with more care.

Let $j \in (i_{n_0}, i) \backslash S$. Then s^α_j (the same s^β_j) contains ordinals above $\kappa_i^{+\omega+2}$ but $\mathrm{dom} a^\alpha_j \subseteq \kappa_i^{+\omega+2}$. Recall that such ordinals are needed in order to "communicate" with j''s in S below j. For each $j' \in (i_{n_0}, j) \in S$, the assignment function $a^\alpha_{jj'} : s^\alpha_j \cap \kappa^{+\omega+2} \to s^\alpha_{j'}$. The same holds with α replaced by β. Also $s^\alpha_{j'}$ and $s^\beta_{j'}$ realize the same ω-type. The point here is that there is no need to keep inside a_{ij} the image of $rnga^\beta_{ij}$ to realize the same type as $rnga^\beta_{ij}$ inside s^β_j. See clause 3.1(e).

We use commutativity

$$a^\alpha_{jj'}(a^\alpha_{ij}(\tau_\alpha)) = a^\alpha_{ij'},\ (\tau_\alpha) \text{ and } a^\beta_{jj'}(a^\beta_{ij}(\tau_\alpha)) = a^\beta_{ij'}(\tau_\alpha)$$

to preserve the connection between levels j and j' in common extension of p_α and p_β. See the diagram:



Finally we add the maximal coordinate for s_j (in the common extension of p_α and p_β) to one above (in the extender order) every element of $\cup\{s_{j'} \mid i_{n_0} < j' < j\}$. □

Now let us deal with $i \in S$. The argument is very similar but with ordinals replaced by models.

Lemma 3.6. *Suppose that $i \in S$ and the value of κ_i is decided. Then, in $V^{\mathcal{P}}$, $\langle \mathcal{P}^*_{\leq i}, \rightarrow \rangle$ satisfies $\kappa_i^{+\omega+2}$ – c.c.*

Proof. Suppose otherwise. The forcing $\mathcal{P}$ splits into $\mathcal{P}_{<\kappa_i} * \mathcal{P}_{\kappa_i} * \mathcal{P}_{>\kappa_i}$ with $\mathcal{P}_{>\kappa_i}$ strategically closed much above $\kappa_i^{+\omega+3}$. So we will work in $V_1 = V^{\mathcal{P}_{<\kappa_i}}$. Let $\langle \underset{\sim}{p}_\alpha \mid \alpha < \kappa_i^{+\omega+2} \rangle$ be a name of an anti-chain in $\langle \mathcal{P}^*_{\leq i}, \longrightarrow \rangle$ over V_1. Using $\kappa_i^{+\omega+2}$ – strategic closure of $\mathcal{P}_{\kappa_i}$, we define by induction an increasing sequence $\langle t_\alpha \mid \alpha < \kappa_i^{+\omega+2} \rangle$ of members of $\mathcal{P}_{\kappa_i}$ and a sequence $\langle p_\alpha \mid \alpha < \kappa_i^{+\omega+2} \rangle$ so that for every $\alpha < \kappa_i^{+\omega+2}$

$$t_\alpha \Vdash \underset{\sim}{p}_\alpha = \check{p}_\alpha \ .$$

Thus let t_0 and p_0 be arbitrary such that

$$t_0 \| \!\!- \underset{\sim}{p}_0 = \check{p}_0 \ .$$

Suppose now that $\alpha < \kappa_i^{+\omega+2}$ and for every $\beta < \alpha$ t_β and p_β are defined. Let

$$t_\beta = \langle \langle A^{00}_\beta, A^{10}_\beta \rangle, \langle c_\nu \mid \nu \in \lim A^{10}_\beta \rangle \rangle$$

If $\alpha = \alpha' + 1$, then we pick

$$t_\alpha = \langle \langle A^{00}_\alpha, A^{10}_\alpha \rangle, \langle c_\nu \mid \nu \in \lim A^{10}_\alpha \rangle \rangle$$

to be an extension of $t_{\alpha'}$ deciding $\underset{\sim}{p}_\alpha$ and so that $\langle t_\beta \mid \beta \leq \alpha' \rangle \in A^{00}_\alpha$. If α is a limit ordinal, then using strategic closure of $\mathcal{P}_{\kappa_i}$ we find t_α so that

(1) $t_\alpha \geq t_\beta \quad (\beta < \alpha)$.

(2) $t_\alpha \| \underset{\sim}{p}_\alpha$.

(3) $\langle t_\beta \mid \beta < \alpha \rangle \in A^{00}_\alpha$.

(4) $\bigcup_{\beta < \alpha} A^{00}_\beta \in A^{10}_\alpha$.

(5) $c_{(\bigcup_{\beta<\alpha} A^{00}_\beta) \cap \kappa_i^{+\omega+3}} = \{ A^{00}_\beta \cap \kappa_i^{+\omega+3} \mid \beta < \alpha \}$.

This completes the inductive definition of $\langle t_\alpha \mid \alpha < \kappa_i^{+\omega+2}\rangle$ and $\langle p_\alpha \mid \alpha < \kappa^{+\omega+2}\rangle$.

Set $A^{00} = \bigcup_{\alpha<\kappa_i^{+\omega+2}} A^{00}_\alpha$, $A^{10} = \bigcup_{\alpha<\kappa_i^{+\omega+2}} A^{10}_\alpha \cup \{A^{00}\}$ and

$$c_{A^{00}\cap\kappa_i^{+\omega+2}} = \{A^{00}_\alpha \mid \alpha < \kappa_i^{+\omega+2}\}\ .$$

Extend each t_α to t'_α by replacing in it $\langle\langle A^{00}_\alpha, A^{10}_\alpha\rangle, \langle c_\nu \mid \nu \in \lim A^{10}_\alpha\rangle\rangle$ by

$$\langle\langle A^{00}, A^{10}\rangle, \langle c_\nu \mid \nu \in \lim A^{10}\rangle\rangle\ .$$

Let $\alpha < \kappa_i^{+\omega+2}$ be a limit ordinal. Pick a limit α^*, $\alpha \leq \alpha^* < \kappa_i^{+\omega+2}$ such that $\bigcup_{\beta<\alpha^*} A^{00}_\beta$ includes all the models appearing in p_α.

Extend each of p_α's, for a limit α, by adding $\bigcup_{\beta<\alpha} A^{00}_\beta$, $\bigcup_{\beta<\alpha^*} A^{00}_\beta$, A^{00}_α and A^{00}. Denote the resulting extension of p_α by q_α.

Now we continue as in [3, 5.21] (with obvious adaptions due to the present form of p and the definition of $\longleftrightarrow$) use Δ-system argument which will give compatible conditions in the set $\{q_\alpha \mid \alpha < \kappa^{+\omega+2}\}$ forced by t_α's. $\square$

Combining together the lemmas we obtain that the following holds in $V^{\mathcal{P}*\langle\mathcal{P}^*,\to\rangle}$

(1) for every $i \in S$
$$2^{\kappa_i} = \kappa_i^{+3} (= (\kappa_i^{+\omega+3})^V)$$

(2) for every $i \notin S$
$$2^{\kappa_i} = \kappa_i^{++}\ .$$

4 Collapsing Successors of Singulars

In this section we describe how using supercompacts to collapse κ_i^+'s one can obtain a model satisfying

(1) $2^{\kappa_i} = \kappa_i^{++}$, if $i \in S$.

(2) $2^{\kappa_i} = \kappa_i^{+}$, if $i \notin S$.

The construction repeats the previous one, but instead of using the usual extender sequence, we shall use here a $\mathcal{P}_\kappa(\kappa^+)$ extender sequence of the length $\kappa^{+\omega+3}$. Let us define such a sequence. Assume that κ is $\kappa^{+\omega+3}$ – supercompact. Let $j : V \longrightarrow M$ be a witnessing embedding. Define from j a $\mathcal{P}_\kappa(\kappa^+)$ – extender sequence $\langle E_\tau \mid \tau < \kappa^{+\omega+3}\rangle$ of the length $\kappa^{+\omega+3}$ as follows: for every $X \subseteq \mathcal{P}_\kappa(\kappa^+) \times V_\kappa$

$$X \in E_\tau \text{ iff } \langle j''\kappa^+, \tau\rangle \in j(X)\ .$$

Let $N_\tau = Ult(V, E_\tau)$, $N = Ult(V, \langle E_\tau \mid \tau < \kappa^{+\omega+3}\rangle)$ and

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[d, "i_\tau"'] \arrow[drr, "i"] & & M \\
N_\tau \arrow[rr, "k_\tau"'] & & N \arrow[u, "k"']
\end{tikzcd}$$

be the corresponding diagram with embeddings defined in the usual way.

Lemma 4.1 *$i_\tau{}''\kappa^+ \in N_\tau, {}^{\kappa^+}N_\tau \subseteq N_\tau$, $i''\kappa^+ \in N$, ${}^{\kappa^+}N \subseteq N$, $crit(k) = \kappa^{+3}$, $H_{\kappa^{+\omega+3}} = (H_{\kappa^{+\omega+3}})^N$.*

Proof. Just note that $i_\tau{}''\kappa^+$ is represented by the function $(P, \alpha) \mapsto P$. $\square$

The extender based Prikry forcing with such extender $\langle E_\tau \mid \tau < \kappa^{+\omega+3}\rangle$ will blow up the power of κ to $\kappa^{+\omega+3}$ but also will collapse κ^+ to κ changing its cofinality to ω, due to the $\mathcal{P}_\kappa(\kappa^+)$ – supercompact ingredient of the extender.

Here we will use a version of Magidor extended based forcing defined in previous sections, with only change to $\mathcal{P}_\kappa(\kappa^+)$ – extenders. Thus we assume that

$$\vec{E} = \langle E(\alpha, \beta) \mid \alpha \leq \kappa, \alpha \in \mathrm{dom}\vec{E}\ ,\ \beta < \omega_1\rangle$$

is a coherent sequence satisfying condition (a) – (c) of $\vec{E}$ of Section 1. Only in (a) we require here that $E(\alpha, \beta)$ is a $(\mathcal{P}_\alpha(\alpha^+), \alpha^{+\omega+3})$ extender, i.e one of the type considered above. Also, $E(\alpha, \beta)(\tau)$ will be now the set

$$\{X \subseteq \mathcal{P}_\alpha(\alpha^+) \times \alpha \mid (j_{E(\alpha,\beta)}{}''(\alpha^+), \tau) \in j_{E(\alpha,\beta)}(X)\}\ .$$

The rest of the construction is without changes. The supercompact part of the forcing will change cofinality of each $(\kappa_i^+)^V (i < \omega_1)$ to ω by adding to it a cofinal sequence of order type i.

5 Concluding Remarks

5.1 Cofinality Above $\aleph_1$

We like to describe changes in the previous construction that will allow us to deal with cofinality above $\aleph_1$.

Let $\delta < \kappa$ be regular cardinal above $\aleph_1$. We replace ω_1 of the previous construction by δ. The main idea here is to satisfy 2.1(2.5), which states that for every $i < \delta$, $\alpha \in s$ and $k < \omega$

the set $\{j < i \mid a_{ij}(\alpha)$ is not k-good$\}$ is finite. With $\delta = \omega_1$ this was not problematic since then each i is countable and so is a countable union of finite sets. Here the relevant i's are still of countable cofinality but they may be uncountable ordinals.

The following simple observation will be crucial.

Lemma 5.1. *Let $j < \delta$, $k \leq \omega$ and suppose that some $\rho < \kappa^{+\omega+3}$ is k-good in $M_j = Ult(V, E(\kappa, j))$. Let ρ' be an ordinal realizing the same 3-type (in V) as ρ. Then ρ' is k-good in M_j.*

Proof. The set $A = \{\nu < \kappa \mid \nu$ is k–good$\}$ belongs to the ultrafilter $E(\kappa, j)(\rho)$. Since ρ' realizes the same 3-tupe as ρ does, $A \in E(\kappa, j)(\rho')$. But then, in M_j, ρ' is k–good. $\square$

So, passing to equivalent ordinals at certain level will not effect smaller levels. A typical situation in which we identify certain conditions under $\longleftrightarrow$ is as follows:
i is limit, $\alpha \in s$, $\langle k_n \mid n < \omega \rangle$ a nondecreasing converging to infinity sequence and we identify each $a_{ii_n}(\alpha)$ with k_n–equivalent to its ordinal. Then for each j, $i_{n+1} > j > i_n$ (for some $n < \omega$) $a_{ij}(\alpha)$ usually also will be identified with k_n–equivalent to its ordinal. So, if there are uncountably many j's between i_n and i_{n+1} (and this will be certainly the case once $\delta > \omega_1$) all of them drop in their types and further identifying of such j's looks problematic. The way around this problem will be to use 5.1 and consider the types of $a_{ij}(\alpha)$'s in $M_{i_{n+1}}$ instead of V. Then, once $\kappa_{i_{n+1}}$ is decided, we will have enough room to identify ordinals for j.

The following requirement describes this situation precisely, only instead of a single i we will allow finitely many such i's.

$a_{ij}(\alpha)$ is ω–good with probably the following exceptions: there are finitely many ordinals of countable cofinality $i^0, i^1, \ldots, i^{n-1}$ such that

(1) $\delta > i^0 > i^1 > \cdots > i^{n-1}$

(2) for every $j < i^0$ if i^t_m is the least element of the set $\{i^\ell_r \mid \ell < n, r < \omega\}$ above j, then we require $a_{i^0 j}(\alpha)$ to be ω-good in $Ult(V, E(\kappa, i^t_m))$, where $\langle i^\ell_r \mid r < \omega \rangle$ is the fixed ω-sequence for i^ℓ.

(3) for every $m < n$ there is nondecreasing converging to infinity sequence of natural numbers $\langle k^m_\ell \mid \ell < \omega \rangle$ so that for every $j < i^0$, if i^t is the least above j and i^t_m is the least which is at most j, then $a_{i^t,j}(\alpha)$ is k^t_m-good in $Ult(V, E(\kappa, i^t))$.

Incorporating these changes, the rest of the construction is as in the previous sections.

Finally we will have model with a stationary $S \subseteq \{i < \delta \mid cf i = \omega\}$ satisfying, $2^{\kappa_i} = \kappa_i^+$ for every $i \in S$ and $2^{\kappa_i} = \kappa_i^{++}$, for every $i \in \{j < \delta \mid cf j = \omega\} \backslash S$.

5.2 Down to $\aleph_{\omega_1}$

Combining the present construction with the techniques for collapsing cardinals of Merimovich [5] or Segal [6] it is possible to turn κ into $\aleph_{\omega_1}$. For $i < \omega_1$, we start collapses from $\kappa_i^{+\omega+5}$ and insure by this that they will depend only on the normal measure of the extender $E(\kappa, i)$. This way the equivalence relation $\leftrightarrow$ will not effect them.

5.3 Other Stationary Sets

Recall that S was a subset of a club. Outside of a club we are basically free. Only, as in 5.2, for each $i < \omega_1$ we need to start changes above $\kappa_i^{+\omega+5}$ in order to make the final thing work.

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